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Partial differential equations

Stabilization in three-dimensional chemotaxis-growth model with indirect attractant production

Stabilisation d'un modèle tridimensionnel de croissance chimiotaxique avec production d'attracteur indirecte

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A R T I C L E   I N F O

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A B S T R A C T

This paper deals with the chemotaxis-growth system: $u_t = \Delta u - \nabla \cdot (u\nabla v) + \mu u(1-u)$, $v_t = \Delta v - v + w$, $\tau w_t + \delta w = u$ in a smooth bounded domain $\Omega \subset \mathbb{R}^3$ with zero-flux boundary conditions, where μ, δ, and τ are given positive parameters. It is shown that the solution (u, v, w) exponentially stabilizes to the constant stationary solution $(1, \frac{1}{\delta}, \frac{1}{\delta})$ in the norm of $L^\infty(\Omega)$ as $t \to \infty$ provided that $\mu > 0$ and any given nonnegative and suitably smooth initial data (u_0, v_0, w_0) fulfills $u_0 \not\equiv 0$, which extends the condition $\mu > \frac{1}{8\delta^2}$ in [8].

R É S U M É

Cette Note traite du système de croissance chimiotaxique : $u_t = \Delta u - \nabla(u\nabla v) + \mu u(1-u)$, $v_t = \Delta v - v + w$, $\tau w_t + \delta w = u$ dans un domaine borné lisse $\Omega \subset \mathbf{R}^3$ avec une condition de flux zéro au bord et où μ, δ et τ sont des paramètres positifs donnés. Nous montrons que la solution (u, v, w) se stabilise exponentiellement vers la solution constante stationnaire $(1, 1/\delta, 1/\delta)$ en norme $L^\infty(\Omega)$ lorsque t tend vers l'infini, pourvu que $\mu > 0$ et que toute donnée initiale positive ou nulle suffisamment lisse satisfasse $u_0 \not\equiv 0$. Ces hypothèses relaxent la condition $\mu > 1/8\delta^2$ de [8].

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1. Introduction

This paper deals with the chemotaxis-growth system with indirect signal production

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla v) + \mu u(1-u), & x \in \Omega, \quad t > 0, \\ v_t = \Delta v - v + w, & x \in \Omega, \quad t > 0, \\ \tau w_t + \delta w = u, & x \in \Omega, \quad t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, \quad t > 0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad \tau w(x,0) = \tau w_0(x) & x \in \Omega \end{cases} \tag{1.1}$$

which was proposed [15] to model the spread and aggregative behavior of the Mountain Pine Beetle (MPB) in forest habitat. $\Omega \subset \mathbb{R}^n$ ($n \le 3$) is a smooth bounded domain with zero-flux boundary conditions. The parameters μ, δ, and τ are positive. Here $u = u(x,t)$ denotes the density of flying MPBs, which can bias their movement in response to MPB pheromone with concentration $w = w(x,t)$, the latter being secreted only by those nesting MPB, mathematically represented through its density $v = v(x,t)$. In addition to random dispersal, chemotaxis movement, the flying MPBs are assumed to undergo birth and death following a standard logistic law. The nesting MPBs increase through transition from the flying to the nesting state, and they decrease due to death measured by parameter δ.

Eq. (1.1) originates in, but essentially differs from the celebrated chemotaxis model with logistic source

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla v) + \mu u(1-u), & x \in \Omega, \quad t > 0, \\ v_t = \Delta v - v + u, & x \in \Omega, \quad t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, \quad t > 0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in \Omega, \end{cases} \tag{1.2}$$

