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Partial differential equations/Harmonic analysis

On the representation as exterior differentials of closed forms with L^1-coefficients

Sur la représentation comme différentielles extérieures des formes fermées à coefficients L^1

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ABSTRACT

Let $N \geq 2$. If $g \in L^1_c(\mathbf{R}^N)$ has zero integral, then the equation $\operatorname{div} X = g$ need not have a solution $X \in W^{1,1}_{\text{loc}}(\mathbf{R}^N;\mathbf{R}^N)$ [6] or even $X \in L^{N/(N-1)}_{\text{loc}}(\mathbf{R}^N;\mathbf{R}^N)$ [2]. Using these results, we prove that, whenever $N \geq 3$ and $2 \leq \ell \leq N-1$, there exists some ℓ-form $f \in L^1_c(\mathbf{R}^N;\Lambda^\ell)$ such that $\mathrm{d}f = 0$ and the equation $\mathrm{d}\lambda = f$ has no solution $\lambda \in W^{1,1}_{\text{loc}}(\mathbf{R}^N;\Lambda^{\ell-1})$. This provides a negative answer to a question raised by Baldi, Franchi, and Pansu [1].

RÉSUMÉ

Soit $N \geq 2$. Si $g \in L^1_c(\mathbf{R}^N)$ est d'intégrale nulle, alors en général il n'est pas possible de résoudre l'équation $\operatorname{div} X = g$ avec $X \in W^{1,1}_{\text{loc}}(\mathbf{R}^N;\mathbf{R}^N)$ [6], ou même $X \in L^{N/(N-1)}_{\text{loc}}(\mathbf{R}^N;\mathbf{R}^N)$ [2]. En utilisant ces résultats, nous prouvons que, pour $N \geq 3$ et $2 \leq \ell \leq N-1$, il existe une ℓ-forme $f \in L^1_c(\mathbf{R}^N;\Lambda^\ell)$ avec $\mathrm{d}f = 0$ et telle que l'équation $\mathrm{d}\lambda = f$ n'a pas de solution $\lambda \in W^{1,1}_{\text{loc}}(\mathbf{R}^N;\Lambda^{\ell-1})$. Ceci donne une réponse négative à une question posée par Baldi, Franchi et Pansu [1].

Version française abrégée

Répondant à une question posée par Baldi, Franchi et Pansu ([1]), nous montrons le résultat suivant :

Théorème 1. *Soit N, ℓ des entiers tels que $N \geq 3$ et $2 \leq \ell \leq N-1$. Il existe une ℓ-forme différentielle fermée à coefficients L^1, $f \in L^1_c(\mathbf{R}^N;\Lambda^\ell)$, telle que l'équation $\mathrm{d}\lambda = f$ n'a pas de solution $\lambda \in W^{1,1}_{\text{loc}}(\mathbf{R}^N;\Lambda^{\ell-1})$.*

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La preuve est faite par l'absurde et repose sur un résultat de non-existence bien connu pour l'équation $\operatorname{div} X = g$. Plus précisément, il existe des fonctions $g \in L^1_c((0,1)^\ell; \mathbf{R})$ avec $\int g = 0$ et telles que l'équation $\operatorname{div} X = g$ n'a pas de solution $X \in L^{\ell/(\ell-1)}_{\mathrm{loc}}(\mathbf{R}^\ell; \mathbf{R}^\ell)$ (voir Bourgain et Brézis [2]). À partir d'une telle fonction g, nous construisons une forme différentielle explicite f, fermée, à support compact et à coefficients L^1. En supposant le Théorème 1 faux, nous obtenons l'existence d'une fonction $G \in C^2_c((0,1)^\ell; \mathbf{R})$ et d'un champ $Y \in L^{\ell/(\ell-1)}_{\mathrm{loc}}(\mathbf{R}^\ell; \mathbf{R}^\ell)$ tels que $\operatorname{div} Y = g + G$, ce qui contredit les propriétés de la fonction g.

En combinant ce résultat avec les résultats de [6], [2], nous obtenons la conséquence suivante.

Corollaire 2. *Soient $N \geq 2$ et $1 \leq \ell \leq N$. Soit $\mathcal{A}$ la classe des ℓ-formes $f \in L^1_c(\mathbf{R}^N; \Lambda^\ell)$ satisfaisant la condition de compatibilité $\mathrm{d}f = 0$ (si $1 \leq \ell \leq N-1$), respectivement $\int f = 0$ (si $\ell = N$). Alors, nous avons l'équivalence $1 \iff 2$, où*

(1) l'équation $\mathrm{d}\lambda = f$ a une solution $\lambda \in W^{1,1}_{\mathrm{loc}}(\mathbf{R}^N; \Lambda^{\ell-1})$ pour tout $f \in \mathcal{A}$.
(2) $\ell = 1$.

