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Mathematical analysis

A note on maximal commutators with rough kernels ☆

Une note sur les commutateurs maximaux avec des noyaux grossiers

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ABSTRACT

This paper gives a characterization of compactness for maximal commutators with rough kernels in weighted Lebesgue spaces, which is new and interesting even in un-weighted cases. Meanwhile, a new characterization of weighted boundedness for such operators is also established.

RÉSUMÉ

Cet article donne une caractérisation de la compacité des commutateurs maximaux avec des noyaux grossiers dans des espaces de Lebesgue pondérés, ce qui est nouveau et intéressant, même dans les cas non pondérés. Entretemps, une nouvelle caractérisation de la limite pondérée pour ces opérateurs est également établie.

1. Introduction and main results

Let $\mathbb{R}^n$, $n \geq 2$, be the n-dimensional Euclidean space and S^{n-1} be the unit sphere in $\mathbb{R}^n$. Let Ω be a homogeneous function of degree zero on $\mathbb{R}^n$ and $\Omega \in L^1(S^{n-1})$. For $0 \leq \beta < n$, $k \in \mathbb{Z}^+$ and a locally integrable function b, we consider the maximal operator

$$M_{\Omega,\beta} f(x) := \sup_{r>0} \frac{1}{r^{n-\beta}} \int\limits_{|x-y|\leq r} |\Omega(x-y)||f(y)|\,\mathrm{d}y,$$

and the corresponding maximal commutator

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$$(M_{\Omega,\beta})_b^k f(x) := \sup_{r>0} \frac{1}{r^{n-\beta}} \int\limits_{|x-y|\le r} |b(x)-b(y)|^k |\Omega(x-y) f(y)| \,\mathrm{d}y,$$

which play key roles in studying the boundedness of the following commutators of singular and fractional integrals

$$\left(T_{\Omega,\beta}\right)_b^k f(x) := \int\limits_{\mathbb{R}^n} [b(x)-b(y)]^k \frac{\Omega(x-y)}{|x-y|^{n-\beta}} f(y) \,\mathrm{d}y,$$

where, for $\beta = 0$, the integral is in the principal value sense and Ω satisfies the vanishing property in S^{n-1}.

Obviously, when $\Omega \equiv 1$, $M_{1,0}$, denoted by M, is the Hardy–Littlewood maximal operator, and $M_{1,\beta}$, denoted by M_β for $0 < \beta < n$, is the fractional maximal operators. Also, we denote $(M_{0,1})_b^k$ by M_b^k, and $(M_{1,\beta})_b^k$ by $M_{\beta,b}^k$. In 1991, García-Cuerva and Harboure et al. [12] showed that M_b^k is bounded on $L^p(\omega)$ for $\omega \in A_p$, $1 < p < \infty$, if and only if $b \in \mathrm{BMO}(\mathbb{R}^n)$. Segovia and Torrea [19] proved that $M_{\beta,b}^k$ is bounded from $L^p(\omega^p)$ to $L^q(\omega^q)$ if and only if $b \in \mathrm{BMO}(\mathbb{R}^n)$, provided that $0 < \beta < n$, $1 < p < q < \infty$ with $1/q = 1/p - \beta/n$, $\omega \in A_{p,q}$. Recently, Zhang [22] (for $\beta = 0$) and Guliyev, Deringoz, and Hasanov [13] (for $0 < \beta < n$) showed that, for $0 < \alpha \le 1$, $M_{\beta,b}^1$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ for $1 < p < q < \infty$ with $1/q = 1/p - (\alpha + \beta)/n$, if and only if $b \in \mathrm{BMO}_\alpha(\mathbb{R}^n)$ (the Lipschitz spaces, see Definition 2.1 in Section 2). In addition, Ding and Lu [10] proved that, for $0 < \beta < n$ and $b \in \mathrm{BMO}(\mathbb{R}^n)$, $(M_{\Omega,\beta})_b^k$ is bounded from $L^p(\omega^p)$ to $L^q(\omega^q)$, provided that $\Omega \in L^s(S^{n-1})$ for some $s > 1$ with $1 \le s' < p < n/\beta$, $1/q = 1/p - \beta/n$, and $\omega^{s'} \in A_{p/s',q/s'}$, where $s' = s/(s-1)$ denotes the conjugate number of s throughout this paper.

In this paper, we will focus on the compactness of $(M_{\Omega,\beta})_b^k$. We first recall the definition of compact operators.

Definition 1.1. (cf. [2]) Let X, Y be Banach spaces. A mapping T from X to Y is compact if T is continuous and maps bounded subsets of X into precompact subsets of Y.

The investigation on the compactness of commutators dates back to Uchiyama's work [20], in which the author proved that, for $\Omega \in Lip_1(S^{n-1})$, $1 < p < \infty$, $(T_{\Omega,0})_b^1$ is compact on $L^p(\mathbb{R}^n)$ if and only if $b \in \mathrm{CMO}(\mathbb{R}^n)$, where $\mathrm{CMO}(\mathbb{R}^n)$ is the closure of $C_c^\infty(\mathbb{R}^n)$ in the $\mathrm{BMO}(\mathbb{R}^n)$ topology. Afterwards, this result was extensively improved and extended: see, for example, [5,4,14–16,21] et al. In particular, inspired by Lerner–Ombrosi–Rivera–Ríos [17], our second and third authors and others recently gave in [14,15] some new characterizations of the compactness of $(T_{\Omega,\beta})_b^k$ via $\mathrm{CMO}_\alpha(\mathbb{R}^n)$, which can be regarded as the generalization of $\mathrm{CMO}(\mathbb{R}^n)$, see Definition 2.6 in Section 2.

Although many authors studied the compactness of linear operators, the literature is not so rich regarding the compactness of nonlinear operators, one can see [3,11] for the commutators of Littlewood–Paley operators and the maximal truncated commutators for singular integrals, etc. The main purpose of this paper is to establish the characterization theorem on the compactness of $(M_{\Omega,\beta})_b^k$ via $\mathrm{CMO}_\alpha(\mathbb{R}^n)$, which can be formulated as follows.

