



Partial differential equations/Harmonic analysis

Frequency decay for Navier–Stokes stationary solutions

Décroissance fréquentielle pour les équations de Navier–Stokes stationnaires

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ARTICLE INFO

Article history:

Received 10 May 2017

Accepted 1 December 2018

Available online 17 January 2019

Presented by Haïm Brézis

ABSTRACT

We consider stationary Navier–Stokes equations in \mathbb{R}^3 with a regular external force and we prove the exponential frequency decay of the solutions. Moreover, if the external force is small enough, we give a pointwise exponential frequency decay for such solutions. If a damping term is added to the equation, a pointwise decay is obtained without the smallness condition over the force.

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R É S U M É

Pour une force extérieure quelconque, mais suffisamment régulière, on démontre la décroissance fréquentielle des solutions de ces équations. Si, de plus, la force est petite, on peut décrire ponctuellement cette décroissance. La condition de petitesse de la force peut être supprimée si l'on rajoute un terme d'amortissement.

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Version française abrégée

Dans cette note, on s'intéresse aux équations de Navier–Stokes stationnaires (1) avec une force extérieure indépendante du temps. Si cette dernière est suffisamment régulière (Gevrey régulière), on montre dans le Théorème 1.2 que les solutions de ce problème stationnaire possèdent une décroissance exponentielle en variable de Fourier. Dans le cas d'une force extérieure *petite* (qui correspond à un cadre laminaire), il est possible d'être plus précis et, dans le Théorème 1.3, nous donnons avec l'inégalité (4) une estimation ponctuelle devant la décroissance du spectre de ces solutions stationnaires. Si, de plus, on considère un terme d'amortissement (voir Éq. (5)), on récupère avec le Théorème 1.4 l'estimation ponctuelle, sans condition de petitesse sur la force extérieure. Ces résultats étendent au cadre de l'espace tout entier et à des forces plus générales quelques travaux précédents concernant la régularité Gevrey des solutions des équations de Navier–Stokes stationnaires, voir [1], [6], [2] et [4].

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<https://doi.org/10.1016/j.crma.2018.12.007>

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1. Introduction

Gevrey’s regularity for solutions to the Navier–Stokes equations has been studied in many different frameworks: for example, in the periodic setting with external force (see [1], [6]) for the stationary problem in \mathbb{T}^3 with frequently localized forces; see [2]. For the evolution problem in \mathbb{R}^3 (with a null force), a pointwise analysis is obtained in [4]. In this note, we generalize some of these previous results in the framework of stationary Navier–Stokes equations in \mathbb{R}^3

$$-\nu \Delta \vec{U} + \mathbb{P}(\operatorname{div}(\vec{U} \otimes \vec{U})) = \vec{F}, \quad \operatorname{div}(\vec{U}) = 0, \quad \operatorname{div}(\vec{F}) = 0, \tag{1}$$

where $\nu > 0$ is the fluid’s viscosity parameter, $\vec{U} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is the velocity, \mathbb{P} is Leray’s projector and $\vec{F} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is a time-independent external force.

The existence of a solution for large data $\vec{F} \in \dot{H}^{-1}(\mathbb{R}^3)$ goes back to Leray, Finn and Ladyzhenskaya (see [3]) and is based on a priori energy estimates:

Lemma 1.1. *If $\vec{F} \in \dot{H}^{-1}(\mathbb{R}^3)$ with $\operatorname{div}(\vec{F}) = 0$, then there exists at least one solution $\vec{U} \in \dot{H}^1(\mathbb{R}^3)$ to the stationary Navier–Stokes equation (1) such that $\nu \|\vec{U}\|_{\dot{H}^1} \leq \|\vec{F}\|_{\dot{H}^{-1}}$.*

For a proof see [5], Theorem 16.2.

In this note, we study the behavior of such solutions; if the external force is regular enough, we prove in Theorem 1.2 an exponential frequency decay. Moreover, if the external force is small enough, we give in Theorem 1.3 a pointwise exponential frequency decay for such solutions. Finally, if a damping term is added to the equation, a pointwise decay is obtained in Theorem 1.4 without the smallness condition over the force.

