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BILINEAR SPACE-TIME ESTIMATES FOR HOMOGENEOUS WAVE EQUATIONS

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ABSTRACT. – In this paper, we pursue a systematic treatment of the regularity theory for products and bilinear forms of solutions of the homogeneous wave equation. We discuss necessary and sufficient conditions for the validity of bilinear estimates, based on L^2 norms in space and time, of derivatives of products of solutions. Also, we give necessary conditions and formulate some conjectures for similar estimates based on $L_t^q L_x^r$ norms. © 2000 Éditions scientifiques et médicales Elsevier SAS

RÉSUMÉ. – Dans cet article, nous effectuons une étude systématique de la régularité nécessaire pour obtenir des estimations de type produit ou formes bilinéaires de solutions d'une équation d'onde homogène. Nous formulons des conditions nécessaires et suffisantes à la validité de telles estimations, dans le cas de normes L^2 en espace-temps, pour des dérivées de produits de solutions. De plus, nous donnons des conditions nécessaires et posons diverses conjectures pour des estimations similaires basées sur des normes $L_t^q L_x^r$. © 2000 Éditions scientifiques et médicales Elsevier SAS

1. Introduction

The goal of this paper is to investigate bilinear space-time estimates for solutions to homogeneous wave equations. The main part of the paper concerns L^2 estimates where we give a complete set of necessary and sufficient conditions for their validity. In the second part of the paper we discuss more general estimates in $L_t^q L_x^r$ spaces for which we find necessary conditions. These lead us to make a few selected conjectures concerning estimates for quadratic null forms whose solution, we feel, will be important in applications to nonlinear wave equations. We expect however that the complete solution will require an entirely new set of techniques than those now available.

Consider two solutions, ϕ and ψ, of the homogeneous wave equation in $\mathbb{R}^{1+n}$,

$$\square\phi = 0, \qquad \square\psi = 0, \qquad \left(\square = -\partial_t^2 + \Delta_x, \quad t \in \mathbb{R}, x \in \mathbb{R}^n\right), \tag{1}$$

subject to the initial conditions at $t = 0$,

$$\phi(0,\cdot) = \phi_0, \quad \partial_t\phi(0,\cdot) = \phi_1, \qquad \psi(0,\cdot) = \psi_0, \quad \partial_t\psi(0,\cdot) = \psi_1. \tag{2}$$

We want to investigate the space-time regularity properties of the product $\phi\psi$ in terms of the regularity of the initial data (ϕ_0,ϕ_1) and (ψ_0,ψ_1). In particular we want to find the set of exponents $\alpha_1,\alpha_2,\beta_0,\beta_+,\beta_- \in \mathbb{R}$ for which we have the estimate: [1]

$$\begin{aligned} &\left\| D^{\beta_0} D_+^{\beta_+} D_-^{\beta_-}(\phi\psi)\right\|_{L^2(\mathbb{R}^{1+n})} \\ &\lesssim \left(\|D^{\alpha_1}\phi_0\|_{L^2(\mathbb{R}^n)} + \|D^{\alpha_1-1}\phi_1\|_{L^2(\mathbb{R}^n)}\right)\left(\|D^{\alpha_2}\psi_0\|_{L^2(\mathbb{R}^n)} + \|D^{\alpha_2-1}\psi_1\|_{L^2(\mathbb{R}^n)}\right). \end{aligned} \tag{3}$$

Here, D and D_+ are homogeneous "elliptic" operators of fractional differentiation on $\mathbb{R}^n$ and $\mathbb{R}^{1+n}$, corresponding to the multipliers $|\xi|$ and $(|\tau|+|\xi|)$, with $\tau \in \mathbb{R}$ and $\xi \in \mathbb{R}^n$,

$$\widehat{D^\alpha f}(\xi) = |\xi|^\alpha \hat{f}(\xi),$$
$$\widetilde{D_+^\alpha F}(\tau,\xi) = \big(|\tau|+|\xi|\big)^\alpha \tilde{F}(\tau,\xi).$$

The operator D_-, instead, corresponds to the degenerate symbol $\big||\tau|-|\xi|\big|$, and reflects the "hyperbolic" features of the wave operator $\Box$,

$$(D_-^\alpha F)\tilde{}\,(\tau,\xi) = \big||\tau|-|\xi|\big|^\alpha \tilde{F}(\tau,\xi).$$

With the signs $\hat{}$ and $\tilde{}$ we denote the Fourier transform in $\mathbb{R}^n$ and in $\mathbb{R}^{1+n}$, respectively.

