C. R. Acad. Sci. Paris, Ser. I 356 (2018) 886–890

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Partial differential equations/Game theory

Mean-field games with a major player

Jeux à champ moyen avec agent dominant

Jean-Michel Lasry [a], Pierre-Louis Lions [b]

[a] *Université Paris-Dauphine–PSL, place du Maréchal-de Lattre-de-Tassigny, 75775 Paris cedex 16, France*
[b] *Collège de France–PSL, 3 rue d'Ulm, 75005 Paris, France*

ARTICLE INFO

Article history:
Received 6 June 2018
Accepted 7 June 2018
Available online 22 June 2018

Presented by Pierre-Louis Lions

ABSTRACT

We introduce and study mathematically a new class of mean-field-game systems of equations. This class of equations allows us to model situations involving one major player (or agent) and a "large" group of "small" players.

RÉSUMÉ

Nous introduisons et étudions mathématiquement une classe nouvelle de jeux à champ moyen. Les systèmes d'équations que nous présentons permettent de modéliser les situations faisant intervenir un joueur dominant et un « grand » groupe de « petits » joueurs.

Version française abrégée

Dans cette note, nous introduisons et étudions une nouvelle classe de modèles mathématiques pour des situations où interviennent un « grand » groupe de « petits » joueurs indistinguables et un joueur dominant. En d'autres termes, nous introduisons et étudions une classe nouvelle de modèles de jeux à champ moyen (MFG en abrégé) avec agent dominant. Nous renvoyons le lecteur à la version anglaise pour une brève introduction aux MFG et pour quelques références de travaux antérieurs.

Les situations que nous considérons ici à savoir les situations avec un agent dominant sont très fréquentes dans les applications en particulier à l'Économie. Ces modèles sont délicats à manipuler et à étudier puisque, entre autres difficultés, tout aléa s'appliquant au joueur dominant induit un aléa commun à tous les joueurs, et la partie purement MFG des systèmes est donc de type « bruit commun ».

Dans les cas où les petits joueurs sont décrits par des variables prenant un nombre fini de valeurs notées $i \in \{1, \dots, k\}$ ($k \geqslant 1$), les systèmes sont de la forme suivante

E-mail addresses: 2007lasry@gmail.com (J.-M. Lasry), pierre-louis.lions@college-de-france.fr (P.-L. Lions).

https://doi.org/10.1016/j.crma.2018.06.001
1631-073X/

$$\begin{cases} \frac{\partial U}{\partial t} + (A(X,y,U,\alpha^*).\nabla_X)U \ + \alpha^*.\nabla_y U - \nu\Delta_y U - \varepsilon\Delta_X U + \\ \qquad + \lambda U = B(X,y,U,\alpha^*) \end{cases} \tag{1}$$

$$\begin{cases} \frac{\partial \varphi}{\partial t} + F(X,y,U,\nabla_y\varphi,\alpha^*) \ + A(X,y,U,\alpha^*)\cdot\nabla_X\varphi + \\ \qquad -\nu\Delta_y\varphi - \varepsilon\Delta_X\varphi + \mu\varphi = 0 \end{cases} \tag{2}$$

$$\alpha^* = \frac{\partial F}{\partial p}(X,y,U,\nabla_y\varphi,\alpha^*) \tag{3}$$

pour $X \in \mathbb{R}^k$, $y \in \mathbb{R}^d$, $t \geqslant 0$. Les notations et les hypothèses sur les coefficients et les données (H, F, A, B) sont détaillées dans la version anglaise.

Le système (1)–(3) est complété par des conditions initiales sur U et φ, à savoir

$$U|_{t=0} = U_0(X,y), \varphi|_{t=0} = \varphi_0(X,y) \ \text{sur} \ \mathbb{R}^k \times \mathbb{R}^d \tag{4}$$

où U_0 et φ_0 sont des fonctions « régulières » données.