Theorem 1.2. *Let $k \in \mathbb{Z}^+$, $0 \le \alpha \le 1$, $0 \le \beta < n$ with $k\alpha + \beta < n$, $1 < p, q < \infty$ with $1/q = 1/p - (k\alpha + \beta)/n$. Assume that Ω is a homogeneous function of degree 0 on $\mathbb{R}^n$, $\Omega \in L^s(S^{n-1})$ for some $s > 1$ with $s' < p$, $\omega^{s'} \in A_{p/s',q/s'}$. If Ω does not change sign and is not equivalent to zero on some open set of S^{n-1}, then $(M_{\Omega,\beta})_b^k$ is a compact operator from $L^p(\omega^p)$ to $L^q(\omega^q)$ if and only if $b \in \mathrm{CMO}_\alpha(\mathbb{R}^n)$.*

Our main novelty will be embodied in the arguments on compactness of $(M_{\Omega,\beta})_b^k$. To achieve our goal, we prefer to choose a soft but elegant way. First, we use the Cauchy integral trick to get the boundedness of a maximal iterated commutator. Then, combining this boundedness result and some basic properties derived from the definition of $(M_{\Omega,\beta})_b^k$, we give a reduction of such a deep degree, that the part of "checking the conditions of the Fréchet–Kolmogorov theorem" becomes very concise. We would like to point out that, in every previous article, the process of "checking the conditions of the Fréchet–Kolmogorov theorem" is the most tedious part. In order to prove our theorem, the necessity of bounded maximal commutators will also be established, which can be deduced by means of the known estimate in [14,17]. Then, we use the same ideas from [14,15] to obtain further lower and upper estimates of $(M_{\Omega,\beta})_b^k$. These estimates yield the necessity of compactness of $(M_{\Omega,\beta})_b^k$.

Take $\Omega \equiv 1$, we obtain the characterization of compactness for the maximal commutators corresponding to the Hardy–Littlewood maximal function and fractional maximal functions.

Corollary 1.3. *Let $k \in \mathbb{Z}^+$, $0 \le \alpha \le 1$, $0 \le \beta < n$ with $k\alpha + \beta < n$, $1 < p, q < \infty$ with $1/q = 1/p - (k\alpha + \beta)/n$, and $\omega \in A_{p,q}$. Then the maximal commutator $(M_\beta)_b^k$ is a compact operator from $L^p(\omega^p)$ to $L^q(\omega^q)$ if and only if $b \in \mathrm{CMO}_\alpha(\mathbb{R}^n)$.*

The rest of the paper is organized as follows. In Section 2, we will recall some relevant definitions and auxiliary lemmas. We will prove the sufficiency of Theorem 1.2 in Section 3. Finally, the proof of the necessity of Theorem 1.2 and the characterization of weighted boundedness will be given in Section 4.

Throughout the rest of our paper, we will denote positive constants by C, which may change at each occurrence. If $f \le Cg$ and $f \lesssim g \lesssim f$, we denote $f \lesssim g$, $f \sim g$, respectively. For a given cube Q, we use c_Q, l_Q, χ_Q and $A \triangle B$ to denote the center, side length, characteristic function of Q and $(A \setminus B) \cup (B \setminus A)$, respectively.

2. Preliminaries

In this section, we will recall some relevant concepts and auxiliary lemmas.

A weight ω is a nonnegative and locally integrable function on $\mathbb{R}^n$. For $1 < p < \infty$, we say that $\omega \in A_p$ if there exists a constant $C > 0$ such that

$$\sup_Q \Big(\frac{1}{|Q|}\int_Q \omega(y)\,\mathrm{d}y\Big)\Big(\frac{1}{|Q|}\int_Q \omega(y)^{1-p'}\,\mathrm{d}y\Big)^{p-1} \le C,$$

where $1/p + 1/p' = 1$ and the supremum is taken over all cubes $Q \subset \mathbb{R}^n$. We call $\omega \in A_{p,q}$ for $1 < p, q < \infty$ if there exists a constant $C > 0$ such that:

$$\sup_Q \Big(\frac{1}{|Q|}\int_Q \omega(x)^q\,\mathrm{d}x\Big)\Big(\frac{1}{|Q|}\int_Q \omega(x)^{-p'}\,\mathrm{d}x\Big)^{q/p'} \le C.$$

From the definition of $A_{p,q}$, we know that $\omega \in A_{p,q}$ implies $\omega^q \in A_q$ and $\omega^p \in A_p$. Define the A_∞ class of weights by $A_\infty := \cup_{p>1} A_p$, the A_∞ constant is given by

$$[\omega]_{A_\infty} := \sup_Q \int_Q M(\chi_Q \omega)(x)\,\mathrm{d}x.$$

We will frequently use the doubling property of weight: for $\lambda > 1$, and all cubes Q, if $\omega \in A_p$, we have $\omega(\lambda Q) \le \lambda^{np}[\omega]_{A_p}\omega(Q)$. We proceed to the definition of BMO_α.

Definition 2.1. Let Q be a cube, $\alpha \in [0,1]$, for any $f \in L^1_{loc}(\mathbb{R}^n)$, we denote $\mathrm{BMO}_\alpha(\mathbb{R}^n)$ the space of functions with $\|f\|_{\mathrm{BMO}_\alpha(\mathbb{R}^n)} < \infty$, where

$$\|f\|_{\mathrm{BMO}_\alpha(\mathbb{R}^n)} := \sup_{Q\subset\mathbb{R}^n} \mathcal{O}_\alpha(f;Q) := \sup_{Q\subset\mathbb{R}^n} \frac{1}{|Q|^{1+\frac{\alpha}{n}}}\int_Q |f(x) - f_Q|\,\mathrm{d}x,$$

where $f_Q = \frac{1}{|Q|}\int_Q f(y)\,\mathrm{d}y$. We also define

$$\widetilde{\mathcal{O}}_\alpha(f;Q) := \inf_{c\in\mathbb{C}} \frac{1}{|Q|^{1+\frac{\alpha}{n}}}\int_Q |f(x) - c|\,\mathrm{d}x.$$

It is easy to check that $\widetilde{\mathcal{O}}_\alpha(f;Q) \le \mathcal{O}_\alpha(f;Q) \le 2\widetilde{\mathcal{O}}_\alpha(f;Q)$. Clearly, $\mathrm{BMO}_0(\mathbb{R}^n)$ is $\mathrm{BMO}(\mathbb{R}^n)$. As mentioned in Section 1, $\mathrm{CMO}(\mathbb{R}^n)$ denotes the closure of $C_c^\infty(\mathbb{R}^n)$ in $\mathrm{BMO}(\mathbb{R}^n)$. In order to prove our results, we will use the following new characterization of CMO established in [15].

Definition 2.2. By a median value of a real-valued measurable function f over a measure set E of positive finite measure, we mean a possibly non-unique, real number $m_f(E)$ such that

$$\max(|\{x \in E : f(x) > m_f(E)\}|,\ |\{x \in E : f(x) < m_f(E)\}|) \le \frac{|E|}{2}.$$

Definition 2.3. For a real-valued measurable function f, we define the local oscillation of f over a cube Q by

$$\omega_\lambda(f;Q) := ((f - m_f(Q))\chi_Q)^*(\lambda|Q|) \quad (0 < \lambda < 1),$$

where f^* is the non-increasing rearrangement.

We recall the new characterization of CMO as follows.

