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Symbolic summation methods and congruences involving harmonic numbers

Méthodes de sommation symbolique et congruences impliquant les nombres harmoniques

Guo-Shuai Mao [a], Chen Wang [b], Jie Wang [b]

[a] *Department of Mathematics, Nanjing University of Information Science and Technology, Nanjing 210044, People's Republic of China*
[b] *Department of Mathematics, Nanjing University, Nanjing 210093, People's Republic of China*

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ABSTRACT

In this paper, we establish some combinatorial identities involving harmonic numbers via the package `Sigma`, by which we confirm some conjectural congruences of Z.-W. Sun. For example, for any prime $p>3$, we have

$$\sum_{k=0}^{(p-3)/2}\frac{\binom{2k}{k}^2}{(2k+1)16^k}H_k^{(2)}\equiv -7B_{p-3}\ (\mathrm{mod}\ p),$$
$$\sum_{k=1}^{p-1}\frac{\binom{2k}{k}^2}{k16^k}H_{2k}^{(2)}\equiv B_{p-3}\ (\mathrm{mod}\ p),$$
$$\sum_{k=1}^{(p-1)/2}\frac{\binom{2k}{k}^2}{k16^k}(H_{2k}-H_k)\equiv -\frac{7}{3}pB_{p-3}\ (\mathrm{mod}\ p^2),$$

where $H_n^{(m)}=\sum_{k=1}^{n}1/k^m$ $(m\in\mathbb{Z}^+=\{1,2,\ldots\})$ is the n-th harmonic numbers of order m and B_n is the n-th Bernoulli number.

RÉSUMÉ

Nous montrons ici, à l'aide du progiciel Sigma, quelques identités combinatoires faisant intervenir les nombres harmoniques. Nous établissons ainsi des congruences conjecturées par Z.-W. Sun. Par exemple, pour $p>3$ premier, on a

$$\sum_{k=0}^{(p-3)/2}\frac{\binom{2k}{k}^2}{(2k+1)16^k}H_k^{(2)}\equiv -7B_{p-3}\ (\mathrm{mod}\ p),$$
$$\sum_{k=1}^{p-1}\frac{\binom{2k}{k}^2}{k16^k}H_{2k}^{(2)}\equiv B_{p-3}\ (\mathrm{mod}\ p),$$

E-mail addresses: maogsmath@163.com (G.-S. Mao), cwang@smail.nju.edu.cn (C. Wang), 576297794@qq.com (J. Wang).

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$$\sum_{k=1}^{(p-1)/2}\frac{\binom{2k}{k}^2}{k16^k}(H_{2k}-H_k)\equiv-\frac{7}{3}pB_{p-3}\pmod{p^2},$$

où $H_n^{(m)}=\sum_{k=1}^n 1/k^m$ ($m\in\{1,2,\ldots\}$) désigne le n-ième nombre harmonique d'ordre m et B_n est le n-ième nombre de Bernoulli.

1. Introduction

For $m\in\mathbb{Z}^+$, define the n-th harmonic numbers of order m as follows,

$$H_n^{(m)}:=\sum_{k=1}^n\frac{1}{k^m}\quad(n\in\mathbb{N}=\{0,1,\ldots\}),$$

where $H_0^{(m)}:=0$. For the sake of convenience, we often use H_n instead of $H_n^{(1)}$. So far, lots of number-theoretic properties of harmonic numbers have been discovered by mathematicians. For instance, the following celebrated congruences hold for any odd prime p,

$$H_{p-1}\equiv 0\pmod{p^2}\text{ and }H_{p-1}^{(2)}\equiv 0\pmod{p},$$

which were showed by J. Wolstenholme [15] in 1862.

The Bernoulli numbers $\{B_n\}_{n\ge0}$ and Bernoulli polynomials $\{B_n(x)\}_{n\ge0}$ are defined as follows:

$$B_0=1,\ \sum_{k=0}^{n-1}\binom{n}{k}B_k=0\quad(n=2,3,\ldots),$$

$$B_n(x)=\sum_{k=0}^{n}\binom{n}{k}B_kx^{n-k}\quad(n=0,1,\ldots).$$

The Bernoulli polynomials play an important role in the study of congruences involving harmonic numbers. In the past two decades, Z.-H. Sun [8,9] investigated the congruence relations between harmonic numbers and Bernoulli polynomials systematically. For example, he showed that, for any prime $p>5$,