Il est à noter que nous avons renversé le sens du temps dans les équations précédentes de façon à simplifier les notations.

Dans la version anglaise, nous donnons quelques exemples de résultats mathématiques concernant la solvabilité (i.e. l'existence et l'unicité de solutions régulières globales ou locales en temps) de tels systèmes, et nous indiquons brièvement l'analogue de ces systèmes dans le cas de variables d'états continues, auquel cas les équations sont alors en dimension infinie.[1]

1. Introduction

In this note, we introduce and study a new class of mathematical models for situations involving a large group of indistinguishable "small" players and one major player. In other words, we introduce and study a new class of mean-field games (MFG in short) models with a major player.

MFG models have been introduced by the authors first in the case without common noise for the players ([6], [7], [8]) and next in the general case with the introduction of the so-called master equation (see [9]), which is in general an infinite-dimensional nonlinear partial differential equation. Let us also recall that the essential structure conditions (monotonicity) and most of the existing mathematical tools have also been introduced by the authors (see [6], [7], [8] and [9]). Let us mention that the particular case of MFG models with no common noise was independently considered in [4] and that some particular cases have been discussed previously in the Economics literature (anonymous games in the discrete time case without common noise, or a heuristic description in a Macroeconomics example in [5]). Finally, since their introduction, there is now a huge scientific literature on MFG concerning the mathematical theory and also many applications to Economics, Finance, Social Sciences, Communication Networks, Engineering Sciences, Computer Sciences, etc. We refer the reader to the online courses [9], to the book [2] (and the references therein) for further information on the mathematical theory.

The issue studied and solved here is of paramount importance in many applicative contexts such as economical ones (for instance). Indeed, MFG models allow one to describe the average (in mean-field sense like in Statistical Mechanics and Physics) behavior of a large group of interacting indistinguishable players. However, in many realistic situations, there is also at least one major or dominant player, and this is precisely this type of situation we address here. Let us mention that we could consider as well a "large group of players" dominating another "large group of players", but we shall restrict ourselves to only one major player for the sake of simplicity. There have been a few attempts in this direction (see [1], [3]). However, we believe that the models and the theory we present is new, in particular because the "MFG part" of the models automatically falls in the delicate category of models with common noise as soon as the dynamics of the major player involve some "noise" (or random terms). Indeed, this noise automatically impacts all the other (small) players and becomes a "common noise" for them. As a consequence, it is not possible, in general, to write meaningful "forward–backward" systems of equations, as is the case for MFG without common noise.

At this point, we also wish to emphasize that our theory is obviously reminiscent of Stackelberg equilibria in Game Theory, an important concept for static or iterated games, which however is delicate to extend to continuous time situations such as the ones we consider. Our models rely on two simplifications, which seem to be well accepted in Economics Theory by now, namely: (i) we only consider feedbacks and (ii) the major player does not anticipate in the way the strategy of the other players is taken into account in its dynamic optimization. These simplifications imply some recursivity features, which, in turn, allow us to write (simpler) partial differential equations models.

In what follows, we first present our models in the case when the state variables for the "small" players lie in a finite set. We next give a few samples of the mathematical results that can be proved for such models. And we finally introduce the models for continuous state variables.

[1] Ce travail a été réalisé avec le support partiel de la chaire FDD (Institut Louis-Bachelier).

2. Models for discrete state variables

We consider a continuum of agents in a finite state space denoted by $\{1,\dots,k\}$ with $k \geqslant 1$. The state variable for the major agent, denoted by y, belongs to $\mathbb{R}^d$. According to MFG theory for finite state spaces, the continuum of agents is represented by $X = (X_1,\dots,X_k) \in \mathbb{R}^k$ (unnormalized histogram).

