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Statistics

Asymptotic normality of a robust kernel estimator of the regression function for functional ergodic data: Case of an unknown scale parameter

Normalité asymptotique de l'estimateur à noyau de la fonction de régression robuste du paramètre d'échelle pour des données fonctionnelles ergodiques : cas d'un paramètre d'échelle inconnu

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ABSTRACT

Under a nonparametric robust regression model, we consider the problem of estimating the score function ψ_x for a fixed x in a functional space and with unknown scale parameter. The principal aim of this work is to establish the asymptotic normality of this estimator for a stationary ergodic process without any use of traditional mixing conditions.

RÉSUMÉ

Sous un modèle de régression non paramétrique robuste, nous considérons le problème d'estimation de la fonction de score ψ_x pour x fixé dans un espace fonctionnel et quand le paramètre d'échelle est inconnu. L'objectif principal est d'établir la normalité asymptotique de cet estimateur pour un processus ergodique stationnaire sans hypothèse de mélange.

1. Introduction

Let $(X_i, Y_i)_{i=1,\ldots,n}$ be a sequence of strictly stationary dependent random variables that are valued in $\mathcal{F} \times \mathbb{R}$, where $\mathcal{F}$ is a semi-metric space equipped with a semi-metric d.

In this contribution, we deal with the nonparametric estimation of the robust regression, say $\theta(x)$, when the scale parameter is unknown and data are ergodic. In fact, for any $x \in \mathcal{F}$, $\theta(x)$ is defined as a zero with respect (w.r.t.) to the parameter a of the following equation:

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$$\Psi(x, a, \sigma) = E\left[\psi_x\left(\frac{Y-a}{\sigma}\right)|X=x\right] = 0 \tag{1.1}$$

where ψ_x is a real-valued function that satisfies some regularity conditions, to be stated below, and σ is a robust measure of conditional scale. In what follows, we assume, for all $x \in \mathcal{F}$, that the robust regression $\theta(x)$ exists and is unique (see, for instance, Boente and Fraiman [5]).

Recall that the robustification method is an old topic in statistics. The latter was investigated first by Huber [11], who studied an estimation of a location parameter (see also Collomb and Härdle [7], Laïb and Ould-Saïd [14]) for some results including the multivariate time series case under a mixing or an ergodic condition). Motivated by its flexibility when data are affected by outliers, the robust regression was widely studied in nonparametric functional statistics. Indeed, it was firstly introduced by Azzedine et al. [3], who proved the almost-complete convergence of this model in the independent and identically distributed (i.i.d.) case. Since this work, several results on the nonparametric robust functional regression were realized (see, for instance, Crambes et al. [8], Attouch et al. [1,2], Gheriballah et al. [10], Boente et al. [6] and references therein for some key references on this topic). Notice that all these results are obtained when the scale parameter is fixed.

In this paper, we consider the more general case when the scale parameter is unknown and data come from an ergodic functional time series. It should be noticed that the ergodicity hypothesis is less restrictive than the mixing condition, which is usually assumed in functional time series studies. The literature on ergodic functional time series data is still limited and the few existing results are due to Laïb and Louani [12,13], Gheriballah et al. [10], Benziadi et al. [4] and references therein.

Then, we aim to generalize results of Boente et al. [6] that were obtained in the i.i.d case to the ergodic one. Precisely, we prove the asymptotic normality of the same constructed estimator under standard conditions allowing us to explore different structural axes of the topic. We emphasize that, contrary to the usual case where the scale parameter is fixed, it must be estimated here, which makes it more difficult to establish the asymptotic properties of the estimator. But, although this difference is more important in the context of this work, we have been able to overcome it.

The rest of this paper is organized as follows. Section 2 is devoted to the presentation of the robust estimator. Then, the main result is given in Section 3. Finally, interpretations and some research comments are given in the last section.