When $n=3$, the case $\phi=\psi$, $\alpha_1=\alpha_2=1/2$, $\beta_0=\beta_+=\beta_-=0$ reduces to the classical inequality of Strichartz [18],

$$\|\phi\|_{L^4(\mathbb{R}^{1+3})} \lesssim \|D^{1/2}\phi_0\|_{L^2(\mathbb{R}^3)} + \|D^{-1/2}\phi_1\|_{L^2(\mathbb{R}^3)}.$$

Bilinear homogeneous estimates similar to (3) have first appeared in [6] and formulated as estimates for null forms. That paper contains some estimates of type (3) in dimension $n=3$, corresponding to cases where $\beta_-=1/2$ or $\beta_-=1$. In [11], Klainerman and Machedon have proved the following symmetric cases of (3):

$$\begin{aligned} &n \geqslant 3, \qquad \alpha_1=\alpha_2=\frac{n-1}{4}+\frac{\beta_+}{2}, \qquad 1-\frac{n}{2}<\beta_+\leqslant 0, \qquad \beta_0=\beta_-=0,\\ &n=2, \qquad \alpha_1=\alpha_2=\frac{3}{8}+\frac{\beta_+}{2}, \qquad -\frac{1}{4}<\beta_+\leqslant 0, \qquad \beta_0=0, \qquad \beta_-=\frac{1}{4}. \end{aligned}$$

In [14], Klainerman and Selberg have obtained, and made use of, the following cases:

$$\begin{aligned} &n \geqslant 2, \qquad \alpha_1=0, \qquad \alpha_2=\frac{n}{2}, \qquad \beta_0=\beta_+=0, \qquad \beta_-=\frac{1}{2},\\ &n \geqslant 2, \qquad \alpha_1=0, \qquad \alpha_2=1, \qquad \beta_+=1-\frac{n}{2}, \qquad \beta_-=\frac{1}{2}, \qquad \beta_0=0. \end{aligned}$$

Further special cases [2] of estimates of type (3) have appeared in [8–10,13,12,15]. In this paper we pursue a systematic analysis of the estimates (3), which can be summarized in the following theorem.

THEOREM 1.1. – *Let $n \geqslant 2$. Let ϕ,ψ be the solutions of* (1), (2). *Then the estimate* (3) *holds if and only if $\alpha_1,\alpha_2,\beta_0,\beta_+,\beta_-$ satisfy the following conditions*:

[1] For an explanation of the notation see Section 2.
[2] See also the work of Beals and Bezard in [1].

$$\beta_0 + \beta_+ + \beta_- = \alpha_1 + \alpha_2 - \frac{n-1}{2}, \tag{4}$$

$$\beta_- \geqslant -\frac{n-3}{4}, \tag{5}$$

$$\beta_0 > -\frac{n-1}{2}, \tag{6}$$

$$\alpha_i \leqslant \beta_- + \frac{n-1}{2}, \quad i = 1, 2, \tag{7}$$

$$\alpha_1 + \alpha_2 \geqslant \frac{1}{2}, \tag{8}$$

$$(\alpha_i, \beta_-) \neq \left(\frac{n+1}{4}, -\frac{n-3}{4}\right), \quad i = 1, 2, \tag{9}$$

$$(\alpha_1 + \alpha_2, \beta_-) \neq \left(\frac{1}{2}, -\frac{n-3}{4}\right). \tag{10}$$

Remark 1.2. – The necessity of Eq. (4) follows easily by a straightforward scaling argument. This restricts our parameters to a four dimensional polyhedral region in the $(\alpha_1, \alpha_2, \beta_0, \beta_+, \beta_-)$ space.

Remark 1.3. – The best way to understand the structure of the conditions in Theorem 1.1 is to start by fixing the values of β_- and β_0 in the range allowed by (5) and (6). With these values fixed, the conditions (7) and (8) constrain the pair (α_1, α_2) to stay inside the triangle determined by the vertices

$$A_0 = \left(\beta_- + \frac{n-1}{2}, \beta_- + \frac{n-1}{2}\right),$$
$$A_1 = \left(\beta_- + \frac{n-1}{2}, -\beta_- - \frac{n-2}{2}\right),$$
$$A_2 = \left(-\beta_- - \frac{n-2}{2}, \beta_- + \frac{n-1}{2}\right).$$

Finally, for (α_1, α_2) fixed, we determine β_+ from the scaling condition (4). The sides of the triangle are allowed, except in the most critical case when we have equality in (5), $\beta_- = -(n-3)/4$. In that case the sides are entirely forbidden by the conditions (9) and (10).

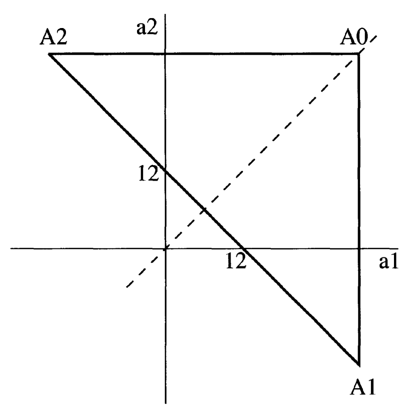

Fig. 1. Allowed region for α_1, α_2.

Remark 1.4. – In applications to nonlinear wave equations the most useful cases seem to be those with $\beta_- = 0$ or $\beta_- = 1/2$. These appear in connection with nonlinear equations which contain null bilinear forms [7–10,12,14,15].