Next, given the strategy α^* of the major player which will be characterized below, the crowd of small players evolves according to a MFG system (in the finite space case) given by

$$\begin{cases} \frac{\partial U}{\partial t} + (A(X,y,U,\alpha^*)\cdot\nabla_X)U \; +\alpha^*\cdot\nabla_y U - \nu\Delta_y U - \varepsilon\Delta_X U + \\ \qquad\qquad +\lambda U = B(X,y,U,\alpha^*) \end{cases} \tag{1}$$

for $t \geqslant 0$, $X \in \mathbb{R}^k$, $y \in \mathbb{R}^d$. Note that we have taken for simplicity Brownian noises ($\nu > 0$, $\varepsilon \geqslant 0$), that the MFG nonlinearities A and B depend as usual on X and U but also on y and α^*, and that time is reversed to simplify notation. Finally, $\lambda \geqslant 0$ corresponds to the intertemporal preference rate. Precise assumptions on A and B will be made later on. Let use emphasize the fact that $\varepsilon > 0$ is a strong assumption on the noise (unless $k = 1\dots$).

The major player solves some stochastic control problem that depends on the state of the crowd (X) and the actions of the small players (U, A). Its value function solves the following equation

$$\begin{cases} \frac{\partial \varphi}{\partial t} + F(X,y,U,\nabla_y\varphi,\alpha^*) \; +A(X,y,U,\alpha^*)\cdot\nabla_X\varphi + \\ \qquad\qquad -\nu\Delta_y\varphi - \varepsilon\Delta_X\varphi + \mu\varphi = 0 \end{cases} \tag{2}$$

for $t \geqslant 0$, $X \in \mathbb{R}^k$, $y \in \mathbb{R}^d$ and F (the Hamiltonian) depends as usual on y and $\nabla_y\varphi$ but also on X and (U,α^*). This allows in particular for an Hamiltonian that depends on $(X, y, \nabla_y\varphi)$ and also on $A = A(X,y,U,\alpha^*)$. And the optimal strategy α^* solves

$$\alpha^*(X,y,t) = \frac{\partial F}{\partial p}(X,y,U(X,y,t),\nabla_y\varphi(X,y,t),\alpha^*) \tag{3}$$

where $A(X,y,U,\alpha)$, $B(X,y,U,\alpha)$, $F(X,y,U,p,\alpha)$ are smooth functions of their arguments (to simplify). Let us observe that we could consider as well time-dependent nonlinearities and more importantly nonlinearities that depend on $\alpha^* = \alpha^*(t)$ in a functional manner, while we consider here only local nonlinearities. The extension is in fact straightforward! Finally, $\mu \geqslant 0$ is the intertemporal preference rate of the major player. Again, time is reversed to simplify notation.

Finally, we prescribe some initial conditions on U and φ, namely

$$U|_{t=0} = U_0(X,y), \varphi|_{t=0} = \varphi_0(X,Y) \text{ in } \mathbb{R}^k \times \mathbb{R}^d \tag{4}$$

where U_0 and φ_0 are given, smooth (to simplify) functions over $\mathbb{R}^k \times \mathbb{R}^d$.

Before we present some examples of mathematical results for the above system of equations, we wish to discuss the equilibrium condition (3). Indeed, without some structure condition, it is quite clear that there may be no solution. Indeed, choose for instance $F = \frac{1}{2}|p|^2 + p\cdot\alpha$, then (3) becomes: $\alpha^* = \nabla_y\varphi + \alpha^*$, i.e. $\nabla_y\varphi = 0$, which is impossible as soon as φ_0 is not constant. This is why we assume in all that follows the following condition:

$$I - \frac{\partial^2 F}{\partial p\,\partial\alpha} \text{ is invertible} \tag{5}$$

This condition means, roughly speaking, that the influence of the actions of the crowd in reaction to the strategy of the major player is not strong enough to prevent the latter to determine an optimal strategy. It is thus a very natural structure condition.