Theorem 1.2. *Let $\vec{F} \in \dot{H}^{-1}(\mathbb{R}^3)$ with $\operatorname{div}(\vec{F}) = 0$ be such that for some $\varepsilon_0 > 0$ we have*

$$\int_{\mathbb{R}^3} e^{2\varepsilon_0|\xi|} |\widehat{\vec{F}}(\xi)|^2 |\xi|^{-2} d\xi < +\infty.$$

Then, for every solution \vec{U} to the stationary Navier–Stokes equations (1) such that $\vec{U} \in \dot{H}^1(\mathbb{R}^3)$, the following exponential frequency decay holds for some $\varepsilon_1 > 0$ (depending on \vec{U}):

$$\int_{\mathbb{R}^3} e^{2\varepsilon_1|\xi|} |\widehat{\vec{U}}(\xi)|^2 |\xi|^2 d\xi < +\infty. \tag{2}$$

In the laminar setting, we obtain a sharper pointwise exponential frequency decay.

Theorem 1.3. *There exists a (small) constant $\eta > 0$ such that if $\vec{F} \in \dot{H}^{-2} \cap \dot{H}^{-1}(\mathbb{R}^3)$ with $\|\vec{F}\|_{\dot{H}^{-1}} \|\vec{F}\|_{\dot{H}^{-2}} < \eta$, $\operatorname{div}(\vec{F}) = 0$ and if*

$$\sup_{\xi \in \mathbb{R}^3} \frac{1}{|\xi|} e^{\varepsilon_0|\xi|} |\widehat{\vec{F}}(\xi)| \leq c_0 < +\infty, \tag{3}$$

for some $c_0, \varepsilon_0 > 0$, then there exists a solution $\vec{U} \in L^2 \cap \dot{H}^1(\mathbb{R}^3)$ to the stationary Navier–Stokes equations (1) such that \vec{U} verifies the following pointwise exponential frequency decay for constants $c_1, \varepsilon_1 > 0$ that depend on \vec{U} :

$$|\widehat{\vec{U}}(\xi)| \leq c_1 e^{-\varepsilon_1|\xi|} |\xi|^{-1}, \quad \text{for all } \xi \neq 0. \tag{4}$$

If a damping term is added to the stationary Navier–Stokes system, we have the following result.

Theorem 1.4. *Let $\vec{F} \in \dot{H}^{-1}(\mathbb{R}^3)$ with $\operatorname{div}(\vec{F}) = 0$ and, for $\alpha > 0$, let us consider the damped stationary Navier–Stokes equations*

$$-\nu \Delta \vec{U} + \mathbb{P}(\operatorname{div}(\vec{U} \otimes \vec{U})) = \vec{F} - \alpha \vec{U}, \quad \operatorname{div}(\vec{U}) = 0. \tag{5}$$

If the external force \vec{F} satisfies the frequency decay (3), for some $c_0, \varepsilon_0 > 0$, then every stationary solution $\vec{U} \in \dot{H}^1(\mathbb{R}^3)$ satisfies the following pointwise exponential frequency decay for some $c_1, \varepsilon_1 > 0$ that depend on \vec{U} :

$$|\widehat{\vec{U}}(\xi)| \leq c_2 |\xi| e^{-\varepsilon_1|\xi|}. \tag{6}$$

2. Proof of Theorem 1.2

Lemma 2.1. *Let $T_0 > 0$. For $\vec{u}_0 \in \dot{H}^1(\mathbb{R}^3)$ a divergence-free initial data and $\vec{f} \in C([0, T_0[, \dot{H}^1(\mathbb{R}^3))$ a divergence-free external force, there exists a time $0 < T_1 < T_0$ and a function $\vec{u} \in C([0, T_1[, \dot{H}^1(\mathbb{R}^3))$, which is a unique solution to the Navier–Stokes equations*

$$\partial_t \vec{u} - \nu \Delta \vec{u} + \mathbb{P}(\text{div}(\vec{u} \otimes \vec{u})) = \vec{f}, \quad \text{div}(\vec{u}) = 0, \quad \vec{u}(0, \cdot) = \vec{u}_0. \tag{7}$$

Existence and uniqueness issues are classical, see [5] for details. In the following proposition, we prove the frequency decay for the solution \vec{u} obtained in Lemma 2.1.

