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Probability theory

Slow convergence in generalized central limit theorems

Convergence lente dans les théorèmes centraux limites généralisés

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A R T I C L E I N F O

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A B S T R A C T

We study the central limit theorem in the non-normal domain of attraction to symmetric α-stable laws for $0 < \alpha \le 2$. We show that for i.i.d. random variables X_i, the convergence rate in L^∞ of both the densities and distributions of $\sum_i^n X_i/(n^{1/\alpha}L(n))$ is at best logarithmic if L is a non-trivial slowly varying function. Asymptotic laws for several physical processes have been derived using convergence of $\sum_{i=1}^n X_i/\sqrt{n \log n}$ to Gaussian distributions. Our result implies that such asymptotic laws are accurate only for exponentially large n.

R É S U M É

Nous étudions le théorème central limite dans le domaine d'attraction non normal, vers des limites symétriques et α-stables, $0 < \alpha \le 2$. Nous montrons que, pour les suites X_i i.i.d., les taux de convergence en L^∞ des densités et des distributions de $\sum_i^n X_i/(n^{1/\alpha}L(n))$ sont au plus logarithmiques si L est une fonction non triviale de variation lente. Plusieurs lois physiques asymptotiques sont basées sur la convergence des suites $\sum_{i=1}^n X_i/\sqrt{n \log n}$ vers des distributions gaussiennes. Nos résultats montrent que ces lois ne sont précises que pour n d'une grandeur exponentielle.

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1. Introduction

In the kinetic theory of gases and related areas, there are a number of limit laws that involve logarithmic scaling. Our interest in the subject was initiated by Börgers et al. [2], where we studied the evolution of free molecular flow in a region bounded by parallel plates with Maxwellian reflection on the boundaries. Using probabilistic methods, we showed that in the limit of vanishing gap height, diffusion occurs on the "anomalous" time scale $1/(h|\log h|)$, where h is the properly non-dimensionalized separation of the plates. More recently, Chumley et al. [5] have obtained diffusion results, both standard and anomalous, for related kinetic problems in a very broad class of geometries. Anomalous diffusion results

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involving logarithmic scaling have also been obtained for Birkhoff sums in the stadium billiard problem [1], for diffusion in a Lorentz gas [7], for iterations of certain one-dimensional maps [8, page 88, remark 2], and in two-dimensional turbulence models [9].

In the research cited above, at the core of the results are central limit theorems for random variables of infinite variance. The limiting distributions are Gaussian, as in the classical central limit theorem, but the infinite variance of the summands necessitates scaling by $\sqrt{n \log n}$ instead of $\sqrt{n}$. The summands are independent and identically distributed, or, in the deterministic problems, have asymptotically vanishing correlations. The log term in the diffusion time scale is a consequence of the logarithmic scaling in the central limit theorem.

Here we consider, more generally, central limit theorems for random variables with infinite variance in which the limiting distribution is symmetric and α-stable, with $0 < \alpha \leq 2$ [4,15]; see [13] for an extensive bibliography on stable distributions and applications. We consider summands outside the normal domain of attraction; that is, we assume scaling not by $n^{1/\alpha}$ but by $n^{1/\alpha}L(n)$, where L is a slowly varying function such that $L(n)$ tends to 0 or to ∞ as $n \to \infty$. In the examples discussed earlier, $\alpha = 2$ and $L(n) = \sqrt{\log n}$.

A natural question is how long must one wait in order for the asymptotic result to provide a good approximation of the physical phenomenon being modeled. We will not attempt to review the considerable body of literature concerning rates of convergence *inside* the normal domain of attraction here. In this paper, we use a scaling argument to prove that convergence of the distribution and density functions in L^∞ is always slow *outside* the normal domain of attraction. We show that the rate of convergence cannot be better than inverse logarithmic. Hence any approximation based on a central limit theorem outside the normal domain of attraction of a stable distribution will be accurate only for an exponentially large number of summands.

