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Homological algebra

Derived invariance of the Tamarkin–Tsygan calculus of an algebra

Invariance dérivée du calcul de Tamarkin–Tsygan d'une algèbre

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ABSTRACT

We prove that derived equivalent algebras have isomorphic differential calculi in the sense of Tamarkin–Tsygan.

RÉSUMÉ

On montre que deux algèbres équivalentes par dérivation ont des calculs différentiels (au sens de Tamarkin–Tsygan) isomorphes.

1. Introduction

Let k be a commutative ring and A an associative k-algebra projective as a module over k. We write $\otimes$ for the tensor product over k. We point out that all the constructions and proofs of this paper extend to small dg categories cofibrant over k. The Hochschild homology $HH_\bullet(A)$ and cohomology $HH^\bullet(A)$ are derived invariants of A, see [3,4,9,10,12]. Moreover, these k-modules come with operations, namely the cup product

$$\cup : HH^n(A) \otimes HH^m(A) \to HH^{n+m}(A),$$

the Gerstenhaber bracket

$$[-,-] : HH^n(A) \otimes HH^m(A) \to HH^{n+m-1}(A),$$

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the cap product

$$\cap : HH_n(A) \otimes HH^m(A) \to HH_{n-m}(A)$$

and Connes' differential

$$B : HH_n(A) \to HH_{n+1}(A),$$

such that $B^2 = 0$ and

$$[Bi_\alpha - (-1)^{|\alpha|} i_\alpha, i_\beta] = i_{[\alpha,\beta]}, \tag{1}$$

where $i_\alpha(z) = (-1)^{|\alpha||z|} z \cap \alpha$. This is the first example [2,11] of a *differential calculus* or a *Tamarkin–Tsygan calculus*, which is by definition a collection

$$(\mathcal{H}^\bullet, \cup, [-,-], \mathcal{H}_\bullet, \cap, B),$$

such that $(\mathcal{H}^\bullet, \cup, [-,-])$ is a Gerstenhaber algebra, the cap product $\cap$ endows $\mathcal{H}_\bullet$ with the structure of a graded Lie module over the Lie algebra $(\mathcal{H}^\bullet[1], \cup, [-,-])$ and the map $B : \mathcal{H}_n \to \mathcal{H}_{n+1}$ squares to zero and satisfies the equation (1). The Gerstenhaber algebra $(HH^\bullet(A), \cup, [-,-])$ has been proved to be a derived invariant [8,7]. The cap product is also a derived invariant [1]. In this work, we use an isomorphism induced from the cyclic functor [6] to prove derived invariance of Connes' differential and of the ISB-sequence. To obtain derived invariance of the differential calculus, we need to prove that this isomorphism equals the isomorphism between Hochschild homologies used in [1] to prove derived invariance of the cap product.

2. The cyclic functor

Let **Alg** be the category whose objects are the associative dg (= differential graded) k-algebras cofibrant over k (i.e. 'closed' in the sense of section 7.5 of [6]) and whose morphisms are morphisms of dg k-algebras that do not necessarily preserve the unit. Let $\mathrm{rep}(A, B)$ be the full subcategory of the derived category $D(A^{op} \otimes B)$ whose objects are the dg bimodules X such that the restriction X_B is compact in $D(B)$, i.e. lies in the thick subcategory generated by the free module B_B. Define **ALG** to be the category whose objects are those of **Alg** and whose morphisms from A to B are the isomorphism classes in $\mathrm{rep}(A, B)$. The composition of morphisms in **ALG** is given by the total derived tensor product [6]. The identity of A is the isomorphism class of the bimodule ${}_A A_A$. There is a canonical functor **Alg** $\to$ **ALG** that associates with a morphism $f : A \to B$ the bimodule ${}_f B_B$ with underlying space $f(1)B$ and A-B-action given by $a.f(1)b.b' = f(a)bb'$.