Proposition 2.1. *Let $\beta > 0$ and consider the Poisson kernel $e^{\beta\sqrt{t}\sqrt{-\Delta}}$. Within the framework of Lemma 2.1, if the external force \vec{f} is such that $e^{\beta\sqrt{t}\sqrt{-\Delta}}\vec{f} \in C([0, T_0[, \dot{H}^1(\mathbb{R}^3))$, then the unique solution to Eq. (7) satisfies $e^{\beta\sqrt{t}\sqrt{-\Delta}}\vec{u} \in C([0, T_1[, \dot{H}^1(\mathbb{R}^3))$ for all time $t \in [0, T_1[$ where $0 < T_1 < T_0$ is small enough.*

Proof. Consider the space $E = \left\{ \vec{u} \in C([0, T_1[, \dot{H}^1(\mathbb{R}^3)) : e^{\beta\sqrt{t}\sqrt{-\Delta}}\vec{u} \in C([0, T_1[, \dot{H}^1(\mathbb{R}^3)) \right\}$, endowed with the norm $\|\cdot\|_E = \|e^{\beta\sqrt{t}\sqrt{-\Delta}}(\cdot)\|_{L_t^\infty \dot{H}_x^1}$. We study the quantity

$$\|\vec{u}_1\|_E = \left\| h_{\nu t} * \vec{u}_0 + \int_0^t h_{\nu(t-s)} * \vec{f}(s, \cdot) ds - \int_0^t h_{\nu(t-s)} * \mathbb{P}(\text{div}(\vec{u}_1 \otimes \vec{u}_1))(s, \cdot) ds \right\|_E, \tag{8}$$

where $h_{\nu t}$ is the heat kernel. The two first terms of this expression are easy to estimate and we have

$$\left\| h_{\nu t} * \vec{u}_0 + \int_0^t h_{\nu(t-s)} * \vec{f}(s, \cdot) ds \right\|_E \leq c(\nu, \beta, T_0) \left(\|\vec{u}_0\|_{\dot{H}_x^1} + \|e^{\beta\sqrt{t}\sqrt{-\Delta}}\vec{f}\|_{L_t^\infty \dot{H}_x^1} \right). \tag{9}$$

For the last term of (8), by definition of the norm $\|\cdot\|_E$, by the Plancherel formula and by the boundedness of the Leray projector, we have

$$\begin{aligned} (I) &= \left\| \int_0^t h_{\nu(t-s)} * \mathbb{P}(\text{div}(\vec{u}_1 \otimes \vec{u}_1)) ds \right\|_E = \sup_{0 < t < T_1} \left\| e^{\beta\sqrt{t}\sqrt{-\Delta}} \left(\int_0^t h_{\nu(t-s)} * \mathbb{P}(\text{div}(\vec{u}_1 \otimes \vec{u}_1)) ds \right) \right\|_{\dot{H}_x^1} \\ &\leq \sup_{0 < t < T_1} c \left\| |\xi|^2 \int_0^t e^{-\nu(t-s)|\xi|^2} e^{\beta\sqrt{t}|\xi|} |(\mathcal{F}[\vec{u}_1] * \mathcal{F}[\vec{u}_1])(s, \cdot)| ds \right\|_{L_x^2}. \end{aligned}$$

Since we have the pointwise inequality

$$e^{\beta\sqrt{t}|\xi|} |(\mathcal{F}[\vec{u}_1] * \mathcal{F}[\vec{u}_1])(s, \xi)| \leq \left[\left(e^{\beta\sqrt{t}|\xi|} |\mathcal{F}[\vec{u}_1]| \right) * \left(e^{\beta\sqrt{t}|\xi|} |\mathcal{F}[\vec{u}_1]| \right) \right] (s, \xi), \tag{10}$$

due to the fact that $e^{\beta\sqrt{t}|\xi|} \leq e^{\beta\sqrt{t}|\xi-\eta|} e^{\beta\sqrt{t}|\eta|}$ for all $\xi, \eta \in \mathbb{R}^3$, then we obtain

$$(I) \leq \sup_{0 < t < T_1} c \int_0^t \left\| |\xi|^{\frac{3}{2}} e^{-\nu(t-s)|\xi|^2} |\xi|^{\frac{1}{2}} \left[\left(e^{\beta\sqrt{t}|\xi|} |\mathcal{F}[\vec{u}_1]| \right) * \left(e^{\beta\sqrt{t}|\xi|} |\mathcal{F}[\vec{u}_1]| \right) \right] \right\|_{L_x^2} ds.$$