1. Introduction

We consider the Hodge system

$$\mathrm{d}\lambda = f \text{ in } \mathbf{R}^N, \tag{1}$$

where f and λ are ℓ and $(\ell-1)$-forms respectively, f being given and satisfying the compatibility condition $\mathrm{d}f = 0$. We focus on the case where f has L^1 coefficients.

To start with, let us recall some known facts about the cases $\ell = N$ and $\ell = 1$.

In the case $\ell = N$, (1) reduces to the divergence equation. It was first shown by Wojciechowski [6] that there exists $g \in L^1_c(\mathbf{R}^N)$, with zero integral, such that the equation $\operatorname{div} X = g$ has no solution $X \in W^{1,1}_{\mathrm{loc}}(\mathbf{R}^N; \mathbf{R}^N)$. On the other hand, Bourgain and Brézis [2] proved, using a different method, the following: there exists $g \in L^1_c(\mathbf{R}^N)$ with zero integral, such that the equation $\operatorname{div} X = g$ has no solution $X \in L^{N/(N-1)}_{\mathrm{loc}}(\mathbf{R}^N; \mathbf{R}^N)$. In view of the embedding $W^{1,1}_{\mathrm{loc}} \hookrightarrow L^{N/(N-1)}_{\mathrm{loc}}$, this improves [6].

In the case $\ell = 1$, (1) reduces to the following "gradient" equation

$$\nabla\lambda = f, \tag{2}$$

where f is a vector field satisfying the compatibility condition $\nabla \times f = 0$ and λ is a function. Unlike the case $\ell = N$, this time (2) has a solution $\lambda \in W^{1,1}_{\mathrm{loc}}(\mathbf{R}^N)$. Actually, any solution to (2) belongs to $W^{1,1}_{\mathrm{loc}}$ and, moreover, if f is compactly supported, then we may choose $\lambda \in W^{1,1}$.

The question of the solvability in $W^{1,1}_{\mathrm{loc}}$ of the system (1) with datum in L^1 in the remaining cases, i.e. $2 \leq \ell \leq N-1$, has been recently raised by Baldi, Franchi, and Pansu [1]. Our main result settles this problem.

Theorem 3. *Let $N \geq 3$. Let $2 \leq \ell \leq N-1$. Then there exists some $f \in L^1_c(\mathbf{R}^N; \Lambda^\ell)$ such that $\mathrm{d}f = 0$ and the equation $\mathrm{d}\lambda = f$ has no solution $\lambda \in W^{1,1}_{\mathrm{loc}}(\mathbf{R}^N; \Lambda^{\ell-1})$.*

The proof of Theorem 3 that we present is a simplification, communicated to the author by P. Mironescu, of the original one. This simplified version has the advantage of being relatively self-contained and elementary.

2. Proof of Theorem 3

We start with some auxiliary results.

Lemma 4. *Let $1 \leq \kappa \leq N-1$ and $f \in L^1_c(\mathbf{R}^N; \Lambda^\kappa)$ be such that $df = 0$. Then there exists some $\omega \in L^q_{\mathrm{loc}}(\mathbf{R}^N; \Lambda^{\kappa-1})$, for all $1 \leq q < N/(N-1)$, such that $\mathrm{d}\omega = f$.*

Proof. Let E be "the" fundamental solution to Δ and set $\eta := E * f$. Let $\omega := \mathrm{d}^*\eta$. First, $\eta \in W^{1,q}_{\mathrm{loc}}(\mathbf{R}^N)$ (by elliptic regularity) and thus $\omega \in L^q_{\mathrm{loc}}(\mathbf{R}^N)$, $1 \leq q < N/(N-1)$. Next, $\mathrm{d}\eta = E * \mathrm{d}f = 0$. Finally,

$$\mathrm{d}\omega = \mathrm{d}\mathrm{d}^*\eta = (\mathrm{d}\mathrm{d}^* + \mathrm{d}^*\mathrm{d})\eta = \Delta\eta = f.$$

Hence, ω has the required properties. □

A similar argument leads to the following.

Lemma 5. *Let* $1 < r < \infty$, $k \in \mathbf{N}$. *Let* $1 \le \kappa \le N-1$. *Let* $f \in W^{k,r}_c(\mathbf{R}^N;\Lambda^\kappa)$ *be such that* $df = 0$. *Then there exists some* $\omega \in W^{k+1,r}_{\rm loc}(\mathbf{R}^N;\Lambda^{\kappa-1})$ *such that* $\mathrm{d}\omega = f$.

