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Topology

Symplectic and orthogonal K-groups of the integers ☆

K-groupes symplectiques et orthogonaux de l'anneau des entiers

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A R T I C L E I N F O

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A B S T R A C T

We explicitly compute the homotopy groups of the topological spaces $B\,\mathrm{Sp}(\mathbb{Z})^+$, $BO_{\infty,\infty}(\mathbb{Z})^+$, and $BO_\infty(\mathbb{Z})^+$.

R É S U M É

Nous calculons explicitement les groupes d'homotopie des espaces topologiques $B\,\mathrm{Sp}(\mathbb{Z})^+$, $BO_{\infty,\infty}(\mathbb{Z})^+$ et $BO_\infty(\mathbb{Z})^+$.

Version française abrégée

Soient $\mathrm{Sp}(\mathbb{Z})$, $O_{\infty,\infty}(\mathbb{Z})$ et $O_\infty(\mathbb{Z})$ le groupe symplectique infini, le $\langle 1,-1\rangle$-groupe orthogonal infini et le groupe orthogonal hyperbolique sur l'anneau des entiers $\mathbb{Z}$. Ils sont obtenus comme réunion des sous-groupes $\mathrm{Sp}_{2n}(\mathbb{Z})$, $O_{n,n}(\mathbb{Z})$ et $O_{2n}(\mathbb{Z})$ de $GL_{2n}(\mathbb{Z})$ laissant invariantes les formes bilinéaires des matrices de Gram :

$$\begin{pmatrix} \begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} & & \\ & \ddots & \\ & & \begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} \end{pmatrix}, \quad \begin{pmatrix} \begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix} & & \\ & \ddots & \\ & & \begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} & & \\ & \ddots & \\ & & \begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} \end{pmatrix}.$$

Les groupes $\mathrm{Sp}(\mathbb{Z})$, $O_{\infty,\infty}(\mathbb{Z})$ et $O_\infty(\mathbb{Z})$ ont des sous-groupes de commutateurs parfaits. Rappelons que, pour un tel groupe G, la construction plus de Quillen BG^+ appliquée à l'espace classifiant BG de G est munie d'une application continue $BG \to BG^+$ qui induit un isomorphisme sur les groupes d'homologie intégrale et vaut $G \to G/[G,G]$ sur π_1.

Le but de cet article est de calculer explicitement les groupes d'homotopie des espaces topologiques $B\,\mathrm{Sp}(\mathbb{Z})^+$, $BO_{\infty,\infty}(\mathbb{Z})^+$ et $BO_\infty(\mathbb{Z})^+$. Ces espaces sont des espaces de lacets infinis, puisqu'ils sont les composants connexes des

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espaces de la K-théorie $K\operatorname{Sp}(\mathbb{Z})$, $GW(\mathbb{Z})$ et $KQ(\mathbb{Z})$ des formes non dégénérées symplectiques, bilinéaires symétriques et quadratiques sur $\mathbb{Z}$. On sait que les groupes d'homotopie de ces espaces sont des groupes abéliens de génération finie.

Pour un groupe abélien A, on note A_{odd} le sous-groupe des éléments d'ordre impair fini.

Théorème 0.1. *Les groupes d'homotopie des espaces $B\operatorname{Sp}(\mathbb{Z})^+$ et $BO_{\infty,\infty}(\mathbb{Z})^+$ pour $n \geq 1$ sont donnés dans le tableau du Théorème 1.1.*

Théorème 0.2. *L'application qui envoie une forme quadratique sur sa forme bilinéaire symétrique associée induit un morphisme d'espaces de K-théorie $KQ(\mathbb{Z}) \to GW(\mathbb{Z})$ qui est un isomorphisme*

$$\pi_n BO_\infty(\mathbb{Z})^+ \xrightarrow{\cong} \pi_n BO_{\infty,\infty}(\mathbb{Z})^+ \quad \text{en degré } n \geq 2$$

et le monomorphisme $(\mathbb{Z}/2)^2 \subset (\mathbb{Z}/2)^3$ en degré $n = 1$.

