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Continuity of a surface in Fréchet spaces

Continuité d'une surface dans des espaces de Fréchet

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ABSTRACT

We establish the continuity of a surface as a function of its first two fundamental forms for several Fréchet topologies, which include in particular those of the space $W^{1,p}_{\text{loc}}$ for the first fundamental form and of the space L^p_{loc} for the second fundamental form, for any $p > 2$.

RÉSUMÉ

On établit la continuité d'une surface en fonction de ses deux premières formes fondamentales pour plusieurs topologies de Fréchet, qui incluent en particulier celles de l'espace $W^{1,p}_{\text{loc}}$ pour la première forme et de l'espace L^p_{loc} pour la deuxième forme, où $p > 2$.

1. Introduction

In this Note, Greek indices and exponents range in the set $\{1,2\}$, Latin indices and exponents range in the set $\{1,2,3\}$ (save when they are used for indexing sequences or when otherwise indicated), and the summation convention for repeated indices and exponents is used. Boldface letters denote vector and matrix fields.

For each integer $n \geqslant 2$, the n-dimensional Euclidean space is denoted by $\mathbb{E}^n$, the space of real square matrices of order n is denoted by $\mathbb{M}^n$, the subspace of $\mathbb{M}^n$ formed by all symmetric matrices is denoted by $\mathbb{S}^n$, and the subset of $\mathbb{S}^n$ formed by all positive-definite matrices is denoted by $\mathbb{S}^n_>$. The inner product, exterior product, and norm, in $\mathbb{E}^n$ are respectively denoted by $\cdot$, $\wedge$, and $|\cdot|$.

The set of all *proper isometries* of $\mathbb{E}^n$ is denoted and defined by

$$\mathbf{Isom}_+(\mathbb{E}^n) := \{\boldsymbol{r} : \mathbb{E}^n \to \mathbb{E}^n, \boldsymbol{r}(x) = \boldsymbol{a} + \boldsymbol{R}x, x \in \mathbb{E}^n;\ \boldsymbol{a} \in \mathbb{E}^n, \boldsymbol{R} \in \mathbb{O}^n_+\},$$

where $\mathbb{O}^n_+$ denotes the subset of $\mathbb{M}^n$ formed by all proper orthogonal matrices.

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Given an open subset ω of $\mathbb{R}^2$, we denote by $y = (y_\alpha)$ a generic point in ω, and we let $\partial_\alpha := \partial/\partial y_\alpha$ and $\partial_{\alpha\beta} := \partial^2/\partial y_\alpha \partial y_\beta$. The notation $K \Subset \omega$ means that K is a compact subset of ω.

Given any integer $m \geqslant 1$ and any finite-dimensional real space $\mathbb{X}$, the notation $\mathcal{C}^m(\omega)$, resp. $\mathcal{C}^m(\omega; \mathbb{X})$, denotes the space of continuously differentiable functions, resp. $\mathbb{X}$-valued fields, up to the order m over ω. The space $\mathcal{C}^m(\omega)$, resp. $\mathcal{C}^m(\omega; \mathbb{X})$, is equipped with the *Fréchet topology* (see, e.g., Rudin [11]) associated with the semi-norms $\|\cdot\|_{\mathcal{C}^m(K)}$, resp. $\|\cdot\|_{\mathcal{C}^m(K;\mathbb{X})}$, $K \Subset \omega$.

Given any integer $m \geqslant 1$ and any real number $p \geqslant 1$, $L^p(\omega)$, resp. $W^{m,p}(\omega)$, designates the usual Lebesgue, resp. Sobolev, space. The notation $L^p_{\rm loc}(\omega)$, resp. $W^{m,p}_{\rm loc}(\omega)$, denotes the space of all (equivalence classes of) measurable functions $f : \omega \to \mathbb{R}$ such that $f|_U \in L^p(U)$, resp. $f|_U \in W^{m,p}(U)$, for all open sets U such that $\overline{U} \Subset \omega$, where $f|_U$ denotes the restriction of f to U. The notation $L^p_{\rm loc}(\omega; \mathbb{X})$, resp. $W^{m,p}_{\rm loc}(\omega; \mathbb{X})$, denotes the space of $\mathbb{X}$-valued fields with components in $L^p_{\rm loc}(\omega)$, resp. $W^{m,p}_{\rm loc}(\omega)$. The spaces $L^p_{\rm loc}(\omega)$, $W^{m,p}_{\rm loc}(\omega)$, $L^p_{\rm loc}(\omega; \mathbb{X})$, and $W^{m,p}_{\rm loc}(\omega; \mathbb{X})$ are equipped with the *Fréchet topologies* associated with the semi-norms $\|\cdot\|_{L^p(U)}$, $\|\cdot\|_{W^{m,p}(U)}$, $\|\cdot\|_{L^p(U;\mathbb{X})}$, and $\|\cdot\|_{W^{m,p}(U;\mathbb{X})}$, respectively, where U is any open set such that $\overline{U} \Subset \omega$.