We next recall the following "inversion of d with loss of regularity". It is folklore, and one possible proof consists in using Bogovskii's formula (see for example [4, Corollary 3.3 and Corollary 3.4] for related arguments).

Lemma 6. *Let* $1 \le \kappa \le N-1$. *Let* Q *be an open cube in* $\mathbf{R}^N$. *Then there exists some integer* $m = m(N,\kappa)$ *such that if* $f \in C^k_c(Q;\Lambda^{\kappa-1})$, *with* $k \in \{m, m+1, \ldots\} \cup \{\infty\}$, *satisfies* $\mathrm{d}f = 0$, *then there exists some* $\omega \in C^{k-m}_c(Q;\Lambda^{\kappa-1})$ *such that* $\mathrm{d}\omega = f$.

Combining Lemmas 4–6, we obtain the following proposition.

Proposition 7. *Let* $1 \le \kappa \le N-1$. *Let* Q *be an open cube in* $\mathbf{R}^N$. *Let* $f \in L^1_c(Q;\Lambda^\kappa)$ *be such that* $\mathrm{d}f = 0$. *Then there exists some* $\omega \in L^q_c(Q;\Lambda^{\kappa-1})$, *for all* $1 \le q < N/(N-1)$, *such that* $\mathrm{d}\omega = f$.

Proof. Set $f_0 := f$. We consider a sequence $(\zeta_j)_{j\geqslant 0}$ in $C^\infty_c(Q;\mathbf{R})$ such that $\zeta_0 = 1$ on $\operatorname{supp} f_0$ and, for $j \geqslant 1$, $\zeta_j = 1$ on $\operatorname{supp}\zeta_{j-1}$. We let η_0 be a solution to $\mathrm{d}\eta_0 = f_0$, constructed as in Lemma 4. We set $\omega_0 := \zeta_0\eta_0$, so that $\omega_0 \in L^q_c(Q;\Lambda^{\kappa-1})$, $1 \le q < N/(N-1)$ and

$$\mathrm{d}\omega_0 = \mathrm{d}\zeta_0 \wedge \eta_0 + \zeta_0 \mathrm{d}\eta_0 = \mathrm{d}\zeta_0 \wedge \eta_0 + \zeta_0 f_0 = \underbrace{\mathrm{d}\zeta_0 \wedge \eta_0}_{-f_1} + f_0.$$

Let us note that $\mathrm{d}f_1 = -\mathrm{d}^2\omega_0 + \mathrm{d}f_0 = 0$ and that $f_1 \in L^q_c(Q;\Lambda^\kappa)$, $1 \le q < N/(N-1)$.

Fix some $1 < r < N/(N-1)$. By Lemma 5, there exists some $\eta_1 \in W^{1,r}_{\rm loc}(\mathbf{R}^N;\Lambda^{\kappa-1})$ such that $\mathrm{d}\eta_1 = f_1$. Set $\omega_1 := \zeta_1\eta_1$. Then $\omega_1 \in W^{1,r}_c(Q;\Lambda^{\kappa-1})$ and, as above, $f_2 := f_1 - \mathrm{d}\omega_1$ satisfies $\mathrm{d}f_2 = 0$ and $f_2 \in W^{1,r}_c(Q;\Lambda^\kappa)$. Applying again Lemma 5, we may find $\eta_2 \in W^{2,r}_{\rm loc}(\mathbf{R}^N)$ such that $\mathrm{d}\eta_2 = f_2$.

Iterating the above, we have

$$\omega_0 + \cdots + \omega_j \in L^q_c(Q;\Lambda^{\kappa-1}),\ 1 \le q < N/(N-1),$$
$$d(\omega_0 + \cdots + \omega_j) = f_0 - f_j,\ \text{with } \mathrm{d}f_j = 0 \text{ and } f_j \in W^{j,r}_c(Q;\Lambda^\kappa).$$

Let now j be such that $W^{j,q}(Q) \hookrightarrow C^m(Q)$, with m as in Lemma 6. Let $\xi \in C^0_c(Q;\Lambda^{\kappa-1})$ be such that $\mathrm{d}\xi = -f_j$. Set $\omega := \omega_0 + \cdots + \omega_j + \xi$. Then ω has all the required properties. □

Let us note the following consequence of hypoellipticity of Δ and of the proofs of Proposition 7 and Lemmas 4 and 5 (but *not* of their statements).