which was proposed as a first model about chemotaxis by Keller and Segel [9] to describe the aggregation phase in the slime mold formation of *Dictyostelium discoideum* in response to the chemical cyclic adenosine monophosphate (cAMP). Chemotaxis as a significant mechanism of directional migration of cells is the movement of cells in response to concentration gradients of a chemical signal emitted by the cells themselves in many biological processes. During the past four decades, a main issue of the investigation was whether the solutions are bounded or blow up (see, e.g., [1,2,4,3,5,7,6,10–14,16,20,21,23,24,22,25–27,29] and references therein for detailed results). For example, if $\mu \equiv 0$, a striking feature of system (1.2) is that some of its solutions blow up in finite time for $n \ge 2$, where n represents the spatial dimension (see [2,3,6,12,22,25,29]). If $n \ge 2$, solutions to (1.2) without the logistic term (i.e. $\mu \equiv 0$) may blow up in finite time (see [4,11]); however, if $n = 2$ and $\mu > 0$ is arbitrarily small, the authors [13] investigated that all of the solutions to (1.2) are global and bounded. In the case $n \ge 3$, the unbounded solution can be excluded for the appropriately large μ which is compared with the chemotactic coefficient χ (see [24]). Recently, it is worth mentioning that if the term $\mu u(1-u)$ was replaced by the term $u - \mu u^r$ in (1.2), where $\mu > 0$ and $r \ge 2$, Lin et al. [10] investigated that the solution of (1.2) is global in time and bounded under the explicit condition on the parameters χ, μ and r. Furthermore, Lin et al. [10] also established the solution (u, v) of system (1.2) converges to the steady state $(1, 1)$ as $t \to \infty$ for $n = 2, 3$.

In drastic contrast to the corresponding three-dimensional chemotaxis-growth system (1.2), in which the global existence or blow-up of classical solutions largely remains open if $\mu > 0$ is small, it is worth mentioning that the authors [8] found out that an arbitrarily small quadratic degradation term $\mu > 0$ is sufficient to suppress any blow-up phenomenon in (1.1) when $n = 3$ and that the solutions to (1.1) exponentially stabilize to the constant stationary solution $(1, \frac{1}{\delta}, \frac{1}{\delta})$ in the norm of $L^\infty(\Omega)$ as time tends to infinity, provided that $\mu > \frac{1}{8\delta^2}$ by using the methods in [17,18]. More results about the Cauchy problem (1.1) were obtained (see [19]).

The purpose of this paper is to extend the condition $\mu > \frac{1}{8\delta^2}$ in [8] to $\mu > 0$. Throughout the paper, we assume that the initial data (u_0, v_0, w_0) satisfies

$$u_0 \in C^0(\overline{\Omega}), v_0 \in C^1(\overline{\Omega}), w_0 \in C^1(\overline{\Omega}), u_0 \ge 0, v_0 \ge 0, w_0 \ge 0 \text{ and } u_0 \not\equiv 0. \tag{1.3}$$

The following solutions to system (1.1) have been known. The first two are on the local existence and global boundedness of solutions to the model (1.1) by B. Hu and Y. Tao [8].

Lemma 1.1. *(Lemma 2.1 in [8]) Let $\mu > 0$, $\delta > 0$ and $\tau > 0$, and suppose that (u_0, v_0, w_0) fulfills (1.3). Then there exist $T_{\max} \in (0, \infty]$ and a unique triple of (u, v, w) nonnegative functions*

$$\begin{aligned} &u \in C(\overline{\Omega} \times [0, T_{\max})) \cap C^{2,1}(\overline{\Omega} \times (0, T_{\max})), \\ &v \in C(\overline{\Omega} \times [0, T_{\max})) \cap C^{2,1}(\overline{\Omega} \times (0, T_{\max})) \cap L^\infty_{loc}([0, T_{\max}); W^{1,q}(\Omega)), \\ &w \in C^{0,1}(\overline{\Omega} \times [0, T_{\max})) \end{aligned} \tag{1.4}$$

that solves (1.1) classically in $\Omega \times (0, T_{\max})$ and that are such that, if $T_{\max} < \infty$, then

$$\lim_{t \to T_{\max}} ||u(\cdot, t)||_{L^\infty(\Omega)} = \infty \text{ as } t \nearrow T_{\max}. \tag{1.5}$$