The rates of convergence in central limit theorems depend critically on the shapes of the distributions and the corresponding scalings of the partial sums. The main results in this paper provide general best-case bounds on the rates of convergence. There are several results providing *exact* rates of convergence for special families of distributions and scalings outside the normal domain of attraction; see for instance Juozulynas and Paulauskas [10], Kuske and Keller [11], and Nándori [12]. We conclude this paper with an example of a family of random variables which have an $O(\log(\log n))/\log n$ rate of convergence under a natural scaling and the improved $O(1/\log n)$ rate of convergence under an alternative scaling.

We note that Cristadoro et al. [6] gave an analysis, supported by numerical simulations, of anomalous diffusion in the Lorentz gas, and found slow convergence to the limiting distribution.

2. Results

The two theorems in this paper demonstrate that rates of convergence outside the normal domain of attraction, with scaling $n^{1/\alpha}L(n)$, are at best of order $1 - L(n)/L(2n)$. The proposition following the theorems makes clear why we call this convergence slow and "at best logarithmic."

The first theorem provides a best-case bound on the rate of convergence of the error as measured by the Kolmogorov metric [14], d, which is defined as follows. Let X, Y be random variables, with corresponding distribution functions F_X, F_Y. The Kolmogorov distance between X and Y is

$$d(X, Y) = \|F_X - F_Y\|_\infty.$$

The proof of Theorem 1 relies on subadditivity properties of the Kolmogorov distance [3, Section 4.2]. We recall these properties in the following two lemmas.

Lemma 1. *Let X, Y, and Z be random variables, and assume that Z is independent of X and of Y. Then*

$$d(X + Z, Y + Z) \leq d(X, Y).$$

Proof. The distribution function of $X + Z$ is given by the Lebesgue–Stieltjes integral

$$F_{X+Z}(s) = \int_{-\infty}^{\infty} F_X(s - z)\, \mathrm{d}F_Z(z),$$

and similarly

$$F_{Y+Z}(s) = \int_{-\infty}^{\infty} F_Y(s - z)\, \mathrm{d}F_Z(z).$$

Hence,

$$\begin{aligned}
d(X+Z, Y+Z) &= \sup_{s\in\mathbb{R}} |F_{X+Z}(s) - F_{Y+Z}(s)| \\
&= \sup_{s\in\mathbb{R}} \left| \int_{-\infty}^{\infty} F_X(s-z)\,\mathrm{d}F_Z(z) - \int_{-\infty}^{\infty} F_Y(s-z)\,\mathrm{d}F_Z(z) \right| \\
&\le \sup_{s\in\mathbb{R}} \int_{-\infty}^{\infty} |F_X(s-z) - F_Y(s-z)|\,\mathrm{d}F_Z(z) \\
&\le \int_{-\infty}^{\infty} \sup_{s\in\mathbb{R}} |F_X(s-z) - F_Y(s-z)|\,\mathrm{d}F_Z(z) \\
&= \int_{-\infty}^{\infty} d(X,Y)\,\mathrm{d}F_Z(z) = d(X,Y). \quad \square
\end{aligned}$$

Lemma 2. *Let (X_1, Y_1) and (X_2, Y_2) be independent pairs of random variables. Then*

$$d(X_1 + X_2, Y_1 + Y_2) \le d(X_1, Y_1) + d(X_2, Y_2).$$

Proof. By the triangle inequality and Lemma 1, we have

$$\begin{aligned}
d(X_1 + X_2, Y_1 + Y_2) &\le d(X_1 + X_2, Y_1 + X_2) + d(Y_1 + X_2, Y_1 + Y_2) \\
&\le d(X_1, Y_1) + d(X_2, Y_2). \quad \square
\end{aligned}$$