Of course, such a condition implies that (3) is equivalent to

$$\alpha^* = \Psi\Big(X, y, U(X,y,t), \nabla\varphi(X,y,t)\Big) \tag{6}$$

for some smooth function on $\mathbb{R}^k \times \mathbb{R}^d \times \mathbb{R}^k \times \mathbb{R}^d$.

And we denote by

$$\tilde{F}(X,y,U,p) = F\Big(X,y,U,\Psi(X,y,U,p)\Big) \tag{7}$$

3. A few mathematical results

We first present two results with full coupling, namely (i) the existence and uniqueness of global solutions when $\varepsilon > 0$ and (ii) the existence and uniqueness of local (in time) solutions when $\varepsilon = 0$. In both cases, we did not try to present the most general results. Indeed, we present some typical results explaining the various expected behaviors. And we conclude with a brief discussion of the situation corresponding to a weak coupling between the dominating agent and the others.

When $\varepsilon > 0$, we assume (for instance) that U_0 and $\varphi_0 \in W^{2,p}_{\text{unif}}$ $(\forall p < \infty)$; that $\frac{\partial B}{\partial U}(1+|U|)^{-1}$ is bounded; that $\frac{\partial F}{\partial p}$ is bounded for U bounded; that A and B are bounded for U and α bounded, and that F is bounded for U, p, and α bounded. In addition, we assume that A and B are Lipschitz continuous in (U,α) for U and α bounded; that F is Lipschitz continuous in (U,p,α) for U, p and α bounded and that $(Id - \frac{\partial F}{\partial p})^{-1}(0)$ is Lipschitz in (U,p) for U and p bounded.

Theorem 3.1. *Under the above conditions, when $\varepsilon > 0$, there exists a unique solution (U,φ,α^*) to (1)–(4) such that $(U,\varphi) \in W^{2,1,p}_{\text{unif}}(\mathbb{R}^k \times \mathbb{R}^d \times (0,T))$ (for all $p < \infty$, $T \in (0,\infty)$) and $\alpha^* \in L^\infty \cap W^{1,0,p}_{\text{unif}}$.*

Let us remark that it is easy to check that such solutions are smooth whenever U_0, φ_0 and all the nonlinearities are smooth.

When $\varepsilon = 0$, we assume that, for some $k \geqslant 1$, DU_0, $D^2\varphi_0$, DA, DB, D^2F, and $D(I - \frac{\partial^2 F}{\partial p \partial \alpha})^{-1}(0)$ are bounded with bounded derivatives up to order k.

Theorem 3.2. *Under the preceding conditions, when $\varepsilon = 0$, there exists $T_0 \in (0,\infty]$ such that, for all $T \in (0,T_0)$, (1)–(4) admits a unique solution (U,φ,α^*) on $[0,T]$ such that DU, $D^2\varphi$, $D\alpha^*$ are bounded with bounded derivatives up to order k. In addition, if $T_0 < \infty$, then $\|DU(t)\|_{L^\infty} + \|D^2\varphi(t)\|_{L^\infty} \to +\infty$ as $t \to T_0$.*

The proof of Theorem 3.2 is straightforward, although rather tedious and technical. And we only present the key arguments needed in our proof of Theorem 3.1. First of all, in view of the assumptions made on B and $\frac{\partial F}{\partial p}$, we obtain easily a priori bounds on U and α^* in L^∞ on $(0,T)$, where T is fixed in $(0,+\infty)$, hence L^∞ bounds on A and B. Using next classical estimates on parabolic equations (and the bound on $\frac{\partial F}{\partial p}$), we deduce bounds on U and φ in $W^{2,1,p}_{\text{unif}}$ that yield a bound on α^* in $W^{1,0,p}_{\text{unif}}$. Thanks to these a priori estimates, it is possible to prove the existence of a global solution as in Theorem 3.1. The uniqueness may be checked using the following steps, where we denote by C various positive constants, and we wish to prove the uniqueness on $(0,T)$, where $T > 0$ is fixed in $(0,\infty)$. We first observe that the conditions listed above imply that we have

