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Optimal control

Kalman's criterion on the uniqueness of continuation for the nilpotent system of wave equations

Critère de Kalman sur l'unicité de la continuation pour le système nilpotent d'équations des ondes

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A B S T R A C T

In this Note, we consider a system of wave equations coupled by a nilpotent matrix with homogeneous Dirichlet boundary condition. We establish the uniqueness of the solution when partial Neumann observation satisfies Kalman's rank condition.

R É S U M É

Dans cette Note, nous considérons un système d'équations des ondes couplées par une matrice nilpotente avec la condition aux limites homogène de Dirichlet. Nous établissons l'unicité de la solution si l'observation partielle de Neumann satisfait le critère de Kalman.

Version française abrégée

Soit $\Omega \subset \mathbb{R}^n$ un ouvert borné de frontière régulière $\Gamma = \Gamma_1 \cup \Gamma_0$. Soit A une matrice d'ordre N et D une matrice de rang plein d'ordre $N \times M$ avec $M \leqslant N$.

Soit $\Phi = (\phi^{(1)}, \cdots, \phi^{(N)})^{\intercal}$. Considérons le système d'équations des ondes suivant :

$$\begin{cases} \Phi'' - \Delta\Phi + A^{\intercal}\Phi = 0 & \text{dans} \quad \Omega, \\ \Phi = 0 & \text{sur} \quad \Gamma. \end{cases} \tag{0.1}$$

Le système (0.1) est D-observable sur un intervalle fini $[0, T]$ si

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$$D^\intercal \partial_\nu \Phi \equiv 0 \quad \text{sur} \quad [0,T]\times\Gamma_1 \text{ implique } \Phi\equiv 0. \tag{0.2}$$

Dans [6], nous avons montré que le critère de Kalman

$$\operatorname{rank}(D, AD, \cdots, A^{N-1}D) = N \tag{0.3}$$

est nécessaire pour la D-observabilité du système (0.1). Comme le rang M de la matrice D peut être très faible, et même largement inférieur au nombre des équations N, la D-observation (0.2), étant partielle, n'implique pas que $\partial_\nu\Phi \equiv 0$ sur $[0,T]\times\Gamma_1$. Ainsi, le théorème classique de Holmgren ne s'applique pas pour conclure à l'unicité de la solution du système (0.1) avec l'observation (0.2).

Seul, le critère de Kalman n'est pas suffisant pour l'unicié de la solution du système (0.1) avec l'observation (0.2). Ceci dépend de nombreux facteurs, par exemple, le temps d'observation, la condition spectrale de la matrice A, l'observabilité de l'opératur $-\Delta$ et la condition géométrique sur Ω, etc.

La D-observabilité est un sujet important. Cependant, il y a très peu de résultats connus dans la litérature. Dans [6], nous avons montré que le critère de Kalman est suffisant pour des systèmes 2×2 et des systèmes en dimension un d'espace.

Lorsque A est un bloc de Jordan avec des zéros sur la diagonale, la D-observabilité du systéme appelé « en cascade » a été établie par Alabau dans [1] pour une matrice d'observation D spécifique, puis généralisée par Li and Rao dans [6] pour toutes les matrices satisfaisant le critère de Kalman (1.6). L'objectif de cette Note est de généraliser ce résultat à une classe de systèmes nilpotents. En voici le résultat principal.

Theorem 0.1. *Supposons que $\Omega\subset\mathbb{R}^n$ satisfait la condition géométrique de multiplicateurs usuelle. Supposons de plus que la matrice du couplage A est nilpiotente et que A et D satisfont le critère de Kalman (0.3). Alors, le système (0.1) est D-observable pourvu que $T>0$ soit assez grand.*

1. Introduction

Let $\Omega\subset\mathbb{R}^n$ be a bounded domain with a smooth boundary $\Gamma=\Gamma_1\cup\Gamma_0$ such that $\overline{\Gamma}_1\cap\overline{\Gamma}_0=\emptyset$. Assume that $\Omega\subset\mathbb{R}^n$ satisfies the usual multiplier geometrical condition (see [7]).