Lemma 2.4. (cf. [15]) *Let $f \in \mathrm{BMO}$. Then $f \in \mathrm{CMO}$ if and only if the following three conditions hold:*

(1) $\lim_{a\to 0} \sup_{|Q|=a} \omega_\lambda(f; Q) = 0$,
(2) $\lim_{a\to +\infty} \sup_{|Q|=a} \omega_\lambda(f; Q) = 0$,
(3) $\lim_{b\to +\infty} \sup_{|Q|\cap[-b,b]^n=\emptyset} \omega_\lambda(f; Q) = 0$.

For $0 < \alpha \le 1$, for a continuous function f on $\mathbb{R}^n$, the (homogeneous) α-order Lipschitz norm is defined by

$$\|f\|_{Lip_\alpha(\mathbb{R}^n)} := \sup_{x\neq y} \frac{|f(x) - f(y)|}{|x-y|^\alpha},$$

and we denote the space of all continuous functions on $\mathbb{R}^n$ such that $\|f\|_{Lip_\alpha(\mathbb{R}^n)} < \infty$ by $Lip_\alpha(\mathbb{R}^n)$. Meyers [18] showed that $Lip_\alpha(\mathbb{R}^n) = \mathrm{BMO}_\alpha(\mathbb{R}^n)$. Moreover, he gave the following lemma.

Lemma 2.5. (cf. [15]) *Let $\alpha \in (0, 1]$. Then*

$$Lip_\alpha(\mathbb{R}^n) = \mathrm{BMO}_\alpha(\mathbb{R}^n).$$

Moreover, if $f \in Lip_\alpha(\mathbb{R}^n)$, $p \in [1, \infty]$, we have

$$\|f\|_{Lip_\alpha(\mathbb{R}^n)} \sim \sup_Q \mathcal{O}_\alpha(f; Q) \sim \sup_Q \frac{1}{|Q|^{\alpha/n}} \Big(\frac{1}{|Q|} \int_Q |f(y) - f_Q|^p \, dy \Big)^{1/p}.$$

Now we give the definition of $\mathrm{CMO}_\alpha(\mathbb{R}^n)$ in [14].

Definition 2.6. Let $\alpha \in [0, 1]$. A BMO_α function f belongs to CMO_α if it satisfies the following three conditions:

(1) $\lim_{r\to 0} \sup_{|Q|=r} \mathcal{O}_\alpha(f; Q) = 0$;
(2) $\lim_{r\to \infty} \sup_{|Q|=r} \mathcal{O}_\alpha(f; Q) = 0$;
(3) $\lim_{d\to \infty} \sup_{Q\cap[-d,d]^n=\emptyset} \mathcal{O}_\alpha(f; Q) = 0$.

Remark 2.7. When $\alpha = 0$, the characterization of Uchiyama [20] yields that CMO_0 is just the CMO space. When $0 < \alpha \le 1$, in [14] our second and third authors et al. proved that CMO_α is the appropriate function space to characterize the compactness of the commutator. Denote by $\widetilde{\mathrm{CMO}}_\alpha(\mathbb{R}^n)$ the $C_c^\infty(\mathbb{R}^n)$ closure in $\mathrm{BMO}_\alpha(\mathbb{R}^n)$; in [14], it showed that $\widetilde{\mathrm{CMO}}_\alpha(\mathbb{R}^n) = \mathrm{CMO}_\alpha$ when $\alpha \in [0, 1)$, and $\mathrm{CMO}_\alpha(\mathbb{R}^n) \subsetneq \widetilde{\mathrm{CMO}}_\alpha(\mathbb{R}^n)$ when $\alpha = 1$, in fact, $\mathrm{CMO}_1(R^n)$ is equal to the constant space $\mathbb{C}$ containing all complex numbers with usual norm.

Finally, we give some necessary lemmas, which will be used in our proofs.

Lemma 2.8. (cf. [8]) *Suppose that $s > 1$ with $s' \le p < \infty$, $p \neq 1$. If Ω is a homogeneous function of degree 0 on $\mathbb{R}^n$, $\Omega \in L^s(S^{n-1})$ and $\omega \in A_{p/s'}$, then*

$$\|M_{\Omega,0}(f)\|_{L^p(\omega)} \le C\|\Omega\|_{L^s(S^{n-1})}\|f\|_{L^p(\omega)}.$$

Lemma 2.9. (cf. [9]) *Suppose that $0 < \beta < n$, $s > 1$ with $s' < p < n/\beta$ and $1/q = 1/p - \beta/n$. If Ω is a homogeneous function of degree 0 on $\mathbb{R}^n$, $\Omega \in L^s(S^{n-1})$ and $\omega^{s'} \in A_{p/s',q/s'}$, then there exists a constant C independent of f such that*

$$\|M_{\Omega,\beta}(f)\|_{L^q(\omega^q)} \le C\|\Omega\|_{L^s(S^{n-1})}\|f\|_{L^p(\omega^p)}.$$

Lemma 2.10. (cf. [15]) *Let $\lambda \in (0, 1)$, b be a real-valued measurable function. Suppose that Ω is a measurable function on S^{n-1} and does not change sign and is not equivalent to zero on some open set of S^{n-1}. Then there exist $\epsilon_0 > 0$ and $K_0 > 10\sqrt{n}$ depending only on Ω and n such that the following holds: for every cube Q, there exists another cube P with the same length of Q satisfying $|c_Q - c_P| = K_0 l_Q$, and measurable sets $E \subset Q$ with $|E| = \frac{\lambda}{2}|Q|$, and $F \subset P$ with $|F| = |Q|/2$, and $G \subset E \times F$ with $|G| \ge \frac{\lambda|Q|^2}{8}$ such that*

(1) $\omega_\lambda(b; Q) \le |b(x) - b(y)|$ *for all* $(x, y) \in E \times F$;
(2) $\Omega(x - y)$ *and* $b(x) - b(y)$ *do not change sign in* $E \times F$;
(3) $\Omega(x - y) \ge \epsilon_0$ *for all* $(x, y) \in G$.

Lemma 2.11. (cf. [14]) *Let b be a real-valued measurable function. Suppose that Ω is a measurable function on S^{n-1} and does not change sign and is not equivalent to zero on some open set of S^{n-1}. For every $\gamma \in (0,1)$, there exist $\epsilon_0 > 0$ and $K_0 > 10\sqrt{n}$ depending only on Ω, γ and n such that the following holds. For every cube Q, there exists another cube P with the same length of Q satisfying $|c_Q - c_P| = K_0 l_Q$, and measurable sets $E_1, E_2 \subset Q$ with $Q = E_1 \cup E_2$, and $F_1, F_2 \subset P$ with $|F_1| = |F_2| = |Q|/2$, such that*

(1) *$b(x) - b(y)$ do not change sign in $E_i \times F_i$, $i = 1, 2$;*
(2) *$|b(x) - m_b(P)| \le |b(x) - b(y)|$ in $E_i \times F_i$, $i = 1, 2$;*
(3) *$\Omega(x - y)$ does not change sign in $Q \times P$;*
(4) *$|N_x \cap P| \le \gamma |Q|$ for all $x \in Q$, where $N_x := \{y \in \mathbb{R}^n : |\Omega(x - y)| < \epsilon_0\}$.*