Let Λ be the dg algebra $k[\epsilon]/(\epsilon^2)$ where $|\epsilon| = -1$ and the differential vanishes. As in [5,6], we will identify the category of dg Λ-modules with the category of mixed complexes. Denote by DMix the derived category of dg Λ-modules. Let C : **Alg** $\to$ DMix be the *cyclic functor* [6], that is, the underlying dg k-module of $C(A)$ is the mapping cone over $(1-t)$ viewed as a morphism of complexes $(A^{\otimes *+1}, b') \to (A^{\otimes *}, b)$ and the first and second differentials of the mixed complex $C(A)$ are

$$\begin{bmatrix} b & 1-t \\ 0 & -b' \end{bmatrix}$$

and

$$\begin{bmatrix} 0 & 0 \\ N & 0 \end{bmatrix}.$$

Clearly, a dg algebra morphism $f : A \to B$ (even if it does not preserve the unit) induces a morphism $C(f) : C(A) \to C(B)$ of dg Λ-modules. Let X be an object of $\mathrm{rep}(A, B)$. We assume, as we may, that X is cofibrant (i.e. 'closed' in the sense of section 7.5 of [6]). This implies that X_B is cofibrant as a dg B-module and thus that morphism spaces in the derived category with source X_B are isomorphic to the corresponding morphism spaces in the homotopy category. Consider the morphisms

$$A \xrightarrow{\alpha_X} \mathrm{End}_B(B \oplus X) \xleftarrow{\beta_X} B$$

where $\mathrm{End}_B(B \oplus X)$ is the differential graded endomorphism algebra of $B \oplus X$, the morphism α_X be given by the left action of A on X and β_X is induced by the left action of B on B. Note that these morphisms do not preserve the units. The second author proved in [6] that $C(\beta_X)$ is invertible in DMix and defined $C(X) = C(\beta_X)^{-1} \circ C(\alpha_X)$. We recall that C is well defined on **ALG** and that this extension of C from **Alg** to **ALG** is unique by Theorem 2.4 of [6].

Let $X : A \to B$ be a morphism of **ALG** where X is cofibrant. Put $X^\vee = \mathrm{Hom}_B(X, B)$. We can choose morphisms $u_X : A \to X \overset{\mathbf{L}}{\otimes}_B X^\vee$ and $v_X : X^\vee \overset{\mathbf{L}}{\otimes}_B X \to B$ such that the following triangles commute

$$\begin{array}{ccc} X & \xrightarrow{u_X \otimes 1} & X \overset{\mathbf{L}}{\otimes}_B X^\vee \overset{\mathbf{L}}{\otimes}_A X \\ & \searrow^{=} & \downarrow{\scriptstyle 1\otimes v_X} \\ & & X \end{array} \qquad \begin{array}{ccc} X^\vee & \xrightarrow{1 \otimes u_X} & X^\vee \overset{\mathbf{L}}{\otimes}_A X \overset{\mathbf{L}}{\otimes}_B X^\vee \\ & \searrow^{=} & \downarrow{\scriptstyle v_X\otimes 1} \\ & & X^\vee. \end{array}$$

Then the functors

$$? \overset{\mathbf{L}}{\otimes}_{A^e} (X \otimes X^\vee) : D(A^e) \to D(B^e)$$

and

$$? \overset{\mathbf{L}}{\otimes}_{B^e} (X^\vee \otimes X) : D(B^e) \to D(A^e)$$

form an adjoint pair. We will identify $X \overset{\mathbf{L}}{\otimes}_B X^\vee \overset{\sim}{\to} (X \otimes X^\vee) \overset{\mathbf{L}}{\otimes}_{B^e} B$ and $X^\vee \overset{\mathbf{L}}{\otimes}_A X \overset{\sim}{\to} (X^\vee \otimes X) \overset{\mathbf{L}}{\otimes}_{A^e} A$, and still call u_X and v_X the same morphisms when composed with this identification. Since k is a commutative ring, the tensor product over k is symmetric. We will denote the symmetry isomorphism by τ. Let $D(k)$ denote the derived category of k-modules. We define a functor $\psi : \mathbf{Alg} \to D(k)$ by putting $\psi(A) = A \overset{\mathbf{L}}{\otimes}_{A^e} A$, and $\psi(f) = f \otimes f$ for a morphism $f : A \to B$. There is a canonical quasi-isomorphism $\psi(A) \to \varphi(A)$ for any algebra A, where $\varphi(A)$ is the underlying complex of $C(A)$. Therefore, the functors φ and ψ take isomorphic values on objects. We now define ψ on morphisms of **ALG** as follows: Let X be a cofibrant object of $\mathrm{rep}(A, B)$. Define $\psi(X)$ to be the composition

$$\begin{aligned} A \overset{\mathbf{L}}{\otimes}_{A^e} A &\to A \overset{\mathbf{L}}{\otimes}_{A^e} X \otimes X^\vee \overset{\mathbf{L}}{\otimes}_{B^e} B \\ &\overset{\sim}{\to} B \overset{\mathbf{L}}{\otimes}_{B^e} X^\vee \otimes X \overset{\mathbf{L}}{\otimes}_{A^e} A \\ &\to B \overset{\mathbf{L}}{\otimes}_{B^e} B. \end{aligned}$$

That is, we put $\psi(X) = (1 \otimes v_X) \circ \tau \circ (1 \otimes u_X)$.