Let $U=(u^{(1)},\cdots,u^{(N)})^\intercal$. Consider the following coupled system of wave equations:

$$\begin{cases} U''-\Delta U+AU=0 & \text{in } (0,+\infty)\times\Omega,\\ U=0 & \text{on } (0,+\infty)\times\Gamma_0,\\ U=DH & \text{in } (0,+\infty)\times\Gamma_1\end{cases} \tag{1.1}$$

with the initial condition

$$t=0: \quad U=\widehat{U}_0,\ U'=\widehat{U}_1 \quad \text{in } \Omega, \tag{1.2}$$

where A is the coupling matrix of order N, and D is a full column-rank matrix of order $N\times M$ with $M\leqslant N$, called the boundary control matrix, both A and D being with constant elements.

It was shown in [4] that the system (1.1) is exactly controllable if and only if the boundary control matrix D is invertible, namely, if $M=N$. In order to reduce the number of boundary controls, we consider the approximate boundary controllability. Accordingly, consider the following adjoint system for $\Phi=(\phi^{(1)},\cdots,\phi^{(N)})^\intercal$:

$$\begin{cases} \Phi''-\Delta\Phi+A^\intercal\Phi=0 & \text{in } (0,+\infty)\times\Omega,\\ \Phi=0 & \text{on } (0,+\infty)\times\Gamma\end{cases} \tag{1.3}$$

with the initial data

$$t=0: \quad \Phi=\Phi_0,\ \Phi'=\Phi_1 \quad \text{in } \Omega, \tag{1.4}$$

where $A^\intercal$ denotes the transpose of A.

Definition 1.1. The adjoint system (1.3) is D-observable in the space $(H_0^1(\Omega))^N\times(L^2(\Omega))^N$ on a finite interval $[0,T]$ if, for any given initial data $(\Phi_0,\Phi_1)\in(H_0^1(\Omega))^N\times(L^2(\Omega))^N$, the following partial Neumann observation:

$$D^\intercal\partial_\nu\Phi\equiv 0 \quad \text{on } [0,T]\times\Gamma_1 \tag{1.5}$$

implies $\Phi\equiv 0$, where ∂_ν denotes the outward normal derivative operator.

In [5] and [6], we established the equivalence between the approximate boundary controllability of the system (1.1) and the D-observability of the adjoint system (1.3). Moreover, the following Kalman's criterion

$$\mathrm{rank}(D, AD, \cdots, A^{N-1}D) = N \tag{1.6}$$

is necessary for the D-observability of the adjoint system (1.3).

Although Kalman's criterion (1.6) provides an elegant way to characterize the uniqueness of continuation, it is not sufficient in general even for the D-observation on the infinite time interval $[0, +\infty)$ (Theorem 3.2 in [6]). In fact, under the rank condition (1.6), we have the sharp lower bound estimate: $\mathrm{rank}(D) \geqslant \mu$, where

$$\mu = \max_{\lambda \in Sp(A^\intercal)} \dim \mathrm{Ker}(A^\intercal - \lambda I) \tag{1.7}$$

is the largest geometrical multiplicity of eigenvalues of $A^\intercal$. So, the rank M of the matrix D may be substantially smaller than the number N of equations. Thus, the D-observation (1.5) cannot guarantee the vanishing of all the components of the normal derivative

$$\partial_\nu \Phi \equiv 0 \quad \text{on} \quad [0, T] \times \Gamma_1. \tag{1.8}$$

Therefore, the classic Holmgren's uniqueness theorem ([3], [7]) cannot be applied to obtain the desired uniqueness of continuation.

The D-observability is an important subject to be studied. There are several approaches for parabolic systems (see [2] and the reference therein). However there are fewer results for the hyperbolic systems in the literature. In [6], the sufficiency of Kalman's condition was established for 2×2 systems and for one-space-dimensional systems.