A smooth enough mapping $\boldsymbol{\theta} : \omega \to \mathbb{E}^3$ is an *immersion* if the two vector fields $\partial_\alpha \boldsymbol{\theta} : \omega \to \mathbb{E}^3$ are linearly independent at each point of ω. Given an immersion $\boldsymbol{\theta} : \omega \to \mathbb{E}^3$, define the matrix fields

$$\boldsymbol{A}(\boldsymbol{\theta}) := (a_{\alpha\beta}(\boldsymbol{\theta})) : \omega \to \mathbb{S}^2_>, \text{ where } a_{\alpha\beta}(\boldsymbol{\theta}) := \partial_\alpha \boldsymbol{\theta} \cdot \partial_\beta \boldsymbol{\theta},$$

and

$$\boldsymbol{B}(\boldsymbol{\theta}) := (b_{\alpha\beta}(\boldsymbol{\theta})) : \omega \to \mathbb{S}^2, \text{ where } b_{\alpha\beta}(\boldsymbol{\theta}) := \partial_{\alpha\beta} \boldsymbol{\theta} \cdot \frac{\partial_1 \boldsymbol{\theta} \wedge \partial_2 \boldsymbol{\theta}}{|\partial_1 \boldsymbol{\theta} \wedge \partial_2 \boldsymbol{\theta}|}.$$

The image $\boldsymbol{\theta}(\omega)$ is then a *surface* in $\mathbb{E}^3$ and the functions $a_{\alpha\beta}(\boldsymbol{\theta})$ and $b_{\alpha\beta}(\boldsymbol{\theta})$ are the covariant components of the *first* and *second fundamental forms* of the surface $\boldsymbol{\theta}(\omega)$.

The reconstruction of a surface from its first two fundamental forms, which constitutes the object of the *fundamental theorem of surface theory*, is classically established under the assumptions that the given forms are functions of class $\mathcal{C}^m(\omega)$ for m "large enough", where ω is a simply-connected open subset of $\mathbb{R}^2$ that satisfies the *Gauss and Codazzi–Mainardi equations*; the surface is then uniquely determined up to proper isometries of $\mathbb{E}^3$ (see, e.g., [3]). In addition, the mapping that defines in this fashion a surface (up to proper isometries) in terms of its fundamental forms is *continuous* when the spaces $\mathcal{C}^m(\omega)$ are equipped with their natural *Fréchet topologies*: cf. Ciarlet [2].

S. Mardare has shown (cf. [10]) that the fundamental theorem of surface theory holds as well in function spaces with little regularity, such as the spaces $W^{m,p}_{\rm loc}(\omega)$, for any $p > 2$. The objective of this Note consists in showing that a surface considered as a function of its fundamental forms is *continuous* in these Fréchet spaces.

The proofs of the results announced here will be found in [7].

2. Continuity of a surface as a function of its fundamental forms in $W^{1,p}_{\rm loc} \times L^p_{\rm loc}$, $p > 2$

Our *first result* is a significant complement to the generalization of the fundamental theorem of surface theory established by S. Mardare in [10], by showing that the mapping that defines a surface of class $W^{2,p}_{\rm loc}(\omega; \mathbb{E}^3)$ in terms of its fundamental forms in $W^{1,p}_{\rm loc}(\omega; \mathbb{S}^2_>) \times L^p_{\rm loc}(\omega; \mathbb{S}^2)$ is *continuous* when these spaces are equipped with their natural *Fréchet topologies*.