Getting back to the spatial variable, we can write

$$\begin{aligned} (I) &\leq \sup_{0 < t < T_1} c \int_0^t \left\| (-\Delta)^{\frac{3}{4}} h_{\nu(t-s)} * (-\Delta)^{\frac{1}{4}} \left\{ \left(\mathcal{F}^{-1} \left[e^{\beta\sqrt{t}|\xi|} |\mathcal{F}[\vec{u}_1]| \right] \right) \otimes \left(\mathcal{F}^{-1} \left[e^{\beta\sqrt{t}|\xi|} |\mathcal{F}[\vec{u}_1]| \right] \right) \right\} \right\|_{L_x^2} ds \\ &\leq \left(c \int_0^{T_1} \left\| (-\Delta)^{\frac{3}{4}} h_{\nu(t-s)} \right\|_{L^1} ds \right) \left\| \left(\mathcal{F}^{-1} \left[e^{\beta\sqrt{t}|\xi|} |\mathcal{F}[\vec{u}_1]| \right] \right) \otimes \left(\mathcal{F}^{-1} \left[e^{\beta\sqrt{t}|\xi|} |\mathcal{F}[\vec{u}_1]| \right] \right) \right\|_{L_t^\infty \dot{H}_x^{\frac{1}{2}}} \\ &\leq c \frac{T^{\frac{1}{4}}}{\nu^{\frac{3}{4}}} \left\| \mathcal{F}^{-1} \left[e^{\beta\sqrt{t}|\xi|} |\mathcal{F}[\vec{u}_1]| \right] \right\|_{L_t^\infty \dot{H}_x^1} \left\| \mathcal{F}^{-1} \left[e^{\beta\sqrt{t}|\xi|} |\mathcal{F}[\vec{u}_1]| \right] \right\|_{L_t^\infty \dot{H}_x^1} = c \frac{T^{\frac{1}{4}}}{\nu^{\frac{3}{4}}} \|\vec{u}_1\|_E \|\vec{u}_1\|_E. \end{aligned} \tag{11}$$

With estimates (9) and (11) at hand, we fix T_1 small enough in order to apply Picard's contraction principle and we obtain a solution $\vec{u}_1 \in E$ to (7). Since $E \subset C([0, T_1[, \dot{H}^1(\mathbb{R}^3))$, we have $\vec{u}_1 \in C([0, T_1[, \dot{H}^1(\mathbb{R}^3))$ and, by uniqueness of the solution \vec{u} , we have $\vec{u}_1 = \vec{u}$, and thus $\vec{u} \in E$. \square

Now, we come back to the stationary Navier–Stokes equations (1), and we will prove that the solution $\vec{U} \in \dot{H}^1(\mathbb{R}^3)$ (given by Lemma 1.1) satisfies the exponential frequency decay given in (2). In the space $C([0, 1[, \dot{H}^1(\mathbb{R}^3))$, we consider the evolution problem (7) with the initial data $\vec{u}_0 = \vec{U}$, where the external force \vec{f} is now given by the expression $\vec{f} = e^{-\beta\sqrt{t}\sqrt{-\Delta}}(e^{\beta\sqrt{t}\sqrt{-\Delta}}\vec{F})$ for the particular value $\beta = \frac{2}{3}\varepsilon_0 > 0$, where $\varepsilon_0 > 0$ is given in the hypothesis of the force \vec{F} . To obtain a unique solution $\vec{u} \in C([0, 1[, \dot{H}^1(\mathbb{R}^3))$ to Eq. (7) such that $e^{\beta\sqrt{t}\sqrt{-\Delta}}\vec{u} \in C([0, 1[, \dot{H}^1(\mathbb{R}^3))$, we prove that the external force \vec{f} verifies the hypotheses of Lemma 2.1 and Proposition 2.1 above:

$$\begin{aligned} \left\| e^{\beta\sqrt{t}\sqrt{-\Delta}}\vec{F} \right\|_{L_t^\infty \dot{H}_x^1}^2 &= \sup_{0 < t < 1} \int_{\mathbb{R}^3} |\xi|^2 e^{2\beta\sqrt{t}|\xi|} |\widehat{\vec{F}}(\xi)|^2 d\xi \leq \frac{1}{\beta^4} \int_{\mathbb{R}^3} (\beta|\xi|)^4 e^{2\beta|\xi|} |\widehat{\vec{F}}(\xi)|^2 |\xi|^{-2} d\xi \\ &\leq \frac{1}{\beta^4} \int_{\mathbb{R}^3} e^{3\beta|\xi|} |\widehat{\vec{F}}(\xi)|^2 |\xi|^{-2} d\xi = \frac{1}{\beta^4} \int_{\mathbb{R}^3} e^{2\varepsilon_0|\xi|} |\widehat{\vec{F}}(\xi)|^2 |\xi|^{-2} d\xi < +\infty. \end{aligned}$$