Theorem 2.1. *The assignments $A \mapsto \psi(A)$, $X \mapsto \psi(X)$ define a functor on **ALG** that extends the functor $\varphi : \mathbf{Alg} \to D(k)$.*

Corollary 2.2. *The functors φ and $\psi : \mathbf{ALG} \to D(k)$ are isomorphic.*

Proof of the Corollary. This is immediate from Theorem 2.4 of [6] and the remark following it. □

Proof of the Theorem. Let $f : A \to B$ be a morphism of **Alg**. The associated morphism in **ALG** is $X = {}_f B_B$. Note that $X^\vee = {}_B B_f$. The diagrams

$$\begin{array}{ccc} A \overset{\mathbf{L}}{\otimes}_{A^e} ({}_f B \overset{\mathbf{L}}{\otimes}_B B_f) & & \\ \downarrow{\scriptstyle \simeq} & \searrow^{\simeq} & \\ A \overset{\mathbf{L}}{\otimes}_{A^e} ({}_f B \overset{\mathbf{L}}{\otimes} B_f) \overset{\mathbf{L}}{\otimes}_{B^e} B & \xrightarrow[\simeq]{} & A \overset{\mathbf{L}}{\otimes}_{A^e} {}_f B_f \end{array}$$

and

$$\begin{array}{ccc} A \overset{\mathbf{L}}{\otimes}_{A^e} ({}_f B \overset{\mathbf{L}}{\otimes} B_f) \overset{\mathbf{L}}{\otimes}_{B^e} B & \xrightarrow[\simeq]{} & A \overset{\mathbf{L}}{\otimes}_{A^e} {}_f B_f \\ \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle \tau} \\ B \overset{\mathbf{L}}{\otimes}_{B^e} B_f \overset{\mathbf{L}}{\otimes} {}_f B \overset{\mathbf{L}}{\otimes}_{A^e} A & \xrightarrow{\simeq} & {}_f B_f \overset{\mathbf{L}}{\otimes}_{A^e} A \end{array}$$

are commutative. Since

$$\begin{array}{ccc} {}_f B_f \overset{\mathbf{L}}{\otimes}_{A^e} A & \xrightarrow{\tau} & A \overset{\mathbf{L}}{\otimes}_{A^e} {}_f B_f \\ \downarrow{\scriptstyle 1 \overset{\mathbf{L}}{\otimes} f} & & \downarrow{\scriptstyle f \overset{\mathbf{L}}{\otimes} 1} \\ B \overset{\mathbf{L}}{\otimes}_{B^e} B & \xrightarrow{\tau} & B \overset{\mathbf{L}}{\otimes}_{B^e} B \end{array}$$

is also commutative and the bottom morphism equals the identity, we get that $\psi({}_f B_B)$ is the morphism induced by f from $A \overset{\mathbf{L}}{\otimes}_{A^e} A$ to $B \overset{\mathbf{L}}{\otimes}_{B^e} B$. Therefore $\psi({}_f B_B) = \varphi({}_f B_B)$. Let $X : A \to B$ and $Y : B \to C$ be morphisms in **ALG**. We have the canonical isomorphisms

$$\begin{aligned} \mathrm{RHom}_C(Y, C) \overset{\mathbf{L}}{\otimes}_B \mathrm{RHom}_B(X, B) &\xrightarrow{\sim} \mathrm{RHom}_B(X, \mathrm{RHom}_C(Y, C)) \\ &\xrightarrow{\sim} \mathrm{RHom}_C(X \overset{\mathbf{L}}{\otimes}_B Y, C). \end{aligned}$$

Whence the identification

$$(X \overset{\mathbf{L}}{\otimes}_B Y)^\vee = Y^\vee \overset{\mathbf{L}}{\otimes}_B X^\vee.$$