Before stating and proving our theorems, we introduce some notation that we will use in the rest of this paper. When G is the distribution function of a random variable X and $a \in \mathbb{R}$, we denote by $T_a G$ the distribution function of the scaled random variable aX, so

$$T_a G(x) = G(x/a).$$

Similarly, if q is the density function of X, we denote by $\tau_a q$ the density function of aX, so

$$\tau_a q(x) = (1/a)\, q(x/a).$$

Theorem 1. *Let $X_1, X_2, \ldots$ be a sequence of independent, identically distributed, random variables. Denote by S_n the sum $S_n = \sum_{i=1}^n X_i$. Assume that for some α, $0 < \alpha \le 2$,*

$$\frac{S_n}{n^{1/\alpha} L(n)} \Longrightarrow Z,$$

where Z is symmetric and α-stable with distribution function F, L is a slowly varying function, and the symbol $\Rightarrow$ denotes convergence in distribution. Denote by F_n the distribution function

$$F_n(x) = P\left(\frac{S_n}{n^{1/\alpha} L(n)} \le x \right).$$

Then for all $C < 1$ and $z \in \mathbb{R}$,

$$2\|F_n - F\|_\infty + \|F_{2n} - F\|_\infty \ge C \left| z F'(z) \right| \left| 1 - \frac{L(n)}{L(2n)} \right|.$$

Hence, provided $L(n) \ne L(2n)$ for all sufficiently large n,

$$\limsup_{n\to\infty} \frac{\|F_n - F\|_\infty}{|1 - L(n)/L(2n)|} > 0 \quad or \quad \limsup_{n\to\infty} \frac{\|F_{2n} - F\|_\infty}{|1 - L(n)/L(2n)|} > 0.$$

Proof. Define $\tilde{S}_n = \sum_{i=n+1}^{2n} X_i$ and let Z_1, Z_2 be independent copies of Z. From Lemma 2 and the α-stability and symmetry of Z, we have that

$$\begin{aligned}
\left\| T_{L(2n)/L(n)} F_{2n} - F \right\|_\infty &= d\left(\frac{L(2n)}{L(n)} \frac{S_{2n}}{(2n)^{1/\alpha} L(2n)}, Z \right) \\
&= d\left(\frac{S_n}{(2n)^{1/\alpha} L(n)} + \frac{\tilde{S}_n}{(2n)^{1/\alpha} L(n)}, \frac{Z_1}{2^{1/\alpha}} + \frac{Z_2}{2^{1/\alpha}} \right) \\
&\le 2d\left(\frac{S_n}{(2n)^{1/\alpha} L(n)}, \frac{Z}{2^{1/\alpha}} \right) \\
&= 2d\left(\frac{S_n}{n^{1/\alpha} L(n)}, Z \right) \\
&= 2 \left\| F_n - F \right\|_\infty . \qquad (1)
\end{aligned}$$

On the other hand,

$$\| T_{L(2n)/L(n)} F_{2n} - T_{L(2n)/L(n)} F \|_\infty = \| F_{2n} - F \|_\infty . \qquad (2)$$

Let $C < 1$ and $z \in \mathbb{R}$. Using (1) and (2), then the triangle inequality and the smoothness of F,

$$\begin{aligned}
2\|F_n - F\|_\infty + \|F_{2n} - F\|_\infty &\ge \left\| T_{L(2n)/L(n)} F_{2n} - F \right\|_\infty + \| T_{L(2n)/L(n)} F_{2n} - T_{L(2n)/L(n)} F \|_\infty \\
&\ge \left\| T_{L(2n)/L(n)} F - F \right\|_\infty \\
&= \sup_{x \in \mathbb{R}} \left| F\left(\frac{L(n)}{L(2n)} x \right) - F(x) \right| \\
&\ge \left| F\left(\frac{L(n)}{L(2n)} z \right) - F(z) \right| \\
&= \left| F'(z) \left(\frac{L(n)}{L(2n)} - 1 \right) z + O\left(\frac{L(n)}{L(2n)} - 1 \right)^2 \right| \\
&\ge C |zF'(z)| \left| 1 - \frac{L(n)}{L(2n)} \right|
\end{aligned}$$

for all sufficiently large n, as claimed. □

A similar argument shows slow convergence of the density functions.

