C. R. Acad. Sci. Paris, Ser. I 357 (2019) 681–685

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

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Geometry/Differential geometry

A centro-projective inequality

Une inégalité centro-projective

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ARTICLE INFO

Article history:
Received 24 July 2018
Accepted after revision 16 July 2019
Available online 27 August 2019

Presented by the Editorial Board

ABSTRACT

We give a new integral formula for the centro-projective area of a convex body, which was first defined by Berck–Bernig–Vernicos. We then use the formula to prove that it is bounded from above by the centro-projective area of an ellipsoid and that equality occurs if and only if the convex set is an ellipsoid.

RÉSUMÉ

Nous présentons une nouvelle formule pour l'aire centro-projective d'un corps convexe. Cette aire a été préalablement définie par Berck–Bernig–Vernicos. Nous utilisons cette formule pour montrer qu'elle est majorée par l'aire centro-projective d'une ellipse, l'égalité caractérisant les ellipsoïdes.

0. Introduction and statement of results

Let V be an n-dimensional vector space with origin o. Given a convex body K containing o in the interior, we define the function $a : \partial K \to (0, \infty)$ such that, for each $p \in \partial K$, $-a(p)p \in \partial K$. The letter a stands for *antipodal*. Given a Euclidean scalar product $\langle \cdot, \cdot \rangle$ on V, let $k(p)$ be the Gauss curvature and $\nu_K(p)$ the outer unit normal at each $p \in \partial K$, whenever they are well and uniquely defined (which, by A.D. Alexandroff [1], holds almost everywhere).

Definition 1. The *centro-projective* area of K is

$$\mathcal{C}_o(K) := \int_{\partial K} \frac{\sqrt{k}}{\langle \nu_K(p), p \rangle^{\frac{n-1}{2}}} \left(\frac{2a}{1+a} \right)^{\frac{n-1}{2}} \mathrm{d}A(p), \tag{1}$$

where $\mathrm{d}A$ is the $(n-1)$-dimensional Hausdorff measure on ∂K.

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https://doi.org/10.1016/j.crma.2019.07.005

As shown in [2], this does not depend on the choice of the scalar product. In fact, the centro-projective area is invariant under *projective transformations* fixing the origin. It is also upper semi-continuous with respect to the Hausdorff topology.

It is worth comparing the definitions of centro-affine surface area with those of centro-projective area. The two are similar, except that the latter has an additional factor (containing the function a). A reader familiar with the theory of valuations will recognize that centro-projective area is not a valuation, but is in a sense the closest possible projective analogue of a valuation. In particular, if the body is origin-symmetric, then centro-projective area equals centro-affine surface area and therefore is a valuation on the space of origin-symmetric convex bodies.

We will prove the following centro-projective inequality.

Theorem 2. *Let K be a convex body containing the origin in its interior, then one has*

$$\mathcal{C}_o(K) \leq \mathcal{C}_o(B),$$

where $B \subset \mathbb{R}^n$ is the standard unit ball. Equality holds if and only if K is an ellipsoid that contains the origin in its interior, but is not necessarily centered at the origin.

Let us point out that $\mathcal{C}_o(B)$ does not depend on the particular choice of the origin inside the ellipsoid B. Indeed, if p, q are points inside B, then there is a projective map g such that $g(B) = B$ and $g(p) = q$, and therefore $\mathcal{C}_o(B - p) = \mathcal{C}_o(B - q)$. In other words, the projective group acts transitively on any fixed ellipsoid B.

1. Preliminaries

We recall here some basic definitions in convexity used in our paper. More details can be found, for example, in the books by Gruber [3], Schneider [12], or Thompson [13].

A subset $K \subset V$ is *convex* if the line segment joining any two points $x, y \in K$ also lies in K.

A non-empty compact convex set $K \subset V$ is uniquely determined by its *support function* denoted here by $h_K : V^* \to \mathbb{R}$ and defined by

$$h_K(\xi) = \max_{x \in K} \langle \xi, x \rangle.$$

Indeed, we have then

$$K = \{x \mid \langle \xi, x \rangle \leq h_K(\xi) \text{ for all } \xi \in V^*\}. \tag{2}$$

Note that h_K is a positively homogeneous function of degree 1.

