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Differential geometry

A Bernstein theorem for affine maximal-type hypersurfaces

Un théorème de Bernstein pour les hypersurfaces de type affine maximal

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A R T I C L E I N F O

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A B S T R A C T

We obtain, in any dimension N and for a large range of values of θ, a Bernstein theorem for the fourth-order partial differential equation of affine maximal type

$$u^{ij}D_{ij}w = 0, \quad w = [\det D^2u]^{-\theta}$$

assuming the completeness of Calabi's metric. This contains the results of Li–Jia [A.M. Li, F. Jia, Ann. Glob. Anal. Geom. 23 (2003)] for affine maximal equations and of Zhou [B. Zhou, Calc. Var. Partial Differ. Equ. 43 (2012)] for Abreu's equation. In particular, we extend the result of Zhou from $2 \le N \le 4$ to $2 \le N \le 5$.

R É S U M É

Nous obtenons, en toute dimension N et pour un large spectre de valeurs θ, un théorème de Bernstein pour l'équation différentielle partielle d'ordre quatre, de type affine maximal

$$u^{ij}D_{ij}w = 0, \quad w = [\det D^2u]^{-\theta}$$

sous l'hypothèse de complétude de la métrique de Calabi. Ceci contient les résultats de Li–Jia [A.M. Li, F. Jia, Ann. Glob. Anal. Geom. 23 (2003)] pour les équations affines maximales et de Zhou [B. Zhou, Calc. Var. Partial Differ. Equ. 43 (2012)] pour l'équation d'Abreu. En particulier, nous généralisons les résultats de Zhou pour $2 \le N \le 4$ à $2 \le N \le 5$.

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1. Introduction

In this paper, we study the entire locally uniform convex solution u to the affine maximal-type equation

$$D_{ij}(U^{ij}w)=0, \tag{1.1}$$

where U^{ij} is the co-factor matrix of $[u_{ij}]$ and

$$w\equiv[\det D^2u]^{-\theta}.$$

In the whole paper, we always assume that $\theta\neq 0$. Equation (1.1) is the Euler–Lagrange equation of the affine area functional

$$\begin{aligned}\mathcal{A}(u,\Omega)&\equiv\int_{\Omega}[\det D^2u]^{1-\theta}\\&=\int_{\mathcal{M}_\Omega}K^{1-\theta}(1+|Du|^2)^{\vartheta}\,\mathrm{d}V_{g_{\mathcal{M}}},\quad \vartheta=\frac{N+1}{2}-\frac{N+2}{2}\theta,\end{aligned}$$

for $\theta\neq 1$, and

$$\mathcal{A}(u,\Omega)\equiv\int_{\Omega}\log\det D^2u$$

for $\theta=1$, where u is a convex function over $\Omega\subset\mathbb{R}^N$, $g_{\mathcal{M}}$ and K are the induced metric and the Gauss curvature of the graph $\mathcal{M}_\Omega\equiv\left\{(x,z)\in\mathbb{R}^{N+1}|\ z=u(x),x\in\Omega\right\}$ respectively. Noting that

$$D_jU^{ij}=0,\quad\forall i=1,2,\cdots,N,$$

(1.1) can be written as

$$U^{ij}D_{ij}w=0$$

which is equivalent to

$$u^{ij}D_{ij}w=0, \tag{1.2}$$

here $[u^{ij}]$ denotes the inverse of the metric $[u_{ij}]$ of graph $\mathcal{M}=\mathcal{M}_\Omega$.

The classical affine maximal case $\theta\equiv\frac{N+1}{N+2}$ has been studied extensively; for more details, please check the references [15,11]. As one introduces the affine metric

$$A_{ij}=\frac{u_{ij}}{[\det D^2u]^{1/(N+2)}}$$

on $\mathcal{M}$ and $H\equiv[\det D^2u]^{-1/(N+2)}$, it is not hard to see that (1.2) turns to be here

$$\triangle_{\mathcal{M}}H=0, \tag{1.3}$$

the Laplace–Beltrami operator

$$\triangle_{\mathcal{M}}\equiv\frac{1}{\sqrt{A}}D_i(\sqrt{A}A^{ij}D_j)=HD_i(H^{-2}u^{ij}D_j)$$

with respect to this affine metric, where A is the determinant of $[A_{ij}]$ and $[A^{ij}]$ stands for the inverse of $[A_{ij}]$. So the hypersurface $\mathcal{M}$ is affine maximal if and only if H is harmonic on $\mathcal{M}$.