Corollary 8. *Assume, in addition to the hypotheses of Proposition 7, that* $f \in C^\infty(U)$ *for some open set* $U \subset Q$. *Let* $s \in \mathbf{N}$. *Then we may choose* ω *such that, in addition,* $\omega \in C^s(U)$.

Proof of Theorem 3. We write the variables in $\mathbf{R}^N$ as follows: $x = (x', x'')$, with $x' \in \mathbf{R}^\ell$ and $x'' \in \mathbf{R}^{N-\ell}$.

Pick some $g \in L^1_c((0,1)^\ell;\mathbf{R})$ with zero integral, such that the equation $\operatorname{div} X = g$ has no solution $X \in L^{\ell/(\ell-1)}_{\rm loc}(\mathbf{R}^\ell;\mathbf{R}^\ell)$ (see [6], [2]). Clearly, for any $G \in C^2_c((0,1)^\ell;\mathbf{R})$,

$$\text{the equation } \operatorname{div} Y = g + G \text{ has no solution } Y \in L^{\ell/(\ell-1)}_{\rm loc}(\mathbf{R}^\ell;\mathbf{R}^\ell). \tag{3}$$

Let $\psi \in C^\infty_c((0,1)^{N-\ell})$ be such that $\psi \equiv 1$ in some nonempty open set $V \subset (0,1)^{N-\ell}$. Set $Q := (0,1)^N$ and $\eta := g(x')\psi(x'')\,\mathrm{d}x' \in L^1_c(Q;\Lambda^\ell)$. We note that $\mathrm{d}\eta = g(x')\,\mathrm{d}\psi(x'') \wedge \mathrm{d}x' \in L^1_c(Q;\Lambda^{\ell+1})$. Let us also note that $\mathrm{d}\eta = 0$ in $\mathbf{R}^\ell \times V$. By Corollary 8 with $U = (0,1)^\ell \times V$, there exists some $\omega \in L^q_c(Q;\Lambda^\ell)$, $1 \le q < N/(N-1)$, such that $\mathrm{d}\omega = \mathrm{d}\eta$ and $\omega \in C^2((0,1)^\ell \times V)$.

Consider now the closed form $f := \eta - \omega \in L^1_c(Q;\Lambda^\ell)$. We claim that there exists no $\lambda \in W^{1,1}_{\rm loc}(\mathbf{R}^N;\Lambda^{\ell-1})$ such that $\mathrm{d}\lambda = f$. Argue by contradiction and let λ_i denote the coefficient, in λ, of $\mathrm{d}x_1 \wedge \mathrm{d}x_2 \cdots \wedge \mathrm{d}x_{i-1} \wedge \mathrm{d}x_{i+1} \wedge \cdots \wedge \mathrm{d}x_\ell$, $1 \le i \le \ell$. Let ω_0 denote the coefficient of $\mathrm{d}x'$ in ω. Then, in $\mathbf{R}^\ell \times V$, we have:

$$\sum_{i=1}^{\ell} (-1)^{i+1}\partial_i\lambda_i(x', x'') = g(x')\psi(x'') - \omega_0(x', x'') = g(x') - \omega_0(x', x''). \tag{4}$$

Hence, for a.e. $x'' \in V$, the following equation is satisfied in $\mathcal{D}'(\mathbf{R}^\ell)$:

$$\sum_{i=1}^{\ell} (-1)^{i+1}\partial_i\lambda_i' = g - \omega_0', \tag{5}$$

with

$$\lambda_i' := \lambda_i(\cdot, x'') \in W^{1,1}_{\text{loc}}(\mathbf{R}^\ell) \text{ and } \omega_0' = \omega_0(\cdot, x'') \in C^2_c((0,1)^\ell). \tag{6}$$

The above properties (5) and (6), combined with the embedding $W^{1,1}_{\text{loc}}(\mathbf{R}^\ell) \hookrightarrow L^{\ell/(\ell-1)}(\mathbf{R}^\ell)$, contradict (3). □

Remark 1. We have actually proved the following improvement of Theorem 3. Let $N \geq 3$ and $2 \leq \ell \leq N-1$. Then there exists some $f \in L^1_c(\mathbf{R}^d; \Lambda^\ell)$ satisfying $\mathrm{d}f = 0$ and such that the system $\mathrm{d}\lambda = f$ has no solution

$$\lambda \in L^1_{\text{loc}}(\mathbf{R}^{(N-\ell)}; L^{\ell/(\ell-1)}_{\text{loc}}(\mathbf{R}^\ell; \Lambda^{\ell-1})).$$

Remark 2. A similar question can be raised in L^∞. We have the following analogue of Theorem 3.