Remarque 0.1. Notons par B_k le k-ième nombre de Bernoulli [12, Exemple 24] et par d_n le dénominateur de $\frac{1}{n+1}B_{(n+1)/4}$ pour $n = 3 \mod 4$. Selon [12, Introduction, Lemme 27], on a $K_n(\mathbb{Z}) = \mathbb{Z}/2d_n$ pour $n = 3 \mod 8$ et $K_n(\mathbb{Z}) = \mathbb{Z}/d_n$ pour $n = 7 \mod 8$. En outre, pour $k > 0$, les groupes $K_{4k}(\mathbb{Z})$ sont finis d'ordre impair et conjecturés zéro [12, Introduction]. Par exemple, $K_4(\mathbb{Z}) = 0$ [5]. Donc, on a pour $n \geq 1$ le tableau de groupes d'homotopie présenté dans la Remarque 1.3.

1. Statement of results

Let $\operatorname{Sp}(\mathbb{Z})$, $O_{\infty,\infty}(\mathbb{Z})$ and $O_\infty(\mathbb{Z})$ be the infinite symplectic, infinite $\langle 1, -1\rangle$-orthogonal and infinite hyperbolic orthogonal groups over the integers. They are obtained as the union of subgroups $\operatorname{Sp}_{2n}(\mathbb{Z})$, $O_{n,n}(\mathbb{Z})$ and $O_{2n}(\mathbb{Z})$ of $GL_{2n}(\mathbb{Z})$ fixing the bilinear forms with Gram matrix

$$\begin{pmatrix} \begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} & & \\ & \ddots & \\ & & \begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} \end{pmatrix}, \quad \begin{pmatrix} \begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix} & & \\ & \ddots & \\ & & \begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} & & \\ & \ddots & \\ & & \begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} \end{pmatrix}.$$

The groups $\operatorname{Sp}(\mathbb{Z})$, $O_{\infty,\infty}(\mathbb{Z})$ and $O_\infty(\mathbb{Z})$ have perfect commutator subgroups. Recall that, for such groups G, Quillen's plus construction BG^+ applied to the classifying space BG of G comes with a continuous map $BG \to BG^+$, which induces an isomorphism on integral homology groups and is $G \to G/[G,G]$ on π_1.

The purpose of this article is to compute explicitly the homotopy groups of the topological spaces $B\operatorname{Sp}(\mathbb{Z})^+$, $BO_{\infty,\infty}(\mathbb{Z})^+$ and $BO_\infty(\mathbb{Z})^+$. These spaces are infinite loop spaces since they are the connected components of the spaces $K\operatorname{Sp}(\mathbb{Z})$, $GW(\mathbb{Z})$, and $KQ(\mathbb{Z})$, which are the K-theory spaces of non-degenerate symplectic, symmetric bilinear and quadratic forms over $\mathbb{Z}$. It is known that the homotopy groups of these spaces are finitely generated abelian groups.

For an abelian group A, denote by A_{odd} the subgroup of elements of finite odd order.

Theorem 1.1. *The homotopy groups of the spaces $B\operatorname{Sp}(\mathbb{Z})^+$ and $BO_{\infty,\infty}(\mathbb{Z})^+$ for $n \geq 1$ are given in the following table:*

$n \mod 8$	0	1	2	3	4	5	6	7
$\pi_n B\operatorname{Sp}(\mathbb{Z})^+$	$K_n(\mathbb{Z})$	0	$\mathbb{Z}$	$K_n(\mathbb{Z})$	$\mathbb{Z}/2 \oplus K_n(\mathbb{Z})$	$\mathbb{Z}/2$	$\mathbb{Z}$	$K_n(\mathbb{Z})$
$\pi_n BO_{\infty,\infty}(\mathbb{Z})^+$	$\mathbb{Z} \oplus \mathbb{Z}/2 \oplus K_n(\mathbb{Z})$	$(\mathbb{Z}/2)^3$	$(\mathbb{Z}/2)^2$	$\mathbb{Z}/8 \oplus K_n(\mathbb{Z})_{\text{odd}}$	$\mathbb{Z} \oplus K_n(\mathbb{Z})$	0	0	$K_n(\mathbb{Z})$

Theorem 1.2. *The map that sends a quadratic form to its associated symmetric bilinear form induces a map of K-theory spaces $KQ(\mathbb{Z}) \to GW(\mathbb{Z})$ which is an isomorphism*

$$\pi_n BO_\infty(\mathbb{Z})^+ \xrightarrow{\cong} \pi_n BO_{\infty,\infty}(\mathbb{Z})^+ \quad \text{in degree } n \geq 2$$

and the monomorphism $(\mathbb{Z}/2)^2 \subset (\mathbb{Z}/2)^3$ in degree $n = 1$.