Much of efforts were done toward the Bernstein problem of (1.2), see for examples [2–4,10,15]. In particular, Trudinger–Wang [15] proved the Chern conjecture in affine geometry [5], which states that each entire graph of locally convex solution to (1.2) must be a paraboloid for $\theta=\frac{3}{4}$ and $N=2$. So it is natural to consider the higher-dimensional case. When $\theta=\frac{N+1}{N+2}$ and $2\le N\le 3$, Li–Jia [11] proved that any locally convex affine maximal hypersurfaces that are complete under Calabi's metric must be a paraboloid. If $\theta=1$, (1.2) is called Abreu's equation (see, e.g., [1,6–8,11,17]). Zhou proved in [17] that if $2\le N\le 4$, Bernstein theorem holds under completeness of Calabi's metric.

We will study the generalized Bernstein problem under completeness of θ'-affine metric, which will be defined in Section 3, and prove the following theorem.

Theorem 1.1 *(Main Theorem). (1) For $N = 2$ and $\theta \neq 0, 1/2$, if the graph of u is complete under θ'-affine metric for some θ' satisfying $(2\theta - 1)\theta' \geq 1/8$, then it must be a paraboloid.*

(2) For $N \geq 2$, if

$$0 \neq \theta \in \left(-\infty, \frac{1}{2} - \frac{N-1}{4N}\sqrt{N}\right) \bigcup \left(\frac{1}{2} + \frac{N-1}{4N}\sqrt{N}, +\infty\right)$$

and the Calabi metric is complete, then the convex entire solution to (1.2) must be a quadratic function.

From the second part of above theorem, we can get the following corollary. As mentioned above, partial results have been proved in [11] and [17], respectively.

Corollary 1.1. *Suppose that the Calabi metric is complete, and $\theta = \frac{N+1}{N+2}$ for $2 \leq N \leq 3$, or $\theta = 1$ for $2 \leq N \leq 5$, then the entire graph of convex solutions to (1.2) must be a paraboloid.*

In the next section, we will derive some elementary equations of third derivatives, then prove the main theorem in Section 3.

2. Equations of third derivatives

In this section, we will deduce some equations related to the third partial derivatives of u, and an important inequality for proving the second part of the main theorem. Throughout this section, all of the covariant derivatives is with respect to the Calabi metric on the graph of u.

For any convex solution u to (1.2) and $N \geq 2$, we use lower indexes like $u_i \equiv D_i u$ to denote the partial derivatives of u, and use upper indexes $T^i_j \equiv u^{ik} T_{kj}$ to denote the conjugate ones, where $[u^{ij}]$ is the inverse of $[u_{ij}]$. Elementary computations show that

$$D_k u^{ij} = -u^{ip} u^{jq} u_{pqk} = -u^{ij}_k \tag{2.1}$$

and

$$w_k = -\theta\, d^{-\theta-1} U^{pq} u_{pqk} = -\theta\, w\, u^{pq} u_{pqk} = -\theta\, w\, u^a_{ak}. \tag{2.2}$$

Denoting $g_{ij} = u_{ij}$ as the Calabi metric, we have its Ricci tensor

$$R_{ij} = \frac{1}{4} B_{i,j} - \frac{1}{4} u_{ijp} u^{ap}_a$$

and the Laplace–Beltrami operator

$$\triangle_g = u^{ij}\left(\frac{\partial^2}{\partial x^i \partial x^j} - \frac{1}{2} u^k_{ij} \frac{\partial}{\partial x^k}\right),$$

where

$$B_{i,j} \equiv u^{ab}_i u_{abj}.$$

For any $\theta \neq 0$, we set

$$h = w^{1-\frac{1}{2\theta}}.$$

As above, it is not hard to verify that h is a harmonic function under metric g. Taking derivatives on h, we have