$$\frac{B_{2p-4}}{2p-4}\equiv\frac{B_{p-3}}{p-3}\pmod{p},\tag{1.1}$$

$$H_{\lfloor p/4\rfloor}^{(2)}\equiv 4(-1)^{(p-1)/2}(2E_{p-3}-E_{2p-4})+\frac{14}{3}pB_{p-3}\pmod{p^2},\tag{1.2}$$

$$H_{\lfloor p/4\rfloor}^{(3)}\equiv-9B_{p-3}\pmod{p},\tag{1.3}$$

$$H_{(p-1)/2}^{(2)}\equiv 7p\left(\frac{B_{2p-4}}{2p-4}-2\frac{B_{p-3}}{p-3}\right)\pmod{p^3},\tag{1.4}$$

$$H_{(p-1)/2}^{(3)}\equiv 6\left(2\frac{B_{p-3}}{p-3}-\frac{B_{2p-4}}{2p-4}\right)\pmod{p^2},\tag{1.5}$$

where $\{E_n\}_{n\ge0}$ are the Euler numbers defined by

$$\frac{2}{e^x+e^{-x}}=\sum_{n=0}^{\infty}E_n\frac{x^n}{n!}\ \left(|x|<\frac{\pi}{2}\right).$$

We can check directly that the above congruences also hold for $p=5$.

For more interesting properties of harmonic numbers, one may consult [1–5,8–14,16].

In 2015, Z.-W. Sun [12] showed that, if $p>3$ is a prime,

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}}{4^k}H_k\equiv 2-2p+4p^2q_p(2)\pmod{p^3},$$

$$\sum_{k=0}^{(p-1)/2}\binom{2k}{k}^2\frac{H_k^{(2)}}{16^k}\equiv -4E_{p-3} \pmod p$$

and

$$\sum_{k=0}^{(p-1)/2}\binom{2k}{k}^2\frac{H_k}{k16^k}\equiv 4\left(\frac{-1}{p}\right)E_{p-3} \pmod p,$$

where $q_p(2)=(2^{p-1}-1)/p$ is the Fermat quotient and $(\frac{\cdot}{\cdot})$ is the Legendre symbol. In the same paper, Z.-W. Sun posed some conjectures involving harmonic numbers. The first author [5] confirmed two of the conjectures.

With the above backgrounds, we first establish the following results, which confirm some conjectures of Z.-W. Sun [12] by using the package `Sigma`. (The readers may consult [6,7] to see how to use such package to find and prove identities.)

Theorem 1.1. *For each prime $p>3$, we have*

$$\sum_{k=0}^{(p-3)/2}\frac{\binom{2k}{k}^2}{(2k+1)16^k}H_k^{(2)}\equiv -7B_{p-3} \pmod p. \tag{1.6}$$

Theorem 1.2. *Let $p>3$ be a prime. Then*

$$\sum_{k=1}^{(p-1)/2}\frac{\binom{2k}{k}^2}{k16^k}H_k^{(2)}\equiv -12\frac{H_{p-1}}{p^2} \pmod {p^2}, \tag{1.7}$$

$$\sum_{k=1}^{(p-1)/2}\frac{\binom{2k}{k}^2}{k16^k}H_{2k}^{(2)}\equiv -\frac{5}{2}B_{p-3} \pmod p, \tag{1.8}$$

$$\sum_{k=1}^{p-1}\frac{\binom{2k}{k}^2}{k16^k}H_{2k}^{(2)}\equiv B_{p-3} \pmod p. \tag{1.9}$$

Theorem 1.3. *For each prime $p>3$, we have:*

$$\sum_{k=1}^{p-1}\frac{\binom{2k}{k}H_{2k}}{k^24^k}\equiv \frac{5}{2}B_{p-3} \pmod p, \tag{1.10}$$

$$\sum_{k=1}^{p-1}\frac{\binom{2k}{k}}{k^24^k}\equiv -\frac{H_{(p-1)/2}^2}{2}-\frac{7}{4}\frac{H_{p-1}}{p} \pmod {p^2}. \tag{1.11}$$

Remark 1.1. Actually, there are some more conjectures in [12] that might be proved. For example, for any prime $p>3$,

$$\sum_{k=1}^{p-1}\frac{\binom{2k}{k}H_{2k}}{k4^k}\equiv \frac{7}{3}pB_{p-3} \pmod {p^2}$$

and for each prime $p>5$,

$$\sum_{k=1}^{p-1}\frac{\binom{2k}{k}}{k^24^k}\equiv -\frac{H_{(p-1)/2}^2}{2}-\frac{7}{4}\frac{H_{p-1}}{p} \pmod {p^3}.$$

We can prove the first one modulo p in the same way as the proof of (1.10).