If K contains the origin o in its interior, we define its *polar body* $K^* \subset V^*$ with respect to the origin by

$$K^* := \{\xi \mid \langle \xi, x \rangle \leq 1 \text{ for all } x \in K\}. \tag{3}$$

One can also show that $K^* = \{\xi \mid h_K(\xi) \leq 1\}$. We can also define for each point $x \neq o$ in K the positive number $\rho_K(x)$ such that $\rho_K(x)x \in \partial K$. The function ρ_K is called the *radial function* and satisfies

$$\rho_K(x) = \frac{1}{h_{K^*}(x)}. \tag{4}$$

The antipodal function a defined in the introduction is given by

$$a(p) = \rho_K(-p) \text{ for all } p \in \partial K. \tag{5}$$

For convex sets K and L in V, the *Minkowski sum* $K + L$ is the convex set defined by

$$K + L := \{x + y \mid x \in K \text{ and } y \in L\}, \tag{6}$$

and, for $\alpha \in \mathbb{R}$, one can define the convex set αK by

$$\alpha K := \{\alpha x \mid x \in K\}.$$

Recall that $h_{K+L} = h_K + h_L$ and $h_{\alpha K} = \alpha h_K$ if $\alpha \geq 0$.

From now on, we will fix a Euclidean scalar product on V. We denote by S^{n-1} the corresponding unit sphere in V and by dm the induced volume form on V and V^*. We also will use the following notation: For an integrable homogeneous function of degree $-n$, $f : V \setminus \{0\} \to \mathbb{R}$,

$$\oint f \, \mathrm{d}m := \int_{S^{n-1}} f(\theta) \, \mathrm{d}\theta, \tag{7}$$

where dθ is the standard spherical measure on S^{n-1}. The value of this integral depends only on the volume measure dm and is otherwise independent of the Euclidean scalar product chosen (see [14] for details).

2. Proof of Theorem 2

Let $K \in V$ be a bounded open convex body containing the origin and let $K^* \subset V^*$ be its polar with respect to the origin.

Notice that for any point p on the boundary ∂K at which there exists a unique outer unit normal $\nu_K(p) \in V^*$ to K, the following holds:

$$h_K(\nu_K(p)) = \langle \nu_K(p), p \rangle. \tag{8}$$

Recall that the *curvature function* of K, $f_K : V \backslash \{o\} \to \mathbb{R}^+$ is defined as follows: for each $\theta \in S^{n-1}$ where h_K is twice differentiable, the curvature function $f_K(\theta)$ is the sum of the determinants of the principal $(n-1)$-minors of the Hessian of h_K (viewed as a function on $V^* \backslash \{0\}$). It is then extended as a function homogeneous of degree $-n-1$. Recall that, for each $\theta \in S^{n-1}$ where the radial function ρ_K is twice differentiable and the Gauss curvature $\kappa(p)$ is positive, where $p = \rho_K(\theta)\theta \in \partial K$,

$$f_K(\nu_K(p)) = \frac{1}{\kappa_K(p)}.$$

The volume of K is given by

$$V(K) = \frac{1}{n} \oint \rho_K^n \, \mathrm{d}m = \frac{1}{n} \oint \frac{1}{h_{K^*}^n} \, \mathrm{d}m$$

and the *affine surface area* of K is defined as

$$S(K) = \oint f_K^{\frac{n}{n+1}} \, \mathrm{d}m.$$

See Schneider's book [12] for a more detailed discussion of affine surface area, which was defined by Blaschke for smooth convex bodies. The definition above, valid for all convex bodies, is due to Leichtweiss [6]. Lutwak [8] gave a different but equivalent definition. Also, see Santalo [11], Hug [4,5].

The following is straightforward if the boundary ∂K is C^2 and has strictly positive Gauss curvature. The general case is due to Hug [5].