Remark 1.3. Denote by B_k the k-th Bernoulli number [12, Example 24] and let d_n denote the denominator of $\frac{1}{n+1}B_{(n+1)/4}$ for $n = 3 \mod 4$. By [12, Introduction, Lemma 27], we have $K_n(\mathbb{Z}) = \mathbb{Z}/2d_n$ for $n = 3 \mod 8$ and $K_n(\mathbb{Z}) = \mathbb{Z}/d_n$ for $n = 7 \mod 8$. Moreover, for $k > 0$, the groups $K_{4k}(\mathbb{Z})$ are finite of odd order and are conjectured to be zero [12, Introduction]. For example, $K_4(\mathbb{Z}) = 0$ [5]. In particular, for $n \geq 1$, we have the following table of homotopy groups

$n \mod 8$	0	1	2	3	4	5	6	7
$\pi_n B\,\mathrm{Sp}(\mathbb{Z})^+$	(0?)	0	$\mathbb{Z}$	$\mathbb{Z}/2d_n$	$\mathbb{Z}/2 \oplus (0?)$	$\mathbb{Z}/2$	$\mathbb{Z}$	$\mathbb{Z}/d_n$
$\pi_n B O_{\infty,\infty}(\mathbb{Z})^+$	$\mathbb{Z} \oplus \mathbb{Z}/2 \oplus (0?)$	$(\mathbb{Z}/2)^3$	$(\mathbb{Z}/2)^2$	$\mathbb{Z}/d_n$	$\mathbb{Z} \oplus (0?)$	0	0	$\mathbb{Z}/d_n$

where (0?) denotes a finite group of odd order conjectured to be zero.

2. Proof part 1: odd torsion

Lemma 2.1. *Let R be the ring of integers in a number field F. Then, for all $n \geq 0$, there are isomorphisms*

$$KQ_n(R)_{\text{odd}} \cong GW_n(R)_{\text{odd}} \cong K\,\mathrm{Sp}_n(R)_{\text{odd}} \cong (K_n(R)_{\text{odd}})^{C_2}$$

where the action of C_2 on K-theory is induced by $GL(R) \to GL(R) : M \mapsto {}^tM^{-1}$.

Proof. The natural map $KQ_n(R)_{\text{odd}} \to GW_n(R)_{\text{odd}}$ is an isomorphism with inverse the cup product with the quadratic space associated with the Leech lattice Γ_8 [4, Ch. 2, §6]. Write $GW^{[0]}(R)$ and $GW^{[2]}(R)$ for $GW(R)$ and $K\,\mathrm{Sp}(R)$; see Section 3 below for general $GW^{[n]}$. The hyperbolic and forgetful maps factor as $K^{[r]}(R)_{hC_2} \to GW^{[r]}(R) \to K^{[r]}(R)^{hC_2}$; see [7, (7.3) and Lemma 7.4], which does not use $1/2 \in R$. Here $K^{[n]}$ denotes the K-theory spectrum K with C_2-action induced by the n-th shifted duality $\mathrm{Hom}(\ \ , R[n])$. On the spectrum level, this action depends on $n = 0, 2$. However, on homotopy groups, the actions agree for $n = 0, 2$. Denote by $L^{[r]}$ the homotopy cofibre of the map of spectra[1] $K^{[r]}(R)_{hC_2} \to GW^{[r]}(R)$, then $L_i^{[r]} = L_{i-1}^{[r-1]}$ only depends on the difference $n - i$, $i \geq 1$ [9] and

$$GW_n^{[r]}(R)[1/2] \cong K_n^{[r]}(R)[1/2]^{C_2} \oplus L_n^{[r]}(R)[1/2]$$

since the composition $K^{[r]}(R)[1/2]_{hC_2} \to GW^{[r]}(R)[1/2] \to K^{[r]}(R)[1/2]^{hC_2}$ is an equivalence [7, Lemma B.14]. Strictly speaking, we define a non-connective version of $L^{[r]}$ as the homotopy colimit of the sequence