$$h_i = \tau\, w^{\tau-1} w_i = -\tau\, \theta\, h\, u^b_{bi} \text{ for } \tau = 1 - \frac{1}{2\theta}, \tag{2.3}$$

and consequently

$$\tau^2 \theta^2 v = \frac{||\nabla h||^2}{h^2}, \tag{2.4}$$

where

$$v \equiv u^{ai}_a u^b_{bi}$$

and ∇, ∇^2 denote the covariant and Hessian operators, respectively, while $||\cdot|| \equiv ||\cdot||_g$ denotes the norm with respect to the metric g. Differentiating (2.4) twice, we get

$$\tau^2\theta^2\nabla_i v = \frac{2\nabla h \nabla_i \nabla h}{h^2} - 2\frac{||\nabla h||^2 \nabla_i h}{h^3} \tag{2.5}$$

and the Bochner-type identity

$$\tau^2\theta^2 \triangle_g v = \frac{2||\nabla^2 h||^2 + 2\,\mathrm{Ric}(\nabla h, \nabla h)}{h^2} - 8\frac{\nabla h \nabla^2 h \nabla h}{h^3} + 6\frac{||\nabla h||^4}{h^4}. \tag{2.6}$$

The following fundamental inequality can be found in [17]. For completeness of the paper, we give a proof here.

Lemma 2.1. *Using the above notations $B_{i,j} \equiv u_i^{ab} u_{abj}$ and $v \equiv u_a^{ai} u_{bi}^b$, we have*

$$4R_{ij} \equiv B_{i,j} - u_{ijp} u_a^{ap} \ge -\frac{N-1}{4N} v u_{ij}. \tag{2.7}$$

Proof. Taking any vector $\xi \in \mathbb{R}^N$, after scaling and rotations, we may assume that $u_{ij} = \delta_{ij}$ and $\xi = (1, 0, \cdots, 0)$. So we can get

$$\begin{aligned}
4\,R_{ij}\xi^i\xi^j &= R_{11} = \Sigma_{a,b} u_{ab1}^2 - \Sigma_{a,p} u_{11p} u_{aap} \\
&\ge \Sigma_a u_{aa1}^2 - \Sigma_a u_{111} u_{aa1} \\
&= u_{111}^2 - u_{111}\Sigma_a u_{aa1} + \Sigma_{a\ge 2} u_{aa1}^2 \\
&\ge u_{111}^2 - u_{111}\Sigma_a u_{aa1} + \frac{1}{N-1}\Big(\Sigma_{a\ge 2} u_{aa1}\Big)^2 \\
&= u_{111}^2 - u_{111}\Sigma_a u_{aa1} + \frac{1}{N-1}\Big(\Sigma_a u_{aa1} - u_{111}\Big)^2 \\
&= \frac{N}{N-1} u_{111}^2 \frac{N+1}{N-1} u_{111}\Sigma_a u_{aa1} + \frac{1}{N-1}\Big(\Sigma_a u_{aa1}\Big)^2 \\
&\ge -\frac{N-1}{4N}\Big(\Sigma_a u_{aa1}\Big)^2.
\end{aligned}$$

Hence (2.7) holds. □

Now we estimate the last three terms in (2.6) as follows. The first one is estimated by

$$\begin{aligned}
\frac{\mathrm{Ric}(\nabla h, \nabla h)}{h^2} &= \frac{\tau^2\theta^2}{4}\Big(B_{i,j} - u_{ijp} u_a^{ap}\Big) u_b^{bi} u_c^{cj} \\
&\ge -\frac{N-1}{16N}\tau^2\theta^2 v^2
\end{aligned} \tag{2.8}$$

using Lemma 2.1. Secondly, we estimate

$$\begin{aligned}
-\frac{\nabla h \nabla^2 h \nabla h}{h^3} &= -\frac{1}{2h}\nabla h \cdot \left(\tau^2\theta^2 \nabla v + 2\frac{||\nabla h||^2 \nabla h}{h^3}\right) \\
&= -\frac{\tau^2\theta^2}{2h}\nabla h \nabla v - \tau^4\theta^4 v^2
\end{aligned} \tag{2.9}$$