In [11], Z.-W. Sun also raised some conjectures involving harmonic numbers, for example, if $p>3$ is a prime, then

$$\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^2}{k16^k}H_{2k}\equiv 4\left(\frac{-1}{p}\right)E_{p-3} \pmod p, \tag{1.12}$$

$$\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^2}{k16^k}(H_{2k}-H_k)\equiv -\frac{7}{3}pB_{p-3} \pmod {p^2}. \tag{1.13}$$

The first one was confirmed by the first author [4]. We will prove (1.13) by establishing the following results.

Theorem 1.4. *Let $p > 3$ be a prime. Then we have*

$$\sum_{k=1}^{(p-1)/2} \frac{\binom{2k}{k}^2}{k16^k} H_k \equiv 4\left(\frac{-1}{p}\right)(2E_{p-3} - E_{2p-4}) \pmod{p^2}, \tag{1.14}$$

$$\sum_{k=1}^{(p-1)/2} \frac{\binom{2k}{k}^2}{k16^k} H_{2k} \equiv 4\left(\frac{-1}{p}\right)(2E_{p-3} - E_{2p-4}) - \frac{7}{3}pB_{p-3} \pmod{p^2}. \tag{1.15}$$

Remark 1.2. In light of [9, (3.1)], (1.15) is an extension of (1.12).

We now give the outline of this paper. We will prove Theorem 1.1 in Section 2. Theorems 1.2–1.4 will be confirmed in Sections 3–5, respectively. The key idea in this paper is that some identities can be found via `Sigma` when you divide the sum into two cases with n odd or even, while you cannot find the identities if you just deal with the sum for n. Another idea is that we use $\sum_{j=1}^{n} 1/(2j-1)$ and H_n to replace H_{2n} because we find that the identities of the sum with H_{2n} is difficult to deal with, while the identities of the sum involving $\sum_{j=1}^{n} 1/(2j-1)$ and H_n are easier.

2. Proof of Theorem 1.1

Recall that for any $k \in \mathbb{N}$ the Pochhammer symbol $(x)_k$ is defined by

$$(x)_k := \begin{cases} x(x+1)\cdots(x+k-1) & \text{if } k \geq 1, \\ 1 & \text{if } k = 0. \end{cases}$$

In order to prove Theorem 1.1, we first show the following two lemmas.

Lemma 2.1. *Let n be a nonnegative integer. Then*

$$\sum_{k=0}^{n} \frac{(-n)_k(n+1)_k}{(2k+1)(1)_k^2} H_k^{(2)} = \begin{cases} -\dfrac{2}{2n+1} + \dfrac{1}{2(2n+1)} \displaystyle\sum_{k=1}^{(n-1)/2} \frac{-8k^2-4k-1}{k^2(2k+1)^2} & n \equiv 1 \pmod 2, \\ \dfrac{1}{2(2n+1)} \displaystyle\sum_{k=1}^{n/2} \frac{-8k^2+4k-1}{k^2(2k-1)^2} & n \equiv 0 \pmod 2. \end{cases} \tag{2.1}$$

Proof. The proofs of these two cases are similar, so we only show the case when n is odd. Set

$$S_n := \sum_{k=1}^{2n+1} \frac{(-1-2n)_k(2+2n)_k}{(1+2k)(1)_k^2} H_k^{(2)}.$$

We shall complete the proof by making use of the Mathematica package `Sigma` [7]. Here we list the procedure of computation in Mathematica.

Step 1: load `Sigma` and input S_n;

Step 2: use the command `GenerateRecurrence` to find that S_n satisfies

$$(3+4n)S_n + (-7-4n)S_{n+1} = (13+20n+8n^2)/(2(1+n)^2(3+2n)^2);$$

Step 3: use the command `SolveRecurrence` to solve the above recurrence relation and obtain a particular solution

$$\frac{\sum_{l=1}^{n} \frac{-1-4l-8l^2}{l^2(1+2l)^2}}{2(3+4n)}$$

and the basic system of solutions $\{1/(4n+3)\}$ of the homogeneous version;

Step 4: use the command `FindLinearCombination` to get another form of S_n as follows:

$$S_n = -\frac{2}{4n+3} + \frac{\sum_{l=1}^{n} \frac{-1-4l-8l^2}{l^2(1+2l)^2}}{2(3+4n)}.$$

