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Stokes and Navier–Stokes equations with Navier boundary condition

Équations de Stokes et de Navier–Stokes avec la condition de Navier

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ABSTRACT

In this paper, we study the stationary Stokes and Navier–Stokes equations with non-homogeneous Navier boundary condition in a bounded domain $\Omega \subset \mathbb{R}^3$ of class $\mathcal{C}^{1,1}$ from the viewpoint of the behavior of solutions with respect to the friction coefficient α. We first prove the existence of a unique weak solution (and strong) in $\boldsymbol{W}^{1,p}(\Omega)$ (and $\boldsymbol{W}^{2,p}(\Omega)$) to the linear problem for all $1 < p < \infty$ considering minimal regularity of α, using some inf–sup condition concerning the rotational operator. Furthermore, we deduce uniform estimates of the solutions for large α, which enables us to obtain the strong convergence of Stokes solutions with Navier slip boundary condition to the one with no-slip boundary condition as $\alpha \to \infty$. Finally, we discuss the same questions for the non-linear system.

RÉSUMÉ

Dans cette note, nous étudions les équations stationnaires de Stokes et de Navier–Stokes avec une condition aux limites non homogène de Navier dans un domaine borné $\Omega \subset \mathbb{R}^3$ de classe $\mathcal{C}^{1,1}$, et envisageons le comportement des solutions par rapport au coefficient de friction α. Nous prouvons, d'abord dans le cas linéaire, l'existence d'une solution faible (et d'une solution forte) unique dans $\boldsymbol{W}^{1,p}(\Omega)$ (et $\boldsymbol{W}^{2,p}(\Omega)$) pour tout $1 < p < \infty$ en supposant α le moins régulier possible et en utilisant une condition inf–sup concernant l'opérateur rotationnel. De plus, nous déduisons des estimations uniformes des solutions pour α grand, qui nous permettent d'obtenir la convergence forte des solutions de Stokes

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avec la condition de glissement vers les solutions vérifiant la condition d'adhérence lorsque $\alpha \to \infty$. Finalement, nous étudions les mêmes questions pour le système non linéaire.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^3$ with boundary Γ, possibly not connected, of class $\mathcal{C}^{1,1}$. Consider the stationary Stokes equation with Navier boundary condition

$$\begin{cases} -\Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{f} + \operatorname{div} \mathbb{F}, \quad \operatorname{div} \boldsymbol{u} = 0 & \text{in } \Omega \\ \boldsymbol{u} \cdot \boldsymbol{n} = 0, \quad [(2\mathbb{D}\boldsymbol{u} + \mathbb{F})\boldsymbol{n}]_{\boldsymbol{\tau}} + \alpha \boldsymbol{u}_{\boldsymbol{\tau}} = \boldsymbol{h} & \text{on } \Gamma \end{cases} \tag{S}$$

and the stationary Navier–Stokes equation with Navier boundary condition

$$\begin{cases} -\Delta \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla \pi = \boldsymbol{f} + \operatorname{div} \mathbb{F}, \quad \operatorname{div} \boldsymbol{u} = 0 & \text{in } \Omega \\ \boldsymbol{u} \cdot \boldsymbol{n} = 0, \quad [(2\mathbb{D}\boldsymbol{u} + \mathbb{F})\boldsymbol{n}]_{\boldsymbol{\tau}} + \alpha \boldsymbol{u}_{\boldsymbol{\tau}} = \boldsymbol{h} & \text{on } \Gamma \end{cases} \tag{NS}$$

where $\boldsymbol{u}$ and π are the velocity field and the pressure of the fluid respectively, $\boldsymbol{f}$ and $\mathbb{F}$ are the external forces acting on the fluid, $\boldsymbol{h}$ is a given tangential vector field, $\boldsymbol{n}$ and $\boldsymbol{\tau}$ are the unit outward normal and tangent vectors on Γ respectively and $\mathbb{D}\boldsymbol{u} = \frac{1}{2}(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^{\mathsf{T}})$ is the rate of strain tensor. Here, α is the coefficient which measures the tendency of the fluid to slip on the boundary, called *friction coefficient*.