Given any $p > 2$ and any open subset ω of $\mathbb{R}^2$, let $\mathbb{T}^p_{\rm loc}(\omega)$ be the subset of the set $W^{1,p}_{\rm loc}(\omega; \mathbb{S}^2_>) \times L^p_{\rm loc}(\omega; \mathbb{S}^2)$ defined by

$$\mathbb{T}^p_{\rm loc}(\omega) := \left\{ (\boldsymbol{A}(\boldsymbol{\theta}), \boldsymbol{B}(\boldsymbol{\theta})) \in W^{1,p}_{\rm loc}(\omega; \mathbb{S}^2_>) \times L^p_{\rm loc}(\omega; \mathbb{S}^2);\ \boldsymbol{\theta} \in W^{2,p}_{\rm loc}(\omega; \mathbb{E}^3) \right\}.$$

We also define the *quotient set*

$$\dot{W}^{2,p}_{\rm loc}(\omega; \mathbb{E}^3) := \left\{ \dot{\boldsymbol{\theta}};\ \boldsymbol{\theta} \in W^{2,p}_{\rm loc}(\omega; \mathbb{E}^3) \right\}, \text{where } \dot{\boldsymbol{\theta}} := \{\boldsymbol{r} \circ \boldsymbol{\theta};\ \boldsymbol{r} \in \mathbf{Isom}_+(\mathbb{E}^3)\},$$

and its subset

$$\dot{V}^{2,p}_{\rm loc}(\omega; \mathbb{E}^3) := \{\dot{\boldsymbol{\theta}} \in \dot{W}^{2,p}_{\rm loc}(\omega; \mathbb{E}^3);\ \partial_1 \boldsymbol{\theta} \wedge \partial_2 \boldsymbol{\theta} \neq \mathbf{0} \text{ in } \omega\}.$$

In Theorem 1 below, which constitutes our *first* main result, we establish that the mapping

$$(\boldsymbol{A}(\boldsymbol{\theta}), \boldsymbol{B}(\boldsymbol{\theta})) \in \mathbb{T}^p_{\rm loc}(\omega) \to \dot{\boldsymbol{\theta}} \in \dot{V}^{2,p}_{\rm loc}(\omega; \mathbb{E}^3),$$

is *continuous*.

Theorem 1. *Let ω be a connected open subset of $\mathbb{R}^2$ and let $p > 2$. Let there be given immersions $\boldsymbol{\theta}^k \in W^{2,p}_{\rm loc}(\omega; \mathbb{E}^3)$, $k \geqslant 1$, and $\boldsymbol{\theta} \in W^{2,p}_{\rm loc}(\omega; \mathbb{E}^3)$ such that*

$$\boldsymbol{A}(\boldsymbol{\theta}^k) \to \boldsymbol{A}(\boldsymbol{\theta}) \text{ in } W^{1,p}_{\rm loc}(\omega; \mathbb{S}^2_>) \text{ as } k \to \infty,$$
$$\boldsymbol{B}(\boldsymbol{\theta}^k) \to \boldsymbol{B}(\boldsymbol{\theta}) \text{ in } L^{p}_{\rm loc}(\omega; \mathbb{S}^2) \text{ as } k \to \infty.$$

Then there exist proper isometries $\boldsymbol{r}^k$, $k \geqslant 1$, of $\mathbb{E}^3$ such that

$$\boldsymbol{r}^k \circ \boldsymbol{\theta}^k \to \boldsymbol{\theta} \text{ in } W^{2,p}_{\rm loc}(\omega; \mathbb{E}^3) \text{ as } k \to \infty.$$

Sketch of the proof. With any immersion $\boldsymbol{\theta} \in W^{2,p}_{\rm loc}(\omega; \mathbb{E}^3)$, we associate the matrix fields

$$\boldsymbol{F}(\boldsymbol{\theta}) := \Big(\partial_1\boldsymbol{\theta} \,\Big|\, \partial_2\boldsymbol{\theta} \,\Big|\, \boldsymbol{a}_3(\boldsymbol{\theta})\Big) \in W^{1,p}_{\rm loc}(\omega; \mathbb{M}^3) \text{ and } \boldsymbol{C}(\boldsymbol{\theta}) := \boldsymbol{F}(\boldsymbol{\theta})^{\rm T}\boldsymbol{F}(\boldsymbol{\theta}) \in W^{1,p}_{\rm loc}(\omega; \mathbb{S}^3),$$

where

$$\boldsymbol{a}_3(\boldsymbol{\theta}) := \frac{\partial_1\boldsymbol{\theta} \wedge \partial_2\boldsymbol{\theta}}{|\partial_1\boldsymbol{\theta} \wedge \partial_2\boldsymbol{\theta}|} \in W^{1,p}_{\rm loc}(\omega; \mathbb{E}^3).$$

