



Mathematical analysis/Partial differential equations

An Ikehara-type theorem for functions convergent to zero

Un théorème de type Ikehara pour les fonctions convergeant vers zéro

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ABSTRACT

We establish an analogue of the Ikehara theorem for positive non-increasing functions convergent to zero. In particular, this provides a complete proof of the results formulated in Diekmann & Kaper (1978) [5] and Carr & Chmaj (2004) [1], which are widely used nowadays to prove the uniqueness of traveling waves for various reaction–diffusion equations.

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R É S U M É

Nous établissons un analogue du théorème de Ikehara pour les fonctions positives non croissantes qui tendent vers zéro. En particulier, cela fournit une démonstration complète des énoncés formulés par Diekmann et Kaper (1978) [5] et par Carr et Chmaj (2004) [1], qui sont maintenant largement utilisés pour démontrer l'unicité des ondes progressives dans diverses équations de réaction–diffusion.

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1. Introduction

The Ikehara theorem and its extensions are the so-called complex Tauberian theorems, inspired, in particular, by the number theory, see, e.g., the review [12]. The following version of the Ikehara theorem can be found in [8, Subsection 2.5.7]:

Theorem 1. *Let ϕ be a positive monotone increasing function, and let there exist $\mu > 0$, $j > 0$, such that*

$$\int_0^{\infty} e^{-tz} \phi(t) dt = \frac{F(z)}{(z - \mu)^j}, \quad \operatorname{Re} z > \mu,$$

where F is holomorphic on $\{\operatorname{Re} z \geq \mu\}$. Then,

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$$\phi(t) \sim \frac{F(\mu)}{\Gamma(j)} t^{j-1} e^{\mu t}, \quad t \rightarrow \infty.$$

The case of $j = 1$ goes back to [11] and the general case was firstly considered in [4]. For more recent results in the case $j = 1$, see, e.g., [3,14]. Alternatively, Theorem 1 may be formulated for the Stieltjes measure $d\phi(t)$ instead of $\phi(t) dt$ obtaining similar asymptotics for ϕ (see, e.g., Proposition 2.1 below).

In Theorem 1, ϕ increases to ∞ . In [5, Lemma 6.1] (for $j = 1$) and in [1, Proposition 2.3] (for $j > 0$), similar results were stated for positive monotone decreasing φ (cf., correspondingly, Propositions 2.4 and 2.2 below). The aim of both generalizations was to find an *a priori* asymptotics for solutions to a class of nonlinear integral equations. In [1], in particular, it was applied to the study of the uniqueness of traveling wave solutions to certain nonlocal reaction–diffusion equations; see also, e.g., [17,16,13,2,19,6,7]. Note also that then the case $j = 2$ corresponded to the traveling wave with the minimal speed.

In both papers [5,1], no proof was given, mentioning that it is supposed to be analogous to the case of increasing ϕ without any further details. In Theorem 2 below, we prove an analogue of Theorem 1 for non-increasing functions, and in Proposition 2.2 we apply it to prove the mentioned result of [1]. We require, however, an *a priori* regular decaying of φ ; namely, we assume that there exists $\nu > 0$ such that $\varphi(t) e^{\nu t}$ is an increasing function. We require also the convergence of $\int_0^\infty e^{zt} d\varphi(t)$ for $0 < \operatorname{Re} z < \mu$ instead of the weaker corresponding assumption for $\int_0^\infty e^{zt} \varphi(t) dt$.

Beside the aim to present a proof, the reason for the generalization we provide was to omit the requirement on the function F to be analytical on the line $\{\operatorname{Re} z = \mu\}$ keeping the general case $j > 0$. We were motivated by the integro-differential equation that we studied in [10] (which covers the equations considered in [1]), where the Laplace-type transform of the traveling wave with the minimal speed (which requires, recall, $j = 2$) might be not analytical at $z = \mu$.

Our result is based on a version of the Ikehara–Ingham theorem proposed in [15], see Proposition 2.1 below. Using the latter result, we prove also in Proposition 2.4 a generalization of [5, Lemma 6.1] (under the regularity assumptions on φ mentioned above).