Theorem 2. *Let $X_1, X_2, \ldots$ be a sequence of independent random variables with a common density. Denote by S_n the sum $S_n = \sum_{i=1}^n X_i$. Denote by ρ_n the density of $S_n/(n^{1/\alpha} L(n))$, where $0 < \alpha \le 2$ and L is slowly varying. Assume that there is a symmetric and α-stable random variable Z with density ρ such that $\|\rho_n - \rho\|_\infty \to 0$ as $n \to \infty$. Then for all $C < 1$ and $z \in \mathbb{R}$,*

$$2^{(\alpha+1)/\alpha} \|\rho_n - \rho\|_\infty + \|\rho_{2n} - \rho\|_\infty \ge C \left| z\rho'(z) + \rho(z) \right| \left| 1 - \frac{L(n)}{L(2n)} \right| \qquad (3)$$

for all sufficiently large n. Hence, provided $L(n) \ne L(2n)$ for all sufficiently large n,

$$\limsup_{n\to\infty} \frac{\|\rho_n - \rho\|_\infty}{|1 - L(n)/L(2n)|} > 0 \quad \text{or} \quad \limsup_{n\to\infty} \frac{\|\rho_{2n} - \rho\|_\infty}{|1 - L(n)/L(2n)|} > 0. \qquad (4)$$

Proof. From the α-stability of Z, we have that $\rho = \tau_{2^{-1/\alpha}} \rho * \tau_{2^{-1/\alpha}} \rho$. As earlier, we write $\tilde{S}_n = \sum_{i=n+1}^{2n} X_i$. Since $\tau_{L(2n)/L(n)} \rho_{2n}$ is the density of

$$\frac{L(2n)}{L(n)} \frac{S_{2n}}{(2n)^{1/\alpha} L(2n)} = 2^{-1/\alpha} \frac{S_n}{n^{1/\alpha} L(n)} + 2^{-1/\alpha} \frac{\tilde{S}_n}{n^{1/\alpha} L(n)},$$

we have that

$$\begin{aligned}
\left\| \tau_{L(2n)/L(n)} \rho_{2n} - \rho \right\|_\infty &= \left\| \tau_{2^{-1/\alpha}} \rho_n * \tau_{2^{-1/\alpha}} \rho_n - \tau_{2^{-1/\alpha}} \rho * \tau_{2^{-1/\alpha}} \rho \right\|_\infty \\
&= 2^{1/\alpha} \|\rho_n * \rho_n - \rho * \rho\|_\infty
\end{aligned}$$

$$\begin{aligned}
&\le 2^{1/\alpha}\left(\|\rho_n*\rho_n-\rho_n*\rho\|_\infty+\|\rho_n*\rho-\rho*\rho\|_\infty\right)\\
&\le 2^{1/\alpha}\left(\|\rho_n-\rho\|_\infty+\|\rho_n-\rho\|_\infty\right)\\
&=2^{(\alpha+1)/\alpha}\,\|\rho_n-\rho\|_\infty\,, \qquad (5)
\end{aligned}$$

since $\int\rho=\int\rho_n=1$. On the other hand,

$$\left\|\tau_{L(2n)/L(n)}\rho_{2n}-\tau_{L(2n)/L(n)}\rho\right\|_\infty=\left|\frac{L(n)}{L(2n)}\right|\|\rho_{2n}-\rho\|_\infty\,. \qquad (6)$$

Let $z\in\mathbb{R}$. Using (5) and (6), then the triangle inequality, and then the smoothness of ρ, we have

