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Complex analysis/Analytic geometry

# A new proof of Kiselman's minimum principle for plurisubharmonic functions ☆

# *Une nouvelle démonstration du principe du minimum de Kiselman pour les fonctions pluri-sous-harmoniques*

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## A R T I C L E   I N F O

## A B S T R A C T

We give a new proof of Kiselman's minimum principle for plurisubharmonic functions, inspired by Demailly's regularization of plurisubharmonic functions by using Ohsawa–Takegoshi's extension theorem.

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## R É S U M É

Nous donnons une nouvelle démonstration du principe du minimum de Kiselman pour les fonctions pluri-sous-harmoniques. Elle s'inspire de la régularisation des fonctions pluri-sous-harmoniques de Demailly, en utilisant le théorème d'extension d'Ohsawa–Takegoshi.

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## 1. Introduction

The aim of this note is to provide a new proof of Kiselman's minimum principle for plurisubharmonic functions, inspired by Demailly's regularization of plurisubharmonic functions by using the Ohsawa–Takegoshi extension theorem about holomorphic functions.

**Theorem 1.1** *([6]). Let $\Omega \subset \mathbb{C}_t^r \times \mathbb{C}_z^n$ be a pseudoconvex domain and let $p : \Omega \to U := p(\Omega) \subset \mathbb{C}^r$ be the natural projection from $\Omega$ to $\mathbb{C}^r$. Let $\varphi$ be a plurisubharmonic function on $\Omega$. Assume that all fibers $\Omega_t := p^{-1}(t)$ $(t \in U)$ are connected tube domains and $\varphi(t, z)$ is independent of the image part of $z$ for all $(t, z) \in \Omega$. Then the function $\varphi^*$ defined as*

$$\varphi^*(t) := \inf_{z \in \Omega_t} \varphi(t, z)$$

*is a plurisubharmonic function on $U$.*

Let $T^n$ be the $n$-dimensional torus group. The natural action of $T^n$ on $\mathbb{C}^n$ is given by $(e^{i\theta_1}, \cdots, e^{i\theta_n}) \cdot (z_1, \cdots, z_n) := (e^{i\theta_1} z_1, \cdots, e^{i\theta_n} z_n)$. A domain $D$ in $\mathbb{C}^n$ is called a Reinhardt domain if it is invariant under the action of $T^n$.

Taking exponential map, it is obvious that Theorem 1.1 is a consequence of the following one.

**Theorem 1.2.** *Let $\Omega \subset \mathbb{C}_t^r \times \mathbb{C}_z^n$ be a pseudoconvex domain and let $p : \Omega \to U := p(\Omega) \subset \mathbb{C}^r$ be the natural projection from $\Omega$ to $\mathbb{C}^r$ such that all fibers $\Omega_t := p^{-1}(t)$ $(t \in U)$ are (connected) Reinhardt domains. Let $\varphi$ be a plurisubharmonic function on $\Omega$ such that $\varphi(t, \alpha z) = \varphi(t, z)$ for $\alpha \in T^n$. Then the function $\varphi^*$ defined as*

$$\varphi^*(t) := \inf_{z \in \Omega_t} \varphi(t, z)$$

*is a plurisubharmonic function on $U$.*

The argument in the proof of Theorem 1.2 can be generalized to more general settings considered in [4,5]. For example, in the same way one can show the following theorem.

**Theorem 1.3.** *Let $\Omega \subset \mathbb{C}_t^r \times \mathbb{C}_z^n$ be a pseudoconvex domain and let $p : \Omega \to U := p(\Omega) \subset \mathbb{C}^r$ be the natural projection from $\Omega$ to $\mathbb{C}^r$ such that all fibers $\Omega_t := p^{-1}(t)$ $(t \in U)$ are connected. Let $\varphi$ be a plurisubharmonic function on $\Omega$. Assume that there exists a compact Lie group $K$ acting on $\mathbb{C}^n$ holomorphically such that:*

(a) *the action of $K$ on $\mathbb{C}^n$ preserves the Lebesgue measure;*
(b) *all fibers $\Omega_t := p^{-1}(t)$ $(t \in U)$ are invariant under the action of $K$;*
(c) *$K$-invariant holomorphic functions on $\Omega_t$ are constant for $t \in U$;*
(d) *$\varphi$ is invariant under the action of $K$,*

*then the function $\varphi^*$ defined as*

$$\varphi^*(t) := \inf_{z \in \Omega_t} \varphi(t, z)$$

*is a plurisubharmonic function on $U$.*

Our proof of Theorem 1.2 is inspired by the method of Demailly on regularization of plurisubharmonic functions [2]. Applying Ohsawa–Takegoshi's extension theorem to extending holomorphic functions from discrete points, Demailly showed that a plurisubharmonic function can be approximated by certain Bergman kernels. In the recent work [3], the idea was developed to give a new characterization of plurisubharmonic functions.