Lemma 3. *For continuous function $\psi : \partial K \to \mathbb{R}$,*

$$\int_{\partial K} \psi(p) \left(\frac{\kappa_K(p)}{\langle \nu_K(p), p \rangle^{n-1}} \right)^{1/2} \mathrm{d}A(p) = \int_{S^{n-1}} \psi(\rho_K(\theta)\theta) \left(\frac{f_{K^*}(\theta)}{h_{K^*}^{n-1}(\theta)} \right)^{1/2} \mathrm{d}\theta \tag{9}$$

Proof. By Theorem 3.2 and Eq. (1) in [5], Hug established that

$$\int_{\partial K} \left(\frac{\kappa_K(p)}{\langle \nu_K(p), p \rangle^{n-1}} \right)^{1/2} \mathrm{d}A(p) = \int_{S^{n-1}} \left(\frac{f_{K^*}(\theta)}{h_{K^*}^{n-1}(\theta)} \right)^{1/2} \mathrm{d}\theta. \tag{10}$$

However, in the proof of Theorem 3.2, Hug in fact proves that the two measures are equal via the bilipschitz map $\theta \mapsto \rho_K(\theta)\theta$. □

Generalizations of Hug's result can also be found in Ludwig [7]. In particular, using Theorem 4 and 5 in [7] applied with $\phi(t) = t^{1/2}$ one gets Eq. (10).

The new formula for the centro-projective area of a convex body K is given by the following lemma.

Lemma 4. *The centro-projective area of K is equal to*

$$\mathcal{C}_o(K) = \oint \left(\frac{2}{h_{K^*} + h_{-K^*}} \right)^{\frac{n-1}{2}} f_{K^*}^{1/2} \, \mathrm{d}m. \tag{11}$$

Proof. If a is the antipodal function defined by (5), then by Eqs. (4) and (5), we have

$$a(p) = \frac{1}{h_{K^*}(-p)},$$

and therefore, for each $\theta \in S^{n-1}$,

$$a\big(\rho_K(\theta)\theta\big) = \frac{1}{h_{K^*}\big(-\rho_K(\theta)\theta\big)} = \frac{h_{K^*}(\theta)}{h_{K^*}(-\theta)}$$

Hence,

$$\frac{2a\big(p(\theta)\big)}{1+a\big(p(\theta)\big)} = 2\frac{h_{K^*}(\theta)}{h_{K^*}(-\theta)} \cdot \frac{1}{1+\dfrac{h_{K^*}(\theta)}{h_{K^*}(-\theta)}} = \frac{2h_{K^*}(\theta)}{h_{K^*}(-\theta)+h_{K^*}(\theta)}. \tag{12}$$

The lemma now follows from Lemma 3 by setting

$$\psi = \left(\frac{2a}{1+a}\right)^{\frac{n-1}{2}}. \quad \square$$

To prove the theorem, we first apply the Hölder inequality to $\mathcal{C}_o(K)$:

$$\begin{aligned}\mathcal{C}_o(K) &\le \left(\oint\left(\frac{2}{h_{K^*}(x)+h_{K^*}(-x)}\right)^n \mathrm{d}m(x)\right)^{\frac{n-1}{2n}} \cdot \left(\oint f_{K^*}^{\frac{n}{n+1}}\,\mathrm{d}m\right)^{\frac{n+1}{2n}} \\ &= n^{\frac{n-1}{2n}} V(\pi(K))^{\frac{n-1}{2n}} \times S(K^*)^{\frac{n+1}{2n}},\end{aligned} \tag{13}$$

where

$$\pi(K) = \left[\frac{1}{2}(K^* + (-K^*))\right]^*.$$

By the affine isoperimetric inequality (see, for example, [9] or [10]),

$$S(K^*)^{n+1} \le n^{(n+1)} V(K^*)^{n-1} V(B_n)^2, \tag{14}$$

where equality holds if and only if K is an ellipsoid centered at the origin. Applying this to inequality (13) gives

$$\mathcal{C}_o(K) \le n \cdot \Big(V\big(\pi(K)\big)\cdot V(K^*)\Big)^{\frac{n-1}{2n}} V(B_n)^{\frac{1}{n}} \tag{15}$$

Next, we use the Blaschke–Santaló inequality, which states that, for any convex body $C \subset V$ that is symmetric with respect to the origin,

$$V(C) \times V(C^*) \le V(B_n)^2. \tag{16}$$

Again, equality holds if and only if C is an ellipsoid.