by (2.5). Thirdly,

$$\frac{||\nabla h||^4}{h^4} = \tau^4\theta^4 v^2. \tag{2.10}$$

Finally, a reforming of (2.5) yields that

$$\begin{aligned}
\tau^4\theta^4 u^{ij}\frac{v_i v_j}{v} &= \frac{u^{ij}}{v}\left(2\frac{\nabla h \nabla_i \nabla h}{h^2} - 2\,\tau^2\theta^2\frac{v\nabla_i h}{h}\right)\left(2\frac{\nabla h \nabla_j \nabla h}{h^2} - 2\,\tau^2\theta^2\frac{v\nabla_j h}{h}\right) \\
&= \frac{u^{ij}}{v}\left\{4\frac{(\nabla h \nabla_i \nabla h)(\nabla h \nabla_j \nabla h)}{h^4} - 8\,\tau^2\theta^2\frac{v(\nabla h \nabla_i \nabla h \nabla_j h)}{h^3}\right\} + \frac{u^{ij}}{v}\left(4\,\tau^4\theta^4\frac{v^2 \nabla_i h \nabla_j h}{h^2}\right) \\
&= 4\frac{\nabla h \nabla^2 h \nabla^2 h \nabla h}{v\, h^4} - 8\,\tau^2\theta^2\frac{\nabla h \nabla^2 h \nabla h}{h^3} + 4\,\tau^6\theta^6 v^2.
\end{aligned} \tag{2.11}$$

Now let us prove the following inequality.

Lemma 2.2. *Using the same notations as above, we have*

$$\frac{||\nabla^2 h||^2}{h^2} \geq 2\frac{\nabla h\nabla^2 h\nabla^2 h\nabla h}{\tau^2\theta^2 v\, h^4} - \frac{N-2}{N-1}\frac{(\nabla h\nabla^2 h\nabla h)^2}{\tau^4\theta^4 v^2 h^6} \geq \frac{N}{N-1}\frac{\nabla h\nabla^2 h\nabla^2 h\nabla h}{\tau^2\theta^2 v\, h^4}. \tag{2.12}$$

Proof. After rotation and translation, we may assume that $u_{ij} = \delta_{ij}$ and $u^b_{bi} = 0$ for $i \geq 2$. Using (2.3), we can get

$$\begin{aligned}
||\nabla^2 h||^2 &= \Sigma_{i,j} h_{ij}^2 = h_{11}^2 + \Sigma_{i\geq 2} h_{ii}^2 + 2\,\Sigma_{i\geq 2} h_{1i}^2 \\
&\geq h_{11}^2 + \frac{1}{N-1}\left(\Sigma_{i\geq 2} h_{ii}\right)^2 + 2\,\Sigma_{i\geq 2} h_{1i}^2 \\
&= 2\,\Sigma_i h_{1i}^2 - \frac{N-2}{N-1} h_{11}^2 \\
&= 2\frac{\nabla h\nabla^2 h\nabla^2 h\nabla h}{\tau^2\theta^2 v\, h^2} - \frac{N-2}{N-1}\frac{(\nabla h\nabla^2 h\nabla h)^2}{\tau^4\theta^4 v^2 h^4} \\
&\geq \frac{N}{N-1}\frac{\nabla h\nabla^2 h\nabla^2 h\nabla h}{\tau^2\theta^2 v\, h^2},
\end{aligned}$$

since h is harmonic. □

Consequently, we can get the following crucial lemma.

Lemma 2.3. *Let u be a convex solution to (1.2) for $\theta \neq 0,\ 1/2$, we have*

$$\triangle_g v + \alpha_1(\theta, N) u_b^{bj} D_j v \geq \frac{N}{2\,(N-1)}\frac{||\nabla v||^2}{v} + \alpha_2(\theta, N) v^2, \tag{2.13}$$

where

$$\begin{cases} \alpha_1(\theta, N) = -\dfrac{2\,(N-2)}{N-1}(\theta - 1/2), \\ \alpha_2(\theta, N) = \dfrac{2}{N-1}(\theta - 1/2)^2 - \dfrac{N-1}{8\,N}. \end{cases}$$

In particular, if $\theta > \frac{1}{2} + \frac{N-1}{4N}\sqrt{N}$ or $\theta < \frac{1}{2} - \frac{N-1}{4N}\sqrt{N}$, then $\alpha_2(\theta, N) > 0$.