$$GW^{[r]} \to S^1 \wedge GW^{[r-1]} \to S^2 \wedge GW^{[r-2]} \to \cdots \tag{2.1}$$

with appropriate delooping of $GW^{[n]}$ as in [7] using the definition of $\mathcal{E}^{[n]}$ as below. The maps in (2.1) are the connecting maps of the homotopy fibration (3.1). Then we have *by definition* $L_i^{[n]} = L_0^{[n-i]}$ and, as in [7], we formally obtain the homotopy fibration whose connected cover we used above:

$$(K^{[n]})_{hC_2} \to GW^{[n]} \to L^{[n]}.$$

By Lemma 3.4 below, the canonical map $L_i^{[r]}(R)[1/2] \to L_i^{[r]}(F)[1/2]$ is an isomorphism for $i \geq r$. By [7, Proposition 7.2] and [1, Theorem 5.6], we have

$$L_i^{[r]}(F)[1/2] = \begin{cases} W(F)[1/2] & r \equiv i \mod 4 \\ 0 & \text{else} \end{cases}$$

where $W(F)$ is the usual Witt group of F. But it is well known that $W(F)[1/2]$ is a free $\mathbb{Z}[1/2]$-module of rank the number of orderings of F. This proves the lemma for K_nQ, GW_n for $n \geq 0$ and $K_n\mathrm{Sp}$ for $n \geq 2$. From the Zariski local-to-global spectral sequence, we see $L_1^{[2]}(R)[1/2] = L_0^{[1]}(R)[1/2] = H^1(R, L_0^0[1/2]) = 0$ since $L_0^{[0]}[1/2]$ is constant (flasque) on a ring of integers R and $L_0^{[1]}$ is Zariski-locally trivial. So, $K_1\,\mathrm{Sp}(R)_{\text{odd}} = (K_1(R)_{\text{odd}})^{C_2}$. Finally, $L_0^{[2]}(R) = 0$ for a ring of integers since $K_0\,\mathrm{Sp}(R) = H^0(R, \mathbb{Z})$, by the Zariski spectral sequence, hence $H : K_0(R) \to K_0\,\mathrm{Sp}(R)$ is surjective and $L_0^{[2]} = 0$. □

Continue to assume that R is a ring of integers in a number field. Let $\ell \in \mathbb{Z}$ be an odd prime and set $R' = R[1/\ell]$. Then the inclusion $R \subset R'$ induces an isomorphism: $K_n(R)\{\ell\} \cong K_n(R')\{\ell\}$ on ℓ-primary torsion subgroups for $n \geq 1$. For $i \geq 1$, the abelian group $K_{2i}(R')$ is finite and the group $K_{2i-1}(R')$ is finitely generated. For all $i \geq 1$ and large ν, we therefore have an exact sequence

$$0 \to K_{2i}(R')\{\ell\} \to K_{2i}(R', \mathbb{Z}/\ell^\nu) \to K_{2i-1}(R')\{\ell\} \to 0 \tag{2.2}$$

[12, Lemma 68]. Since ℓ is invertible in R', which has $\mathrm{cd}_\ell(R') \leq 2$, the proved Quillen–Lichtenbaum conjecture says that the following change of topology map is an isomorphism $K_{2i}(R', \mathbb{Z}/\ell^\nu) \cong K_{2i}^{\text{ét}}(R', \mathbb{Z}/\ell^\nu)$ for $i \geq 1$. The change of topology map is C_2-equivariant. From the etale local-to-global spectral sequence for $K^{\text{ét}}$, we obtain the C_2-equivariant isomorphism

[1] All spectra in this paper are -1-connected, and all homotopy fibrations are in the category of -1-connected spectra unless otherwise stated. In particular, the second map of a homotopy fibration need not be surjective on π_0.

$$K_{2i}(R', \mathbb{Z}/\ell^{\nu}) \cong K_{2i}^{\text{ét}}(R', \mathbb{Z}/\ell^{\nu}) \cong H^0_{\text{ét}}(R', K_{2i}/\ell^{\nu}) \tag{2.3}$$

[12, Proof of Theorem 70], on which the action on the left is $GL(R) \to GL(R) : M \mapsto {}^tM^{-1}$ and on the right-hand side it is multiplication with $(-1)^i$. Combining (2.2) and (2.3), Lemma 2.1 yields the following.