Thus, once we have $e^{\beta\sqrt{t}\sqrt{-\Delta}}\vec{F} \in C([0, 1[, \dot{H}^1(\mathbb{R}^3))$, since the operator $e^{-\beta\sqrt{t}\sqrt{-\Delta}}$ is bounded in the space $C([0, 1[, \dot{H}^1(\mathbb{R}^3))$, we have $\vec{f} = e^{-\beta\sqrt{t}\sqrt{-\Delta}}(e^{\beta\sqrt{t}\sqrt{-\Delta}}\vec{F}) \in C([0, 1[, \dot{H}^1(\mathbb{R}^3))$. Moreover, we have $e^{\beta\sqrt{t}\sqrt{-\Delta}}\vec{f} = e^{\beta\sqrt{t}\sqrt{-\Delta}}\vec{F} \in C([0, 1[, \dot{H}^1(\mathbb{R}^3))$. By Lemma 2.1, there exists a time $0 < T_1 < 1$ and a unique solution $\vec{u} \in C([0, T_1[, \dot{H}^1(\mathbb{R}^3))$ to Eq. (7). Moreover, since $e^{\beta\sqrt{t}\sqrt{-\Delta}}\vec{f} \in C([0, 1[, \dot{H}^1(\mathbb{R}^3))$, by Proposition 2.1 we have $e^{\beta\sqrt{t}\sqrt{-\Delta}}\vec{u} \in C([0, T_1[, \dot{H}^1(\mathbb{R}^3))$. Since the solution $\vec{U} \in \dot{H}^1(\mathbb{R}^3)$ to the stationary Navier–Stokes equations (1) is constant in time, we have $\vec{U} \in C([0, T_1[, \dot{H}^1(\mathbb{R}^3))$ and since $\partial_t \vec{U} \equiv 0$ and $\vec{f} = e^{-\beta\sqrt{t}\sqrt{-\Delta}}(e^{\beta\sqrt{t}\sqrt{-\Delta}}\vec{F}) = \vec{F}$, we find that $\vec{U} \in C([0, T_1[, \dot{H}^1(\mathbb{R}^3))$ is also a solution to Eq. (7) and thus, by uniqueness, we get $\vec{U} = \vec{u}$. Then, since $e^{\beta\sqrt{t}\sqrt{-\Delta}}\vec{u} \in C([0, T_1[, \dot{H}^1(\mathbb{R}^3))$, we have $e^{\beta\sqrt{t}\sqrt{-\Delta}}\vec{U} \in C([0, T_1[, \dot{H}^1(\mathbb{R}^3))$ for all time $t \in [0, T_1[$. Thus, if $\varepsilon_1 = \beta\sqrt{\frac{T_1}{2}} > 0$, we have:

$$\int_{\mathbb{R}^3} e^{2\varepsilon_1|\xi|} |\vec{U}(\xi)|^2 |\xi|^2 d\xi = \left\| e^{\beta\sqrt{\frac{T_1}{2}}\sqrt{-\Delta}}\vec{U} \right\|_{\dot{H}_x^1}^2 \leq \sup_{0 < t < T_1} \left\| e^{\beta\sqrt{t}\sqrt{-\Delta}}\vec{U} \right\|_{\dot{H}_x^1}^2 < +\infty,$$

and we obtain the frequency decay given in (2). \square

3. Proof of Theorem 1.3

In the case of a small force, we do not need to use the Schaefer fixed-point theorem to get the existence of a solution. The condition $\|\vec{F}\|_{\dot{H}^{-1}} \|\vec{F}\|_{\dot{H}^{-2}} < \eta$ ensures that $\|\vec{F}\|_{\dot{H}^{-3/2}}$ is small enough to allow the construction of a solution \vec{U} through Picard's iterative scheme, see [5] for details. Moreover, this solution will belong to $L^2 \cap \dot{H}^1 = H^1$.