$$\|\alpha_1^* - \alpha_2^*\|_{L^\infty} \leqslant C\left\{\|U_1 - U_0\|_{L^\infty} + \|\nabla_y\varphi_1 - \nabla_y\varphi_2\|_{L^\infty}\right\}$$

where (U_1,φ_1), (U_2,φ_2) solve (1)–(4). Observing that U and $D\varphi$ are bounded on $\mathbb{R}^k \times \mathbb{R}^d \times (0,T)$ in view of the regularity of u and φ, we deduce easily that

$$\|\varphi_1 - \varphi_2\|_{W^{2,1,p}_{\text{unif}}} \leqslant C\left\{\|U_1 - U_2\|_{L^\infty} + \|\alpha_1^* - \alpha_2^*\|_{L^\infty}\right\}$$

$$\|U_1 - U_2\|_{W^{2,1,p}_{\text{unif}}} \leqslant C\left\{\|U_1 - U_2\|_{L^\infty} + \|\nabla_y\varphi_1 - \nabla_y\varphi_2\|_{L^\infty} + \|\alpha_1^* - \alpha_2^*\|_{L^\infty}\right\}$$

hence $\|(\varphi_1,U_1) - (\varphi_2,U_2)\|_{W^{2,1,p}_{\text{unif}}} \leqslant C\|D(\varphi_1,U_1) - D(\varphi_2,U_2)\|_{L^\infty}$ where the norms are defined on any time interval $(0,T')$ with $0 < T' \leqslant T$. Next, by Sobolev embeddings, for p large enough fixed, there exists $q < p$ such that for all Ψ we have

$$\|D\Psi\|_{L^\infty} \leqslant C\left\{\|\Psi\|_{L^q(0,T';\,W^{2,p}_{\text{unif}})} + \|\frac{\partial \Psi}{\partial t}\|_{L^q(0,T';\,L^p_{\text{unif}})}\right\}$$

(inequality valid for $p > \frac{D+1}{2}$, $\frac{2p}{2p-D} < q < p$ with $D = k+d\ldots$). Using Holder's inequality, we conclude that $U_1 \equiv U_2$, $\varphi_1 \equiv \varphi_2$ on $[0,T']$ as long as $C(T')^\alpha < 1$ with $\alpha = \frac{1}{q} - \frac{1}{p}$. We conclude by reiterating this argument on $(T',2T')$, $(2T',3T')\ldots$. □

The system of equations (1)–(3) becomes much simpler if we assume that F is independent of X and U, and that φ_0 is independent of X. In that case, φ is independent of X, and (2) reduces to a classical Hamilton–Jacobi–Bellman equation that yields, under general conditions on H, a unique smooth solution $\varphi(y,t)$. Note in that case that $\alpha = \alpha(y,t)$ is determined by (3). And (1) becomes a variant of the standard (MFG) equations (with an extra variable namely y). It is then possible to adapt the existing theory for such systems (see [9]) and to obtain global solutions (even when $\varepsilon = 0$) using monotonicity conditions like $U_0(X,y)$ monotone in X for all y, (B,A) monotone in (X,U) for all y, α. We may then build global solutions that are "monotone" in X and smooth in y.

4. The master equation

In the case when the “small” players are described through continuous variables $x \in \mathbb{R}^k$ ($k \geqslant 1$), the analogue of the system (1)–(3) becomes an extension of the (MFG) master equation (see [9]) involving two unknown functions $U(x, y, m, t)$, $\Phi(y, m, t)$, where $x \in \mathbb{R}^k$, $y \in \mathbb{R}^d$ and $m \in P(\mathbb{R}^k \times \mathbb{R}^d)$ (the set of probability measures on $\mathbb{R}^k \times \mathbb{R}^d$).