2. Main results

Let, for any $D \subset \mathbb{C}$, $\mathcal{H}(D)$ be the class of all holomorphic functions on D .

Theorem 2. Let $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+ := [0, \infty)$ be a non-increasing function such that, for some $\mu > 0$, $\nu > 0$,

$$\text{the function } e^{\nu t} \varphi(t) \text{ is non-decreasing,} \quad (2.1)$$

and

$$\int_0^\infty e^{zt} d\varphi(t) < \infty, \quad 0 < \operatorname{Re} z < \mu. \quad (2.2)$$

Let also the following assumptions hold.

1. There exist a constant $j > 0$ and complex-valued functions

$$H \in \mathcal{H}(0 < \operatorname{Re} z \leq \mu), \quad F \in \mathcal{H}(0 < \operatorname{Re} z < \mu) \cap C(0 < \operatorname{Re} z \leq \mu),$$

such that the following representation holds

$$\int_0^\infty e^{zt} \varphi(t) dt = \frac{F(z)}{(\mu - z)^j} + H(z), \quad 0 < \operatorname{Re} z < \mu. \quad (2.3)$$

2. For any $T > 0$,

$$\lim_{\sigma \rightarrow 0+} g_j(\sigma) \sup_{|\tau| \leq T} |F(\mu - 2\sigma - i\tau) - F(\mu - \sigma - i\tau)| = 0, \quad (2.4)$$

where, for $\sigma > 0$,

$$g_j(\sigma) := \begin{cases} \sigma^{j-1}, & 0 < j < 1, \\ \log \sigma, & j = 1, \\ 1, & j > 1. \end{cases} \quad (2.5)$$

Then φ has the following asymptotic

$$\varphi(t) \sim \frac{F(\mu)}{\Gamma(j)} t^{j-1} e^{-\mu t}, \quad t \rightarrow \infty. \tag{2.6}$$

The proof of Theorem 2 is based on the following result due to Tenenbaum.

Proposition 2.1 (“Effective” Ikehara–Ingham Theorem, cf. [15, Theorem 7.5.11]). *Let $\alpha(t)$ be a non-decreasing function such that, for some fixed $a > 0$, the following integral converges:*

$$\int_0^\infty e^{-zt} d\alpha(t), \quad \operatorname{Re} z > a. \tag{2.7}$$

Let also there exist constants $D \geq 0$ and $j > 0$, such that for the functions

$$G(z) := \frac{1}{a+z} \int_0^\infty e^{-(a+z)t} d\alpha(t) - \frac{D}{z^j}, \quad \operatorname{Re} z > 0, \tag{2.8}$$

$$\eta(\sigma, T) := \sigma^{j-1} \int_{-T}^T |G(2\sigma + i\tau) - G(\sigma + i\tau)| d\tau, \quad T > 0, \tag{2.9}$$

one has that

$$\lim_{\sigma \rightarrow 0^+} \eta(\sigma, T) = 0, \quad T > 0. \tag{2.10}$$

Then

$$\alpha(t) = \left\{ \frac{D}{\Gamma(j)} + O(\rho(t)) \right\} e^{at} t^{j-1}, \quad t \geq 1, \tag{2.11}$$

where

$$\rho(t) := \inf_{T \geq 32(a+1)} \left\{ T^{-1} + \eta(t^{-1}, T) + (Tt)^{-j} \right\}. \tag{2.12}$$

Proof of Theorem 2. We first express $\int_0^\infty e^{\lambda t} \varphi(t) dt$ in the form (2.7). Fix any $a > 0$ such that $\mu + a > \nu$. Then, by (2.1), the function

$$\alpha(t) := e^{(\mu+a)t} \varphi(t), \quad t > 0, \tag{2.13}$$

is increasing. Since φ is monotone, then, for any $0 < \operatorname{Re} z < \mu$, one has

$$\int_0^\infty e^{-(a+z)t} d\alpha(t) = (\mu + a) \int_0^\infty e^{(\mu-z)t} \varphi(t) dt + \int_0^\infty e^{(\mu-z)t} d\varphi(t), \tag{2.14}$$

where both integrals in the right-hand side of (2.14) converge, for $0 < \operatorname{Re} z < \mu$, because of (2.2)–(2.3).