This boundary condition was proposed by C. Navier [8]; therefore, it usually referred to as Navier (slip) boundary condition (NBC). The very first work concerning NBC was done by Solonnikov and Ščadilov [10] for $\alpha = 0$, where the authors considered stationary Stokes system with Dirichlet condition on some part of the boundary and Navier condition on the other part, and showed the existence of a weak solution in $\boldsymbol{H}^1(\Omega)$ that is regular (belongs to $\boldsymbol{H}^2_{loc}(\Omega)$) up to some part of the boundary (except in the neighborhood of the intersection of the two part). From then, several studies have been made on the well-posedness of the problem, for example [4] (with $\alpha = 0$ and flat boundary), [1] (with $\alpha = 0$ and weak, strong and very weak solution), [3] (with $\alpha \geq 0$ constant and for $p = 2$), [7] (for Navier-type boundary conditions). In some sense, this note generalizes the work in [5].

In the current work, we want to study the systems (S) and (NS), where the friction coefficient α is a non-smooth function. It is reasonable to consider α, which rather than being constant depends on the boundary, for example in the case of porous media or of a domain with rough boundary, which occurs in many physical phenomenon. Beside the systematic study of the system (S) or (NS), one of the main goals of this note is to understand how the solutions behave with respect to α. Namely, we can see formally that NBC reduces to the Dirichlet boundary condition as $\alpha \to \infty$ and, in this article, we prove this rigorously by obtaining precise estimates on the solution with respect to α. Therefore, we may hope the possibility to transport some interesting properties, true for the Navier–Stokes problem with NBC, to one with no-slip boundary condition.

2. Linear problem

Since the case $\alpha \equiv 0$ on Γ has already been studied in [1], here we consider that $\alpha \not\equiv 0$. Precisely, we assume

$$\alpha \geq 0 \text{ on } \Gamma \quad \text{and} \quad \alpha > 0 \text{ on some } \Gamma_0 \subset \Gamma \text{ with } |\Gamma_0| > 0.$$

Let us introduce the notations:

$$L^p_0(\Omega) := \left\{ v \in L^p(\Omega); \int_\Omega v = 0 \right\}$$

and

$$\boldsymbol{\beta}(x) = \boldsymbol{b} \times \boldsymbol{x}$$

in the case Ω is axisymmetric with respect to a constant vector $\boldsymbol{b} \in \mathbb{R}^3$. Our first main result is the existence, uniqueness, and the estimates of weak solutions to the Stokes problem (S). For that, we need the following regularity assumption on α:

$$\alpha \in L^{t(p)}(\Gamma) \quad \text{with} \quad \begin{cases} t(p) = 2 & \text{if } p = 2 \\ t(p) > 2 & \text{if } \frac{3}{2} \leq p \leq 3, p \neq 2 \\ t(p) > \frac{2}{3}\max\{p, p'\} & \text{otherwise} \end{cases} \tag{1}$$

and where $t(p) = t(p')$. Moreover, we assume $\mathbb{F} \in \mathbb{L}^p(\Omega)$ is a 3×3 matrix, $\boldsymbol{h} \cdot \boldsymbol{n} = 0$ on Γ and

$$\boldsymbol{f} \in \boldsymbol{L}^{r(p)}(\Omega) \quad \text{with} \quad \begin{cases} r(p) = \max\left\{1, \frac{3p}{p+3}\right\} & \text{if} \quad p \neq \frac{3}{2} \\ r(p) > 1 & \text{if} \quad p = \frac{3}{2}. \end{cases} \tag{2}$$

Note that we can always reduce the non-vanishing divergence problem to the problem with zero divergence condition, considering a suitable Neumann problem.