Thus we deduce the desired identity by substituting n for $2n+1$. □

Lemma 2.2. *Let p be an odd prime. For any $k = 1, 2, \dots, (p-1)/2$, we have:*

$$\frac{(\frac{1-p}{2})_k(\frac{1+p}{2})_k}{(1)_k^2} \equiv \frac{\binom{2k}{k}^2}{16^k}\left(1 - p^2\sum_{j=1}^{k}\frac{1}{(2j-1)^2}\right) \pmod{p^4}. \tag{2.2}$$

Proof. First, by simple computation, we have:

$$\begin{aligned}\left(\frac{1-p}{2}\right)_k\left(\frac{1+p}{2}\right)_k &= \prod_{j=1}^{k}\left(\left(\frac{1-p}{2}+j-1\right)\left(\frac{1+p}{2}+j-1\right)\right)\\ &= \frac{1}{4^k}\prod_{j=1}^{k}(2j-1-p)(2j-1+p)\\ &= \frac{1}{4^k}\prod_{j=1}^{k}(2j-1)^2\prod_{j=1}^{k}\left(1-\frac{p^2}{(2j-1)^2}\right).\end{aligned}$$

Then the desired result follows from

$$\frac{\prod_{j=1}^{k}(2j-1)^2}{(1)_k^2 4^k} = \frac{\binom{2k}{k}^2}{16^k}$$

and

$$\prod_{j=1}^{k}\left(1-\frac{p^2}{(2j-1)^2}\right) \equiv 1 - p^2\sum_{j=1}^{k}\frac{1}{(2j-1)^2} \pmod{p^4}. \qquad \square$$

Proof of Theorem 1.1. We will divide our proof into two cases.

Case 1. If $p \equiv 3 \pmod 4$.

Substituting $(p-1)/2$ for n in (2.1), we have

$$\sum_{k=0}^{(p-1)/2}\frac{(\frac{1-p}{2})_k(\frac{1+p}{2})_k}{(2k+1)(1)_k^2}H_k^{(2)} = -\frac{2}{p} + \frac{1}{2p}\sum_{k=1}^{(p-3)/4}\frac{-8k^2-4k-1}{k^2(2k+1)^2}.$$

Note that

$$\sum_{k=1}^{(p-3)/4}\frac{-8k^2-4k-1}{k^2(2k+1)^2} = -4\sum_{k=1}^{(p-3)/4}\left(\frac{1}{(2k)^2}+\frac{1}{(2k+1)^2}\right) = -4(H_{(p-1)/2}^{(2)}-1)$$

and

$$\frac{(\frac{1-p}{2})_{(p-1)/2}(\frac{1+p}{2})_{(p-1)/2}}{p(1)_{(p-1)/2}^2}H_{(p-1)/2}^{(2)} \equiv \frac{1}{p}H_{(p-1)/2}^{(2)} \pmod p$$

in view of (1.4) and (2.2). Thus we arrive at

$$\sum_{k=0}^{(p-3)/2}\frac{(\frac{1-p}{2})_k(\frac{1+p}{2})_k}{(2k+1)(1)_k^2}H_k^{(2)} \equiv -\frac{3}{p}H_{(p-1)/2}^{(2)} \equiv -7B_{p-3} \pmod p$$

by (1.4) and (1.1). Therefore, we obtain:

$$\sum_{k=0}^{(p-3)/2}\frac{\binom{2k}{k}^2}{(2k+1)16^k}H_k^{(2)} \equiv -7B_{p-3} \pmod p$$

in light of (2.2).

Case 2. If $p \equiv 1 \pmod 4$.

By setting $n = (p-1)/2$ in (2.1), we obtain that

$$\sum_{k=0}^{(p-1)/2}\frac{(\frac{1-p}{2})_k(\frac{1+p}{2})_k}{(2k+1)(1)_k^2}H_k^{(2)} = \frac{1}{2p}\sum_{k=1}^{(p-1)/4}\frac{-8k^2+4k-1}{k^2(2k-1)^2}.$$

A discussion similar to the one in the proof of *Case 1* gives:

$$\sum_{k=0}^{(p-3)/2}\frac{\binom{2k}{k}^2}{(2k+1)16^k}H_k^{(2)}\equiv -7B_{p-3} \pmod p.$$

The proof of Theorem 1.1 is now complete. □

3. Proof of Theorem 1.2

We need the known result.