The D-observability of the so-called cascade system was first established by Alabau in [1] for a specific matrix D, and then was generalized by Li and Rao in [6] for all matrices D satisfying Kalman's criterion (1.6). The aim of this Note is to generalize furthermore this result to the nilpotent system.

2. Main theorem

A matrix A of order N is called nilpotent if there exists an integer k with $1 \leqslant k \leqslant N$ such that $A^k = 0$. Then it is easy to see that A is nilpotent if and only if all the eigenvalues of A are equal are zero. Thus, in a suitable basis, a nilpotent matrix A can be written in a diagonal form of Jordan blocs:

$$B^{-1}AB = \begin{pmatrix} J_p & & \\ & J_q & \\ & & \ddots \end{pmatrix}, \tag{2.9}$$

where J_p is a Jordan bloc of order p:

$$J_p = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \cdot & \cdot & \\ & & & 0 & 1 \\ & & & & 0 \end{pmatrix}. \tag{2.10}$$

When A is a single Jordan bloc, it was shown in [1] that the observation on the last component of the adjoint variable Φ of the adjoint system (1.3) is sufficient for the corresponding D-observability. In this Note, we will generalize this result to the nilpotent system with general boundary control matrix D. The following theorem is the main result.

Theorem 2.2. *Assume that $\Omega \subset \mathbb{R}^n$ satisfies the usual multiplier geometrical condition. Assume furthermore that the coupling matrix A is nilpotent and A and D satisfy Kalman's criterion (1.6). Then the adjoint system (1.3) is D-observable, provided that $T > 0$ is large enough.*

Proposition 2.3. *Let P be an invertible matrix such that $PA = AP$. Then the adjoint problem (1.3) is D-observable if and only if it is PD-observable.*

Proof. Let $\widetilde{\Phi} = P^{-T}\Phi$. Since $PA = AP$, the new variable $\widetilde{\Phi}$ satisfies the same system as the system (1.3). On the other hand, since

$$D^\intercal \partial_\nu \Phi = (PD)^\intercal \partial_\nu \widetilde{\Phi} \qquad \text{on } \Gamma_1, \tag{2.11}$$

the D-observability on Φ is equivalent to the PD-observability on $\widetilde{\Phi}$. $\square$

Proposition 2.4. *Let P be an invertible matrix. Define*

$$\tilde{A} = PAP^{-1} \text{ and } \widetilde{D} = PD.$$

Then the matrices A and D satisfy Kalman's criterion (1.6) if and only if the matrices $\tilde{A}$ and $\widetilde{D}$ also do.

Proof. It is sufficient to note that

$$(\widetilde{D}, \tilde{A}\widetilde{D}, \cdots, \tilde{A}^{N-1}\widetilde{D}) = P(D, AD, \cdots, A^{N-1}D)$$

and that P is invertible. □

Proof of Theorem 2.2 (i) Case where A is a Jordan bloc (the cascade case):

$$A = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \cdot & \cdot & \\ & & & 0 & 1 \\ & & & & 0 \end{pmatrix} =: J_N. \tag{2.12}$$

Recall that Kalman's criterion (1.6) is equivalent to the following Hautus test:

$$\operatorname{rank}(D, A - \lambda I) = N, \qquad \forall \lambda \in \mathbb{C}. \tag{2.13}$$

Since $E = (0, \cdots, 0, 1)^\top$ is the only eigenvector of $A^\top$, the Hautus test (2.13) shows that A and D satisfy Kalman's criterion (1.6) if and only if

$$D^\top E \neq 0, \tag{2.14}$$

namely, if and only if the last row of D is not a zero vector. We then denote by $d = (d_1, d_2, \cdots, d_N)^\top$ a column of D with $d_N \neq 0$ and let

$$P = \begin{pmatrix} d_N & d_{N-1} & \cdots & d_1 \\ 0 & d_N & \cdots & d_2 \\ \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & d_N & d_{N-1} \\ 0 & 0 & 0 & d_N \end{pmatrix}. \tag{2.15}$$