In particular, when $n = 3$ the value $\beta_- = 0$ is critical and the range for the parameters α_1, α_2 is restricted to the interior of the triangle,

$$-\frac{1}{2} < \alpha_1, \alpha_2 < 1, \quad \alpha_1 + \alpha_2 > \frac{1}{2}.$$

In the case $\beta_- = 1/2$ the sides of the corresponding triangle are also included,

$$-1 \leqslant \alpha_1, \alpha_2 \leqslant \frac{3}{2}, \quad \alpha_1 + \alpha_2 \geqslant \frac{1}{2}.$$

Notice that for $n = 2$ the value $\beta_- = 0$ is ruled out, we need in fact $\beta_- \geqslant 1/4 > 0$. This is related to the fact that in the case of the classical Strichartz inequality,

$$\|\phi^2\|_{L^3(\mathbb{R}^{1+2})} \lesssim \left(\|D^{1/2}\phi_0\|_{L^2(\mathbb{R}^2)} + \|D^{-1/2}\phi_1\|_{L^2(\mathbb{R}^2)}\right)^2,$$

the L^3 norm is optimal, i.e. it cannot be replaced by a L^p norm with $p < 3$.

On the other hand, estimates with negative values for β_- are allowed when $n > 3$, they have not appeared in the previous literature. They are important to treat generic nonlinear wave equations whose nonlinear terms contain derivatives. [3] Consider for example the equation

$$\square u = |Du|^2, \tag{11}$$

with initial data in H^s. Our estimates give a precise description of the regularity of the corresponding first iterate u_1, which satisfies the equation $\square u_1 = |Du_0|^2$, where u_0 solves the homogeneos wave equation with data in H^s. Indeed, for $\theta > 1/2$ we have

$$\|u_1\|_{L_t^\infty H^s} \lesssim \|D_+^s D_-^\theta u_1\|_{L^2} \lesssim \left\|D_+^{s-1} D_-^{\theta-1}(Du_0)^2\right\|_{L^2},$$

and we can use our bilinear estimates with $\beta_- = \theta - 1$, $\beta_+ = s - 1$, $\alpha_1 = \alpha_2 \leqslant s - 1$, to control the first estimate u_1 in terms of the initial data, whenever $s - n/2 \geqslant \theta - 1/2 > 0$ and $\theta \geqslant -(n-7)/4$. This suggests that the correct range for the wellposedness of (11) should be [4] $s > \max\{n/2, (n+5)/4\}$.

The paper is organized as follows:

Section 3 contains some preliminary remarks concerning the L^2 theory. Section 4 contains some simple lemmas which are repeatedly used in the following sections.

In Section 5 we discuss the basic counterexamples, which show the necessity of the conditions (5)–(8). In Section 6 we provide more refined counterexamples in frequency space to justify the need for the exceptions (9), (10) and the strict inequality in (6).

The proof of the L^2 estimates is broken down into several cases according to different types of interactions of solutions, this is done in Sections 7–11. The techniques used here refine, and at the same time simplify, those contained in [6,11] and [14]. In Section 12 we discuss the dyadic version of these estimates. The importance of these lies in the fact that they hold even in those

[3] And don't satisfy the null condition.

[4] Recently this was proved by Tataru [22].

exceptional cases excluded by the conditions (9) and (10). On the other hand they imply most of the cases covered by Theorem 1.1, with the exception of some limiting cases. Such estimates have recently appeared also in [21]. In Section 13 we will apply Theorem 1.1 to study L^2 estimates for null forms.

In Section 14 we take on the question of the existence of similar estimates in L^p spaces. We find a set of inequalities for the indices $q, r, \alpha_1, \alpha_2, \beta_0, \beta_+, \beta_- \in \mathbb{R}$, which are necessary for the validity of estimates of the type,

$$\begin{aligned} &\left\|D^{\beta_0} D_+^{\beta_+} D_-^{\beta_-}(\phi\psi)\right\|_{L_t^q L_x^r} \\ &\quad\lesssim \left(\|D^{\alpha_1}\phi_0\|_{L^2} + \|D^{\alpha_1-1}\phi_1\|_{L^2}\right)\left(\|D^{\alpha_2}\psi_0\|_{L^2} + \|D^{\alpha_2-1}\psi_1\|_{L^2}\right), \end{aligned} \tag{12}$$

and raise the question whether these condition are also sufficients (modulo borderline cases). These lead us to some specific conjectures concerning estimates for null quadratic forms. [5] The special case of (12) with $q = \infty$ and $r = 2$ is relatively easy and is treated in Sections 15 and 16.

Finally, for completeness, we conclude the paper with a section on the so called bilinear restriction conjecture, [6] which generalize the restriction theorem of Stein and Tomas [23]. As the restriction theorem is intimately tied to the Strichartz inequalities, in a similar fashion we expect that the solution of the bilinear restriction conjecture, which is much easier to formulate, will shed light on the above mentioned bilinear conjectures for the wave equation.