Proof. A combination of (2.8)–(2.11) together with Lemma 2.2 yields that

$$\begin{aligned}
\tau^2\theta^2 \triangle_g v &\geq \frac{2\,N}{N-1}\frac{\nabla h\nabla^2 h\nabla^2 h\nabla h}{\tau^2\theta^2 v\, h^4} - 8\frac{\nabla h\nabla^2 h\nabla h}{h^3} + \left(6\,\tau^4\theta^4 - \frac{N-1}{8\,N}\tau^2\theta^2\right) v^2 \\
&\geq -8\frac{\nabla h\nabla^2 h\nabla h}{h^3} + \left(6\tau^4\theta^4 - \frac{N-1}{8\,N}\tau^2\theta^2\right) v^2 \\
&\quad + \frac{N}{2\,(N-1)\tau^2\theta^2}\left\{\tau^4\theta^4\frac{||\nabla v||^2}{v} + 8\,\tau^2\theta^2\frac{\nabla h\nabla^2 h\nabla h}{h^3} - 4\,\tau^6\theta^6 v^2\right\} \\
&= \frac{N\tau^2\theta^2}{2\,(N-1)}\frac{||\nabla v||^2}{v} + \left\{\frac{2}{N-1}(\theta - 1/2)^4 - \frac{N-1}{8\,N}(\theta - 1/2)^2\right\} v^2 + \left(\frac{2\,N}{N-1} - 4\right)\tau^2\theta^2 \nabla\log h \nabla v.
\end{aligned}$$

Dividing $\tau^2\theta^2 = (\theta - 1/2)^2$ on both sides of the above inequality, and reorganizing it, one can get the result of the lemma. □

3. Bernstein theorem for complete θ'-affine metrics

In this section, we will prove the main theorem of this paper. Let us define the θ'-affine metric by

$$g'_{ij} = \frac{u_{ij}}{[\det D^2 u]^{\theta'}}$$

for some $\theta' \in \mathbb{R}$ when $\det D^2 u > 0$. As usually, the θ'-affine metric is called Calabi's metric [17] for $\theta' = 0$ and called affine maximal metric [15] if $\theta' = \frac{1}{N+2}$ (or Blaschke metric, see [11]). Since

$$\Gamma^k_{ij} = \frac{1}{2} u^k_{ij} - \frac{\theta'}{2}(\delta^k_j u^a_{ai} + \delta^k_j u^a_{aj}) + \frac{\theta'}{2} u_{ij} u^{ak}_a,$$

the Laplace–Beltrami operator under θ'-affine metric is given by

$$\begin{aligned}\triangle_{g'} &= g'^{ij}\left(\frac{\partial^2}{\partial x^i \partial^j} - \Gamma^k_{ij}\frac{\partial}{\partial x^k}\right)\\ &= [\det D^2 u]^{-\theta'}\left\{u^{ij}\frac{\partial^2}{\partial x^i \partial x^j} - \frac{\theta'(N-2)+1}{2} u^{ak}_a \frac{\partial}{\partial x^k}\right\}.\end{aligned}$$

Given any $\beta \neq 0$, we have

$$\begin{aligned}-u^{ij} D_{ij} w^\beta &= -\beta\,(\beta - 1)\, u^{ij} w^{\beta-2} D_i\, w\, D_j\, w\\ &= (\beta - 1)\,\theta\;\; u^{ij}\, u^b_{bi}\, D_j\, w^\beta.\end{aligned}$$

So, if we take $\beta = -\frac{\theta'(N-2)+1}{2\theta} + 1$, then

$$\triangle_{g'} w^\beta = 0 \tag{3.1}$$

on graph $\mathcal{M} \equiv \left\{(x, u(x)) \,\middle|\, x \in \mathbb{R}^N\right\}$ equipped with the θ'-affine metric $[g'_{ij}]$.