Then, by [18, Corollary II.1.1a], the integral in the left-hand side of (2.14) converges, for all $\operatorname{Re} z > 0$. Therefore, by [18, Theorem II.2.3a], one gets another representation for the latter integral, for $\operatorname{Re} z > 0$:

$$\int_0^\infty e^{-(a+z)t} d\alpha(t) = -\varphi(0) + (a+z) \int_0^\infty e^{(\mu-z)t} \varphi(t) dt. \tag{2.15}$$

Let G be given by (2.8) with $\alpha(t)$ as above and $D := F(\mu)$. Combining (2.15) with (2.3) (where we replace z by $\mu - z$), we obtain, for $0 < \operatorname{Re} z < \mu$,

$$G(z) = \frac{F(\mu - z) - F(\mu)}{z^j} + K(z),$$

$$K(z) := H(\mu - z) - \frac{\varphi(0)}{a+z}.$$

We check the condition (2.10); one can assume, clearly, that $0 < \sigma < \frac{\mu}{2}$. Since $K \in \mathcal{H}(0 \leq \operatorname{Re} z < \mu)$, one easily gets that

$$\begin{aligned} & \lim_{\sigma \rightarrow 0+} \sigma^{j-1} \int_{-T}^T |G(2\sigma + i\tau) - G(\sigma + i\tau)| \, d\tau \\ & \leq \lim_{\sigma \rightarrow 0+} \sigma^{j-1} \int_{-T}^T \left| \frac{F(\mu - 2\sigma - i\tau) - F(\mu)}{(2\sigma + i\tau)^j} - \frac{F(\mu - \sigma - i\tau) - F(\mu)}{(\sigma + i\tau)^j} \right| \, d\tau \\ & \leq \lim_{\sigma \rightarrow 0+} \sigma^{j-1} \int_{-T}^T \left| \frac{F(\mu - 2\sigma - i\tau) - F(\mu - \sigma - i\tau)}{(\sigma + i\tau)^j} \right| \, d\tau \\ & \quad + \lim_{\sigma \rightarrow 0+} \sigma^{j-1} \int_{-T}^T |F(\mu - 2\sigma - i\tau) - F(\mu)| \left| \frac{1}{(2\sigma + i\tau)^j} - \frac{1}{(\sigma + i\tau)^j} \right| \, d\tau, \\ & =: \lim_{\sigma \rightarrow 0+} A_j(\sigma) + \lim_{\sigma \rightarrow 0+} B_j(\sigma). \end{aligned} \tag{2.16}$$

We now prove that both limits in (2.16) are equal to 0. For each $j > 0$, we define the function

$$h_j(\sigma) := \sigma^{j-1} \int_{-T}^T \frac{1}{(\sigma^2 + \tau^2)^{\frac{j}{2}}} \, d\tau, \quad \sigma > 0. \tag{2.17}$$

We have then

$$A_j(\sigma) \leq \sup_{|\tau| \leq T} |F(\mu - 2\sigma - i\tau) - F(\mu - \sigma - i\tau)| h_j(\sigma). \tag{2.18}$$

It is straightforward to check that

$$h_1(\sigma) = 2 \log \frac{\sqrt{T^2 + \sigma^2} + T}{\sigma} \sim -2 \log \sigma, \quad \sigma \rightarrow 0+.$$

For $j \neq 1$, we make the substitution $\tau = \sigma u$ in (2.17), then

$$h_j(\sigma) = \int_{-T/\sigma}^{T/\sigma} \frac{du}{(1 + u^2)^{j/2}}. \tag{2.19}$$

Then, for $j > 1$, $\lim_{\sigma \rightarrow 0} h_j(\sigma) < \infty$, since the corresponding integral is convergent, and we obtain $h_j(\sigma) = O(1)$, as $\sigma \rightarrow 0+$. For $0 < j < 1$,

$$h_j(\sigma) \leq O(1) + 2 \int_1^{T/\sigma} \frac{du}{u^j} = O(\sigma^{j-1}), \quad \sigma \rightarrow 0+.$$

Combining the results, we have that

$$h_j(\sigma) = O(|g_j(\sigma)|), \quad \sigma \rightarrow 0+, \quad j > 0,$$

that, together with (2.18) and (2.4), yield $\lim_{\sigma \rightarrow 0+} A_j(\sigma) = 0$.