3. Compactness of the maximal commutators

This section is devoted to the proof of the sufficiency of Theorem 1.2. We first establish the boundedness of $(M_{\Omega,\beta})^k_{\vec{b}}$ defined by

$$(M_{\Omega,\beta})^k_{\vec{b}}(f)(x) := \sup_{r>0} \frac{1}{r^{n-\beta}} \int_{B(x,r)} |\Omega(x-y)||f(y)| \prod_{j=1}^{k} |b_j(x) - b_j(y)| \, \mathrm{d}y,$$

which will be used to give the first reduction of the compactness of $(M_{\Omega,\beta})^k_{\vec{b}}$. We also point out that this boundedness result has its own interest since it improves and extends the corresponding results in [13,22]. Here, we will use the Cauchy integral trick to prove this boundedness result. This idea can be tracked back to the pioneering work of Coifman–Rochberg–Weiss [7]. Recently, there is a comprehensive study originating from this idea, see [1]. One can also see [15] for a specific application for the iterated commutator of the Calderón–Zygmund operator.

Theorem 3.1. *Let $k \in \mathbb{Z}^+$, $0 \le \alpha \le 1$, $0 \le \beta < n$ with $k\alpha + \beta < n$, $1 < p, q < \infty$ with $1/q = 1/p - (k\alpha + \beta)/n$. Assume that Ω is a homogeneous function of degree 0 on $\mathbb{R}^n$, $\Omega \in L^s(S^{n-1})$ for some $s > 1$ with $s' < p$, $\omega^{s'} \in A_{p/s',q/s'}$. Then, for $\vec{b} = (b_1, \ldots, b_k)$, $b_j \in \mathrm{BMO}_\alpha$,*

$$\|(M_{\Omega,\beta})^k_{\vec{b}}(f)\|_{L^q(\omega^q)} \lesssim \prod_{j=1}^{k} \|b_j\|_{\mathrm{BMO}} \|\Omega\|_{L^s(S^{n-1})} \|f\|_{L^p(\omega^p)}.$$

Before giving the proof, we first recall the following lemma in [15].

Lemma 3.2. (cf. [15]) *Let $p, q \in (1,\infty)$, $\omega \in A_{p,q}$, $b_j \in \mathrm{BMO}(\mathbb{R}^n)$ for $j = 1, \ldots, k$. There exists a constant $\kappa_{n,p,q,k}$ depending only on the indicated parameters such that*

$$[\mathrm{e}^{\mathrm{Re}(\sum_{j=1}^{k} b_j z_j)} \omega]_{A_{p,q}} \le 4^{1+q/p'} [\omega]_{A_{p,q}}$$

for all z_j with

$$|z_j| \le \frac{\kappa_{n,p,q,k}}{\|b_j\|_{\mathrm{BMO}(\mathbb{R}^n)} (1 + (\omega)_{A_\infty})},$$

where $(\omega)_{A_\infty} := \max\{[\omega]_{A_\infty}, [\omega^{1-p'}]_{A_\infty}\}$.

Proof of Theorem 3.1. When $0 < \alpha \le 1$, since $b_j \in BMO_\alpha$, we have

$$(M_{\Omega,\beta})^k_{\vec{b}}(f)(x) \le \prod_{j=1}^{k} \|b_j\|_{\mathrm{BMO}_\alpha} \sup_{r>0} \frac{1}{r^{n-(\beta+k\alpha)}} \int_{B(x,r)} |\Omega(x-y) f(y)| \, \mathrm{d}y.$$

Then the result follows from Lemma 2.9. In the following, we prove the case for $\alpha = 0$. For $z_j \in \mathbb{C}$, $j = 1, \ldots, k$, set $F(z_j) = \mathrm{e}^{z_j [b_j(x) - b_j(y)]}$. Then for any $\epsilon_j > 0$, by the Cauchy integral formula,

$$b_j(x) - b_j(y) = F'(0) = \frac{1}{2\pi i} \int_{|z_j| = \epsilon_j} \frac{F(z_j)}{z_j^2} \, \mathrm{d}z_j,$$

it follows that

$$\prod_{j=1}^{k}(b_j(x)-b_j(y))=\frac{1}{(2\pi i)^k}\int\limits_{|z_k|=\epsilon_k}\cdots\int\limits_{|z_1|=\epsilon_1}\frac{e^{\Sigma_j z_j(b_j(x)-b_j(y))}}{\Pi_{j=1}^{k}z_j^2}\,dz_1\cdots dz_k.$$

From this, we have:

$$\begin{aligned}(M_{\Omega,\beta})_b^k(f)(x)&=\sup_{r>0}\frac{1}{r^{n-\beta}}\int\limits_{B(x,r)}|f(y)\Omega(x-y)|\prod_{j=1}^{k}|b_j(x)-b_j(y)|\,dy\\&\le\frac{e^{\mathrm{Re}(\Sigma_{j=1}^k z_j b_j(x))}}{\Pi_{j=1}^k\epsilon_j}\sup_{r>0}\frac{1}{r^{n-\beta}}\int\limits_{B(x,r)}|f(y)\Omega(x-y)e^{-\mathrm{Re}(\Sigma_{j=1}^k z_j b_j(y))}|\,dy\\&=\frac{e^{\mathrm{Re}(\Sigma_{j=1}^k z_j b_j(x))}}{\Pi_{j=1}^k\epsilon_j}M_{\Omega,\beta}(fe^{-\mathrm{Re}(\Sigma_{j=1}^k z_j b_j(\cdot))})(x).\end{aligned}$$

Invoking Lemma 3.2 and taking $|z_j|\le\frac{\kappa_{n,p,q,k,s'}}{\|b_j\|_{\mathrm{BMO}}(1+(\omega^{s'})_{A_\infty})}$ yield that $(e^{\mathrm{Re}(\Sigma_{j=1}^k z_j b_j)}\omega)^{s'}\in A_{p/s',q/s'}$. This, together with Lemmas 2.8 and 2.9, allows us to deduce that

$$\begin{aligned}\|(M_{\Omega,\beta})_b^k(f)\|_{L^q(\omega^q)}&\le\frac{1}{\prod_{j=1}^k\epsilon_j}\|M_{\Omega,\beta}(fe^{-\mathrm{Re}(\Sigma_j b_j z_j)})\|_{L^q(\omega^q e^{q\mathrm{Re}(\Sigma_j z_j b_j)})}\\&\lesssim\prod_{j=1}^{k}\|b_j\|_{\mathrm{BMO}}\|\Omega\|_{L^s(S^{n-1})}\|f\|_{L^p(\omega^p)}.\end{aligned}$$

Theorem 3.1 is proved. □

Next, we recall the weighted Fréchet–Kolmogorov theorem on compact sets.