Let us prove the first part of the main theorem.

Theorem 3.1. *Let u be a locally convex solution to (1.2) for any $\theta \neq 0, 1/2$ on $\mathbb{R}^2$. If the graph of u is complete under a θ'-affine metric for some θ' satisfying*

$$(2\theta - 1)\,\theta' \geq \frac{1}{8}, \tag{3.2}$$

then it must be a paraboloid.

Proof. Noting that $\theta \neq 1/2$ and $N = 2$ imply $\beta \neq 0$. Using (3.1), we can get w^β is a positive harmonic function on surface $(\mathcal{M}, g')$. On the other hand, the Ricci curvature tensor with respect to g' can be rewritten as

$$R_{ij}(g') = \frac{1}{4} B_{i,j} - \frac{1}{4} u_{ijp} u^{ap}_a + \frac{(2\theta - 1)\,\theta'}{4} u_{ij} v.$$

By Lemma 2.1, we can obtain that the tensor is semi-positive definite if (3.2) holds and $N = 2$.

Appealing a result of Liouville theorem by Yau [16], we conclude that if the surface is θ'-affine complete, then $\det D^2 u$ is a constant, and therefore u is a quadratic function by Jörgens' theorem [9], which finishes the proof of the theorem. □

Now we want to prove the second part of the main theorem, Calabi's metric case.

Theorem 3.2. *Let u be a locally convex solution to (1.2) for any $N \geq 2$. Suppose that*

$$0 \neq \theta \in \left(-\infty, \frac{1}{2} - \frac{N-1}{4N}\sqrt{N}\right) \bigcup \left(\frac{1}{2} + \frac{N-1}{4N}\sqrt{N}, +\infty\right) \tag{3.3}$$

and the Calabi metric is complete on the graph, then the hypersurface u must be a paraboloid.

Before proving this theorem, let us recall the following Laplace comparison lemma in differential geometry; see, for example, [14].

Lemma 3.1. *Let $\mathcal{M} \equiv \{(x, u(x)| \, x \in \mathbb{R}^N\}$ be the graph of solution u to (1.2) and set $g_{ij} = u_{ij}$ to be the Calabi metric of $\mathcal{M}$ as before. Suppose that the Ricci curvature*

$$R_{ij} \geq -(N-1)\, K^2 g_{ij}$$

for some nonnegative constant K, then lying outside the cut-locus of given point $o = (0, u(0))$, the distance function $r(x) \equiv dist(p, o)$, $p \equiv (x, u(x))$ satisfying $|Dr|_g = 1$ and

$$r \triangle_g r = u^{ij}\left(r_{ij} - 1/2\, u^k_{ij} u_k\right) \leq (N-1)\,(1 + Kr). \tag{3.4}$$

Proof of Theorem 3.2. Under the assumption (3.3), by Lemma 2.3, we have (2.13) holds for some positive number $\alpha_2(\theta, N)$. After an approximation argument as in [14] or [12,13], we may assume that the distance function $r(x)$ from Lemma 3.1 is smooth. For any $R > 1$, we define a cut-off function $\eta \equiv [(R^2 - r^2(x))_+]^\kappa$ for some large constant $\kappa > 1$. Suppose that $v\,\eta$ attains its maximum at $p_0 \in B_R(o)$, we set $R' \equiv r(p_0) = \mathrm{dist}(p_0, o)$. Then, it is clear that

$$\max_{B_{R'}(o)} v = v(p_0).$$

Noting that

$$R_{ij} = \frac{1}{4} B_{i,j} - \frac{1}{4} u_{ijp} u^{ap}_a \geq -\frac{N-1}{16\,N} v(p_0) u_{ij},$$

we have

$$K^2 = \frac{v(p_0)}{16\,N}$$

in Lemma 3.1. On the other hand, direct computations show that

$$\begin{aligned} \eta_j &= -2\kappa\,\eta^{\frac{\kappa-1}{\kappa}} r\, D_j r, \text{ and} \\ \frac{\partial^2 \eta}{\partial x^i \partial x^j} &= -2\kappa\,\eta^{\frac{\kappa-1}{\kappa}} (r\, D_{ij} r + r_i r_j) + 2\kappa^2 \eta^{\frac{\kappa-2}{\kappa}} r^2 r_i r_j. \end{aligned}$$