Take now an arbitrary $\beta \in (0, \mu)$ and consider, for each $T > 0$, the set

$$K_{\beta, \mu, T} := \{z \in \mathbb{C} \mid \beta \leq \operatorname{Re} z \leq \mu, \quad |\operatorname{Im} z| \leq T\}. \tag{2.20}$$

Let $0 < \sigma < \mu^2$; since $F \in C(K_{\sqrt{\sigma}, \mu, T})$, there exists $C_1 > 0$ such that $|F(z)| \leq C_1, z \in K_{\sqrt{\sigma}, \mu, T}$. Therefore,

$$\begin{aligned} B_j(\sigma) & \leq \sigma^{j-1} \sup_{|\tau| \leq \sqrt{\sigma}} |F(\mu - 2\sigma - i\tau) - F(\mu)| \int_{|\tau| \leq \sqrt{\sigma}} \left| \frac{1}{(2\sigma + i\tau)^j} - \frac{1}{(\sigma + i\tau)^j} \right| \, d\tau \\ & \quad + 2C_1 \sigma^{j-1} \int_{\sqrt{\sigma} \leq |\tau| \leq T} \left| \frac{1}{(2\sigma + i\tau)^j} - \frac{1}{(\sigma + i\tau)^j} \right| \, d\tau. \end{aligned} \tag{2.21}$$

Next, since

$$j \left| \frac{1}{(2\sigma + i\tau)^j} - \frac{1}{(\sigma + i\tau)^j} \right| = \left| \int_{\sigma+i\tau}^{2\sigma+i\tau} \frac{1}{z^{j+1}} dz \right| = \left| \int_0^1 \frac{\sigma}{((1+t)\sigma + i\tau)^{j+1}} dt \right|$$

$$\leq \int_0^1 \frac{\sigma}{((1+t)^2\sigma^2 + \tau^2)^{\frac{j+1}{2}}} dt \leq \frac{\sigma}{(\sigma^2 + \tau^2)^{\frac{j+1}{2}}},$$

we can continue (2.21) as follows, cf. (2.17),

$$jB_j(\sigma) \leq \sup_{|\tau| \leq \sqrt{\sigma}} |F(\mu - 2\sigma - i\tau) - F(\mu)| h_{j+1}(\sigma)$$

$$+ 4C_1 \int_{\sqrt{\sigma} \leq \tau \leq T} \frac{\sigma^j}{(\sigma^2 + \tau^2)^{\frac{j+1}{2}}} d\tau. \tag{2.22}$$

By (2.19), functions h_{j+1} are bounded on $(0, \infty)$ for all $j > 0$. Next, since F is uniformly continuous on $K_{\sqrt{\sigma}, \mu, T}$, we have that, for any $\varepsilon > 0$, there exists $\delta > 0$ such that $f(\mu, \sigma, \tau) := |F(\mu - 2\sigma - i\tau) - F(\mu)| < \varepsilon$, if only $4\sigma^2 + \tau^2 < \delta$. Therefore, if $\sigma > 0$ is such that $4\sigma^2 + \sigma < \delta$, then $\sup_{|\tau| \leq \sqrt{\sigma}} f(\mu, \sigma, \tau) < \varepsilon$, hence

$$\sup_{|\tau| \leq \sqrt{\sigma}} |F(\mu - 2\sigma - i\tau) - F(\mu)| h_{j+1}(\sigma) \rightarrow 0, \quad \sigma \rightarrow 0+. \tag{2.23}$$

Finally, making the substitution $\tau = \sigma u$ in the integral in (2.22), we obtain that it is equal to

$$I_j := \int_{1/\sqrt{\sigma}}^{T/\sigma} \frac{du}{(1+u^2)^{\frac{j+1}{2}}} \leq \int_{1/\sqrt{\sigma}}^{T/\sigma} \frac{du}{u^{j+1}} = O(\sigma^{j/2}), \quad \sigma \rightarrow 0+.$$

As a result, $I_j \rightarrow 0$ as $\sigma \rightarrow 0+$, which, together with (2.23) and (2.22), proves that $B_j(\sigma) \rightarrow 0, \sigma \rightarrow 0+$.