Obviously, P is invertible. Noting that

$$P = d_N I + d_{N-1} J_N + \cdots + d_1 J_N^{N-1}, \tag{2.16}$$

it is easy to see that $PA = AP$. On the other hand, under the multiplier geometrical condition, it was shown in [1] that the adjoint system (1.3) with the coupling matrix (2.12) is D_0-observable with

$$D_0 = (0, \cdots, 0, 1)^\top. \tag{2.17}$$

Thus, by Proposition 2.3, the same system is PD_0-observable. Then, since

$$PD_0 = (d_1, \cdots, d_N)^\top \tag{2.18}$$

is a sub-matrix of D, the adjoint system (1.3) must be D-observable.

(ii) Case where A is composed of two Jordan blocs of the same size:

$$A = \begin{pmatrix} J_p & 0 \\ 0 & J_p \end{pmatrix}, \tag{2.19}$$

where J_p is the Jordan bloc of order p.

First, let ϵ_i be fined by

$$\epsilon_i = (0, \cdots, 0, \overset{(i)}{1}, 0, \cdots, 0)^\top \tag{2.20}$$

for $i = 1, \cdots, 2p$. Consider the special boundary control matrix

$$D_0 = (\epsilon_p, \epsilon_{2p}). \tag{2.21}$$

Noting that in this situation the adjoint system (1.3) and the observation (1.5) are entirely decoupled into two independent sub-systems, we conclude that each of them satisfies Kalman's criterion (1.6) with $N = p$. Then we conclude that the previous case shows that the adjoint system (1.3) is D_0-observable.

Now, let us consider the general boundary control matrix D of order $2p \times M$:

$$D = \begin{pmatrix} a_1 & c_1 & \cdots\cdots \\ \vdots & \vdots & \\ a_p & c_p & \cdots\cdots \\ b_1 & d_1 & \cdots\cdots \\ \vdots & \vdots & \\ b_p & d_p & \cdots\cdots \end{pmatrix}. \tag{2.22}$$

Since ϵ_p and ϵ_{2p} are the only two eigenvectors of $A^\intercal$ associated with the same eigenvalue zero, it follows that, for any given real numbers α and β with $|\alpha| + |\beta| > 0$, $\alpha\epsilon_p + \beta\epsilon_{2p}$ is also an eigenvector of $A^\intercal$. By the Hautus test (2.13), Kalman's criterion (1.6) holds if and only if $D^\intercal\epsilon_p$ and $D^\intercal\epsilon_{2p}$, namely, the row vectors

$$(a_p, c_p, \cdots\cdots) \quad \text{and} \quad (b_p, d_p, \cdots\cdots) \tag{2.23}$$

are linearly independent. Without loss of generality, we may assume that

$$a_p d_p - c_p b_p \neq 0. \tag{2.24}$$

Let the matrix P of order $2p$ be defined by

$$P = \left(\begin{array}{cccc|cccc} a_p & a_{p-1} & \cdots & a_1 & b_p & b_{p-1} & \cdots & b_1 \\ 0 & a_p & \cdots & a_2 & 0 & b_p & \cdots & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_p & 0 & 0 & \cdots & b_p \\ \hline c_p & c_{p-1} & \cdots & c_1 & d_p & d_{p-1} & \cdots & d_1 \\ 0 & c_p & \cdots & c_2 & 0 & d_p & \cdots & d_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & c_p & 0 & 0 & \cdots & d_p \end{array}\right) = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix}. \tag{2.25}$$

Since P_{11}, P_{12}, P_{21} and P_{22} have a similar structure (2.16) as P, we check easily that

$$PA = \begin{pmatrix} P_{11}J_p & P_{12}J_p \\ P_{21}J_p & P_{22}J_p \end{pmatrix} = \begin{pmatrix} J_pP_{11} & J_pP_{12} \\ J_pP_{21} & J_pP_{22} \end{pmatrix} = AP. \tag{2.26}$$

Moreover, under condition (2.24), P is invertible. Since the adjoint system (1.3) is D_0-observable, by Lemma 2.3, it is PD_0-observable, and therefore D-observable since PD_0 is composed of the first two columns of D.