Lemma 1.2. *(see [8]) Let $n = 3$, $\mu > 0$, $\delta > 0$ and $\tau > 0$, and suppose that (1.3) holds. Then there exists $T_{\max} \in (0, \infty]$ such that the problem (1.1) possesses a unique classical solution (u, v, w), uniformly bounded, i.e. $T_{\max} = \infty$ and there exists a constant $C > 0$ such that*

$$u(x,t) \le C,\ v(x,t) \le C \ \text{ and } \ w(x,t) \le C \ \text{ for all } x \in \Omega \text{ and } t > 0. \tag{1.6}$$

Moreover, there exist $\vartheta \in (0,1)$ and $K > 0$ such that

$$||u||_{C^{2+\vartheta,1+\frac{\vartheta}{2}}(\overline{\Omega}\times[t,t+1])} + ||v||_{C^{2+\vartheta,1+\frac{\vartheta}{2}}(\overline{\Omega}\times[t,t+1])} + ||w||_{C^{2+\vartheta,1+\frac{\vartheta}{2}}(\overline{\Omega}\times[t,t+1])} \le K,\ t \ge \delta_0 > 0. \tag{1.7}$$

Now we state the main result of the present paper.

Theorem 1.1. *Let $n = 3$, $\mu > 0$, $\delta > 0$ and $\tau > 0$, assume that (1.3) holds. Then any global classical solution to (1.1) constructed in Lemma 1.2 has the exponential convergence property*

$$||u(\cdot,t) - 1||_{L^\infty(\Omega)} + ||v(\cdot,t) - \frac{1}{\delta}||_{L^\infty(\Omega)} + ||w(\cdot,t) - \frac{1}{\delta}||_{L^\infty(\Omega)} \le C e^{-\alpha t} \tag{1.8}$$

for all $t > 0$, with some $C > 0$ and $\alpha > 0$.

2. Proof of Theorem 1.1

For any $\mu > 0$, Lemma 1.2 asserts the global existence and boundedness of classical solutions to (1.1). In this section, we shall obtain that the solution (u, v, w) will exponentially stabilize to the constant stationary solution $(1, \frac{1}{\delta}, \frac{1}{\delta})$ for any $\mu > 0$ based on some known smooth estimates for the heat semigroup $(e^{\tau\Delta})_{\tau \ge 0}$ under Neumann boundary conditions [2,23].

Lemma 2.1. *Let $(e^{t\Delta})_{t\ge 0}$ be the Neumann heat semigroup in Ω, and let $\lambda_1 > 0$ denote the first nonzero eigenvalue of $-\Delta$ in Ω under Neumann boundary conditions. Then there exist positive constants η_1, η_2, η_3, and η_4 depending on Ω only, which have the following properties.*

(i) If $1 \le q \le p \le \infty$, then

$$||e^{t\Delta}\phi||_{L^p(\Omega)} \le \eta_1\big(1 + t^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})}\big)\, e^{-\lambda_1 t}||\phi||_{L^q(\Omega)} \ \text{ for all } \ t > 0 \tag{2.1}$$

holds for all $\phi \in L^q(\Omega)$ satisfying $\int_\Omega \phi = 0$.

(ii) If $1 \le q \le p \le \infty$, then

$$||\nabla e^{t\Delta}\phi||_{L^p(\Omega)} \le \eta_2\big(1 + t^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})}\big)\, e^{-\lambda_1 t}||\phi||_{L^q(\Omega)} \ \text{ for all } t > 0 \tag{2.2}$$

is true for each $\phi \in L^q(\Omega)$.

(iii) If $2 \le q \le p < \infty$, then

$$||\nabla e^{t\Delta}\phi||_{L^p(\Omega)} \le \eta_3\big(1 + t^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})}\big)\, e^{-\lambda_1 t}||\nabla\phi||_{L^q(\Omega)} \ \text{ for all } t > 0 \tag{2.3}$$

is valid for all $\phi \in W^{1,q}(\Omega)$.

(iv) If $1 < q \le p \le \infty$, then

$$||e^{t\Delta}\nabla\cdot\phi||_{L^p(\Omega)} \le \eta_4\big(1 + t^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})}\big)\, e^{-\lambda_1 t}||\phi||_{L^q(\Omega)} \ \text{ for all } t > 0 \tag{2.4}$$

holds for all $\phi \in (C_0^\infty(\Omega))^n$, where $\lambda_1 > 0$ denotes the first nonzero eigenvalue of $-\Delta$ in Ω under Neumann boundary conditions.