Theorem 2.1 *(Existence and estimate of weak solution to the Stokes problem). Let $p \in (1, \infty)$ and*

$$\boldsymbol{f} \in \boldsymbol{L}^{r(p)}(\Omega),\ \mathbb{F} \in \mathbb{L}^p(\Omega),\ \boldsymbol{h} \in \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma) \text{ and } \alpha \in L^{t(p)}(\Gamma)$$

where $r(p)$ and $t(p)$ are defined in (2) and (1) respectively. Then the Stokes problem (S) has a unique solution $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p_0(\Omega)$, which satisfies the following estimates:

a) *If Ω is not axisymmetric, then*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\pi\|_{L^p(\Omega)} \le C_p(\Omega)\left(\|\boldsymbol{f}\|_{\boldsymbol{L}^{r(p)}(\Omega)} + \|\mathbb{F}\|_{\mathbb{L}^p(\Omega)} + \|\boldsymbol{h}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)}\right).$$

b) *If Ω is axisymmetric and*

i) $\alpha \ge \alpha_* > 0$ on Γ, then

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\pi\|_{L^p(\Omega)} \le \frac{C_p(\Omega)}{\min\{2, \alpha_*\}}\left(\|\boldsymbol{f}\|_{\boldsymbol{L}^{r(p)}(\Omega)} + \|\mathbb{F}\|_{\mathbb{L}^p(\Omega)} + \|\boldsymbol{h}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)}\right).$$

ii) $\boldsymbol{f}$, $\mathbb{F}$ and $\boldsymbol{h}$ satisfy the condition:

$$\int_\Omega \boldsymbol{f} \cdot \boldsymbol{\beta} - \int_\Omega \mathbb{F} : \nabla \boldsymbol{\beta} + \langle \boldsymbol{h}, \boldsymbol{\beta} \rangle_\Gamma = 0$$

and α is a non-zero constant, then

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\pi\|_{L^p(\Omega)} \le C_p(\Omega)\left(\|\boldsymbol{f}\|_{\boldsymbol{L}^{r(p)}(\Omega)} + \|\mathbb{F}\|_{\mathbb{L}^p(\Omega)} + \|\boldsymbol{h}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)}\right)$$

where $C_p(\Omega) > 0$ is independent of α.

Moreover, if

$$\boldsymbol{f} \in \boldsymbol{L}^p(\Omega),\ \boldsymbol{h} \in \boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma) \text{ and } \alpha \in W^{1-\frac{1}{q},q}(\Gamma)$$

with $q > \frac{3}{2}$ if $p \le \frac{3}{2}$ and $q = p$ otherwise, then the solution $(\boldsymbol{u}, \pi)$ of (S) with $\mathbb{F} = 0$ belongs to $\boldsymbol{W}^{2,p}(\Omega) \times W^{1,p}(\Omega)$, satisfying similar estimates as above.

Idea of the proof. The existence and uniqueness of a weak solution in $\boldsymbol{H}^1(\Omega)$ follows from the Lax–Milgram Lemma. For $p > 2$, we study a more general system where we use the inf–sup condition involving the **curl** operator, deduced from [2]; and then, for $p < 2$, a duality argument is employed as the bilinear form associated with the system (S) is symmetric.

Next, the existence of a strong solution for more regular data is deduced using a bootstrap argument.

For the uniform bounds with respect to α, we first obtain the following Caccioppoli-type inequality up to the boundary, for the Stokes system, where we use some suitable pressure estimate

$$\int_{B\cap\Omega} |\boldsymbol{u}|^2 + |\mathbb{D}\boldsymbol{u}|^2 \le C(\Omega)\left(\frac{1}{r^2}\int_{2B\cap\Omega} |\boldsymbol{u}|^2 + \int_{2B\cap\Omega} |\mathbb{F}|^2\right).$$