(iii) Case where A is composed of two Jordan blocs of different sizes:

$$A = \begin{pmatrix} J_p & 0 \\ 0 & J_q \end{pmatrix} \tag{2.27}$$

with $q < p$. In this case, the adjoint system (1.3) is composed of the first sub-system

$$i = 1, \cdots, p: \quad \begin{cases} \phi_i'' - \Delta\phi_i + \phi_{i-1} = 0 & \text{in } (0, +\infty) \times \Omega, \\ \phi_i = 0 & \text{on } (0, +\infty) \times \Gamma \end{cases} \tag{2.28}$$

with $\phi_0 = 0$, and of the second sub-system

$$j = p - q + 1, \cdots, p: \quad \begin{cases} \psi_j'' - \Delta\psi_j + \psi_{j-1} = 0 & \text{in } (0, +\infty) \times \Omega, \\ \psi_j = 0 & \text{on } (0, +\infty) \times \Gamma \end{cases} \tag{2.29}$$

with $\psi_{p-q} = 0$. These two sub-systems (2.28) and (2.29) are coupled together by the D-observations:

$$\begin{cases} \displaystyle\sum_{i=1}^{p} a_i \partial_\nu \phi_i + \sum_{j=p-q+1}^{p} b_j \partial_\nu \psi_j = 0 & \text{on } (0, T) \times \Gamma_1, \\ \displaystyle\sum_{i=1}^{p} c_i \partial_\nu \phi_i + \sum_{j=p-q+1}^{p} d_j \partial_\nu \psi_j = 0 & \text{on } (0, T) \times \Gamma_1, \\ \cdots\cdots\cdots\cdots \end{cases} \tag{2.30}$$

In order to transfer the problem to the case $p = q$, we expand the second sub-system (2.29) from $\{p-q+1, \cdots, p\}$ to $\{1, \cdots, p\}$:

$$j = 1, \cdots, p: \quad \begin{cases} \psi_j'' - \Delta\psi_j + \psi_{j-1} = 0 & \text{in } (0, +\infty) \times \Omega, \\ \psi_j = 0 & \text{on } (0, +\infty) \times \Gamma \end{cases} \tag{2.31}$$

with $\psi_0 = 0$, so that the two sub-systems (2.28) and (2.31) have the same size.

Accordingly, the D-observations (2.30) can be extended to the $\widetilde{D}$-observations:

$$\begin{cases} \sum_{i=1}^{p} a_i \partial_\nu \phi_i + \sum_{j=1}^{p} b_j \partial_\nu \psi_j = 0 & \text{on } (0, T) \times \Gamma_1, \\ \sum_{i=1}^{p} c_i \partial_\nu \phi_i + \sum_{j=1}^{p} d_j \partial_\nu \psi_j = 0 & \text{on } (0, T) \times \Gamma_1, \\ \cdots\cdots\cdots\cdots \end{cases} \tag{2.32}$$

with arbitrarily given coefficients b_j and d_j for $j = 1, \cdots, p-q$.

Let us write the matrix D of order $(p+q) \times M$ of the observations (2.30) as

$$D = \begin{pmatrix} a_1 & c_1 & \cdots\cdots \\ \vdots & \vdots & \\ a_p & c_p & \cdots\cdots \\ b_{p-q+1} & d_{p-q+1} & \cdots\cdots \\ \vdots & \vdots & \\ b_p & d_p & \cdots\cdots \end{pmatrix}. \tag{2.33}$$

Like in the case (ii), ϵ_p and ϵ_{p+q} are the only two eigenvectors of $A^\top$ associated with the same eigenvalue zero. Then, for any given real numbers α and β with $|\alpha| + |\beta| > 0$, $\alpha\epsilon_p + \beta\epsilon_{p+q}$ is also an eigenvector of $A^\top$. Kalman's criterion (1.6) holds if and only if $D^\top \epsilon_p$ and $D^\top \epsilon_{p+q}$, namely, the row vectors

$$(a_p, c_p, \cdots\cdots) \quad \text{and} \quad (b_p, d_p, \cdots\cdots) \tag{2.34}$$

are linearly independent. Without loss of generality, we may still assume that (2.24) holds.