Lemma 2.2. *Let $\gamma \in (0,1)$. Then, for any positive parameters χ, δ, τ, and μ, and for each solution to (1.1) with initial data fulfilling (1.3), one can find $\alpha \in \big(0, \min\{\mu, \gamma, \frac{\tau}{\delta}\}\big)$ and $C > 0$ such that*

$$||u(\cdot,t) - 1||_{L^\infty(\Omega)} \le C\, e^{-\alpha t} \tag{2.5}$$

and

$$||v(\cdot,t) - \frac{1}{\delta}||_{L^\infty(\Omega)} \le C\, e^{-\alpha t} \tag{2.6}$$

as well as

$$||w(\cdot,t) - \frac{1}{\delta}||_{L^\infty(\Omega)} \le C\, e^{-\alpha t} \tag{2.7}$$

hold for all $t > 0$.

Proof. Since u is bounded in $L^\infty(\Omega\times(0,\infty))$ asserted by Lemma 1.2, we can pick some large constants $C_i>0$ $(i=1,2,3)$ such that

$$\left\|u(\cdot,t)-1\right\|_{L^\infty(\Omega)}\le C_1,\ \left\|v(\cdot,t)-1\right\|_{W^{1,\infty}(\Omega)}\le C_2 \text{ and } \left\|w(\cdot,t)\right\|_{L^\infty(\Omega)}\le C_3 \tag{2.8}$$

for all $t>0$, which implies that the set

$$T:=\sup\left\{T_0\ge t_0\,\middle|\,||u(\cdot,t)-1||_{L^\infty(\Omega)}\le C_1\,\mathrm{e}^{-\gamma(t-t_0)} \text{ for all } t\in[t_0,T_0]\right\} \tag{2.9}$$

is not empty (see [28] for details). In particular, T is well defined and, in order to prove the lemma, it is sufficient to make sure that actually $T=\infty$. Next we abbreviate $U(\cdot,t):=u(\cdot,t)-1$, $V(\cdot,t):=v(\cdot,t)-\frac{1}{\delta}$ and $W(\cdot,t):=w(\cdot,t)-\frac{1}{\delta}$; the system (1.1) becomes

$$\begin{cases} U_t=\Delta U-\nabla\cdot(u\nabla V)-\mu U-\mu U^2, & x\in\Omega,\quad t>0,\\ V_t=\Delta V-V+W, & x\in\Omega,\quad t>0,\\ \tau W_t+\delta W=U, & x\in\Omega,\quad t>0. \end{cases} \tag{2.10}$$

The equality $\tau W_t+\delta W=U$ and (2.9) imply that

$$\begin{aligned} \left\|W(\cdot,t)\right\|_{L^\infty(\Omega)} &\le \mathrm{e}^{-\frac{\tau}{\delta}t}\left\|W_0\right\|_{L^\infty(\Omega)}+\frac{1}{\delta}\,\mathrm{e}^{-\frac{\tau}{\delta}t}\int_0^t \mathrm{e}^{\frac{\tau}{\delta}s}\left\|U(\cdot,s)\right\|_{L^\infty(\Omega)}\mathrm{d}s\\ &\le \mathrm{e}^{-\frac{\tau}{\delta}t}\left\|W_0\right\|_{L^\infty(\Omega)}+\frac{C_1}{\delta}\,\mathrm{e}^{-\frac{\tau}{\delta}t}\int_0^t \mathrm{e}^{\frac{\tau}{\delta}s}\mathrm{e}^{-\gamma(s-t_0)}\mathrm{d}s\\ &\le \mathrm{e}^{-\frac{\tau}{\delta}t}\left\|W_0\right\|_{L^\infty(\Omega)}+\begin{cases}\frac{C_1}{\tau-\delta\gamma}\,\mathrm{e}^{-\gamma(t-t_0)}, & \gamma<\frac{\tau}{\delta};\\ \frac{C_1}{\delta\gamma-\tau}\,\mathrm{e}^{\gamma t_0}\mathrm{e}^{-\frac{\tau}{\delta}t}, & \gamma\ge\frac{\tau}{\delta}\end{cases} \end{aligned} \tag{2.11}$$