Combining this with $A_j(\sigma) \rightarrow 0$, one gets (2.10) from (2.16), and we can apply Proposition 2.1. Namely, by (2.11),

$$\varphi(t) e^{(\mu+a)t} = \left\{ \frac{D}{\Gamma(j)} + O(\rho(t)) \right\} e^{at} t^{j-1}, \quad t \rightarrow \infty.$$

By (2.10) and (2.12), $\rho(t) \rightarrow 0$ as $t \rightarrow \infty$. Therefore,

$$\varphi(t) e^{(\mu+a)t} \sim \frac{D}{\Gamma(j)} e^{at} t^{j-1}, \quad t \rightarrow \infty,$$

which is equivalent to (2.6) and finishes the proof. \square

The following simple proposition shows that, if F in (2.3) is holomorphic on the line $\{\text{Re } z = \mu\}$, then (2.4) holds.

Proposition 2.2. *Let $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a non-increasing function such that, for some $\mu > 0, \nu > 0$, (2.1)–(2.2) hold. Suppose also that there exist $j > 0$ and $F, H \in \mathcal{A}(0 < \text{Re } z \leq \mu)$ such that (2.3) holds. Then φ has the asymptotic (2.6).*

Proof. Take any $\beta \in (0, \mu)$ and $T > 0$. Let $K_{\beta, \mu, T}$ be defined by (2.20). Since $F \in \mathcal{A}(0 < \text{Re } z \leq \mu)$, then $F' \in C(K_{\beta, \mu, T})$, and hence F' is bounded on $K_{\beta, \mu, T}$. Then one can apply a mean-value-type theorem for complex-valued functions, see, e.g., [9, Theorem 2.2], to get that, for some $K > 0$,

$$|F(\mu - 2\sigma - i\tau) - F(\mu - \sigma - i\tau)| \leq K|\sigma|, \quad 2\sigma < \mu - \beta,$$

which yields (2.4) for all $j > 0$, cf. (2.5). Hence, we can apply Theorem 2. \square

Remark 2.3. Note that, for $F \in \mathcal{A}(0 < \text{Re } z \leq \mu)$ in (2.3), the holomorphic function H is redundant there, as we always can replace $F(z)$ by a holomorphic function $F(z) + H(z)(\mu - z)^j$. Therefore, Proposition 2.2 corresponds to [1, Proposition 2.3].

Proposition 2.4. *Let $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a non-increasing function such that, for some $\mu > 0, \nu > 0$, (2.1)–(2.2) hold. Suppose also that there exist $j \geq 1, D > 0$, and $h : \mathbb{R} \rightarrow \mathbb{R}$ such that*

$$H(z) := \int_0^{\infty} e^{zt} \varphi(t) dt - \frac{D}{(\mu - z)^j} \rightarrow h(\operatorname{Im} z), \quad \operatorname{Re} z \rightarrow \mu-, \quad (2.24)$$

uniformly (in $\operatorname{Im} z$) on compact subsets of \mathbb{R} . Then the following asymptotic holds,

$$\varphi(t) \sim \frac{D}{\Gamma(j)} t^{j-1} e^{-\mu t}, \quad t \rightarrow \infty. \quad (2.25)$$

Proof. Let $a > \max\{0, \nu - \mu\}$ and $\alpha(t)$ be given by (2.13). Let G be given by (2.8). Similarly to the proof of Theorem 2, we will get from (2.15) and (2.24), that

$$G(z) = H(\mu - z) - \frac{\varphi(0)}{a + z}, \quad 0 < \operatorname{Re} z < \mu.$$

Next, (2.24) implies (2.10). Hence, by Lemma 2.1, (2.25) holds, which fulfills the proof. \square

Note that the result in [5, Lemma 6.1] corresponds to $j = 1$ in Proposition 2.4.

Remark 2.5. It is worth noting that, for the case $j > 1$, we have, by (2.9), that if G is bounded, then (2.10) holds. Therefore, in this case, it is enough to assume that H in (2.24) is bounded to conclude (2.25).

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