Lemma 3.3. (cf. [6]) *Let $p\in(1,\infty)$, $\omega\in A_p$, a subset E of $L^p(\omega)$ is precompact (or totally bounded) if the following statements hold:*

(a) E is uniformly bounded, i.e., $\sup_{f\in E}\|f\|_{L^p(\omega)}\lesssim 1$;

(b) E uniformly vanishes at infinity, that is,

$$\lim_{N\to\infty}\int\limits_{|x|>N}|f(x)|^p\omega(x)\,dx=0,$$

uniformly for all $f\in E$;

(c) E is uniformly equicontinuous, that is,

$$\lim_{\rho\to 0}\sup_{y\in B(0,\rho)}\int\limits_{\mathbb{R}^n}|f(x+y)-f(x)|^p\omega(x)\,dx=0,$$

uniformly for all $f\in E$.

Now, the sufficiency of Theorem 1.2 can be proved as follows.

The sufficiency of Theorem 1.2. It is obvious that the conclusion holds for $\alpha=1$ by Remark 2.7. In the following, we consider the case for $0\le\alpha<1$. By the definition of a compact operator, we only need to check that the set

$$A(\Omega,b):=\{(M_{\Omega,\beta})_b^k(f):\|f\|_{L^p(\omega^p)}\le 1\}$$

is precompact. Applying Theorem 3.1 and the same method in [14], it suffices to verify the precompactness of $A(\Omega,b)$ for $b\in C_c^\infty$ and $\Omega\in Lip_1(S^{n-1})$. Without loss of generality, we assume that b is supported in a cube Q centered at the origin.

Let us proceed a further reduction. For fixed $r_0,r_1>0$, we claim that it suffices to check that

$$A(\Omega_{r_0,r_1},b):=\{(M_{\Omega,\beta}^{r_0,r_1})_b^k(f):\|f\|_{L^p(\omega^p)}\le 1\}$$

is precompact, where

$$(M_{\Omega,\beta}^{r_0,r_1})_b^k(f)(x)=\sup_{r_0<r<r_1}\frac{1}{r^{n-\beta}}\int\limits_{B(x,r)}|b(x)-b(y)|^k|f(y)\Omega(x-y)|\,dy.$$

By a direct computation,

$$\begin{aligned}(M_{\Omega,\beta})_b^k(f)(x)-(M_{\Omega,\beta}^{r_0,r_1})_b^k(f)(x) &\le \sup_{r\le r_0}\frac{1}{r^{n-\beta}}\int\limits_{B(x,r)}|b(x)-b(y)|^k|f(y)\Omega(x-y)|\,\mathrm{d}y\\ &\quad+\sup_{r\ge r_1}\frac{1}{r^{n-\beta}}\int\limits_{B(x,r)}|b(x)-b(y)|^k|f(y)\Omega(x-y)|\,\mathrm{d}y\\ &=: I(x)+II(x).\end{aligned}\tag{3.1}$$

Since $b\in C_c^\infty$ and $\Omega\in Lip_1(S^{n-1})$, we have

$$\begin{aligned}I(x)&\lesssim \sup_{r\le r_0}\frac{r^k}{r^{n-\beta}}\int\limits_{B(x,r)}|f(y)|\,\mathrm{d}y\\ &\le r_0^{k(1-\alpha)}\sup_{r\le r_0}\frac{1}{r^{n-\beta-k\alpha}}\int\limits_{B(x,r)}|f(y)|\,\mathrm{d}y\le r_0^{k(1-\alpha)}M_{\beta+k\alpha}f(x).\end{aligned}$$

This implies that

$$\|I\|_{L^q(\omega^q)}\lesssim r_0^{k(1-\alpha)}\|M_{\beta+k\alpha}f\|_{L^q(\omega^q)}\lesssim r_0^{k(1-\alpha)}\|f\|_{L^p(\omega^p)},\tag{3.2}$$

which tends to zero as $r_0\to 0$.

Now we deal with $II(x)$. For sufficiently large $R>2\sqrt{n}l_Q$ such that $Q\subset B(0,R)$, we write $II(x)=II_1(x)+II_2(x):=II(x)\chi_{B(0,R)^c}(x)+II(x)\chi_{B(0,R)}(x)$. Observe that if $x\in B(0,R)^c$ with $B(x,r)\cap Q\neq\emptyset$ implies that $r>|x|-\sqrt{n}l_Q$. From this, for $x\in B(0,R)^c$ we have:

$$\begin{aligned}II_1(x)&=\sup_{r\ge r_1}\frac{1}{r^{n-\beta}}\int\limits_{B(x,r)\cap Q}|b(y)|^k|f(y)||\Omega(x-y)|\,\mathrm{d}y\\ &\le \sup_{r\ge|x|-\sqrt{n}l_Q}\frac{1}{r^{n-\beta}}\int\limits_{B(x,r)\cap Q}|b(y)|^k|f(y)||\Omega(x-y)|\,\mathrm{d}y\lesssim\frac{1}{|x|^{n-\beta}}.\end{aligned}$$

Thus,

$$\|II_1\|_{L^q(\omega^q)}\lesssim\Big(\int\limits_{B(0,R)^c}\frac{1}{|x|^{(n-\beta)q}}\omega(x)^q\,\mathrm{d}x\Big)^{1/q}.$$

Observe that $\omega^q\in A_{1+q/p'-\tau}$ for some $\tau>0$. We have

$$\int\limits_{B(0,2^jR)}\omega(x)^q\,\mathrm{d}x\le(2^jR)^{n(1+q/p'-\tau)}[\omega^q]_{A_{1+q/p'-\tau}}\int\limits_{B(0,1)}\omega(x)^q\,\mathrm{d}x.$$

As a result,

$$\int\limits_{B(0,2^{j+1}R)\backslash B(0,2^jR)}\frac{\omega(x)^q}{|x|^{(n-\beta)q}}\,\mathrm{d}x\lesssim\frac{(2^jR)^{n(1+q/p'-\tau)}}{2^{jq(n-\beta)}R^{(n-\beta)q}}=\frac{1}{(2^jR)^{n\tau}}.$$

From above, we have:

$$\begin{aligned}\|II_1\|_{L^q(\omega^q)}&\lesssim\Big(\int\limits_{B(0,R)^c}\frac{\omega(x)^q}{|x|^{(n-\beta)q}}\,\mathrm{d}x\Big)^{1/q}\\ &\le\Big(\sum_{j=0}^{\infty}\int\limits_{B(0,2^{j+1}R)\backslash B(0,2^jR)}\frac{\omega(x)^q}{|x|^{(n-\beta)q}}\,\mathrm{d}x\Big)^{1/q}\\ &\lesssim R^{-n\tau/q}\Big(\sum_{j=0}^{\infty}2^{-jn\tau}\Big)^{1/q}\lesssim R^{-n\tau/q}.\end{aligned}\tag{3.3}$$