Put $Z = X \overset{\mathbf{L}}{\otimes}_B Y$. For u_Z, we choose the composition

$$A \xrightarrow{u_X} X \overset{\mathbf{L}}{\otimes}_B X^\vee \xrightarrow{1 \otimes u_Y \otimes 1} X \overset{\mathbf{L}}{\otimes}_B Y \overset{\mathbf{L}}{\otimes}_C \overset{\mathbf{L}}{\otimes} Y^\vee \overset{\mathbf{L}}{\otimes}_B X^\vee$$

and for v_Z the composition

$$(Y^\vee \overset{\mathbf{L}}{\otimes}_B X^\vee) \overset{\mathbf{L}}{\otimes}_A (X \overset{\mathbf{L}}{\otimes}_B Y) \xrightarrow{1 \otimes v_X \otimes 1} Y^\vee \overset{\mathbf{L}}{\otimes}_B Y \xrightarrow{v_Y} C\ .$$

By definition, the composition $\psi(Y) \circ \psi(X)$ is the composition of $(1 \otimes v_Y) \circ \tau \circ (1 \otimes u_Y)$ with $(1 \otimes v_X) \circ \tau \circ (1 \otimes u_X)$. We first examine the composition $(1 \otimes u_Y) \circ (1 \otimes v_X)$:

$$B \overset{\mathbf{L}}{\otimes}_{B^e} (X^\vee \overset{\mathbf{L}}{\otimes} X) \overset{\mathbf{L}}{\otimes}_{A^e} A \xrightarrow{1 \otimes v_X} B \overset{\mathbf{L}}{\otimes}_{B^e} B \xrightarrow{1 \otimes u_Y} B \overset{\mathbf{L}}{\otimes}_{B^e} (Y \overset{\mathbf{L}}{\otimes} Y^\vee) \overset{\mathbf{L}}{\otimes}_{C^e} C$$

Clearly, the following square is commutative

$$\begin{array}{ccc} B \overset{\mathbf{L}}{\otimes}_{B^e} (X^\vee \overset{\mathbf{L}}{\otimes} X) \overset{\mathbf{L}}{\otimes}_{A^e} A & \xrightarrow{c} & ((X^\vee \overset{\mathbf{L}}{\otimes} X) \overset{\mathbf{L}}{\otimes}_{A^e} A) \overset{\mathbf{L}}{\otimes}_{B^e} B \\ {\scriptstyle 1 \otimes v_X}\downarrow & & \downarrow{\scriptstyle v_X \otimes 1} \\ B \overset{\mathbf{L}}{\otimes}_{B^e} B & \xrightarrow{\tau} & B \overset{\mathbf{L}}{\otimes}_{B^e} B \end{array},$$

where c is the obvious cyclic permutation. Notice that

$$\tau : B \overset{\mathbf{L}}{\otimes}_{B^e} B \to B \overset{\mathbf{L}}{\otimes}_{B^e} B$$

equals the identity. Thus, we have $1 \otimes u_Y = (1 \otimes u_Y) \circ \tau$ and

$$(1 \otimes u_Y) \circ (1 \otimes v_X) = (1 \otimes u_Y) \circ \tau \circ (1 \otimes v_X) = (1 \otimes u_Y) \circ (v_X \otimes 1) \circ c.$$

Let σ

$$((X^\vee \overset{\mathbf{L}}{\otimes} X) \overset{\mathbf{L}}{\otimes}_{A^e} A) \overset{\mathbf{L}}{\otimes}_{B^e} (Y \overset{\mathbf{L}}{\otimes} Y^\vee) \overset{\mathbf{L}}{\otimes}_{C^e} C \xrightarrow{\sim} A \overset{\mathbf{L}}{\otimes}_{A^e} (X \overset{\mathbf{L}}{\otimes}_B Y) \overset{\mathbf{L}}{\otimes} (Y^\vee \overset{\mathbf{L}}{\otimes}_B X^\vee) \overset{\mathbf{L}}{\otimes}_{C^e} C$$