2. Notation

To simplify the expression of our inequalities, we will use the symbols $\lesssim, \simeq, \gtrsim$ to denote relations $\leqslant, =, \geqslant$ up to a multiplicative constant, which may depend on $n, \alpha_1, \alpha_2, \beta_0, \beta_+, \beta_-$, but not on the initial data ϕ_0, ψ_0 (or the L^2 functions f, g). Also, if $X \lesssim Y$ and $X \gtrsim Y$ we will write $X \approx Y$. If in the inequality $\lesssim$ the multiplicative constant is, or can be, much smaller than 1 then we use the symbols $\ll$; similarly, if in $\gtrsim$ the constant is, or can be, much greater than 1 then we use $\gg$.

By $\mathbb{S}^k$ we denote the k-dimensional unit sphere, canonically imbedded in $\mathbb{R}^{k+1}$, and by $\mathrm{d}S$ its standard volume element.

Fourier transforms on $\mathbb{R}^n$ and $\mathbb{R}^{1+n}$ are denoted by $\hat{\cdot}$ and $\tilde{\cdot}$:

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{ix\cdot\xi} f(x)\,\mathrm{d}x, \qquad \tilde{F}(\tau,\xi) = \int_{\mathbb{R}^{1+n}} e^{i(t\tau + x\cdot\xi)} F(t,x)\,\mathrm{d}x\,\mathrm{d}t.$$

The L^p norms are defined in the usual way,

$$\|f\|_{L_x^p} = \left(\int_{\mathbb{R}^n} |f(x)|^p \mathrm{d}x\right)^{1/p}, \quad 1 \leqslant p < \infty,$$

and since we mostly deal with L^2 theory, we will often suppress the subscript and simply write $\|f\| = \|f\|_{L^2}$. We also write $\langle f, g\rangle = \int f(x)\overline{g(x)}\,\mathrm{d}x$ for the inner product on L^2, while with $x \cdot y$ we denote the standard scalar product of vectors in $\mathbb{R}^n$.

[5] Some particular cases of these conjectures were first considered in [6]. See Section 14 for precise references.

[6] The conjecture, which surfaced in discussions between Klainerman and Machedon many years ago, was first announced by Klainerman at a conference on Harmonic Analysis and Applications to PDE's at MSRI, Berkeley, July 1997.

By $L_t^q L_x^r$ we denote the Lebesque space with mixed exponents defined through the norm

$$\|F\|_{L_t^q L_x^r} = \left(\int_{-\infty}^{+\infty} \left(\int_{\mathbb{R}^n} |F(t,x)|^r \, \mathrm{d}x \right)^{q/r} \mathrm{d}t \right)^{1/q}.$$

With $\Re(z)$ and $\Im(z)$ we denote the real and imaginary parts of the complex number z.

3. Preliminaries

The solutions to (1) and (2) can be decomposed into their $(+)$ and $(-)$ parts: $\phi = \phi^+ + \phi^-$, where

$$\widetilde{\phi^\pm}(\tau,\xi) \simeq \delta\bigl(\tau \mp |\xi|\bigr)\widehat{\phi_0^\pm}(\xi),$$

where $\phi_0^\pm$ are linear combinations of ϕ_0 and $D^{-1}\phi_1$, and similarly for ψ. The product then decomposes into four pieces:

$$\phi\psi = \phi^+\psi^+ + \phi^+\psi^- + \phi^-\psi^+ + \phi^-\psi^-.$$

By symmetry, it is enough to prove the estimate only for the $(+\,+)$ and $(+\,-)$ cases, since the $(-\,-)$ becomes $(+\,+)$ reversing the direction of time and $(-\,+)$ becomes $(+\,-)$ exchanging ϕ_0 with ψ_0.

Fourier transforms of products become convolutions, and we have:

$$\widetilde{\phi^+\psi^+}(\tau,\xi) \simeq \int \delta\bigl(\tau - |\eta| - |\xi-\eta|\bigr)\widehat{\phi_0^+}(\eta)\widehat{\psi_0^+}(\xi-\eta)\,\mathrm{d}\eta, \tag{13}$$

$$\widetilde{\phi^+\psi^-}(\tau,\xi) \simeq \int \delta\bigl(\tau - |\eta| + |\xi-\eta|\bigr)\widehat{\phi_0^+}(\eta)\widehat{\psi_0^-}(\xi-\eta)\,\mathrm{d}\eta. \tag{14}$$

The two integrals look similar but have different behaviors: (13) is an integration over the ellipsoid of revolution with foci at 0 and ξ,

$$\mathcal{E}(\tau,\xi) = \bigl\{\eta \in \mathbb{R}^n\colon\ |\eta| + |\xi-\eta| = \tau\bigr\}, \tag{15}$$

which is a compact manifold; (14) is an integration over the hyperboloid of revolution with foci at 0 and ξ,

$$\mathcal{H}(\tau,\xi) = \bigl\{\eta \in \mathbb{R}^n\colon\ |\eta| - |\xi-\eta| = \tau\bigr\}, \tag{16}$$

which is an unbounded manifold with infinite volume. Also, notice that $\widetilde{\phi^+\psi^+}$ is supported on the region $\tau \geqslant |\xi|$, while $\widetilde{\phi^+\psi^-}$ is supported on the region $|\tau| \leqslant |\xi|$.