$$\begin{aligned}
&2^{(\alpha+1)/\alpha}\,\|\rho_n-\rho\|_\infty+\left|\frac{L(n)}{L(2n)}\right|\|\rho_{2n}-\rho\|_\infty \qquad (7)\\
&\ge\left\|\tau_{L(2n)/L(n)}\rho_{2n}-\rho\right\|_\infty+\left\|\tau_{L(2n)/L(n)}\rho_{2n}-\tau_{L(2n)/L(n)}\rho\right\|_\infty\\
&\ge\left\|\tau_{L(2n)/L(n)}\rho-\rho\right\|_\infty\\
&=\sup_{x\in\mathbb{R}}\left|\frac{L(n)}{L(2n)}\rho\left(\frac{L(n)}{L(2n)}x\right)-\rho(x)\right|\\
&\ge\left|\frac{L(n)}{L(2n)}\rho\left(\frac{L(n)}{L(2n)}z\right)-\rho(z)\right|\\
&=\left|\frac{L(n)}{L(2n)}\rho\left(z+\left(\frac{L(n)}{L(2n)}-1\right)z\right)-\rho(z)\right|\\
&=\left|\left(\frac{L(n)}{L(2n)}-1\right)\rho(z)+\frac{L(n)}{L(2n)}\rho'(z)\left(\frac{L(n)}{L(2n)}-1\right)z+O\left(\frac{L(n)}{L(2n)}-1\right)^2\right|\\
&=\left|z\rho'(z)+\rho(z)\right|\left|1-\frac{L(n)}{L(2n)}\right|+O\left(1-\frac{L(n)}{L(2n)}\right)^2.
\end{aligned}$$

This implies the estimate (3); note that the factor of $|L(n)/L(2n)|$ appearing in (7) becomes irrelevant in the limit as $n\to 1$, as its limit equals 1. This in turn implies eq. (4) because $z\rho'(z)+\rho(z)$ is not identically zero, as ρ is not a constant multiple of $1/z$. □

As discussed earlier, scaling by $\sqrt{n\log n}$ is frequently encountered in the literature. As an example of an application of the above theorems, we state the bounds on rates of convergence for a generalization of this scaling.

Corollary. *Let $X_1, X_2, \dots$ be a sequence of independent, identically distributed, random variables. Denote by S_n the sum $S_n=\sum_{i=1}^n X_i$, and by F_n and ρ_n the distribution and density, respectively, of $S_n/\left(n^{1/\alpha}(\log n)^r\right)$, for some α with $0<\alpha\le 2$ and $r>0$. Let F and ρ be symmetric, α-stable distribution and density functions. Then*

$$\limsup_{n\to\infty}\log n\|F_n-F\|_\infty>0$$

and

$$\limsup_{n\to\infty}\log n\|\rho_n-\rho\|_\infty>0.$$

Proof. This follows immediately from the two theorems above, since

$$\lim_{n\to\infty}\log n\left(1-\left(\frac{\log n}{\log(2n)}\right)^r\right)=r\log 2. \quad □$$

The following proposition makes clear why we characterize the rates of convergence in the two theorems as being "at best logarithmic."

Proposition. *Let L be a slowly varying function such that $L(x)\to\infty$ or $L(x)\to 0$ as $x\to\infty$. Then for any $\varepsilon>0$,*

$$\limsup_{n\to\infty}(\log n)^{1+\varepsilon}\left|1-\frac{L(n)}{L(2n)}\right|=\infty. \qquad (8)$$

Proof. Assume first that $L(x) \to \infty$ as $x \to \infty$. Then for all integers $n \geq 1$, $K \geq 1$,

$$\sum_{k=0}^{K-1} \log \frac{L\left(2^k n\right)}{L\left(2^{k+1} n\right)} = \log L\,(n) - \log L\left(2^K n\right). \tag{9}$$

Since, for fixed n, $\log L\left(2^K n\right) \to \infty$ as $K \to \infty$, the sum on the left-hand side of (9) goes to negative infinity. Hence, the series tends to infinity absolutely, and we have, for all $n \geq 1$,

$$\sum_{k=0}^{\infty} \left| \log \frac{L\left(2^k n\right)}{L\left(2^{k+1} n\right)} \right| = \infty.$$

Since L is slowly varying, $L\left(2^k n\right)/L\left(2^{k+1} n\right) \to 1$ as $k \to \infty$. It follows that

$$\sum_{k=0}^{\infty} \left| 1 - \frac{L\left(2^k n\right)}{L\left(2^{k+1} n\right)} \right| = \infty. \tag{10}$$