Here B is a ball centered on the boundary with radius r. From this, we then deduce the following weak reverse Hölder inequality

$$\left(\frac{1}{r^3}\int_{B\cap\Omega} \left(|\boldsymbol{u}|^2 + |\mathbb{D}\boldsymbol{u}|^2\right)^{p/2}\right)^{1/p} \le C_p(\Omega)\left[\left(\frac{1}{r^3}\int_{2B\cap\Omega} |\boldsymbol{u}|^2 + |\mathbb{D}\boldsymbol{u}|^2\right)^{1/2} + \left(\frac{1}{r^3}\int_{2B\cap\Omega} |\mathbb{F}|^p\right)^{1/p}\right].$$

This along with the uniform $\boldsymbol{H}^1$-estimate finally enables us to prove the desired estimate. □

The above Caccioppoli inequality has been deduced for the Stokes equation with Dirichlet boundary condition up to the boundary, for example in [6]. But it is new in the case of Navier boundary condition, and the novelty of our work is that we have employed it suitably to obtain the α-independent estimate.

In the following theorem, we derive some inf–sup condition from the above estimate result for a weak solution, which we believe is quite interesting on its own. We use the notation:

$$\mathbf{W}^{1,p}_{\sigma,\boldsymbol{\tau}}(\Omega) := \{\boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega) : \text{div}\, \boldsymbol{v} = 0 \text{ in } \Omega,\ \boldsymbol{v}\cdot\boldsymbol{n} = 0 \text{ on } \Gamma\}$$

endowed with the norm of $\boldsymbol{W}^{1,p}(\Omega)$.

Theorem 2.2. *Let $p \in (1,\infty)$ and $\alpha \in L^{t(p)}(\Gamma)$. We have the following inf–sup condition: when either (i) Ω is not axisymmetric or (ii) Ω is axisymmetric and $\alpha \geq \alpha_* > 0$,*

$$\inf_{\substack{\boldsymbol{u}\in \mathbf{W}^{1,p}_{\sigma,\boldsymbol{\tau}}(\Omega)\\ \boldsymbol{u}\neq 0}} \ \sup_{\substack{\boldsymbol{\varphi}\in \mathbf{W}^{1,p'}_{\sigma,\boldsymbol{\tau}}(\Omega)\\ \boldsymbol{\varphi}\neq 0}} \frac{\left|2\int_\Omega \mathbb{D}\boldsymbol{u} : \mathbb{D}\boldsymbol{\varphi} + \int_\Gamma \alpha \boldsymbol{u}_{\boldsymbol{\tau}}\cdot \boldsymbol{\varphi}_{\boldsymbol{\tau}}\right|}{\|\boldsymbol{u}\|_{\mathbf{W}^{1,p}_{\sigma,\boldsymbol{\tau}}(\Omega)}\ \|\boldsymbol{\varphi}\|_{\mathbf{W}^{1,p'}_{\sigma,\boldsymbol{\tau}}(\Omega)}} \geq \gamma(\Omega, p) \tag{3}$$

where the positive constant $\gamma(\Omega, p)$ does not depend on α.

Idea of the proof. We make use of the relation, for any $\boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega)$ with $\Delta\boldsymbol{v} \in \boldsymbol{L}^{r(p)}(\Omega)$ and $\boldsymbol{v}\cdot\boldsymbol{n} = 0$ on Γ,

$$2\,[(\mathbb{D}\boldsymbol{v})\boldsymbol{n}]_{\boldsymbol{\tau}} = \mathbf{curl}\, \boldsymbol{v}\times\boldsymbol{n} - 2\boldsymbol{\Lambda}\boldsymbol{v} \quad \text{in} \quad \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)$$

to convert the Navier boundary condition into one involving **curl** operator and then use the known inf–sup condition for the operator **curl**:

$$\inf_{\substack{\boldsymbol{\varphi}\in \boldsymbol{V}^{p'}(\Omega)\\ \boldsymbol{\varphi}\neq \mathbf{0}}} \ \sup_{\substack{\boldsymbol{\xi}\in \mathbf{W}^{1,p}_{\sigma,\boldsymbol{\tau}}(\Omega)\\ \boldsymbol{\xi}\neq 0}} \frac{\int_\Omega \mathbf{curl}\,\boldsymbol{\xi}\cdot \mathbf{curl}\,\boldsymbol{\varphi}}{\|\boldsymbol{\xi}\|_{\mathbf{W}^{1,p}_{\sigma,\boldsymbol{\tau}}(\Omega)}\|\boldsymbol{\varphi}\|_{\boldsymbol{V}^{p'}(\Omega)}} \geq C$$