Theorem 9. *Let $N \geq 3$. Let $2 \leq \ell \leq N-1$. Then there exists some $f \in L^\infty_c(\mathbf{R}^N; \Lambda^\ell)$ such that $\mathrm{d}f = 0$ and the equation $\mathrm{d}\lambda = f$ has no solution $\lambda \in W^{1,\infty}_{\text{loc}}(\mathbf{R}^N; \Lambda^{\ell-1})$.*

The proof of Theorem 9 is very similar to the one of Theorem 3. The main difference is the starting point, in dimension ℓ. Here, we use the fact that there exists some $g \in L^\infty_c(\mathbf{R}^\ell)$, with zero integral, such that the equation $\operatorname{div} X = g$ has no solution $X \in W^{1,\infty}_{\text{loc}}(\mathbf{R}^\ell; \mathbf{R}^\ell)$ (see [5]).

3. Solution in $L^{N/(N-1)}$ when $1 \leq \ell \leq N-1$

As mentioned in the introduction, when $\ell = N$, the system (1) with right-hand side $f \in L^1$ need not have a solution $\lambda \in L^{N/(N-1)}_{\text{loc}}$. In view of Theorem 3 and of Proposition 7, it is natural to ask whether, in the remaining cases $1 \leq \ell \leq N-1$, given a closed ℓ-form $f \in L^1_c$, it is possible to solve (1) with $\lambda \in L^{N/(N-1)}_{\text{loc}}$. This is clearly the case when $\ell = 1$ (by the Sobolev embedding $W^{1,1}_{\text{loc}} \hookrightarrow L^{N/(N-1)}_{\text{loc}}$). Moreover, we may pick $\lambda \in W^{1,1}$. The remaining cases are settled by our next result. In what follows, we do not make any support assumption on f, and, therefore, the case where $\ell = 1$ is also of interest.

Proposition 10. *Let $N \geq 2$ and $1 \leq \ell \leq N-1$. Then, for every $f \in L^1(\mathbf{R}^N; \Lambda^\ell)$ with $\mathrm{d}f = 0$, there exists some $\lambda \in L^{N/(N-1)}(\mathbf{R}^N; \Lambda^{\ell-1})$ such that $f = \mathrm{d}\lambda$.*

Proof. Suppose $f \in L^1(\mathbf{R}^N; \Lambda^{\ell-1})$ with $\mathrm{d}f = 0$ as above. According to Bourgain and Brézis [3] (see Corollary 20 in [3] for a very similar statement; see also Theorem 3 in [7]), we have:

$$\left| \int_{\mathbf{R}^d} \langle \psi, f \rangle \right| \lesssim \|f\|_{L^1} \left\| \mathrm{d}^*\psi \right\|_{L^N}, \ \forall \psi \in C^\infty_c(\mathbf{R}^N; \Lambda^\ell). \tag{7}$$

Consider the functional

$$L_f : S = \{\mathrm{d}^*\psi; \quad \psi \in C^\infty_c(\mathbf{R}^N; \Lambda^\ell)\} \to \mathbf{R}, \ L_f\left(\mathrm{d}^*\psi\right) := \int_{\mathbf{R}^d} \langle \psi, f \rangle.$$

Here, S is endowed with the L^N-norm. The inequality (7) shows that L_f is well defined and bounded. By the Hahn–Banach theorem, there exists an extension $\widetilde{L}_f : L^N(\mathbf{R}^N; \Lambda^{\ell+1}) \to \mathbf{R}$ of L_f with $\|\widetilde{L}_f\| = \|L_f\|$. Hence, there exists an $(\ell-1)$-form $\lambda \in L^{N/(N-1)}(\mathbf{R}^N; \Lambda^{\ell-1})$ such that

$$\int_{\mathbf{R}^N} \langle \psi, f \rangle = L_f\left(\mathrm{d}^*\psi\right) = \widetilde{L}_f\left(\mathrm{d}^*\psi\right) = \int_{\mathbf{R}^N} \left\langle \mathrm{d}^*\psi, \lambda \right\rangle = \int_{\mathbf{R}^N} \langle \psi, \mathrm{d}\lambda \rangle$$

for all ℓ-forms $\psi \in C^\infty_c(\mathbf{R}^N; \Lambda^\ell)$.

This implies that $\lambda \in L^{N/(N-1)}(\mathbf{R}^N; \Lambda^{\ell-1})$ satisfies $\mathrm{d}\lambda = f$. □

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