Let $\boldsymbol{\theta}^k \in W^{2,p}_{\rm loc}(\omega; \mathbb{E}^3)$, $k \geqslant 1$, and $\boldsymbol{\theta} \in W^{2,p}_{\rm loc}(\omega; \mathbb{E}^3)$ be immersions that satisfy the assumptions of the theorem. Let a point $y^0 \in \omega$ be fixed once and for all. For each integer $k \geqslant 1$, define the matrix field

$$\boldsymbol{R}^k := \boldsymbol{F}(\boldsymbol{\theta})\boldsymbol{C}(\boldsymbol{\theta})^{-1/2}\boldsymbol{C}(\boldsymbol{\theta}^k)^{-1/2}\boldsymbol{F}(\boldsymbol{\theta}^k)^{\rm T}$$

and the function

$$\boldsymbol{r}^k := \boldsymbol{\theta}(y^0) + \boldsymbol{R}^k(y^0)(\mathbf{id} - \boldsymbol{\theta}^k(y^0)).$$

Note that $\boldsymbol{R}^k(y^0) \in \mathbb{O}^3_+$ and $\boldsymbol{r}^k \in \mathbf{Isom}_+(\mathbb{E}^3)$ for all $k \geqslant 1$.

Then one easily shows that

$$(\boldsymbol{r}^k \circ \boldsymbol{\theta}^k)(y^0) = \boldsymbol{\theta}(y^0) \text{ for all } k \geqslant 1, \text{ and } \nabla(\boldsymbol{r}^k \circ \boldsymbol{\theta}^k)(y^0) \to \nabla\boldsymbol{\theta}(y^0) \text{ as } k \to \infty,$$

since the assumptions of the theorem imply that

$$\boldsymbol{C}(\boldsymbol{\theta}^k) \to \boldsymbol{C}(\boldsymbol{\theta}) \text{ in } W^{1,p}_{\rm loc}(\omega; \mathbb{S}^3) \text{ as } k \to \infty,$$

and thus in $\mathcal{C}^0(\omega; \mathbb{S}^3)$, thanks to the Sobolev inclusion $W^{1,p}_{\rm loc}(\omega) \hookrightarrow \mathcal{C}^0(\omega)$.

By using a nonlinear Korn inequality on a surface in the space $W^{2,p}(\omega; \mathbb{E}^3)$ recently proved by Ciarlet & C. Mardare in [6, Theorem 5.2], and by following a strategy adapted from Ciarlet & S. Mardare in [5, Proof of Theorem 5.2.], one then shows that

$$\|\boldsymbol{r}^k \circ \boldsymbol{\theta}^k - \boldsymbol{\theta}\|_{W^{2,p}(B;\mathbb{E}^3)} \to 0 \text{ as } k \to \infty \text{ for each open ball } B \text{ such that } \overline{B} \Subset \omega.$$

Hence,

$$\boldsymbol{r}^k \circ \boldsymbol{\theta}^k \to \boldsymbol{\theta} \text{ in } W^{2,p}_{\rm loc}(\omega; \mathbb{E}^3) \text{ as } k \to \infty,$$

as claimed. □

3. Continuity of a surface for stronger topologies

Our objective in this section is to show that the mapping defined in the previous section, once restricted to specific subsets of Fréchet spaces with stronger topologies, remains *continuous*.

Our *second result* (Theorem 2 below) is about surfaces of class $\mathcal{C}^{m+2}(\omega)$, $m \geqslant 0$. It contains as particular cases two previous results about the continuity of this mapping, established respectively in Ciarlet [2] and Ciarlet & C. Mardare [4].

Theorem 2. *Let ω be a connected open subset of $\mathbb{R}^2$ and let $m \geqslant 0$ be an integer. Let there be given immersions $\boldsymbol{\theta}^k \in \mathcal{C}^{m+2}(\omega; \mathbb{E}^3)$, $k \geqslant 1$, and $\boldsymbol{\theta} \in \mathcal{C}^{m+2}(\omega; \mathbb{E}^3)$ that satisfy*

$$\boldsymbol{A}(\boldsymbol{\theta}^k) \to \boldsymbol{A}(\boldsymbol{\theta}) \text{ in } \mathcal{C}^{m+1}(\omega; \mathbb{S}^2_>) \text{ as } k \to \infty,$$
$$\boldsymbol{B}(\boldsymbol{\theta}^k) \to \boldsymbol{B}(\boldsymbol{\theta}) \text{ in } \mathcal{C}^{m}(\omega; \mathbb{S}^2) \text{ as } k \to \infty.$$