Remark 3.1. – The delta functions in the integrals (13) and (14) can be viewed as the pull-backs of standard delta distributions, or equivalently as measures supported on hypersurfaces (see [4, Theorem 6.1.5]). Let S be the hypersurface defined by $\phi(x) = 0$, where ϕ is a smooth function with $\nabla\phi(x) \neq 0$ for $x \in S \cap \operatorname{supp} f$, and denote by $\mathrm{d}S_x$ the induced area element on S, then we have

$$\int f(x)\delta\bigl(\phi(x)\bigr)\,\mathrm{d}x = \int_S f(x)\frac{\mathrm{d}S_x}{|\nabla\phi(x)|}. \tag{17}$$

Observe also that if g is a smooth function which doesn't vanish on S then, as a consequence of (17), we have

$$\delta\big(\phi(x)\big) = g(x)\delta\big(g(x)\phi(x)\big). \tag{18}$$

By Plancherel's theorem, the main estimate (3) will follow from the following estimates in frequency space,

$$\big\| |\xi|^{\beta_0}\big(|\tau| + |\xi|\big)^{\beta_+} \big||\tau| - |\xi|\big|^{\beta_-} \widetilde{\phi^+\psi^\pm}(\tau,\xi)\big\|_{L^2_{\tau,\xi}} \lesssim \big\| |\eta|^{\alpha_1}\widehat{\phi}_0(\eta)\big\|_{L^2_\eta} \big\| |\zeta|^{\alpha_2}\widehat{\psi}_0(\zeta)\big\|_{L^2_\zeta},$$

which are equivalent to the L^2 boundedness of the bilinear operators

$$B_{(++)}, B_{(+-)} : L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^{1+n}),$$

defined as the inverse Fourier transforms of

$$\tilde{B}_{(++)}(f,g)(\tau,\xi) = \int \delta\big(\tau - |\eta| - |\xi - \eta|\big) \frac{|\xi|^{\beta_0}\tau^{\beta_+}(\tau - |\xi|)^{\beta_-}}{|\eta|^{\alpha_1}|\xi - \eta|^{\alpha_2}} f(\eta)g(\xi - \eta)\,\mathrm{d}\eta, \tag{19}$$

$$\tilde{B}_{(+-)}(f,g)(\tau,\xi) = \int \delta\big(\tau - |\eta| + |\xi - \eta|\big) \frac{|\xi|^{\beta_0+\beta_+}(|\xi| - |\tau|)^{\beta_-}}{|\eta|^{\alpha_1}|\xi - \eta|^{\alpha_2}} f(\eta)g(\xi - \eta)\,\mathrm{d}\eta. \tag{20}$$

Remark 3.2. – Since these are positive operators, in the sense that $\tilde{B}_{(+\pm)}(f,g) \geqslant 0$ when $f \geqslant 0$ and $g \geqslant 0$, we can always assume, without loss of generality that $f \geqslant 0$ and $g \geqslant 0$ are non-negative functions. This will simplify the notation and we won't have to worry about absolute values.

To construct counterexamples it will be useful to consider also the representation of the operators $B_{(+\pm)}$ in physical space:

$$B_{(+\pm)}(f,g)(t,x) \simeq \iint e^{i\varphi_\pm(t,x;\eta,\zeta)} W_\pm(\eta,\zeta) f(\eta) g(\zeta)\,\mathrm{d}\eta\,\mathrm{d}\zeta, \tag{21}$$

$$\varphi_\pm(t,x;\eta,\zeta) = t\big(|\eta| \pm |\zeta|\big) + x\cdot(\eta + \zeta), \tag{22}$$

$$W_+(\eta,\zeta) = \frac{|\eta+\zeta|^{\beta_0}(|\eta| + |\zeta|)^{\beta_+}(|\eta| + |\zeta| - |\eta+\zeta|)^{\beta_-}}{|\eta|^{\alpha_1}|\xi - \eta|^{\alpha_2}}, \tag{23}$$

$$W_-(\eta,\zeta) = \frac{|\eta+\zeta|^{\beta_0+\beta_+}(|\eta+\zeta| - ||\eta| - |\zeta||)^{\beta_-}}{|\eta|^{\alpha_1}|\xi - \eta|^{\alpha_2}}. \tag{24}$$

4. Integration on ellipsoids and hyperboloids

In this section we collect various results about the geometry of the ellipsoids and hyperboloids defined in (15) and (16), which will be needed in the sequel.

To deal with integrations over the ellipsoid $\mathcal{E}(\tau,\xi)$ we introduce a convenient parameterization.