where

$$\boldsymbol{V}^{p'}(\Omega) := \left\{\boldsymbol{v}\in \mathbf{W}^{1,p'}_{\sigma,\boldsymbol{\tau}}(\Omega);\ \langle \boldsymbol{v}\cdot\boldsymbol{n}, 1\rangle_{\Sigma_j} = 0 \quad \forall\, 1\leq j\leq J\right\}$$

and Σ_j are the cuts in Ω such that the open set $\Omega^0 = \Omega\backslash \bigcup_{j=1}^{J} \Sigma_j$ is simply connected (for details, see [2]).

3. Non-linear problem

Now we state our results regarding the Navier–Stokes problem (NS), which are based on the linear problem. In order to do so, we need the following estimates providing some suitable equivalent $\boldsymbol{H}^1(\Omega)$ norm.

Proposition 3.1. *Let Ω be Lipschitz. For Ω axisymmetric, we have the following inequalities: for all $\boldsymbol{u}\in \boldsymbol{H}^1(\Omega)$ with $\boldsymbol{u}\cdot\boldsymbol{n} = 0$ on Γ,*

$$\|\boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)} \leq C\left[\|\mathbb{D}\boldsymbol{u}\|^2_{\mathbb{L}^2(\Omega)} + \left(\int_\Omega \boldsymbol{u}\cdot\boldsymbol{\beta}\right)^2\right]$$

and

$$\|\boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)} \leq C\left[\|\mathbb{D}\boldsymbol{u}\|^2_{\mathbb{L}^2(\Omega)} + \left(\int_\Gamma \boldsymbol{u}\cdot\boldsymbol{\beta}\right)^2\right].$$

Theorem 3.1 *(Existence of a solution to the Navier–Stokes problem and estimate). Let $p \in (\frac{3}{2},\infty)$ and*

$$\boldsymbol{f}\in \boldsymbol{L}^{r(p)}(\Omega),\ \mathbb{F}\in \mathbb{L}^p(\Omega),\ \boldsymbol{h}\in \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma) \text{ and } \alpha\in L^{t(p)}(\Gamma).$$

1. *Then the problem (NS) has a solution $(\boldsymbol{u},\pi)\in \boldsymbol{W}^{1,p}(\Omega)\times L^p_0(\Omega)$.*
2. *Also for any $p\in(1,\infty)$, if $\mathbb{F} = 0$ and*

$$\boldsymbol{f}\in \boldsymbol{L}^p(\Omega),\ \boldsymbol{h}\in \boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma) \text{ and } \alpha\in W^{1-\frac{1}{q},q}(\Gamma)$$

with $q > \frac{3}{2}$ if $p\leq\frac{3}{2}$ and $q = p$ otherwise, then $(\boldsymbol{u},\pi)\in \boldsymbol{W}^{2,p}(\Omega)\times W^{1,p}(\Omega)$.

3. *For* $p = 2$*, the weak solution* $(\boldsymbol{u}, \pi) \in \boldsymbol{H}^1(\Omega) \times L^2_0(\Omega)$ *satisfies the following estimate: if* Ω *is not axisymmetric, then*

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} + \|\pi\|_{L^2(\Omega)} \le C(\Omega) \left(\|\boldsymbol{f}\|_{\boldsymbol{L}^{\frac{6}{5}}(\Omega)} + \|\mathbb{F}\|_{\mathbb{L}^2(\Omega)} + \|\boldsymbol{h}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)} \right) \tag{4}$$

where the constant $C(\Omega) > 0$ *is independent of* α*.*

Remark 1. We also obtain the estimate (4) in the case when Ω is axisymmetric and either (i) $\alpha \ge \alpha_* > 0$ on Γ or (ii) α is a non-zero constant and $\boldsymbol{f}, \mathbb{F}$ and $\boldsymbol{h}$ satisfy the condition:

$$\int_\Omega \boldsymbol{f} \cdot \boldsymbol{\beta} - \int_\Omega \mathbb{F} : \nabla \boldsymbol{\beta} + \langle \boldsymbol{h}, \boldsymbol{\beta} \rangle_\Gamma = 0.$$