Theorem 2.2. *Let R be a ring of integers in a number field, and $\ell \in \mathbb{Z}$ an odd prime. Then for all $n \geq 1$ we have isomorphisms*

$$GW_n(R)\{\ell\} \cong K\,\mathrm{Sp}_n(R)\{\ell\} \cong KQ_n(R)\{\ell\} \cong \begin{cases} K_n(R)\{\ell\} & n \equiv 0, 3 \mod 4 \\ 0 & n \equiv 1, 2 \mod 4. \end{cases}$$

3. Proof part 2: 2-adic computations

For an exact category with weak equivalences and duality $(\mathscr{E}, w, \sharp, \mathrm{can})$, denote by $GW(\mathscr{E}, w, \sharp, \mathrm{can})$ the associated Grothendieck–Witt space of symmetric bilinear forms [6, Definition 3]. If $\mathscr{E}$ has a strong symmetric cone [6, Definition 4], [9] I denote by $\mathscr{E}^{[1]} = (\mathrm{Mor}\,\mathscr{E}, w_{\mathrm{cone}}, \sharp, \mathrm{can})$ the exact category with weak equivalences and duality of morphisms in $\mathscr{E}$ with duality and double dual identification induced by functoriality of $\sharp$ and can and weak equivalences those maps $f \to g$ of arrows in $\mathscr{E}$ such that $\mathrm{cone}(f) \to \mathrm{cone}(g)$ is a weak equivalence in $\mathscr{E}$. By functoriality, $\mathscr{E}^{[1]}$ also has a strong symmetric cone. Set $GW^{[0]}(\mathscr{E}) = GW(\mathscr{E})$ and define inductively for $r \geq 1$

$$GW^{[r+1]}(\mathscr{E}) = GW^{[r]}(\mathscr{E}^{[1]}).$$

By [6, Theorem 6], the sequence

$$\mathscr{E} \xrightarrow{E \mapsto 1_E} \mathrm{Mor}\,\mathscr{E} \xrightarrow{1} \mathscr{E}^{[1]}$$

induces a homotopy fibration $GW(\mathscr{E}) \to K(\mathscr{E}) \to GW^{[1]}(\mathscr{E})$ of -1-connected spectra and by iteration the homotopy fibration

$$GW^{[r]}(\mathscr{E}) \to K(\mathscr{E}) \to GW^{[r+1]}(\mathscr{E}); \tag{3.1}$$

compare [7, Proof of Proposition 4.9]. For details and a generalisation, see [9]. For $r < 0$, we define $GW^{[r]}(\mathscr{E})$ such that (3.1) holds for all $r \in \mathbb{Z}$. For a commutative ring R, we denote by $GW^{[r]}(R)$ the space $GW^{[r]}(\mathrm{Ch}^b\,\mathcal{P}(R), \mathrm{quis}, \mathrm{Hom}(\ \ , R), \mathrm{can})$ where $\mathcal{P}(R)$ is the category of finitely generated projective R-modules and quis is the set of quasi-isomorphisms.

Theorem 3.1 *([10]). Let R be a commutative ring, then*

(1) *$GW^{[0]}(R)$ is the K-theory space $GW(R)$ of the category of non-degenerate symmetric bilinear forms over R,*
(2) *$GW^{[2]}(R)$ is the K-theory space $K\,\mathrm{Sp}(R)$ of the category of non-degenerate symplectic forms over R, and*
(3) *$GW^{[4]}(R)$ is the K-theory space $KQ(R)$ of the category of non-degenerate quadratic forms over R.*