Then, we write the matrix $\widetilde{D}$ of order $2p \times M$ of the extended observations (2.32) as

$$\widetilde{D} = \begin{pmatrix} a_1 & c_1 & \cdots\cdots \\ \vdots & \vdots & \\ a_p & c_p & \cdots\cdots \\ b_1 & d_1 & \cdots\cdots \\ \vdots & \vdots & \\ b_{p-q} & d_{p-q} & \\ b_{p-q+1} & d_{p-q+1} & \cdots\cdots \\ \vdots & \vdots & \\ b_p & d_p & \cdots\cdots \end{pmatrix}. \tag{2.35}$$

The matrix $\widetilde{A}$ of order $2p$ for the extended adjoint system, composed of (2.28) and (2.31), is the same as that given in (2.19). Then $\widetilde{A}, \widetilde{D}$ satisfy the corresponding Kalman's criterion if and only if the condition (2.24) holds. Then by the conclusion of (ii), the extended adjoint system, composed of (2.28) and (2.31), is $\widetilde{D}$-observable in the space $(H_0^1(\Omega))^{2p} \times (L^2(\Omega))^{2p}$. In particular, if we choose the specific initial data such that

$$\psi_1 = \cdots = \psi_{p-q} = 0 \text{ and } \psi_1' = \cdots = \psi_{p-q}' = 0 \quad \text{at } t = 0 \tag{2.36}$$

for the extended sub-system (2.31), then by well-posedness we have

$$\psi_1 \equiv \cdots \equiv \psi_{p-q} \equiv 0 \quad \text{in } (0, +\infty) \times \Omega. \tag{2.37}$$

Thus, the extended adjoint system, composed of (2.28) and (2.31) with the $\widetilde{D}$-observations (2.32), is reduced to the original adjoint system composed of (2.28) and (2.29) with the D-observations (2.30). Hence we conclude that the original adjoint system is D-observable in the space $(H_0^1(\Omega))^{p+q} \times (L^2(\Omega))^{p+q}$.

The case where A is composed of several Jordan blocs can be similarly treated.

Since any given nilpotent matrix can be decomposed in a diagonal form of Jordan blocs under a suitable basis, the previous conclusion is still valid for any given nilpotent matrix A by Proposition 2.4. The proof is complete.

References

[1] F. Alabau-Boussouira, A hierarchic multi-level energy method for the control of bidiagonal and mixed n-coupled cascade systems of PDE's by a reduced number of controls, Adv. Differ. Equ. 18 (2013) 1005–1072.

[2] F. Ammar Khodja, A. Benabdallah, C. Dupaix, Null-controllability of some reaction–diffusion systems with one control force, J. Math. Anal. Appl. 320 (2006) 928–944.

[3] L. Hörmander, Linear Partial Differential Operators, Springer-Verlag, 1976.

[4] T.-T. Li, B. Rao, Exact synchronization for a coupled system of wave equations with Dirichlet boundary controls, Chin. Ann. Math., Ser. B 34 (2013) 139–160.

[5] T.-T. Li, B. Rao, Asymptotic controllability and asymptotic synchronization for a coupled system of wave equations with Dirichlet boundary controls, Asymptot. Anal. 86 (2014) 199–226.

[6] T.-T. Li, B. Rao, Criteria of Kalman's type to the approximate controllability and the approximate synchronization for a coupled system of wave equations with Dirichlet boundary controls, SIAM J. Control Optim. 54 (2016) 49–72.

[7] J.-L. Lions, Contrôlabilité Exacte, Perturbations et Stabilisation de Systèmes Distribués, vol. 1, Masson, Paris, 1986.