It is natural to ask what can we get if we apply Demailly's idea of extending holomorphic functions from submanifolds of positive dimension. In this note, we show that this can lead to Kiselman's minimum principle for plurisubharmonic functions, namely the above theorems and their generalizations.

The above theorems were proved by Berndtsson in [1] (see also [4]) by showing a Prekopa-type theorem for plurisubharmonic functions. The method in this note is quite different from those in [6], [1] and [4].

## 2. Proof of the main result

In this section we give the proof of Theorem 1.3. We recall the Ohsawa–Takegoshi extension theorem for holomorphic functions.

**Lemma 2.1** *([7]). Let $\Omega \subset \mathbb{C}_t^r \times \mathbb{C}_z^n$ be a bounded pseudoconvex domain and let $p : \Omega \to U := p(\Omega) \subset \mathbb{C}^r$ be the natural projection from $\Omega$ to $\mathbb{C}^r$. Let $\varphi$ be a plurisubharmonic function on $\Omega$. Then, for any $t \in U$ and for any holomorphic function $f$ on $\Omega_t := p^{-1}(t)$, there exists a holomorphic function $F$ on $\Omega$ such that $F|_{\Omega_t} = f$, and*

$$\int_\Omega |F|^2 e^{-\varphi} \le C \int_{\Omega_t} |f|^2 e^{-\varphi_t},$$

*where $\varphi_t(z) = \varphi(t, z)$ and $C$ is a constant independent of $\varphi$, $t$, and $f$.*

We will prove the following Proposition 2.2 and show that Theorem 1.3 is a consequence of it.

**Proposition 2.2.** *Let $D \subset \mathbb{C}_z^n$ be a bounded pseudoconvex Reinhardt domain and $U \subset \mathbb{C}_t^r$, and let $\varphi$ be a plurisubharmonic function on $\Omega := U \times D$ that is continuous on the closure of $\Omega$. Assume that $\varphi(t, \alpha z) = \varphi(t, z)$ for $\alpha \in T^n$. Then the function $\varphi^*$ on $U$ defined by*

$$\varphi^*(t) := \inf_{z \in D} \varphi(t, z)$$

*is plurisubharmonic.*

**Proof.** Since $\varphi^*$ is upper-semicontinuous, we can assume that $U$ is a planar domain.

For any positive integer $m$, let $H^2(\Omega, m\varphi) = \{f \in \mathcal{O}(\Omega); \|f\|_m := \int_\Omega |f|^2 e^{-m\varphi} < \infty\}$ be the Hilbert space of holomorphic functions on $\Omega$ that are square integrable with respect to the weight $e^{-m\varphi}$, and let $H^2(\Omega, m\varphi)^I$ be the subspace of $H^2(\Omega, m\varphi)$ consisting of $T^n$-invariant holomorphic functions. We define $E_t^m = H^2(D, m\varphi_t)^I$ with $t \in U$ in the same way.

Let $\tilde{E}^m = U \times H^2(\Omega, m\varphi)^I$ and $E^m = \coprod_{t \in U} H^2(D, m\varphi_t)^I$. Then $\tilde{E}^m$ is a trivial and flat holomorphic hermitian vector bundle (of infinite rank) over $U$, and $E^m$ is a trivial holomorphic line bundle over $U$. Let $\mathbb{I}$ be the canonical holomorphic frame of $E^m$ that restricts to $\Omega_t := \{t\} \times D$ the constant function with value 1, for all $t \in U$. Then a holomorphic section of $E^m$ over $U$ can be naturally identified with a holomorphic function on $U$.

There is a canonical holomorphic bundle morphism $\pi : \tilde{E}^m \to E^m$, with $\pi_t : H^2(\Omega, m\varphi)^I \to E_t^m$ given by $f \mapsto f|_{\Omega_t}$. It is clear that $\pi$ is surjective and hence induces a (singular) hermitian metric on $E^m$, which is denoted by $h_m$. Explicitly, $h_m(\mathbb{I}_t) = \inf\{\|f\|_m; f \in H^2(\Omega, m\varphi)^I, f|_{\Omega_t} \equiv 1\}$.

Let $h^*$ be the metric on $E := E^1$ given by $h^*(\mathbb{I}) = e^{-\varphi^*}$. Let $h_m(\mathbb{I}) = e^{-m\varphi_m}$. Since $\tilde{E}^m$ is a flat bundle and $(E^m, h_m)$ is a quotient bundle of $\tilde{E}^m$, $(E^m, h_m)$ has semipositive curvature current and hence $\varphi_m(z)$ is a subharmonic function on $U$ for all $m$. We want to show that $\varphi_m$ converges to $\varphi^*$ in some sense as $m \to \infty$.