Now assume that eq. (8) is false. Then for some $\varepsilon > 0$, $C > 0$, and all $n \geq 2$,

$$\left| 1 - \frac{L(n)}{L(2n)} \right| \leq \frac{C}{(\log n)^{1+\varepsilon}}.$$

Then

$$\sum_{k=0}^{\infty} \left| 1 - \frac{L\left(2^k n\right)}{L\left(2^{k+1} n\right)} \right| \leq \sum_{k=0}^{\infty} \frac{C}{\left(\log\left(2^k n\right)\right)^{1+\varepsilon}} = \sum_{k=0}^{\infty} \frac{C}{(k \log 2 + \log n)^{1+\varepsilon}} < \infty,$$

which contradicts eq. (10) and thereby establishes eq. (8) for $L(x) \to \infty$.

Now assume that $L(x) \to 0$ as $x \to \infty$. Set $\mathcal{L}(x) = 1/L(x)$. Then $\mathcal{L}$ is slowly varying and $\mathcal{L}(x) \to \infty$ as $x \to \infty$. Hence,

$$\begin{aligned}
\infty &= \limsup_{n\to\infty} (\log n)^{1+\varepsilon} \left| 1 - \frac{\mathcal{L}(n)}{\mathcal{L}(2n)} \right| \\
&= \limsup_{n\to\infty} (\log n)^{1+\varepsilon} \left| 1 - \frac{\mathcal{L}(2n)}{\mathcal{L}(n)} \right| \\
&= \limsup_{n\to\infty} (\log n)^{1+\varepsilon} \left| 1 - \frac{L(n)}{L(2n)} \right|.
\end{aligned}$$

The second equality follows from

$$\lim_{n\to\infty} \frac{\mathcal{L}(2n)/\mathcal{L}(n) - 1}{\mathcal{L}(n)/\mathcal{L}(2n) - 1} = \lim_{n\to\infty} \left(-\frac{\mathcal{L}(2n)}{\mathcal{L}(n)} \right) = -1. \quad \square$$

Note that the hypotheses of the proposition do not imply (8) for $\varepsilon = 0$. For example, in the canonical case discussed above, $L(n) = \sqrt{\log n}$, we find

$$\lim_{n\to\infty} \log n \left| 1 - \frac{L(n)}{L(2n)} \right| = \lim_{n\to\infty} \log n \left(1 - \sqrt{\frac{\log(n)}{\log(2n)}} \right) = \frac{\log 2}{2}.$$

As discussed earlier, our theorems are only best-case estimates. The precise rates of convergence depend on the slowly varying functions in the scalings. As an example, fix a constant $A > 0$ and consider a sequence $X_1, X_2, \ldots$ of identically distributed random variables which have a common smooth, symmetric density function, which for large $|x|$ is equal to $A/(2|x|^3)$. Denote by q_n the density of

$$\frac{\sum_{i=1}^n X_i}{h(n)},$$

where the function h is defined by $h^2 = n \log h$ and $\lim_{n\to\infty} h(n) = \infty$, and by ρ the Gaussian density with mean 0 and variance A. Kuske and Keller [11] proved that $\|q_n - \rho\|_\infty = O\,(1/\log n)$. However, consider the natural scaling $\sqrt{(n \log n)/2}$ and denote by ρ_n the density of

$$\frac{\sum_{i=1}^n X_i}{\sqrt{(n \log n)/2}}.$$

Then it can be shown that the convergence of the scaled partial sums deteriorates slightly, namely we have

$$c\frac{\log\log n}{\log n} \le \|\rho_n - \rho\|_\infty \le C\frac{\log\log n}{\log n}$$

with $0 < c < C < \infty$. This follows from the fact that $h(n) = (1 + c_n)\sqrt{(n\log n)/2}$, with c_n asymptotic to $\log\log n/(2\log n)$. We note that Nándori [12, Theorem 1] proved an analogous result for a random walk on a lattice, with the same rate of convergence.

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