Lemma 3.1. *[11, (2.1)] Let p be an odd prime. Then for $k\in\{1,2,\ldots,p-1\}$, we have:*

$$k\binom{2k}{k}\binom{2(p-k)}{p-k}\equiv (-1)^{\lfloor 2k/p\rfloor-1}2p \pmod{p^2}. \tag{3.1}$$

Proof of Theorem 1.2. First we show (1.7). Via `Sigma`, we find that for any nonnegative integer n, we have:

$$\sum_{k=1}^{n}\frac{(-n)_k(1+n)_k}{k(1)_k^2}H_k^{(2)}=\begin{cases}-2-\frac{1}{4}\sum_{k=1}^{(n-1)/2}\frac{(4k+1)(4k^2+2k+1)}{k^3(2k+1)^3} & n\equiv 1 \pmod 2,\\ -\frac{1}{4}\sum_{k=1}^{n/2}\frac{(4k-1)(4k^2-2k+1)}{k^3(2k-1)^3} & n\equiv 0 \pmod 2.\end{cases} \tag{3.2}$$

If $p\equiv 3 \pmod 4$, substituting $n=(p-1)/2$ into (3.2), we have:

$$\sum_{k=1}^{(p-1)/2}\frac{(\frac{1-p}{2})_k(\frac{1+p}{2})_k}{k(1)_k^2}H_k^{(2)}=-2-\frac{1}{4}\sum_{k=1}^{(p-3)/4}\frac{(4k+1)(4k^2+2k+1)}{k^3(2k+1)^3}.$$

Clearly,

$$\sum_{k=1}^{(p-3)/4}\frac{(4k+1)(4k^2+2k+1)}{k^3(2k+1)^3}=8H_{(p-1)/2}^{(3)}-8.$$

It follows that

$$\sum_{k=1}^{(p-1)/2}\frac{(\frac{1-p}{2})_k(\frac{1+p}{2})_k}{k(1)_k^2}H_k^{(2)}=-2H_{(p-1)/2}^{(3)}.$$

If $p\equiv 1 \pmod 4$, letting $n=(p-1)/2$ in (3.2), we have:

$$\sum_{k=1}^{(p-1)/2}\frac{(\frac{1-p}{2})_k(\frac{1+p}{2})_k}{k(1)_k^2}H_k^{(2)}=-\frac{1}{4}\sum_{k=1}^{(p-1)/4}\frac{(4k-1)(4k^2-2k+1)}{k^3(2k-1)^3}.$$

By a discussion similar to the discussion above, we find that the identity

$$\sum_{k=1}^{(p-1)/2}\frac{(\frac{1-p}{2})_k(\frac{1+p}{2})_k}{k(1)_k^2}H_k^{(2)}=-2H_{(p-1)/2}^{(3)}$$

holds again. Thus, we arrive at

$$\sum_{k=1}^{(p-1)/2}\frac{\binom{2k}{k}^2}{k16^k}H_k^{(2)}\equiv -2H_{(p-1)/2}^{(3)}\equiv -12\frac{H_{p-1}}{p^2} \pmod{p^2}$$

in view of (1.5), (2.2) and [8, Remark 5.1].

We next show (1.8). Write $n=(p-1)/2$. By (2.2), we have

$$\sum_{k=1}^{n}\frac{(-n)_k(n+1)_k}{k(1)_k^2}\equiv\sum_{k=0}^{n}\frac{(1/2)_k^2}{k(1)_k^2}\left(1-p^2\sum_{j=1}^{k}\frac{1}{(2j-1)^2}\right) \pmod{p^3}.$$

Via Sigma, we find that

$$\sum_{k=1}^{n}\frac{(-n)_k(n+1)_k}{k(1)_k^2}=-2H_n.$$

Sun [11, (1.3)] proved that

$$\sum_{k=1}^{n}\frac{(1/2)_k^2}{k(1)_k^2}\equiv -2H_n-\frac{7}{2}p^2B_{p-3} \pmod{p^3}.$$

Thus we deduce that

$$\sum_{k=1}^{n}\frac{(1/2)_k^2}{k(1)_k^2}\sum_{j=1}^{k}\frac{1}{(2j-1)^2}\equiv -\frac{7}{2}B_{p-3} \pmod{p}.$$

So we further obtain

$$\sum_{k=1}^{n}\frac{(1/2)_k^2}{k(1)_k^2}H_{2k}^{(2)}\equiv \frac{1}{4}\sum_{k=1}^{n}\frac{(1/2)_k^2}{k(1)_k^2}H_k^{(2)}-\frac{7}{2}B_{p-3}\equiv -\frac{5}{2}B_{p-3} \pmod{p}$$

as desired in view of (1.7).