be the natural isomorphism given by reordering the factors. Then we have $\psi(Y) \circ \psi(X) = f \circ g$, where $f = \sigma \circ (1 \otimes u_Y) \circ c \circ \tau \circ (1 \otimes u_X)$ and $g = (v_Y \otimes 1) \circ \tau \circ (v_X \otimes 1) \circ \sigma^{-1}$. It is not hard to see that f equals $1 \otimes u_Z$ and g equals $(1 \otimes v_Z) \circ \tau$. Intuitively, the reason is that, given the available data, there is only one way to go from $A \overset{\mathbf{L}}{\otimes}_{A^e} A$ to

$$A \overset{\mathbf{L}}{\otimes}_{A^e} (X \overset{\mathbf{L}}{\otimes}_B Y) \overset{\mathbf{L}}{\otimes} (Y^\vee \overset{\mathbf{L}}{\otimes}_B X^\vee) \overset{\mathbf{L}}{\otimes}_{C^e} C$$

and only one way to go from here to $C \overset{\mathbf{L}}{\otimes}_{C^e} C$. It follows that $\psi(Y) \circ \psi(X) = \psi(Z)$. □

3. Derived invariance

Let A and B be derived equivalent algebras and X be a cofibrant object of $\mathrm{rep}(A, B)$ such that $? \overset{\mathbf{L}}{\otimes}_A X : D(A) \to D(B)$ is an equivalence. Then $C(X)$ is an isomorphism of DMix and $\varphi(X)$ an isomorphism of $D(k)$. There is a canonical short exact sequence of dg Λ-modules

$$0 \to k[1] \to \Lambda \to k \to 0$$

giving rise to a triangle

$$k[1] \xrightarrow{B'} \Lambda \xrightarrow{I} k \xrightarrow{S} k[2] \, .$$

We apply the isomorphism of functors $? \overset{\mathbf{L}}{\otimes}_\Lambda C(A) \xrightarrow{\sim} ? \overset{\mathbf{L}}{\otimes}_\Lambda C(B)$ to this triangle to get an isomorphism of triangles in $D(k)$, where we recall that $\varphi(A)$ is the underlying complex of $C(A)$

$$\begin{array}{ccccccc}
k[1] \overset{\mathbf{L}}{\otimes}_\Lambda C(A) & \xrightarrow{B'} & \varphi(A) & \xrightarrow{I} & k \overset{\mathbf{L}}{\otimes}_\Lambda C(A) & \xrightarrow{S} & k[2] \overset{\mathbf{L}}{\otimes}_\Lambda C(A) \\
\cong \downarrow & & \downarrow \varphi(X) & & \cong \downarrow & & \cong \downarrow \\
k[1] \overset{\mathbf{L}}{\otimes}_\Lambda C(B) & \xrightarrow{B'} & \varphi(B) & \xrightarrow{I} & k \overset{\mathbf{L}}{\otimes}_\Lambda C(B) & \xrightarrow{S} & k[2] \overset{\mathbf{L}}{\otimes}_\Lambda C(B).
\end{array}$$

Taking homology and identifying $H_j(k \overset{\mathbf{L}}{\otimes}_\Lambda C(A)) = HC_j(A)$ as in [5], gives an isomorphism of the ISB-sequences of A and B,

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & HC_{n-1}(A) & \xrightarrow{B'_{n-1}} & HH_n(A) & \xrightarrow{I_n} & HC_n(A) & \xrightarrow{S_n} & HC_{n-2}(A) & \longrightarrow & \cdots \\
& & \cong \downarrow & & \downarrow HH_n(X) & & \cong \downarrow & & \cong \downarrow & & \\
\cdots & \longrightarrow & HC_{n-1}(B) & \xrightarrow{B'_{n-1}} & HH_n(B) & \xrightarrow{I_n} & HC_n(B) & \xrightarrow{S_n} & HC_{n-2}(B) & \longrightarrow & \cdots \quad ,
\end{array}$$

where $HH_n(X)$ is the map induced by $\varphi(X)$. In terms of the differential calculus, Connes' differential is the map

$$B_n : HH_n(A) \to HH_{n+1}(A),$$

given by $B_n = B'_n I_n$. This shows that B_n is a derived invariant via $HH_n(X)$. By Theorem 2.1, the map $HH_n(X)$ is equal to the map induced by $\psi(X)$. It is immediate that this map is precisely the one used in the proof of the derived invariance of the cap product [1]. Therefore, we get the following

Theorem 3.1. *The differential calculus of an algebra is a derived invariant.*

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