2. Estimation of the robust regression estimator when the scale parameter is unknown

Throughout this paper, we will assume that $Z_i = (X_i, Y_i)_{i=1,\ldots,n}$ is a functional stationary ergodic process (see Laïb and Louani [13] for some definitions and examples). When the scale parameter in unknown, the robust estimator may be constructed following the two steps. Firstly, we estimate the scale parameter σ by the local median of the conditional distribution of Y given $X = x$, denoted by $F(y|X=x) = \mathbb{E}\left(\mathbb{I}_{(-\infty,y]}(Y)\,|X=x\right)$, for any $y \in \mathbb{R}$, where $\mathbb{I}_A$ denotes the indicator function on the set A. Then, for $x \in \mathcal{F}$, the kernel estimator $\widehat{s}(x)$ of $\sigma(x)$ is the zero of the following equation

$$\widehat{F}(s|X=x) = \frac{1}{2}$$

where $\widehat{F}(y|X=x)$ is given by

$$\widehat{F}(y|X=x) = \frac{\sum_{i=1}^{n} K\left(h^{-1}d(x, X_i)\right)\mathbb{I}_{(-\infty,y]}(Y_i)}{\sum_{i=1}^{n} K\left(h^{-1}d(x, X_i)\right)}$$

where K is a kernel function and $h = h_n$ is a sequence of positive numbers that goes to zero as n goes to infinity. Next, the kernel estimator $\widehat{\theta}(x)$ of the robust regression $\theta(x)$, is the zero, w.r.t. a, of the equation

$$\widehat{\Psi}(x, a, \widehat{s}) = 0$$

where

$$\widehat{\Psi}(x, a, \widehat{s}) = \frac{\sum_{i=1}^{n} K\left(h^{-1}d(x, X_i)\right)\psi_x\left(\frac{Y_i - a}{\widehat{s}}\right)}{\sum_{i=1}^{n} K\left(h^{-1}d(x, X_i)\right)}.$$

3. Main result

When no confusion is possible, we will denote by C some strictly positive generic constant, by x a fixed point in $\mathcal{F}$, by $\mathcal{N}_x$ a fixed neighborhood of x, and for $r > 0$, $B(x, r) := \left\{x' \in \mathcal{F} \text{ such that } d(x', x) < r\right\}$.

Moreover, for $k = 1, \dots, n$, let $\mathcal{F}_k$ denotes the σ-field generated by $((X_1, Y_1), \dots, (X_k, Y_k))$ and G_k the σ-field generated by $((X_1, Y_1), \dots (X_k, Y_k), X_{k+1})$. In addition, we set

$$\lambda_\gamma (u, a, \sigma) = E\left[\psi_x^\gamma \left(\frac{Y-a}{\sigma}\right)|X = u\right] \text{ and } \Gamma_\gamma (u, a, \sigma) = E\left[\left(\psi_x'\right)^\gamma \left(\frac{Y-a}{\sigma}\right)|X = u\right],$$

for $\gamma \in \{1, 2\}$ and ψ_x' is the derivative function with respect to y of the function ψ_x.

Our basic assumptions are the following.

H1. The process $(X_i, Y_i)_{i\in\mathbb{N}}$ satisfies:

(i) for all $r > 0$

$$\phi (x, r) = P (X \in B (x, r)) > 0 \text{ and } \phi_i (x, r) = P (X_i \in B (x, r) |\mathcal{F}_{i-1}) > 0$$

(ii) for all $r > 0$

$$\frac{1}{n\phi (x, r)} \sum_{i=1}^{n} \phi_i (x, r) \xrightarrow{P} 1 \text{ and } n\phi (x, h) \to \infty \text{ as } h \to 0$$

where $\xrightarrow{P}$ denotes the convergence in probability.

H2. The function Ψ is such that:

(i) the function $\Psi (x, ., \sigma)$ is of class $\mathcal{C}^1$ w.r.t. the second component at a fixed neighborhood $\mathcal{N}_x$ of $\theta (x)$;

(ii) for each fixed a in $\mathcal{N}_x$, the functions $\Psi (., a, \sigma)$ and $\lambda_2 (., a, \sigma)$ are continuous at the point x;

(iii) the derivative of the real function

$$\Phi (x, z, \sigma) = \mathbb{E} [\Psi (X_1, z, \sigma) - \Psi (x, z, \sigma) |d (x, X_1) = s]$$

exists at $s = 0$ and is continuous w.r.t. the second component at $\mathcal{N}_x$.