Now we prove (1.9). In view of (1.8), we have:

$$\begin{aligned}\sum_{k=1}^{p-1}\frac{\binom{2k}{k}^2}{k16^k}H_{2k}^{(2)} &\equiv \sum_{k=1}^{(p-1)/2}\frac{(1/2)_k^2}{k(1)_k^2}H_{2k}^{(2)}+\sum_{k=(p+1)/2}^{p-1}\frac{\binom{2k}{k}^2}{k16^k}H_{2k}^{(2)}\\ &\equiv -\frac{5}{2}B_{p-3}+\sum_{k=(p+1)/2}^{p-1}\frac{\binom{2k}{k}^2}{k16^k}H_{2k}^{(2)} \pmod{p}.\end{aligned}$$

It suffices to show that

$$\sum_{k=(p+1)/2}^{p-1}\frac{\binom{2k}{k}^2}{k16^k}H_{2k}^{(2)}\equiv \frac{7}{2}B_{p-3} \pmod{p}.$$

By Fermat's little theorem and (3.1), we deduce that

$$\sum_{k=(p+1)/2}^{p-1}\frac{\binom{2k}{k}^2}{k16^k}H_{2k}^{(2)}\equiv \frac{4p^2}{16^p}\sum_{k=1}^{(p-1)/2}\frac{16^kH_{2p-2k}^{(2)}}{(p-k)^3\binom{2k}{k}^2}\equiv -\frac{1}{4}\sum_{k=1}^{(p-1)/2}\frac{16^kp^2H_{2p-2k}^{(2)}}{k^3\binom{2k}{k}^2} \pmod{p}.$$

Note that $p^2H_{2p-2k}^{(2)}\equiv 1 \pmod{p}$ for each $k=1,2,\dots,(p-1)/2$. Thus by [11, (1.4)] we have:

$$\sum_{k=(p+1)/2}^{p-1}\frac{\binom{2k}{k}^2}{k16^k}H_{2k}^{(2)}\equiv -\frac{1}{4}\sum_{k=1}^{(p-1)/2}\frac{16^k}{k^3\binom{2k}{k}^2}\equiv -\frac{1}{4}\times(-14B_{p-3})=\frac{7}{2}B_{p-3} \pmod{p}.$$

This concludes the proof. □

4. Proof of Theorem 1.3

Lemma 4.1. *[10, Lemma 4.2] Let p be an odd prime and write $n=(p-1)/2$. Then, for $k\in\{0,\dots,n\}$,*

$$\frac{\binom{n}{k}}{\binom{2k}{k}/(-4)^k}\equiv 1-p\sum_{j=1}^{k}\frac{1}{2j-1} \pmod{p^2}.$$

In order to prove Theorem 1.3, we also need some identities as the following ones, which were found by using the package Sigma:

$$\sum_{k=1}^{n}\binom{n}{k}\frac{(-1)^k}{k^2}\sum_{j=1}^{k}\frac{1}{2j-1}=-\frac{1}{2}H_n\sum_{k=1}^{n}\frac{4^k}{k^2\binom{2k}{k}}+\frac{1}{2}\sum_{k=1}^{n}\frac{4^kH_k}{k^2\binom{2k}{k}}-\frac{1}{2}\sum_{k=1}^{n}\frac{4^k}{k^3\binom{2k}{k}}, \tag{4.1}$$

$$\sum_{k=1}^{n}\frac{(-1)^k}{k^2\binom{n}{k}}=H_n^{(2)}+2\sum_{k=1}^{n}\frac{(-1)^k}{k^2}, \tag{4.2}$$

$$\sum_{k=1}^{n}\frac{(-1)^k}{k^3\binom{n}{k}}=-H_n-\sum_{k=1}^{n}\frac{1}{k}\sum_{j=2}^{k}\frac{j^2-(-1)^j(4j^2-2j+1)}{j^2(j-1)^2}$$
$$=H_n^{(3)}-\sum_{k=1}^{n}\frac{1}{k}\sum_{j=1}^{k}\frac{1+2(-1)^j}{j^2}+3\sum_{k=1}^{n}\frac{(-1)^k}{k^3}, \tag{4.3}$$

$$\sum_{k=1}^{n}\frac{(-1)^kH_k}{k^2\binom{n}{k}}=2\sum_{k=1}^{n}\frac{(-1)^kH_k}{k^2}-\sum_{k=1}^{n}\frac{1}{k}\sum_{j=1}^{k}\frac{1+2(-1)^j}{j^2}. \tag{4.4}$$