The analogue of the system (1)–(3) is given by

$$\begin{cases} \frac{\partial U}{\partial t} + H(x, m, \nabla_x U, \alpha^*) - (\gamma + \beta)\Delta_x U - \nu \Delta_y U + \alpha^* . \nabla_y U + \\ +\lambda U - \beta \, \frac{\partial^2 U}{\partial m^2}(\nabla_x m, \nabla_x m) + \left\langle \frac{\partial U}{\partial m}, -(\gamma + \beta)\Delta_x m - \operatorname{div}\left(\frac{\partial H}{\partial p} m\right)\right\rangle + \\ +2\beta \left\langle \frac{\partial}{\partial m} \nabla_x U, \nabla_x m \right\rangle = 0 \end{cases} \tag{8}$$

$$\begin{cases} \frac{\partial \Phi}{\partial t} + F(m, y, U[m], \nabla_y \Phi, \alpha^*) - \nu \Delta_y \Phi + \mu \Phi + \\ \left\langle \frac{\partial \Phi}{\partial m}, -(\gamma + \beta)\Delta_x m - \operatorname{div}\left(\frac{\partial H}{\partial p} m\right)\right\rangle - \frac{\beta}{2} \frac{\partial^2 \Phi}{\partial m^2}(\nabla_x m, \nabla_x m) = 0 \end{cases} \tag{9}$$

$$\alpha^* = \frac{\partial F}{\partial p}(m, y, U[m], \nabla_y \varphi, \alpha^*) \tag{10}$$

with the following initial conditions

$$\bar{\Phi}|_{t=0} = \Phi_0(y, m) \, ; \; U|_{t=0} = U_0(y, m) \tag{11}$$

Many variants and extensions are possible, which we shall not detail here. In the above equations, all the data H, F, Φ_0, U_0 are assumed to be smooth; $\alpha, \beta, \gamma, \lambda, \nu$ are non-negative parameters, and $U[t, m]$ is the function of $x \in \mathbb{R}^k$ that depends on (y, t), defined by $U[m](x) = U(x, y, m, t)$.

As in [9], it is possible to reformulate the above system of equations using the so-called Hilbertian approach, which consists in writing the above system of equations in a space of random variables ($U = U(X, y, t)$, $\Phi = \Phi(X, y, t)$ whenever the law of X is m). In particular, this formulation allows us to prove, under appropriate conditions, an analogue of the local-in-time Theorem 3.2.

Acknowledgement

This work has been partially supported by the Chair FDD (Institut Louis-Bachelier).

References

[1] A. Bensoussan, M.H.M. Chau, S.C.P. Yan, Mean field games with a dominating player, arXiv:1404.4148v2, 2014.
[2] P. Cardaliaguet, Notes on mean field games, PDF file at http://www.ceremade.dauphine.fr/~cardaliaguet, 2013.
[3] R. Carmona, F. Delarue, Probabilistic Theory of Mean Field Games with Applications I–II, Springer, 2018.
[4] M. Huang, P.P. Malhamé, P.E. Gaines, Large population stochastic dynamics games, Commun. Inf. Syst. 6 (2006) 221–252.
[5] P. Krusell, A.D. Sinith, Income and wealth heterogeneity in the macroeconomy, J. Polit. Econ. 106 (1998) 867–896.
[6] J-M. Lasry, P-L. Lions, Jeux à champ moyen I. Le cas stationnaire, C. R. Acad. Sci. Paris, Ser. I 343 (2006) 619–625.
[7] J-M. Lasry, P-L. Lions, Jeux à champ moyen II. Horizon fini et contrôle optimal, C. R. Acad. Sci. Paris, Ser. I 343 (2006) 679–684.
[8] J-M. Lasry, P-L. Lions, Mean field games, Jpn. J. Math. 2 (2007) 229–260.
[9] P-L. Lions, Videos and abstracts at http://www.college-de-france.fr/site/pierre-louis-lions, 2007–2012.