Applying Theorem 1.2, we find that \vec{U} has an exponential decay

$$\int_{\mathbb{R}^3} e^{2\varepsilon_1|\xi|} |\widehat{\vec{U}}(\xi)|^2 |\xi|^2 d\xi \leq c_1 < +\infty.$$

Moreover, we may skip the weight $|\xi|^2$; as \vec{U} belongs to L^2 , we have

$$\int_{\mathbb{R}^3} e^{2\varepsilon_1|\xi|} |\widehat{\vec{U}}(\xi)|^2 d\xi \leq c_3 < +\infty. \tag{12}$$

Now we write

$$\vec{U} = \frac{1}{\nu} \mathbb{P} \left(\frac{1}{\Delta} \operatorname{div}(\vec{U} \otimes \vec{U}) \right) - \frac{1}{\nu} \frac{1}{\Delta} \vec{F},$$

and take the Fourier transform

$$|\widehat{\vec{U}}(\xi)| \leq c \frac{1}{\nu|\xi|} |\widehat{\vec{U}} * \widehat{\vec{U}}(\xi)| + \frac{1}{\nu|\xi|^2} |\widehat{\vec{F}}(\xi)|,$$

hence, by (12) and (3), we find

$$|\widehat{U}(\xi)| \leq \frac{1}{\nu|\xi|} e^{-\varepsilon_1|\xi|} \int_{\mathbb{R}^3} \left(e^{\varepsilon_1|\eta|} |\widehat{U}(\eta)| \right) \left(e^{\varepsilon_1|\xi-\eta|} |\widehat{U}(\xi-\eta)| \right) d\eta + c_0 \frac{e^{-\varepsilon_0|\xi|}}{\nu|\xi|} \leq \frac{c_3^2 + c_0}{\nu|\xi|} e^{-\varepsilon_2|\xi|},$$

with $\varepsilon_2 = \min(\varepsilon_0, \varepsilon_1) > 0$. \square

4. Proof of Theorem 1.4

For $\alpha > 0$, and under the hypotheses of Theorem 1.4, the existence of solutions to Eq. (5) in $H^1(\mathbb{R}^3)$ such that

$$\nu \|\nabla \otimes \vec{U}\|_2^2 + \alpha \|\vec{U}\|_2^2 \leq \|\vec{F}\|_{\dot{H}^{-1}} \|\nabla \otimes \vec{U}\|_2$$

is given by applying the Schaefer fixed-point theorem [5].

Following essentially the same lines as in Proposition 2.1 and Theorem 1.2 above, we find that \vec{U} has an exponential decay given by

$$\int_{\mathbb{R}^3} e^{2\varepsilon_1|\xi|} |\widehat{U}(\xi)|^2 d\xi \leq c_1 < +\infty. \tag{13}$$

Now, from equation (5), we write

$$\vec{U} = \mathbb{P} \left(\frac{1}{\nu\Delta - \alpha Id} [\operatorname{div}(\vec{U} \otimes \vec{U})] \right) - \frac{1}{\nu\Delta - \alpha Id} [\vec{F}],$$

and take the Fourier transform

$$|\widehat{U}(\xi)| \leq c \frac{|\xi|}{\nu|\xi|^2 + \alpha} |\widehat{U} * \widehat{U}(\xi)| + \frac{1}{\nu|\xi|^2 + \alpha} |\widehat{F}(\xi)|;$$

hence, by (13) and (3), we find

$$|\widehat{U}(\xi)| \leq \frac{|\xi|}{\alpha} e^{-\varepsilon_1|\xi|} \int_{\mathbb{R}^3} \left(e^{\varepsilon_1|\eta|} |\widehat{U}(\eta)| \right) \left(e^{\varepsilon_1|\xi-\eta|} |\widehat{U}(\xi-\eta)| \right) d\eta + c_0 \frac{|\xi| e^{-\varepsilon_0|\xi|}}{\alpha} \leq \frac{c_1^2 + c_0}{\alpha} |\xi| e^{-\varepsilon_2|\xi|},$$

with $\varepsilon_2 = \min(\varepsilon_0, \varepsilon_1) > 0$. \square

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