Next, we consider $II_2(x)$, since $\omega^{-p'} \in A_{1+p'/q-\tau}$ for some $\tau > 0$, we have:

$$\int\limits_{B(0,2r)} \omega(y)^{-p'}\,\mathrm{d}y \le (2r)^{n(1+p'/q-\tau)}[\omega^{-p'}]_{A_{1+p'/q-\tau}} \int\limits_{B(0,1)} \omega(y)^{-p'}\,\mathrm{d}y.$$

Choose r_1 sufficiently large (may depending on R) such that $B(x,r) \subset B(0,2r)$ for any $x \in B(0,R)$, $r \ge r_1$. From this and the L^∞-boundedness of b and Ω, we have

$$\begin{aligned}
\|II_2\|_{L^q(\omega^q)} &\lesssim \sup_{x\in B(0,R)} \sup_{r\ge r_1} \frac{1}{r^{n-\beta}} \int\limits_{B(x,r)} |b(x)-b(y)|^k |f(y)\Omega(x-y)|\,\mathrm{d}y \\
&\lesssim \sup_{r\ge r_1} \frac{1}{r^{n-\beta}} \int\limits_{B(0,2r)} |f(y)|\,\mathrm{d}y \\
&\le \sup_{r\ge r_1} \frac{1}{r^{n-\beta}} \Big(\int\limits_{\mathbb{R}^n} |f(y)|^p \omega(y)^p\,\mathrm{d}y\Big)^{1/p} \Big(\int\limits_{B(0,2r)} \omega(y)^{-p'}\,\mathrm{d}y\Big)^{1/p'} \\
&\lesssim \sup_{r\ge r_1} \frac{1}{r^{n-\beta}} (r^{n(1+p'/q-\tau)})^{1/p'} = r_1^{-n\tau/p'}.
\end{aligned} \tag{3.4}$$

By (3.3), for a fixed large R such that $\|II_1\|_{L^q(\omega^q)} < \epsilon/3$, we choose r_1 sufficiently large such that $\|II_2\|_{L^q(\omega^q)} < \epsilon/3$ by (3.4). Moreover, from (3.2), we can choose a sufficiently small r_0 such that $\|I\|_{L^q(\omega^q)} < \epsilon/3$. Hence, for any $\epsilon > 0$, we have

$$\|(M_{\Omega,\beta})_b^k(f) - (M_{\Omega,\beta}^{r_0,r_1})_b^k(f)\|_{L^q(\omega^q)} < \epsilon \tag{3.5}$$

for sufficiently small r_0 and sufficiently large r_1. Thus, the remaining thing is to prove that $A(\Omega_{r_0,r_1}, b)$ is precompact. We will verify that $A(\Omega_{r_0,r_1}, b)$ satisfies the three conditions in Lemma 3.3.

The condition (a) is automatically valid because of the weighted L^q-boundedness of $(M_{\Omega,\beta}^{r_0,r_1})_b^k$.

Next, for sufficiently large N such that $Q \subset B(0,N)$ and $N > r_1 + \sqrt{n} l_Q$, we have

$$(M_{\Omega,\beta}^{r_0,r_1})_b^k(f)(x) = 0, \quad (|x| \ge N). \tag{3.6}$$

Hence, condition (b) is valid. Finally, it remains to check whether condition (c) holds. Take $z \in \mathbb{R}^n$ with $|z| \le \min\{\delta/8, r_0\}$; note that

$$\begin{aligned}
&\Big|\frac{1}{r^{n-\beta}} \int\limits_{B(x+z,r)} |b(x+z)-b(y)|^k |\Omega(x+z-y)||f(y)|\,\mathrm{d}y \\
&\quad - \frac{1}{r^{n-\beta}} \int\limits_{B(x,r)} |b(x)-b(y)|^k |\Omega(x-y)||f(y)|\,\mathrm{d}y\Big| \\
&\quad \le \frac{1}{r^{n-\beta}} \int\limits_{B(x+z,r)\triangle B(x,r)} |b(x+z)-b(y)|^k |\Omega(x+z-y)f(y)|\,\mathrm{d}y \\
&\qquad + \frac{1}{r^{n-\beta}} \int\limits_{B(x,r)} \big||b(x+z)-b(y)|^k - |b(x)-b(y)|^k\big| |\Omega(x-y)f(y)|\,\mathrm{d}y \\
&\qquad + \frac{1}{r^{n-\beta}} \int\limits_{B(x,r)} |b(x+z)-b(y)|^k |\Omega(x+z-y)-\Omega(x-y)||f(y)|\,\mathrm{d}y \\
&\quad =: D_1(x) + D_2(x) + D_3(x).
\end{aligned}$$

Observe that

$$|\Omega(x+z-y)-\Omega(x-y)| \lesssim |z|, \quad |b(x+z)-b(y)|^k \lesssim 1.$$

We have

$$\sup_{r_0<r<r_1} D_3(x) \lesssim |z| M_{\beta+k\alpha}(f)(x).$$

This and the weighted boundedness of $M_{\beta+k\alpha}$ yield that

$$\|\sup_{r_0<r<r_1} D_3\|_{L^q(\omega^q)} \lesssim |z|\|f\|_{L^p(\omega^p)} \le |z|. \tag{3.7}$$

On the other hand, we have

$$|\Omega(x-y)| \lesssim 1, \quad \big||b(x+z)-b(y)|^k - |b(x)-b(y)|^k\big| \lesssim |z|,$$

then

$$\sup_{r_0<r<r_1} D_2(x) \lesssim |z| M_{\beta+k\alpha}(f)(x),$$

which implies that

$$\|\sup_{r_0<r<r_1} D_2\|_{L^q(\omega^q)} \lesssim |z|\|f\|_{L^p(\omega^p)}. \tag{3.8}$$

Finally, let us consider $D_1(x)$. Note that for $|x-y| \le r_1$ or $|x+z-y| \le r_1$,

$$|b(x+z)-b(y)| \le |b(x+z)-b(y)|\chi_{B(0,R')}(x)\chi_{B(0,R')}(y),$$

where $R' = \sqrt{n}l_Q + \delta + r_1$. From this, we have:

$$\begin{aligned}
&\|\sup_{r_0<r<r_1} D_1\|_{L^q(\omega^q)} \\
&\lesssim \sup_{x\in B(0,R')} \sup_{r_0<r<r_1} \int_{B(x+z,r)\triangle B(x,r)} |f(y)||b(x+z)-b(y)|^k \, \mathrm{d}y \\
&\lesssim \sup_{x\in B(0,R')} \sup_{r_0<r<r_1} \|f\|_{L^p(\omega^p)} \Big(\int_{B(x+z,r)\triangle B(x,r)} \omega(y)^{-p'} \, \mathrm{d}y \Big)^{1/p'}.
\end{aligned}$$