for all $t\in[t_0,T]$. Along with (2.8), this implies that

$$||W(\cdot,t)||_{L^\infty(\Omega)}\le C_4\,\mathrm{e}^{-\min\{\gamma,\frac{\tau}{\delta}\}(t-t_0)} \tag{2.12}$$

for all $t\in[t_0,T]$ with some constants $C_4>0$. Applying the variation-of-constants formula with respect to U and V, we can find that

$$\begin{aligned} U(\cdot,t)&=\mathrm{e}^{(\Delta-\mu)(t-t_0)}U(\cdot,t_0)-\int_{t_0}^t \mathrm{e}^{(\Delta-\mu)(t-s)}\nabla\cdot(u(\cdot,s)\nabla V(\cdot,s))\,\mathrm{d}s\\ &\quad-\mu\int_{t_0}^t \mathrm{e}^{(\Delta-\mu)(t-s)}U^2(\cdot,s)\,\mathrm{d}s \end{aligned} \tag{2.13}$$

and

$$V(\cdot,t)=\mathrm{e}^{(\Delta-1)(t-t_0)}V(\cdot,t_0)+\int_{t_0}^t \mathrm{e}^{(\Delta-1)(t-s)}W(\cdot,s)\,\mathrm{d}s. \tag{2.14}$$

Then by means of the variation-of-constants representation for V and applying Lemma 2.1 and (2.12), we can estimate

$$\begin{aligned} &\left\|\nabla V(\cdot,t)\right\|_{L^p(\Omega)}\\ &\le \left\|\nabla\mathrm{e}^{(t-t_0)(\Delta-1)}V(\cdot,t_0)\right\|_{L^p(\Omega)}+\left\|\int_{t_0}^t \nabla\mathrm{e}^{(t-s)(\Delta-1)}W(\cdot,s)\,\mathrm{d}s\right\|_{L^p(\Omega)}\\ &\le \mathrm{e}^{-(t-t_0)}\left\|\nabla\mathrm{e}^{(t-t_0)\Delta}V(\cdot,t_0)\right\|_{L^p(\Omega)}+\eta_2\int_{t_0}^t\left(1+(t-s)^{-\frac{1}{2}}\right)\mathrm{e}^{-(t-s)}\left\|W(\cdot,s)\right\|_{L^p(\Omega)}\mathrm{d}s\\ &\le \mathrm{e}^{-(t-t_0)}\cdot C_5\left\|\nabla V(\cdot,t_0)\right\|_{L^p(\Omega)} \end{aligned} \tag{2.15}$$

$$
\begin{aligned}
&+\eta_2|\Omega|^{\frac{1}{p}}\cdot\int_{t_0}^{t}\left(1+(t-s)^{-\frac{1}{2}}\right)e^{-(t-s)}\left|\left|W(\cdot,s)\right|\right|_{L^\infty(\Omega)}ds\\
&\le e^{-(t-t_0)}\cdot C_5\cdot C_2+\eta_2|\Omega|^{\frac{1}{p}}C_4\cdot\int_{t_0}^{t}\left(1+(t-s)^{-\frac{1}{2}}\right)e^{-(t-s)}\cdot e^{-\min\{\gamma,\frac{\tau}{\delta}\}(s-t_0)}ds\\
&\le C_2C_5\cdot e^{-(t-t_0)}+\eta_2|\Omega|^{\frac{1}{p}}C_4\cdot\left(\int_{t_0}^{t}\left(1+\rho^{-\frac{1}{2}}\right)e^{-(1-\min\{\gamma,\frac{\tau}{\delta}\})\rho}d\rho\right)\cdot e^{-\min\{\gamma,\frac{\tau}{\delta}\}(t-t_0)}
\end{aligned}
$$