Proof of (1.10). By (3.1) and the fact that $pH_{2p-2k}\equiv 1 \pmod p$ for each $k=1,2,\cdots,(p-1)/2$, we have:

$$\sum_{k=1}^{p-1}\frac{\binom{2k}{k}}{k^24^k}(H_{2k}-\frac{1}{2}H_k)\equiv\sum_{k=1}^{(p-1)/2}\frac{\binom{2k}{k}}{k^24^k}\sum_{j=1}^{k}\frac{1}{2j-1}+\sum_{k=(p+1)/2}^{p-1}\frac{H_{2k}}{k^24^k}\frac{2p}{k\binom{2p-2k}{p-k}}$$
$$\equiv\sum_{k=1}^{(p-1)/2}\frac{\binom{2k}{k}}{k^24^k}\sum_{j=1}^{k}\frac{1}{2j-1}-\frac{1}{2}\sum_{k=1}^{(p-1)/2}\frac{4^k}{k^3\binom{2k}{k}} \pmod p. \tag{4.5}$$

It is known that $\binom{2k}{k}/(-4)^k\equiv\binom{(p-1)/2}{k} \pmod p$. Setting $n=(p-1)/2$ in the above identities (4.1)–(4.4), we have:

$$\sum_{k=1}^{p-1}\frac{\binom{2k}{k}}{k^24^k}\left(H_{2k}-\frac{1}{2}H_k\right)\equiv-\frac{1}{2}H_n\left(H_n^{(2)}+2\sum_{k=1}^{n}\frac{(-1)^k}{k^2}\right)$$
$$+\frac{1}{2}\left(2\sum_{k=1}^{n}\frac{(-1)^kH_k}{k^2}-\sum_{k=1}^{n}\frac{1}{k}\sum_{j=1}^{k}\frac{1+2(-1)^j}{j^2}\right)$$
$$-\left(H_n^{(3)}-\sum_{k=1}^{n}\frac{1}{k}\sum_{j=1}^{k}\frac{1+2(-1)^j}{j^2}+3\sum_{k=1}^{n}\frac{(-1)^k}{k^3}\right) \pmod p.$$

Note that

$$\sum_{k=1}^{n}\frac{1}{k}\sum_{j=1}^{k}\frac{(-1)^j}{j^2}=\sum_{j=1}^{n}\frac{(-1)^j}{j^2}\sum_{k=j}^{n}\frac{1}{k}=H_n\sum_{j=1}^{n}\frac{(-1)^j}{j^2}-\sum_{j=1}^{n}\frac{(-1)^j}{j^2}H_{j-1}$$

and

$$\sum_{k=1}^{n}\frac{1}{k}\sum_{j=1}^{k}\frac{1}{j^2}=\sum_{j=1}^{n}\frac{1}{j^2}\sum_{k=j}^{n}\frac{1}{k}=H_nH_n^{(2)}-\sum_{j=1}^{n}\frac{H_{j-1}}{j^2}.$$

These, together with (1.4), yield that

$$\sum_{k=1}^{p-1}\frac{\binom{2k}{k}}{k^24^k}\left(H_{2k}-\frac{1}{2}H_k\right)\equiv\frac{3}{2}H_n^{(3)}-\frac{1}{2}\sum_{k=1}^{n}\frac{H_k}{k^2}-\frac{1}{2}H_{\lfloor n/2\rfloor}^{(3)} \pmod p. \tag{4.6}$$

In view of (1.3), (1.5) and [5, Lemma 3.2, (1.3)], we immediately get the result as desired.

Proof of (1.11). Let $n=(p-1)/2$. By Lemma 3.1, Lemma 4.1, (4.5) and (4.6), we have:

$$\sum_{k=1}^{p-1}\frac{\binom{2k}{k}}{k^24^k}\equiv\sum_{k=1}^{n}\frac{(-1)^k}{k^2}\binom{n}{k}\left(1+p\sum_{j=1}^{k}\frac{1}{2j-1}\right)-\frac{p}{2}\sum_{k=1}^{n}\frac{4^k}{k^3\binom{2k}{k}}$$
$$=\sum_{k=1}^{n}\frac{(-1)^k}{k^2}\binom{n}{k}+p\left(\sum_{k=1}^{n}\frac{\binom{n}{k}(-1)^k}{k^2}\sum_{j=1}^{k}\frac{1}{2j-1}-\frac{1}{2}\sum_{k=1}^{n}\frac{4^k}{k^3\binom{2k}{k}}\right)$$

$$\equiv \sum_{k=1}^{n} \frac{(-1)^k}{k^2}\binom{n}{k} + p\left(\frac{3}{2}H_n^{(3)} - \frac{1}{2}\sum_{k=1}^{n}\frac{H_k}{k^2} - \frac{1}{2}H_{\lfloor n/2\rfloor}^{(3)}\right) \pmod{p^2}.$$