LEMMA 4.1. – *Consider the integral*

$$I(F)(\tau,\xi) = \int \delta\big(\tau - |\eta| - |\xi - \eta|\big) F\big(|\eta|, |\xi - \eta|\big)\,\mathrm{d}\eta,$$

defined in the space-time region $\tau \geqslant |\xi|$, then

$$(25)\quad I(F)(\tau,\xi) \simeq \left(\tau^2 - |\xi|^2\right)^{\frac{n-3}{2}} \int_{-1}^{1} F\left(\frac{\tau+|\xi|x}{2}, \frac{\tau-|\xi|x}{2}\right)\left(\tau^2 - |\xi|^2 x^2\right)\left(1-x^2\right)^{\frac{n-3}{2}} \mathrm{d}x.$$

Proof. – Using formula (18) we can multiply the argument of the delta function by the quantity $\tau - |\eta| + |\xi - \eta|$,

$$\begin{aligned}\delta\left(\tau - |\eta| - |\xi-\eta|\right) &= \left(\tau - |\eta| + |\xi-\eta|\right)\delta\left((\tau-|\eta|)^2 - |\xi-\eta|^2\right)\\ &= 2\left(\tau - |\eta|\right)\delta\left(\tau^2 - |\xi|^2 - 2\tau|\eta| + 2\xi\cdot\eta\right).\end{aligned}$$

Introduce polar coordinate for η, $\rho = |\eta|$, $\omega = \eta/|\eta|$, then $\mathrm{d}\eta = \rho^{n-1}\,\mathrm{d}S_\omega\,\mathrm{d}\rho$; set also the cosine $a = \omega\cdot\xi/|\xi|$, then $\mathrm{d}S_\omega = (1-a^2)^{\frac{n-3}{2}}\mathrm{d}S_{\omega'}\mathrm{d}a$. With these transformations our integral becomes

$$I(F)(\tau,\xi) \simeq \int_0^\infty \int_{-1}^{1} F(\rho, \tau-\rho)\delta\left(\tau^2 - |\xi|^2 - 2\tau\rho + 2|\xi|\rho a\right)(\tau-\rho)\rho^{n-1}(1-a^2)^{\frac{n-3}{2}}\,\mathrm{d}a\,\mathrm{d}\rho.$$

We use the delta function to set the value of a to

$$(26)\qquad a = -\frac{\tau^2 - |\xi|^2 - 2\tau\rho}{2|\xi|\rho},$$

with the condition $-1 \leqslant a \leqslant 1$ that forces $(\tau - |\xi|)/2 \leqslant \rho \leqslant (\tau + |\xi|)/2$,

$$I(F)(\tau,\xi) \simeq \frac{1}{|\xi|}\int_{\frac{\tau-|\xi|}{2}}^{\frac{\tau+|\xi|}{2}} F(\rho,\tau-\rho)(\tau-\rho)\rho^{n-2}\left[1 - \left(\frac{\tau^2-|\xi|^2-2\tau\rho}{2|\xi|\rho}\right)^2\right]^{\frac{n-3}{2}}\mathrm{d}\rho.$$

With a little bit of algebraic manipulation we see that this is

$$I(F)(\tau,\xi) \simeq \frac{(\tau^2-|\xi|^2)^{\frac{n-3}{2}}}{|\xi|^{n-2}}\int_{\frac{\tau-|\xi|}{2}}^{\frac{\tau+|\xi|}{2}} F(\rho,\tau-\rho)(\tau-\rho)\rho\left[\left(\frac{\tau+|\xi|}{2} - \rho\right)\left(\rho - \frac{\tau-|\xi|}{2}\right)\right]^{\frac{n-3}{2}}\mathrm{d}\rho.$$

As a last step, performing the change of variable $\rho \mapsto x = (2\rho - \tau)/|\xi|$ we reduce the integral to the form (25). □

As a byproduct of this proof, notice that inverting formula (26) we have the following polar coordinate representation for $\eta \in \mathcal{E}(\tau,\xi)$:

$$(27)\qquad \rho = |\eta| = \frac{\tau^2 - |\xi|^2}{2(\tau - \xi\cdot\omega)} \in \left[\frac{\tau-|\xi|}{2}, \frac{\tau+|\xi|}{2}\right].$$

LEMMA 4.2. – *Let $a \in \mathbb{R}$ and $m > -1$. Define*

$$H_m^a(\lambda) = \int_0^1 (\lambda+t)^a t^m\,\mathrm{d}t = \lambda^{a+m+1}\int_0^{1/\lambda}(1+s)^a s^m\,\mathrm{d}s$$

for $\lambda > 0$. Then

$$H_m^a(\lambda) \approx \begin{cases} \lambda^a, & \text{as } \lambda \to \infty; \\ \lambda^{\min(a+m+1,0)}, & \text{as } \lambda \to 0, \text{ if } a+m+1 \neq 0; \\ |\log \lambda|, & \text{as } \lambda \to 0, \text{ if } a+m+1 = 0. \end{cases}$$

In particular, if $a \leqslant b$ then $H_m^a(\lambda) \gtrsim H_m^b(\lambda)$ as $\lambda \to 0$.