for all $t\in(t_0,T)$. Furthermore, we note that, since $\alpha\in\left(0,\min\left\{\mu,\gamma,\frac{\tau}{\delta}\right\}\right)$, $\gamma\in(0,1)$ and $\frac{1}{2}+\frac{n}{2p}<1$, the numbers

$$
C_6:=\int_0^\infty\left(1+\rho^{-\frac{1}{2}}\right)e^{-(1-\min\{\gamma,\frac{\tau}{\delta}\})\rho}d\rho \tag{2.16}
$$

and

$$
C_7:=\int_0^\infty\left(1+\rho^{-\frac{1}{2}-\frac{3}{2p}}\right)e^{-(\mu-\alpha)\rho}d\rho \tag{2.17}
$$

are finite. Now using (2.15) and (2.16), and recalling that $\alpha\in\left(0,\min\left\{\mu,\gamma,\frac{\tau}{\delta}\right\}\right)$ and $\gamma\in(0,1)$, we obtain that

$$
\left|\left|\nabla V(\cdot,t)\right|\right|_{L^p(\Omega)}\le\left(C_5C_2+\eta_2|\Omega|^{\frac{1}{p}}C_4C_6\right)\cdot e^{-\alpha(t-t_0)} \tag{2.18}
$$

for all $t\in(t_0,T)$. In order to show $T=\infty$, we need to obtain that

$$
||u(\cdot,t)-1||_{L^\infty(\Omega)}\le C_0\,e^{-\alpha(t-t_0)}\quad\text{for all }\ t\in(t_0,T) \tag{2.19}
$$

with some $C_0<C_1$. Employing the variation-of-constants formula for U, we see that

$$
\begin{aligned}
\left|\left|U(\cdot,t)\right|\right|_{L^\infty(\Omega)}\le\ & e^{-\mu(t-t_0)}\left|\left|e^{(t-t_0)\Delta}U(\cdot,t_0)\right|\right|_{L^\infty(\Omega)}\\
&+\int_{t_0}^{t}e^{-\mu(t-s)}\left|\left|e^{(t-s)\Delta}\nabla\cdot(u(\cdot,s)\nabla V(\cdot,s))\right|\right|_{L^\infty(\Omega)}ds\\
&+\int_{t_0}^{t}e^{-\mu(t-s)}\left|\left|e^{(t-s)\Delta}U^2(\cdot,s)\right|\right|_{L^\infty(\Omega)}ds
\end{aligned} \tag{2.20}
$$

for all $t\in(t_0,T)$. Next the maximum principle, together with (2.8), (2.9), and the fact that $\alpha<\mu$ yields that

$$
\begin{aligned}
e^{-\mu(t-t_0)}\left|\left|e^{(t-t_0)\Delta}U(\cdot,t_0)\right|\right|_{L^\infty(\Omega)}&\le e^{-\mu(t-t_0)}\left|\left|U(\cdot,t_0)\right|\right|_{L^\infty(\Omega)}\\
&\le C_8e^{-\alpha(t-t_0)}\quad\text{for all }\ t\ge t_0
\end{aligned} \tag{2.21}
$$

with some constants $C_8>0$. Moreover, combining (2.9), (2.15), and (2.18) yields

$$
\begin{aligned}
&\int_{t_0}^{t}e^{-\mu(t-s)}\left|\left|e^{(t-s)\Delta}\nabla\cdot(u(\cdot,s)\nabla V(\cdot,s))\right|\right|_{L^\infty(\Omega)}ds\\
&\le\eta_4\int_{t_0}^{t}\left(1+(t-s)^{-\frac{1}{2}-\frac{3}{2p}}\right)e^{-\mu(t-s)}\left|\left|u(\cdot,s)\right|\right|_{L^\infty(\Omega)}\left|\left|\nabla V(\cdot,s)\right|\right|_{L^p(\Omega)}ds\\
&\le C_1\eta_4\int_{t_0}^{t}\left(1+(t-s)^{-\frac{1}{2}-\frac{3}{2p}}\right)e^{-\mu(t-s)}\cdot\left(C_4C_2+\eta_2|\Omega|^{\frac{1}{p}}C_4C_6\right)\cdot e^{-\alpha(s-t_0)}ds
\end{aligned} \tag{2.22}
$$