Idea of the proof. First we show the existence of a solution for $p = 2$. The problem (NS) is equivalent to the following variational formulation: for all $\boldsymbol{\varphi} \in \mathbf{H}^1_{\sigma,\boldsymbol{\tau}}(\Omega) := \mathbf{W}^{1,2}_{\sigma,\boldsymbol{\tau}}(\Omega)$,

$$2 \int_\Omega \mathbb{D}\boldsymbol{u} : \mathbb{D}\boldsymbol{\varphi} + b(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{\varphi}) + \int_\Gamma \alpha \boldsymbol{u}_{\boldsymbol{\tau}} \cdot \boldsymbol{\varphi}_{\boldsymbol{\tau}} = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{\varphi} - \int_\Omega \mathbb{F} : \nabla \boldsymbol{\varphi} + \langle \boldsymbol{h}, \boldsymbol{\varphi} \rangle_\Gamma .$$

Using standard arguments, i.e. by Galerkin's method, we construct an approximate solution and then pass to the limit to obtain a solution to the above problem; and for $p > 2$, we can improve the integrability using the regularity of the linear problem.

Note that the existence of weak solution in $\boldsymbol{W}^{1,p}(\Omega)$ for $\frac{3}{2} < p < 2$ is not trivial and that we use the construction developed in [9]. Then the regularity for a strong solution follows using the bootstrap argument.

For the α-independent estimates, in the case of Ω not axisymmetric, as $\|\mathbb{D}\boldsymbol{u}\|_{\mathbb{L}^2(\Omega)}$ is an equivalent norm on $\boldsymbol{H}^1(\Omega)$ by Korn's inequality, we obtain the required estimate from the variational formulation. Similarly, the estimates for Ω axisymmetric can be deduced from the inequalities in Proposition 3.1. $\square$

Our last main result is the strong convergence of (NS) to the Navier–Stokes equation with no-slip boundary condition when α grows large. This can be shown using the estimates proved above.

Theorem 3.2 *(Limiting case for the Navier–Stokes problem). Let* $p \ge 2$*,* α *be a constant and* $(\boldsymbol{u}_\alpha, \pi_\alpha)$ *be a solution to* (NS)*, where*

$$\boldsymbol{f} \in \boldsymbol{L}^{r(p)}(\Omega), \;\; \mathbb{F} \in \mathbb{L}^p(\Omega) \text{ and } \boldsymbol{h} \in \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma).$$

Then

$$(\boldsymbol{u}_\alpha, \pi_\alpha) \to (\boldsymbol{u}_\infty, \pi_\infty) \quad \text{in} \quad \boldsymbol{W}^{1,p}(\Omega) \times L^p_0(\Omega) \quad \text{as} \quad \alpha \to \infty$$

where $(\boldsymbol{u}_\infty, \pi_\infty)$ *is a solution to the Navier–Stokes problem with Dirichlet boundary condition,*

$$\begin{cases} -\Delta \boldsymbol{u}_\infty + \boldsymbol{u}_\infty \cdot \nabla \boldsymbol{u}_\infty + \nabla \pi_\infty = \boldsymbol{f} + \operatorname{div} \mathbb{F} & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{u}_\infty = 0 & \text{in } \Omega, \\ \boldsymbol{u}_\infty = \boldsymbol{0} & \text{on } \Gamma. \end{cases}$$

Remark 2. The above technique can also be used to handle the non-linear dependence of α, as in the case of law walls used in turbulence, under suitable modification (work in progress).

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