For any $x \in B(0,R')$, $r_0 < r < r_1$, we have

$$\begin{aligned}
|B(x+z,r)\triangle B(x,r)| &\le 2(|B(0,r)| - |B(0,r-|z|)|) \\
&\lesssim r_1^{n-1}|z|,
\end{aligned}$$

which implies that $|B(x+z,r)\triangle B(x,r)| \to 0$ as $|z| \to 0$. Using this and the fact $\omega^{-p'} \in A_{1+p'/q} \subset L^1(B(0,R'))$, we get

$$\begin{aligned}
&\|\sup_{r_0<r<r_1} D_1\|_{L^q(\omega^q)} \\
&\lesssim \sup_{x\in B(0,R')} \sup_{r_0<r<r_1} \Big(\int_{B(x+z,r)\triangle B(x,r)} \omega(y)^{-p'} \, \mathrm{d}y \Big)^{1/p'} \to 0
\end{aligned}$$

as $|z| \to 0$. This, together with (3.7) and (3.8), yields the desired conclusion and completes the proof. □

4. Necessity of boundedness and compactness for commutators

This section is devoted to the proof of the necessity of Theorem 1.2. As a first step, we give the necessity of boundedness of $(M_{\Omega,\beta})_b^k$. Thanks to the breakthrough work of Lerner–Ombrosi–Rivera–Ríos [17], the following theorem can be presented as it is now.

Theorem 4.1. *Let $k \in \mathbb{Z}^+$, $0 \le \alpha \le 1$, $0 \le \beta < n$ with $k\alpha + \beta < n$, $1 < p, q < \infty$ with $1/q = 1/p - (k\alpha+\beta)/n$. Suppose that Ω is a measurable function on S^{n-1} and does not change sign and is not equivalent to zero on some open set of S^{n-1}, $\omega \in A_{p,q}$. If for any bounded measurable set $E \subset \mathbb{R}^n$,*

$$\|(M_{\Omega,\beta})_b^k(\chi_E)\|_{L^q(\omega^q)} \lesssim (\omega^p(E))^{1/p},$$

then $b \in \mathrm{BMO}_\alpha$.

Remark 4.2. Compared to the previous results, for example, in [12,13,19,22], our theorem relaxes the condition of kernel Ω and restrict the bounded condition of the operator to the characteristic functions. The original idea of the proof of Theorem 4.1 comes from [14,17], in which the corresponding results are established for the singular and fractional integral operators. Here, we only give some pointwise estimates in our proof; then the conclusion will be valid automatically by the same method in [14,17].

Proof of Theorem 4.1. We first prove the case for $\alpha = 0$. Given a cube Q, let E, F, be the sets given in Lemma 2.10. Take $f = (\int_F \omega(x)^p \, dx)^{-1/p} \chi_F$. Then for $x \in E$, by Lemma 2.10,

$$\begin{aligned}(M_{\Omega,\beta})_b^k(f)(x) &= \sup_{r>0} \frac{1}{r^{n-\beta}} \int\limits_{|x-y|\le r} |b(x)-b(y)|^k |\Omega(x-y) f(y)| \, dy \\ &\ge \omega_\lambda(b; Q)^k \Big(\int\limits_F \omega(x)^p \, dx \Big)^{-1/p} \sup_{r>0} \frac{1}{r^{n-\beta}} \int\limits_{\{y:|x-y|\le r\}\cap F} |\Omega(x-y)| \, dy.\end{aligned}$$

Observe that $\{y : |x-y| \le r\} \cap F = F$ when $r \ge (K_0+1)\sqrt{n} l_Q$, which implies that

$$(M_{\Omega,\beta})_b^k(f)(x) \gtrsim \omega_\lambda(b; Q)^k \Big(\int\limits_F \omega(x)^p \, dx \Big)^{-1/p} |Q|^{-1+\beta/n} \int\limits_F |\Omega(x-y)| \, dy.$$

For the rest of the proof, we refer to [15] or [17].

Next, we deal with the case where $0 < \alpha \le 1$. Given a cube Q, let γ, P, E_i, F_i be given in Lemma 2.11, and take $\gamma = 1/4$, $f_i = [\omega^p(F_i)]^{-1/p} \chi_{F_i}$, $i = 1, 2$. For $x \in E_i$, using Hölder's inequality,

$$\begin{aligned}(M_{\Omega,\beta})_b^k(f_i)(x) &= \sup_{r>0} \frac{1}{r^{n-\beta}} \int\limits_{|x-y|\le r} |b(x)-b(y)|^k |\Omega(x-y) f_i(y)| \, dy \\ &= |\omega^p(F_i)|^{-1/p} \sup_{r>0} \frac{1}{r^{n-\beta}} \int\limits_{F_i \cap B(x,r)} |b(x)-b(y)|^k |\Omega(x-y)| \, dy.\end{aligned}$$

Since $F_i \subset P$, $|F_i| = |P|/2$, $\omega^p \in A_p$ and again note that $\{y : |x-y| \le r\} \cap F_i = F_i$ provided that $r \ge (K_0+1)\sqrt{n} l_Q$, which yields that

$$(M_{\Omega,\beta})_b^k(f_i)(x) \sim |\omega^p(F_i)|^{-1/p} |Q|^{-1+\beta/n} \int\limits_{F_i} |b(x)-b(y)|^k |\Omega(x-y)| \, dy.$$

Then, following the method in [14, Proposition 2.5], we complete our proof. □

As a corollary of Theorems 4.1 and 3.1, we give the characterization of the boundedness of $(M_{\Omega,\beta})_b^k$.

Corollary 4.3. *Let $k \in \mathbb{Z}^+$, $0 \le \alpha \le 1$, $0 \le \beta < n$ with $k\alpha + \beta < n$, $1 < p, q < \infty$ with $1/q = 1/p - (k\alpha+\beta)/n$. Suppose that Ω is a homogeneous function of degree 0 on $\mathbb{R}^n$, $\Omega \in L^s(S^{n-1})$ for some $s > 1$ with $s' < p$, $\omega^{s'} \in A_{p/s', q/s'}$. If Ω does not change sign and is not equivalent to zero on some open set of S^{n-1}, then $b \in \mathrm{BMO}_\alpha$ if and only if $(M_{\Omega,\beta})_b^k$ is bounded from $L^p(\omega^p)$ to $L^q(\omega^q)$.*

Next, we prove the necessity of Theorem 1.2. To do this, we still follow the ideals in [14,15] by giving four lemmas about the lower and upper estimates of $(M_{\Omega,\beta})_b^k$. Lemmas 4.4 and 4.5 are used for the proof of the necessity of Theorem 1.2 for the case of $\alpha = 0$, while Lemmas 4.6 and 4.7 are used for the case of $0 < \alpha \le 1$.