$$\leq C_1\eta_4 \cdot \left(C_4C_2 + \eta_2|\Omega|^{\frac{1}{p}} C_4C_6\right) \cdot \left(\int_{t_0}^{t} (1+\rho^{-\frac{1}{2}-\frac{3}{2p}}) e^{-(\mu-\alpha)\rho}\, d\rho\right) \cdot e^{-\alpha(t-t_0)}$$
$$\leq C_1C_7\eta_4 \cdot \left(C_4C_2 + \eta_2|\Omega|^{\frac{1}{p}} C_4C_6\right) \cdot e^{-\alpha(t-t_0)}$$
$$\leq C_9 \cdot e^{-\alpha(t-t_0)}$$

for all $t \in (t_0, T)$ with some $C_9 = C_9(p, |\Omega|) > 0$. Furthermore, we can find some constants $C_{10} > 0$ to estimate

$$\begin{aligned}\int_{t_0}^{t} e^{-\mu(t-s)} \left|\left|e^{(t-s)\Delta} U^2(\cdot,s)\right|\right|_{L^\infty(\Omega)} ds &\leq C_3^2 \int_{t_0}^{t} e^{-\mu(t-s)} e^{-\gamma(s-t_0)}\, ds \\ &\leq C_{10} e^{-\min\{\mu,\gamma\}(t-t_0)} \\ &\leq C_{10} e^{-\alpha(t-t_0)}\end{aligned} \tag{2.23}$$

for all $t \in (t_0, T)$ again because $\alpha \in \left(0, \min\left\{\mu, \gamma, \frac{\tau}{\delta}\right\}\right)$. Thus if suitably enlarging C_1 such that $C_8 + C_9 + C_{10} < C_1$, then the continuity of the function $t \mapsto ||u(\cdot,t) - 1||_{L^\infty(\Omega)}$ implies that indeed T cannot be finite. This yields that $T = \infty$ and hence prove (2.5). Thus we can make use of (2.5) and (2.11) to obtain (2.7) for all $t > t_0$.

According to (2.7), we can apply the maximum principle and (2.10) to obtain that

$$\begin{aligned}\left|\left|V(\cdot,t)\right|\right|_{L^\infty(\Omega)} &\leq e^{-(t-t_0)} \left|\left|e^{(t-t_0)\Delta} V_0\right|\right|_{L^\infty(\Omega)} + \int_{t_0}^{t} e^{-(t-s)} \left|\left|e^{(t-s)\Delta} W(\cdot,s)\right|\right|_{L^\infty(\Omega)} ds \\ &\leq e^{-(t-t_0)} \left|\left|V_0\right|\right|_{L^\infty(\Omega)} + \int_{t_0}^{t} e^{-(t-s)} \left|\left|W(\cdot,s)\right|\right|_{L^\infty(\Omega)} ds \quad \text{for all } t > 0.\end{aligned} \tag{2.24}$$

We abbreviate $C_{12} := ||V_0||_{L^\infty(\Omega)}$ to obtain that

$$\begin{aligned}\left|\left|V(\cdot,t)\right|\right|_{L^\infty(\Omega)} &\leq C_{12} e^{-(t-t_0)} + C_{11} \int_{t_0}^{t} e^{-(t-s)} e^{-\alpha(s-t_0)}\, ds \\ &= C_{12} e^{-(t-t_0)} + \frac{C_{11}}{1-\alpha} (e^{-\alpha(t-t_0)} - e^{-(t-t_0)}) \quad \text{for all } t > 0.\end{aligned} \tag{2.25}$$

This yields (2.6); then we can obtain the desired results in the light of the definition of U, V and W. □

Now, we are in a position to prove Theorem 1.1.

Proof of Theorem 1.1. With the aid of Lemma 2.2, we can obtain the desired result (Theorem 1.1). □

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