Now we are ready to determine the precise asymptotic behaviour of the integrals on $\mathcal{E}(\tau,\xi)$ as $\tau/|\xi| \to 1$, which we will be needed in the following sections.

PROPOSITION 4.3. – *Let $a, b \in \mathbb{R}$, and $\tau > |\xi|$. Define the integral*

$$I(\tau,\xi) = \int \frac{\delta(\tau - |\eta| - |\xi - \eta|)}{|\eta|^a |\xi - \eta|^b} \, \mathrm{d}\eta.$$

We have the following estimate for I:

$$I(\tau,\xi) \approx \tau^A \big(\tau - |\xi|\big)^B, \tag{28}$$

where

$$A = \max\left\{a, b, \frac{n+1}{2}\right\} - a - b, \qquad B = n - 1 - \max\left\{a, b, \frac{n+1}{2}\right\},$$

except when $\max\{a,b\} = \frac{n+1}{2}$, in which case we have

$$I(\tau,\xi) \approx \tau^{-\min\{a,b\}} \big(\tau - |\xi|\big)^{\frac{n-3}{2}} \left(1 + \log\left(\frac{\tau}{\tau - |\xi|}\right)\right). \tag{29}$$

Proof. – We use Lemma 4.1 with $F(s,t) = s^{-a} t^{-b}$ and we reduce to

$$I(\tau,\xi) \simeq |\xi|^{2-a-b} \big(\tau^2 - |\xi|^2\big)^{\frac{n-3}{2}} \int_{-1}^{1} \left(\frac{\tau}{|\xi|} + x\right)^{1-a} \left(\frac{\tau}{|\xi|} - x\right)^{1-b} \big(1 - x^2\big)^{\frac{n-3}{2}} \, \mathrm{d}x.$$

Split the integration in two pieces, $\int_{-1}^{1} = \int_{-1}^{0} + \int_{0}^{1}$. On the interval $-1 \leqslant x \leqslant 0$, set $t = 1 + x$ and use

$$\frac{\tau}{|\xi|} + x \approx \left(\frac{\tau}{|\xi|} - 1\right) + t, \qquad \frac{\tau}{|\xi|} - x \approx \frac{\tau}{|\xi|}, \qquad 1 - x^2 \approx t;$$

while on $0 \leqslant x \leqslant 1$, set $t = 1 - x$ and use

$$\frac{\tau}{|\xi|} + x \approx \frac{\tau}{|\xi|}, \qquad \frac{\tau}{|\xi|} - x \approx \left(\frac{\tau}{|\xi|} - 1\right) + t, \qquad 1 - x^2 \approx t.$$

We obtain

$$\begin{aligned} I(\tau,\xi) \approx{}& |\xi|^{2-a-b} \big(\tau^2 - |\xi|^2\big)^{\frac{n-3}{2}} \\ &\times \left[\left(\frac{\tau}{|\xi|}\right)^{1-a} H^{1-b}_{\frac{n-3}{2}}\left(\frac{\tau}{|\xi|} - 1\right) + \left(\frac{\tau}{|\xi|}\right)^{1-b} H^{1-a}_{\frac{n-3}{2}}\left(\frac{\tau}{|\xi|} - 1\right)\right], \end{aligned}$$

and to conclude use Lemma 4.2 with $\lambda = (\tau/|\xi|) - 1$. Indeed, when $\tau/|\xi|$ is large we have

$$I(\tau,\xi) \approx |\xi|^{2-a-b}\left(\tau^2 - |\xi|^2\right)^{\frac{n-3}{2}} \left(\frac{\tau}{|\xi|}\right)^{1-a+1-b} \approx \tau^{n-1-a-b};$$

while when $1 \leqslant \tau/|\xi| \leqslant 2$, suppose $\max\{a,b\} = a$, we deduce that

$$\begin{aligned} I(\tau,\xi) &\approx |\xi|^{2-a-b}\left(\tau^2 - |\xi|^2\right)^{\frac{n-3}{2}} \left[H^{1-b}_{\frac{n-3}{2}}(\lambda) + H^{1-a}_{\frac{n-3}{2}}(\lambda)\right] \\ &\approx \tau^{\frac{n+1}{2}-a-b}(\tau - |\xi|)^{\frac{n-3}{2}} H^{1-a}_{\frac{n-3}{2}}\left(\frac{\tau - |\xi|}{\tau}\right). \end{aligned}$$

The behavior of the function H is then determined by the sign of the quantity

$$(1-a) + \frac{n-3}{2} + 1 = \frac{n+1}{2} - a,$$

according to the statement of Lemma 4.2. □

We can try to prove a similar proposition for integrations over the hyperboloids $\mathcal{H}(\tau,\xi)$.