For fix $t \in U$, let $f \in H^2(\Omega, m\varphi)$ be the function with minimal norm such that $f|_{\Omega_t} \equiv 1$. By the uniqueness of the minimal element and the $T^n$-invariance of $\varphi$, it is clear that $f \in H^2(\Omega, m\varphi)^I$. By Lemma 2.1, we have the estimate

$$\int_\Omega |f|^2 e^{-m\varphi} \leq C \int_{\Omega_t} e^{-m\varphi_z},$$

where $C$ is a constant independent of $m$ and $t$.

By definition,

$$-\varphi_m(t) = \frac{1}{m} \log \left( \int_\Omega |f|^2 e^{-m\varphi} \right).$$

So we have

$$-\varphi_m(t) \leq \frac{1}{m} \log \left( \int_{\Omega_t} e^{-m\varphi_t} \right) + \frac{\log C}{m}. \tag{1}$$

We apply the mean value inequality to prove another inequality. For any $\epsilon > 0$, there is $0 < r << 1$ independent of $z$ such that $\varphi(t', z) \leq \varphi(t, z) + \epsilon$ for any $(t', z) \in \Omega$ with $|t - t'| < r$. By the mean value inequality, we have

$$\int_\Omega |f|^2 e^{-m\varphi}$$

$$\geq \int_{\Delta(t,r) \times D} |f|^2 e^{-m\varphi}$$

$$\geq \int_D \left( \int_{\Delta(t,r)} |f(\tau, z)|^2 \, d\mu_\tau \right) e^{-m(\varphi_z + \epsilon)} \, d\mu_z$$

$$\geq \pi r^2 \int_D e^{-m(\varphi_z + \epsilon)},$$

where $\Delta(t, r) = \{\tau \in \mathbb{C}; |\tau - t| < r\}$, and $d\mu_\tau$ and $d\mu_z$ are Lebesgue measures on $U$ and $D$, respectively. This implies

$$-\varphi_m(t) \geq \frac{1}{m} \log \left( \int_D e^{-m(\varphi_t + \epsilon)} \right) + \frac{\log \pi r^2}{m}. \tag{2}$$

Combining (1) and (2), we have

$$-\frac{1}{m} \log \left( \int_D e^{-m\varphi_t} \right) - \frac{\log C}{m} \leq \varphi_m \leq -\frac{1}{m} \log \left( \int_D e^{-m(\varphi_t + \epsilon)} \right) - \frac{\log \pi r^2}{m}, \tag{3}$$

which implies that $\varphi_m$ converges to $\varphi^*$ pointwise on $\Delta$.

Let $\tilde{\varphi}_m$ be the upper semicontinuous envelope of $\sup_{j \geq m} \varphi_j$ and let $\tilde{\varphi}^*$ be the limit of $\tilde{\varphi}_m$. Then $\tilde{\varphi}^*$ is plurisubharmonic and hence it suffices to prove $\varphi^* = \tilde{\varphi}^*$. By the first inequality in (3), it is obvious that $\tilde{\varphi}^* \geq \varphi^*$. Note that $\varphi$ is assumed to be uniformly continuous on the closure of $\Omega$, the last term in (3) converges uniformly on $U$ to $\varphi^* + \epsilon$ as $m \to \infty$. So we have $\tilde{\varphi}^* \leq \varphi^* + 2\epsilon$ if $m$ is large enough. Note that $\epsilon$ is arbitrary, $\tilde{\varphi}^* \leq \varphi^*$ and hence $\varphi^* = \tilde{\varphi}^*$. □

We now explain how to deduce Theorem 1.2 from Proposition 2.2. The argument is inspired by the idea in [1]. By approximation, we can assume that $\varphi$ is continuous. By replacing $\Omega$ by a smaller domain, we can assume that $\varphi$ is defined on some neighborhood of $\overline{\Omega}$, and there is a smooth plurisubharmonic function $\rho$ on some neighborhood of $\overline{\Omega}$ such that $\Omega$ is given by $\rho < 0$. By averaging, $\rho$ can be taken to be invariant under the action of $T^n$. Since the theorem is local for $U$, we can contract $U$ and find a pseudoconvex Reinhardt domain $V$ such that $\Omega \subset U \times V$ and $U \times V$ lies in the domain of definition of $\varphi$ and $\rho$. Applying Proposition 2.2 to the domain $U \times V$ and the function $\varphi + M \max\{\rho, 0\}$, and letting $M \to +\infty$, we get Theorem 1.2.

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