Lemma 4.4. *Let $k \in \mathbb{Z}^+$, $\omega \in A_{p,q}$, $1 < p, q < \infty$, $0 \le \beta < n$, $1/q = 1/p - \beta/n$, $\lambda \in (0,1)$ and b be a real-valued measurable function. Given a cube Q, let E, F be the sets associated with Q given as in Lemma 2.10, set $f = (\int_F \omega(x)^p \, dx)^{-1/p} \chi_F$. Suppose that Ω is a homogeneous function of degree 0 on $\mathbb{R}^n$ and a measurable function on S^{n-1}, which does not change sign and is not equivalent to zero on some open set of S^{n-1}. Then there exists a constant $C > 0$, which is independent of Q, such that for any measurable set B with $|B| \le \frac{\lambda}{8}|Q|$,*

$$\Big(\int\limits_{E\setminus B} (M_{\Omega,\beta})_b^k(f)(x)^q \omega(x)^q \, dx \Big)^{1/q} \ge C \omega_\lambda(b; Q)^k.$$

Proof. Similarly to the proof of Theorem 4.1, we can get the desired result. □

Lemma 4.5. *Let $k \in \mathbb{Z}^+$, $b \in \mathrm{BMO}$ and $\omega \in A_{p,q}$. Given a cube Q, let F be the set associated with Q given as in Lemma 2.10, set $f = (\int_F \omega(x)^p\,\mathrm{d}x)^{-1/p}\chi_F$. Suppose that Ω is a homogeneous function of degree 0 on $\mathbb{R}^n$ and $\Omega \in L^\infty(S^{n-1})$. Then there exists a constant $\zeta > 0$ such that*

$$\Big(\int_{2^{d+1}Q\backslash 2^d Q} (M_{\Omega,\beta})_b^k(f)(x)^q \omega(x)^q\,\mathrm{d}x\Big)^{1/q} \lesssim d^k 2^{-\zeta dn/p}$$

holds for d large enough, uniformly for all cubes Q.

Proof. Observe that $(M_{\Omega,\beta})_b^k(f)(x) \le (T_{|\Omega|,\beta})_b^k(|f|)(x)$, where

$$(T_{|\Omega|,\beta})_b^k(|f|)(x) = \int_{\mathbb{R}^n} \frac{|\Omega(x-y)||f(y)|}{|x-y|^{n-\beta}}|b(x)-b(y)|^k\,\mathrm{d}y.$$

By this fact and the known results for $(T_{|\Omega|,\beta})_b^k(f)(x)$ in [15, Proposition 4.4], the desired conclusion for $(M_{\Omega,\beta})_b^k(f)$ follows. $\square$

Lemma 4.6. *Let $1 < p, q < \infty$, $\alpha \in (0,1]$, $0 \le \beta < n$, $k\alpha + \beta < n$, $1/q = 1/p - (k\alpha+\beta)/n$, $\omega \in A_{p,q}$ and Ω be stated as in Theorem 4.1. Given a real-valued function $b \in Lip_\alpha(\mathbb{R}^n)$, for arbitrary given Q with $\widetilde{\mathcal{O}_\alpha}(b;Q) \ge \eta_0 > 0$, let $\tilde{P} := 2P$ be the set associated with $\tilde{Q} := 2Q$ as stated in Lemma 2.11 and $\gamma = 1/2^{n+1}\Big(\min\Big\{\Big(\eta_0/4\|b\|_{Lip_\alpha(\mathbb{R}^n)}\Big)^{1/\alpha}/\sqrt{n}, 1/2\Big\}\Big)^n$. Suppose that there are cubes $E \subset 2Q$ and $F \subset 2P$ satisfy*

$$|E| = |F| \ge \tilde{C}\min\{(\mathcal{O}_\alpha(b;Q))^{n/\alpha}, 1\}|Q|,$$

where $\tilde{C}$ is independent of Q. Set $f := (\int_F \omega(x)^p\,\mathrm{d}x)^{-1/p}\chi_F$. Then for any measurable set B with $|B| \le |E|/2$, we have

$$\|(M_{\Omega,\beta})_b^k(f)\|_{L^q(E\backslash B,\omega^q)} \ge C\min\{(\mathcal{O}_\alpha(b;Q))^{2n/\alpha}, 1\}\mathcal{O}_\alpha(b;Q)^k.$$

Proof. Just as in the proof of Theorem 4.1,

$$\begin{aligned}
&\int_{E\backslash B} (M_{\Omega,\beta})_b^k(f)(x)\,\mathrm{d}x \\
&\ge \Big(\int_F \omega(x)^p\,\mathrm{d}x\Big)^{-1/p} \int_{E\backslash B} \sup_{r \ge (K_0+1)\sqrt{n}l_Q} \frac{1}{r^{n-\beta}} \int_{B(x,r)\cap F} |b(x)-b(y)|^k|\Omega(x-y)|\,\mathrm{d}y\,\mathrm{d}x \\
&\sim \Big(\int_F \omega(x)^p\,\mathrm{d}x\Big)^{-1/p} |Q|^{\beta/n-1} \int_{E\backslash B}\int_F |b(x)-b(y)|^k|\Omega(x-y)|\,\mathrm{d}y\,\mathrm{d}x.
\end{aligned}$$

Then the desired conclusion follows by [14, Proposition 4.1]. $\square$

Lemma 4.7. *Let $p, q, \alpha, \beta, k, \omega$ be stated as in Lemma 4.6. Let $b \in Lip_\alpha$ and $\Omega \in L^\infty(S^{n-1})$. Given a cube Q, let F be the set associated with Q given as in Lemma 2.11, and set $f = (\int_F \omega(x)^p\,\mathrm{d}x)^{-1/p}\chi_F$. Then there is a constant $\zeta > 0$ such that*

$$\Big(\int_{2^{d+1}Q\backslash 2^d Q} (M_{\Omega,\beta})_b^k(f)(x)^q \omega(x)^q\,\mathrm{d}x\Big)^{1/q} \lesssim 2^{-\zeta dn/p}$$

holds for d large enough.

Proof. Observe that $(M_{\Omega,\beta})_b^k(f)(x) \le (T_{|\Omega|,\beta}(|f|))_b^k(x)$, where

$$(T_{|\Omega|,\beta})_b^k(|f|)(x) = \int_{\mathbb{R}^n} \frac{|\Omega(x-y)||f(y)|}{|x-y|^{n-\beta}}|b(x)-b(y)|^k\,\mathrm{d}y.$$

Then, the desired conclusion follows by [14, Proposition 4.2]. $\square$

Necessity of Theorem 1.2. Using the lower and upper estimates for $(M_{\Omega,\beta})_b^k$ in Lemmas 4.4, 4.5, 4.6 and 4.7, the proof of necessity of Theorem 1.2 follows from the same arguments in [14,15]. We omit the details. $\square$

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