By making use of the package `Sigma`, we also find that

$$\sum_{k=1}^{n} \frac{(-1)^k}{k^2}\binom{n}{k} = -\frac{1}{2}H_n^{(2)} - \frac{1}{2}(H_n)^2.$$

This, together with $H_{p-1} \equiv -\frac{1}{3}p^2 B_{p-3} \pmod{p^3}$ in [9], (1.3), (1.5) and [5, Lemma 3.2, (1.3)] yield that

$$\sum_{k=1}^{p-1} \frac{\binom{2k}{k}}{k^2 4^k} \equiv -\frac{H_{(p-1)/2}^2}{2} - \frac{7}{4}\frac{H_{p-1}}{p} \pmod{p^2}.$$

Now the proof of Theorem 1.4 is complete. □

5. Proof of Theorem 1.4

By using the package `Sigma` in the same way as in Section 2, we obtain the following identities:

$$\sum_{k=1}^{n} \frac{(-n)_k(1+n)_k}{k(1)_k^2} H_k = \begin{cases} -2 + 2\sum_{k=1}^{(n-1)/2}\left(\frac{1}{(2k)^2} - \frac{1}{(2k+1)^2}\right) & \text{if } n \equiv 1 \pmod 2, \\ -2\sum_{k=1}^{n/2}\left(\frac{1}{(2k-1)^2} - \frac{1}{(2k)^2}\right) & \text{if } n \equiv 0 \pmod 2. \end{cases} \tag{5.1}$$

$$\sum_{k=1}^{n} \frac{(-n)_k(1+n)_k}{k(1)_k^2} \sum_{j=1}^{k}\frac{1}{2j-1} = \begin{cases} -2\sum_{k=0}^{(n-1)/2}\frac{1}{(2k+1)^2} & \text{if } n \equiv 1 \pmod 2, \\ -2\sum_{k=1}^{n/2}\frac{1}{(2k-1)^2} & \text{if } n \equiv 0 \pmod 2. \end{cases} \tag{5.2}$$

Substituting $n = (p-1)/2$ into (5.1) and (5.2), we have:

$$\sum_{k=1}^{n} \frac{(-n)_k(1+n)_k}{k(1)_k^2} H_k = H_{\lfloor p/4\rfloor}^{(2)} - 2H_{(p-1)/2}^{(2)} \tag{5.3}$$

and

$$\sum_{k=1}^{n} \frac{(-n)_k(1+n)_k}{k(1)_k^2} \sum_{j=1}^{k}\frac{1}{2j-1} = \frac{1}{2}H_{\lfloor p/4\rfloor}^{(2)} - 2H_{(p-1)/2}^{(2)}.$$

Hence,

$$\sum_{k=1}^{n} \frac{(-n)_k(1+n)_k}{k(1)_k^2} H_{2k} = \frac{1}{2}\sum_{k=1}^{n} \frac{(-n)_k(1+n)_k}{k(1)_k^2} H_k + \sum_{k=1}^{n} \frac{(-n)_k(1+n)_k}{k(1)_k^2}\sum_{j=1}^{k}\frac{1}{2j-1}$$
$$= \frac{1}{2}\left(H_{\lfloor p/4\rfloor}^{(2)} - 2H_{(p-1)/2}^{(2)}\right) + \frac{1}{2}H_{\lfloor p/4\rfloor}^{(2)} - 2H_{(p-1)/2}^{(2)} = H_{\lfloor p/4\rfloor}^{(2)} - 3H_{(p-1)/2}^{(2)}.$$

Therefore,

$$\sum_{k=1}^{n} \frac{\binom{2k}{k}^2}{k16^k} H_k \equiv 4\left(\frac{-1}{p}\right)\left(2E_{p-3} - E_{2p-4}\right) \pmod{p^2}$$

and

$$\sum_{k=1}^{n} \frac{\binom{2k}{k}^2}{k16^k} H_{2k} \equiv 4\left(\frac{-1}{p}\right)\left(2E_{p-3} - E_{2p-4}\right) - \frac{7}{3}pB_{p-3} \pmod{p^2}$$

in view of (1.2), (1.4), (2.2) and (5.3). Now the proof of Theorem 1.4 is finished. □

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