Then there exist proper isometries $\boldsymbol{r}^k$, $k \geqslant 1$, of $\mathbb{E}^3$ such that

$$\boldsymbol{r}^k \circ \boldsymbol{\theta}^k \to \boldsymbol{\theta} \text{ in } \mathcal{C}^{m+2}(\omega; \mathbb{E}^3) \text{ as } k \to \infty.$$

Sketch of the proof. Let $\boldsymbol{\theta} \in C^{m+2}(\omega; \mathbb{E}^3)$ and $\boldsymbol{\theta}^k \in C^{m+2}(\omega; \mathbb{E}^3)$, $k \geqslant 1$, be immersions that satisfy the assumptions of the theorem. Let $p > 2$ be given once and for all. Since $\mathcal{C}^m(\omega) \hookrightarrow W^{m,p}_{\rm loc}(\omega)$ for each integer $m \geqslant 0$, Theorem 1 implies that there exist proper isometries $\boldsymbol{r}^k$, $k \geqslant 1$, of $\mathbb{E}^3$ such that

$$\boldsymbol{\varphi}^k := \boldsymbol{r}^k \circ \boldsymbol{\theta}^k \to \boldsymbol{\theta} \ \text{ in } W^{2,p}_{\rm loc}(\omega; \mathbb{E}^3) \text{ as } k \to \infty.$$

We now prove that this convergence also holds in the space $C^{m+2}(\omega; \mathbb{E}^3)$: with any immersion $\boldsymbol{\theta} \in C^{m+2}(\omega; \mathbb{E}^3)$, we associate the vector field

$$\boldsymbol{a}_3(\boldsymbol{\theta}) := \frac{\partial_1 \boldsymbol{\theta} \wedge \partial_2 \boldsymbol{\theta}}{|\partial_1 \boldsymbol{\theta} \wedge \partial_2 \boldsymbol{\theta}|} \in C^{m+1}(\omega; \mathbb{E}^3).$$

By using the Sobolev inclusion $W^{2,p}_{\rm loc}(\omega) \hookrightarrow \mathcal{C}^1(\omega)$, which holds since $p > 2$ and $\omega \subset \mathbb{R}^2$, we get

$$\boldsymbol{\varphi}^k \to \boldsymbol{\theta} \ \text{ in } C^1(\omega; \mathbb{E}^3) \text{ as } k \to \infty,$$

and thus

$$\partial_\alpha \boldsymbol{\varphi}^k \to \partial_\alpha \boldsymbol{\theta} \ \text{ in } C^0(\omega; \mathbb{E}^3) \text{ as } k \to \infty.$$

Furthermore, we also prove that the above convergences in turn imply that

$$\boldsymbol{a}_3(\boldsymbol{\varphi}^k) \to \boldsymbol{a}_3(\boldsymbol{\theta}) \text{ in } C^0(\omega; \mathbb{E}^3) \text{ as } k \to \infty.$$

Since the vector fields $\partial_\alpha \boldsymbol{\varphi}^k$, $\boldsymbol{a}_3(\boldsymbol{\varphi}^k)$, $\partial_\alpha \boldsymbol{\theta}$, and $\boldsymbol{a}_3(\boldsymbol{\theta})$, which belong in particular to the space $C^1(\omega; \mathbb{E}^3)$, satisfy the Gauss and Weingarten equations associated with the immersions $\boldsymbol{\varphi}^k$ and $\boldsymbol{\theta}$ (see, e.g., Klingerberg [8] or Kühnel [9]), the above convergences imply that

$$\partial_\alpha \boldsymbol{\varphi}^k \to \partial_\alpha \boldsymbol{\theta} \text{ and } \boldsymbol{a}_3(\boldsymbol{\varphi}^k) \to \boldsymbol{a}_3(\boldsymbol{\theta}) \text{ in } C^1(\omega; \mathbb{E}^3) \text{ as } k \to \infty.$$

Similarly, the above convergences combined again with the Gauss and Weingarten equations imply that

$$\partial_\alpha \boldsymbol{\varphi}^k \to \partial_\alpha \boldsymbol{\theta} \text{ and } \boldsymbol{a}_3(\boldsymbol{\varphi}^k) \to \boldsymbol{a}_3(\boldsymbol{\theta}) \text{ in } C^2(\omega; \mathbb{E}^3) \text{ as } k \to \infty.$$