Setting

$$C = \frac{1}{2}K^* + \frac{1}{2}(-K^*) \text{ and } C^* = \pi(K)$$

the Blaschke–Santaló inequality and (15) imply

$$\mathcal{C}_o(K) \le nV(B_n)\cdot\left(\frac{V(K^*)}{V\big(\frac{1}{2}K^* + \frac{1}{2}(-K^*)\big)}\right)^{\frac{n-1}{2n}} \tag{17}$$

The theorem now follows by

$$V\left(\frac{1}{2}K^* + \frac{1}{2}(-K^*)\right)^{1/n} \ge \frac{1}{2}V(K^*)^{1/n} + \frac{1}{2}V(-K^*)^{1/n} = V(K^*)^{1/n},$$

which follows from the Brunn–Minkowski inequality, and the identity $\mathcal{C}_o(B_n) = nV(B_n)$.

Let us stress out that the equality conditions of the Brunn–Minkowski inequality, the Blaschke–Santaló inequality, and the affine isoperimetric inequality imply that equality holds in Theorem 2 if and only if K is an ellipsoid that contains the origin in its interior, but is not necessarily centered at the origin.

3. Centro-projective invariance

We remark that the invariance of (11) under centro-projective transformations of K is easy to show. It suffices to show that it is invariant under linear transformations of K and translations of K^*. The invariance of (11) under linear transformations of K is established in [14]. The invariance of f_{K^*} and $h_{K^*} + h_{-K^*}$ under translations of K^* follows directly from their definitions.

4. Application in Hilbert geometries

A *Hilbert geometry* is a metric space structure defined as follows on a *proper* open convex domain of a finite-dimensional affine space. By *proper*, we mean that the domain does not contain any line. The distance between two points in the domain is defined using cross-ratios in the same way one constructs the projective model of the hyperbolic space on a Euclidean ball (see, for example, [2]). Such a metric is called a *Hilbert metric*. The Hausdorff measure associated with that metric is called a *Busemann measure*.

Given an open bounded convex domain $K \subset V$ and a point $p \in K$, let $V_{K,p}(r)$ denote the Busemann measure of the metric ball of radius r centered at p. This defines, for each *pointed* convex domain (K, p) of V, a function $V_{K,p} : \mathbb{R}^+ \to \mathbb{R}^+$. Since the Busemann measure is defined in terms of the Hilbert metric, which in turns is defined using the cross-ratios, the function $(K, p) \to V_{K,p}$ is a projective invariant of K.

One can therefore ask two questions:

- Is it true that for any pointed convex domain (K, p) and $r > 0$ one has

$$V_{K,p}(r) \leq V_{B_n,o}(r)\,? \tag{18}$$

- Is the map $(K, p) \to V_{K,p}$ injective? That is, if (K, p) and (K', p') are pointed convex sets such that $V_{K,p} = V_{K',p'}$, does there exist a projective transformation g such that

$$\big(g(K), g(p)\big) = (K', p')\,?$$

A partial answer can be given if we assume the domain K to have regularity $C^{1,1}$. Geometrically, this means that there exists a ball of some fixed radius that can roll inside K and touch every point on the boundary. It was proved in [2] that, for any convex domain K,

$$\lim_{r\to+\infty} \frac{V_{K,p}(r)}{V_{B_n,o}(r)} = \frac{\mathcal{C}_0(K-p)}{\mathcal{C}_0(B_n)}.$$

Theorem 2 shows that this limit is strictly smaller that 1, when K is not an ellipsoid. In particular, for any $p \in K$, there exists $r_{K,p} > 0$ such that, for all $r > r_{K,p}$, the inequality (18) holds and is strict.

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