So we can get

$$\begin{aligned} -u^{ij} \frac{\partial^2 \eta}{\partial x^i \partial x^j} &= 2k\,\eta^{\frac{\kappa-1}{\kappa}} r\, u^{ij}\left(r_{ij} - 1/2 u^k_{ij} u_k\right) + k\,\eta^{\frac{\kappa-1}{\kappa}} r\, u^{bk}_b r_k + 2k\,\eta^{\frac{\kappa-1}{\kappa}} - 2\kappa^2 \eta^{\frac{\kappa-2}{\kappa}} r^2 \\ &\leq 2\,N k\,\eta^{\frac{\kappa-1}{\kappa}} + \frac{\kappa\,(N-1)}{2\sqrt{N}} \eta^{\frac{\kappa-1}{\kappa}} r \sqrt{v(p_0)} + \kappa\,\eta^{\frac{\kappa-1}{\kappa}} r \sqrt{v} \end{aligned} \tag{3.5}$$

by Lemma 3.1. Consequently,

$$\begin{aligned} -\triangle_g \eta &\leq -\frac{\kappa}{2} \eta^{\frac{\kappa-1}{\kappa}} u^{bj}_b u_j + 2\,N\kappa\,\eta^{\frac{\kappa-1}{\kappa}} + \frac{\kappa\,(N-1)}{2\sqrt{N}} \eta^{\frac{\kappa-1}{\kappa}} r \sqrt{v(p_0)} + \kappa\,\eta^{\frac{\kappa-1}{\kappa}} r \sqrt{v} \\ &\leq C_{\kappa,N}\left(\sqrt{v} + \sqrt{v(p_0)} + 1\right)(r+1). \end{aligned} \tag{3.6}$$

Hence

$$\begin{aligned} & -\triangle_g (v\,\eta) - \alpha_1(\theta, N)\, u^{bj}_b D_j (v\,\eta) \\ &= \eta \left\{ -\triangle_g v - \alpha_1(\theta, N)\, u^{bj}_b D_j v \right\} - 2\nabla v\, \nabla \eta - v\, \triangle_g \eta \\ &\leq -\frac{N}{2\,(N-1)} \frac{||\nabla v||^2}{v} \eta - \alpha_2(\theta, N)\, v^2 \eta + \varepsilon \frac{||\nabla v||^2}{v} \eta + \varepsilon\, v^2 \eta + C_{\varepsilon,\kappa,N}\left(\eta^{\frac{\kappa-4}{\kappa}} r^4 + \eta^{\frac{\kappa-4}{\kappa}}\right) \end{aligned} \tag{3.7}$$

for positive ε. Now taking ε to be small enough and evaluating $v\,\eta$ at its maximum point p_0, we can get

$$v(p_0)^2 \eta \leq \frac{1}{2} v(p_0)^2 \eta + C\left(\eta^{\frac{\kappa-4}{\kappa}} r^4 + \eta^{\frac{\kappa-2}{\kappa}}\right)$$

by Young's inequality, as long as $\kappa \geq 4$, here the positive constant C is independent of R, which implies

$$\begin{aligned} (R/2)^{2\kappa} \sup_{B_{R/2}(p)} v \leq \sup_{B_R(p)} v\,\eta \leq R^\kappa \sqrt{v^2(p_0)\,\eta} \\ \leq C\, R^{2\kappa-2}. \end{aligned} \tag{3.8}$$

Hence there must be $v \equiv 0$ after dividing both sides of (3.8) by $R^{2\kappa}$ and then sending R tends to infinity. Therefore, Theorem 3.2 holds. □

Combining Theorem 3.1 with Theorem 3.2, we can get the results of the Main Theorem.

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