After $(m+1)$ iterations of the same argument, we finally deduce that

$$\partial_\alpha \boldsymbol{\varphi}^k \to \partial_\alpha \boldsymbol{\theta} \text{ and } \boldsymbol{a}_3(\boldsymbol{\varphi}^k) \to \boldsymbol{a}_3(\boldsymbol{\theta}) \text{ in } C^{m+1}(\omega; \mathbb{E}^3) \text{ as } k \to \infty.$$

The conclusion follows by combining the above convergences with the previously established convergence

$$\boldsymbol{\varphi}^k \to \boldsymbol{\theta} \text{ in } C^1(\omega; \mathbb{E}^3) \text{ as } k \to \infty. \quad \square$$

Our *third result* (Theorem 3 below) applies to surfaces of class $W^{m+2,q}_{\rm loc}(\omega)$. Note that it contains as a particular case the continuity result of Theorem 1 (with $m = 0$ and $q = p > 2$).

Theorem 3. *Let ω be a connected open subset of $\mathbb{R}^2$, let $m \geqslant 0$ be an integer, and let $q \in [1, \infty]$ be such that $q > 2/(m+1)$. Let there be given immersions $\boldsymbol{\theta}^k \in W^{m+2,q}_{\rm loc}(\omega; \mathbb{E}^3)$, $k \geqslant 1$, and $\boldsymbol{\theta} \in W^{m+2,q}_{\rm loc}(\omega; \mathbb{E}^3)$ that satisfy*

$$\boldsymbol{A}(\boldsymbol{\theta}^k) \to \boldsymbol{A}(\boldsymbol{\theta}) \ \text{ in } W^{m+1,q}_{\rm loc}(\omega; \mathbb{S}^2_>) \text{ as } k \to \infty,$$
$$\boldsymbol{B}(\boldsymbol{\theta}^k) \to \boldsymbol{B}(\boldsymbol{\theta}) \ \text{ in } W^{m,q}_{\rm loc}(\omega; \mathbb{S}^2) \text{ as } k \to \infty.$$

Then there exist proper isometries $\boldsymbol{r}^k$, $k \geqslant 1$, of $\mathbb{E}^3$ such that

$$\boldsymbol{r}^k \circ \boldsymbol{\theta}^k \to \boldsymbol{\theta} \text{ in } W^{m+2,q}_{\rm loc}(\omega; \mathbb{E}^3) \text{ as } k \to \infty.$$

Sketch of the proof. Let $\boldsymbol{\theta}^k \in W^{m+2,q}_{\rm loc}(\omega; \mathbb{E}^3)$, $k \geqslant 1$, and $\boldsymbol{\theta} \in W^{m+2,q}_{\rm loc}(\omega; \mathbb{E}^3)$ be immersions that satisfy the assumptions of the theorem.

Given an integer $m \geqslant 0$ and $q \geqslant 1$ such that $q > 2/(m+1)$, let $p > 2$ be defined by

$$p := \begin{cases} q & \text{if } m = 0, \\ 2q & \text{if } m = 1, \\ 4q & \text{if } m \geqslant 2. \end{cases}$$

Then, by using classical Sobolev inclusions (see, e.g., Adams [1]), we infer that

$$\boldsymbol{A}(\boldsymbol{\theta}^k) \to \boldsymbol{A}(\boldsymbol{\theta}) \ \text{ in } W^{1,p}_{\text{loc}}(\omega; \mathbb{S}^2_>) \text{ as } k \to \infty,$$
$$\boldsymbol{B}(\boldsymbol{\theta}^k) \to \boldsymbol{B}(\boldsymbol{\theta}) \ \text{ in } L^{p}_{\text{loc}}(\omega; \mathbb{S}^2) \text{ as } k \to \infty,$$

and thus, by Theorem 1, there exist proper isometries $\boldsymbol{r}^k$, $k \geqslant 1$, of $\mathbb{E}^3$ such that

$$\boldsymbol{r}^k \circ \boldsymbol{\theta}^k \to \boldsymbol{\theta} \ \text{ in } W^{2,p}_{\text{loc}}(\omega; \mathbb{E}^3) \text{ as } k \to \infty.$$

It remains to prove that this convergence holds also in the space $W^{m+2,q}_{\text{loc}}(\omega; \mathbb{E}^3)$. To this end, we adapt the strategy used in the proof of Theorem 2 for the spaces $\mathcal{C}^m(\omega)$ to the Sobolev spaces $W^{m,q}_{\text{loc}}(\omega)$ by using classical Sobolev inclusions. □

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