In particular, by [8, Theorem 6.6, Example 3.11 and Remark 2.19], we have

$$\begin{aligned} GW^{[0]}(\mathbb{Z}) &= GW(\mathbb{Z}) \simeq \mathbb{Z} \times \mathbb{Z} \times BO_{\infty,\infty}(\mathbb{Z})^+, \\ GW^{[2]}(\mathbb{Z}) &= K\,\mathrm{Sp}(\mathbb{Z}) \simeq \mathbb{Z} \times B\,\mathrm{Sp}(\mathbb{Z})^+, \\ GW^{[4]}(\mathbb{Z}) &= KQ(\mathbb{Z}) \simeq \mathbb{Z} \times \mathbb{Z} \times BO_{\infty}(\mathbb{Z})^+. \end{aligned}$$

Theorem 3.2 *([11]). Let R be a Dedekind domain and $S \subset R$ a multiplicative set of non-zero divisors. Then there is a natural homotopy fibration*

$$\bigoplus_{\wp \cap S \neq \emptyset} GW^{[-1]}(R/\wp) \to GW^{[0]}(R) \to GW^{[0]}(S^{-1}R).$$

Recall that Friedlander [2] shows that $K_n\,\mathrm{Sp}(\mathbb{F}_2)$ is a finite group of odd order for $n \geq 1$. In particular, its 2-adic completion $K_n\,\mathrm{Sp}(\mathbb{F}_2)^\wedge_2 = 0$ for $n \geq 1$. Since the same is true for $K(\mathbb{F}_2)$, we obtain $GW_n(\mathbb{F}_2)^\wedge_2 = 0$ for $n \geq 1$, $GW_n^{[\pm 1]}(\mathbb{F}_2)^\wedge_2 = 0$ for $n \geq 0$ and the following from Theorems 3.1, 3.2 and the homotopy fibration (3.1).

Theorem 3.3. *Let $\mathbb{Z}' = \mathbb{Z}[1/2]$ then the ring homomorphism $\mathbb{Z} \to \mathbb{Z}'$ induces isomorphisms after 2-adic completion*

$$\begin{aligned} K_n\,\mathrm{Sp}(\mathbb{Z})^\wedge_2 &\cong K_n\,\mathrm{Sp}(\mathbb{Z}')^\wedge_2, & n \geq 0, \\ GW_n(\mathbb{Z})^\wedge_2 &\cong GW_n(\mathbb{Z}')^\wedge_2, & n \geq 1, \\ KQ_n(\mathbb{Z})^\wedge_2 &\cong KQ_n(\mathbb{Z}')^\wedge_2, & n \geq 2. \end{aligned}$$

Finally, the 2-adic homotopy groups of $K\mathrm{Sp}(\mathbb{Z}')$ and $GW(\mathbb{Z}') = KQ(\mathbb{Z}')$ can be found in [3, 4.7.2]. This proves the theorems in Section 1 apart from the following, which was needed in the proof of Lemma 2.1.

Lemma 3.4. *Let R be the ring of integers in a number field F. Then the inclusion $R \subset F$ induces an isomorphism*

$$L_i^{[r]}(R)[1/2] \simeq L_i^{[r]}(F)[1/2], \qquad i \geq r.$$

Proof. It suffices to prove the case $r = 0$ since $L_i^{[r]} = L_{i-r}^{[0]}$. From Theorem 3.2, we deduce the homotopy fibration of -1-connected spectra

$$\bigoplus_{\wp \neq (0)} GW^{[-1]}(R/\wp)[1/2] \to GW^{[0]}(R)[1/2] \to GW^{[0]}(F)[1/2]$$

in which the right horizontal map is also surjective on π_0, by the computations in [4]. Using the analogous statement for K-theory, we obtain the homotopy fibration of spectra

$$\bigoplus_{\wp \neq (0)} L^{[-1]}(R/\wp)[1/2] \to L^{[0]}(R)[1/2] \to L^{[0]}(F)[1/2].$$

The left term in that fibration is trivial since for a finite field $\mathbb{F}_q$, we have

$$L^{[-1]}(\mathbb{F}_q)[1/2] \simeq 0.$$

This is well known for q odd, and for q even, $L^{[-1]}(\mathbb{F}_q)$ is a module spectrum over $L^{[0]}(\mathbb{F}_2)$ whose homotopy groups are 2-primary torsion, since on π_0 it is

$$L_0^{[0]}(\mathbb{F}_2) = W(\mathbb{F}_2) = \mathbb{Z}/2. \quad \square$$

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