H3. For each fixed a in the neighborhood of $\theta (x)$, we have:

$$\mathbb{E}\left[\psi_x^2 \left(\frac{Y-a}{\sigma}\right) |\mathcal{F}_{i-1}\right] = \mathbb{E}\left[\psi_x^2 \left(\frac{Y-a}{\sigma}\right) |X_i\right] < C < \infty, \text{ almost surely.}$$

H4. The function ψ_x is a continuous and monotonous function w.r.t. the second component.

H5. The kernel K is a positive function that is supported on $(0, 1[$. Its first derivative K' exists on $(0, 1)$ and satisfies $K' (a) < 0$ for $0 < a < 1$.

H6. There exists a function $\tau_x (.)$ such that, for all $a \in [0, 1]$

$$\lim_{h\to 0} \frac{\phi (x, ah)}{\phi (x, h)} = \tau_x (a), K^2 (1) - \int_0^1 \left(K^2 (u)\right)' \tau_x (u)\, du > 0 \text{ and } K (1) - \int_0^1 K' (u) \tau_x (u)\, du \neq 0.$$

Some comments on the hypotheses. The assumptions that allowed us to obtain the asymptotic distribution of our estimator are sufficiently moderate. In particular, unlike most studies done on the subject of this work (see H4), we obtained the asymptotic normality without the boundedness condition on the score function. In addition, the hypothesis H1 is the same as that used by Gheriballah et al. [10], while condition H2 is necessary to evaluate the bias term. Finally, conditions H3, H5, and H6 are very similar to those used by Ferraty et al. [9].

We can now formulate our main result.

Theorem 3.1. *Assume that assumptions H1–H6 hold, then $\widehat{\theta} (x)$ exists and is unique for any $x \in \mathcal{A}$, and we have*

$$\left(\frac{n\phi (x, h)}{\sigma^2 (x, \theta (x))}\right)^{1/2} \left(\widehat{\theta} (x) - \theta (x) - B_n (x)\right) \xrightarrow{D} N (0, 1) \text{ as } n \to \infty$$

where $B_n (x) = h\Phi' (0, \theta (x)) \dfrac{\beta_0}{\beta_1} + o (h)$ *and* $\sigma^2 (x, \theta (x)) = \dfrac{\beta_2 \lambda_2 (x, \theta (x), \sigma)}{\beta_1^2 (\Gamma_1 (x, \theta (x), \sigma))^2}$

with $\beta_0 = K(1) - \displaystyle\int_0^1 (sK (s))' \tau_x (s)\, ds$, $\beta_j = K^j(1) - \displaystyle\int_0^1 \left(K^j\right)' (s) \tau_x (s)\, ds$, *for j= 1,2,*

$$\Gamma_1 (x, \theta (x), \sigma) = \frac{\partial \Psi (x, \theta (x), \sigma)}{\partial a} \text{ and } \mathcal{A} = \{z \in \mathcal{F} | \lambda_2 (z, \theta (z), \sigma) \Gamma_1 (z, \theta (z), \sigma) \neq 0\}$$

where $\xrightarrow{D}$ means the convergence in distribution.

Some comments on the main result. Noting that the present contribution consider some generalized robust regression case under the ergodic property. In particular, the unknown scale parameter case covers a more complicated situation when the links between the response variable and the explanatory variable is defined by

$$Y = r(x) + \sigma(x)\epsilon$$

where ϵ is a random variable that is independent of X and the two functions $\sigma(.)$ and $r(.)$ are unknown. So, conversely to the case of a fixed scale parameter, here, we have to estimate the two nonparametric models, which makes the establishment of the asymptotic properties of the robust estimator more difficult. It is based on some additional techniques and tools. The proof of the main result can be obtained on simple request.

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