LEMMA 4.4. – *Consider the integral*

$$I(F)(\tau,\xi) = \int \delta\left(\tau - |\eta| + |\xi - \eta|\right) F\left(|\eta|, |\xi - \eta|\right) \mathrm{d}\eta,$$

defined in the space-time region $|\tau| \leqslant |\xi|$, *then*

$$I(F)(\tau,\xi) \simeq \left(|\xi|^2 - \tau^2\right)^{\frac{n-3}{2}} \int_1^\infty F\left(\frac{|\xi|x + \tau}{2}, \frac{|\xi|x - \tau}{2}\right)\left(|\xi|^2 x^2 - \tau^2\right)\left(x^2 - 1\right)^{\frac{n-3}{2}} \mathrm{d}x. \tag{30}$$

Proof. – The proof follows precisely the same steps as in the proof of Lemma 4.1. We only observe that Eq. (26) remains unchanged,

$$a = \frac{|\xi|^2 - \tau^2 + 2\tau\rho}{2|\xi|\rho}, \tag{31}$$

except that now we have the restriction $\tau/|\xi| \leqslant a \leqslant 1$. □

Inverting (31) we obtain the polar coordinate representation for $\eta \in \mathcal{H}(\tau,\xi)$:

$$\rho = |\eta| = \frac{|\xi|^2 - \tau^2}{2(-\tau + \xi \cdot \omega)} \geqslant \frac{|\xi| + \tau}{2}. \tag{32}$$

The analog of Proposition 4.3 for $\mathcal{H}(\tau,\xi)$ is possible only if we restrict the integration on the "elliptic" portion of the hyperboloid. More precisely, if $\eta \in \mathcal{H}(\tau,\xi)$ is in the region where $|\eta| \ll |\xi|$ then the geometry of $\mathcal{H}(\tau,\xi)$ near η, in terms of curvature, is not too different from that of an ellipsoid.

PROPOSITION 4.5. – *Let* $a, b \in \mathbb{R}$ *and* $|\tau| < |\xi|$. *Define the integral*

$$I(\tau,\xi) = \int\limits_{|\eta| + |\xi - \eta| \leqslant 2|\xi|} \frac{\delta(\tau - |\eta| + |\xi - \eta|)}{|\eta|^a |\xi - \eta|^b} \mathrm{d}\eta.$$

We have the following estimates for I: in the region where $0 \leqslant \tau \leqslant |\xi|$,

$$I(\tau,\xi) \approx |\xi|^A \big(|\xi| - \tau\big)^B, \tag{33}$$

where

$$A = \max\left\{b, \frac{n+1}{2}\right\} - a - b, \qquad B = n - 1 - \max\left\{b, \frac{n+1}{2}\right\},$$

except when $b = (n+1)/2$, in which case we have

$$I(\tau,\xi) \approx |\xi|^{-a} \big(|\xi| - \tau\big)^{\frac{n-3}{2}} \left(1 + \log\left(\frac{|\xi|}{|\xi| - \tau}\right)\right); \tag{34}$$

similarly, in the region $-|\xi| < \tau \leqslant 0$,

$$I(\tau,\xi) \approx |\xi|^A \big(|\xi| + \tau\big)^B,$$

where

$$A = \max\left\{a, \frac{n+1}{2}\right\} - a - b, \qquad B = n - 1 - \max\left\{a, \frac{n+1}{2}\right\},$$

except when $a = (n+1)/2$, in which case we have

$$I(\tau,\xi) \approx |\xi|^{-b} \big(|\xi| + \tau\big)^{\frac{n-3}{2}} \left(1 + \log\left(\frac{|\xi|}{|\xi| + \tau}\right)\right). \tag{35}$$

Proof. – We use Lemma 4.4 with $F(s,t) = s^{-a}t^{-b}$ and we reduce to

$$I(\tau,\xi) \simeq |\xi|^{2-a-b} \big(|\xi|^2 - \tau^2\big)^{\frac{n-3}{2}} \int_1^2 \left(x + \frac{\tau}{|\xi|}\right)^{1-a} \left(x - \frac{\tau}{|\xi|}\right)^{1-b} \big(x^2 - 1\big)^{\frac{n-3}{2}} \,\mathrm{d}x.$$

Let's assume $\tau \geqslant 0$. Set $t = x - 1$ and use

$$x + \frac{\tau}{|\xi|} \approx 1, \qquad x - \frac{\tau}{|\xi|} = \left(1 - \frac{\tau}{|\xi|}\right) + t, \qquad x^2 - 1 \approx t.$$

We obtain

$$I(\tau,\xi) \approx |\xi|^{2-a-b} \big(|\xi|^2 - \tau^2\big)^{\frac{n-3}{2}} H^{1-b}_{\frac{n-3}{2}} \left(1 - \frac{\tau}{|\xi|}\right),$$

and to conclude use Lemma 4.2. $\square$

5. Basic examples

Throughout this section L will denote a large positive parameter, $L > 1$. If $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$ is a generic point, we will use the notations $\xi' = (\xi_2, \dots, \xi_n) \in \mathbb{R}^{n-1}$ and $\xi'' = (\xi_3, \dots, \xi_n) \in \mathbb{R}^{n-2}$ (if $n = 2$ then $\xi' = \xi_2$ and $\xi'' = \emptyset$).

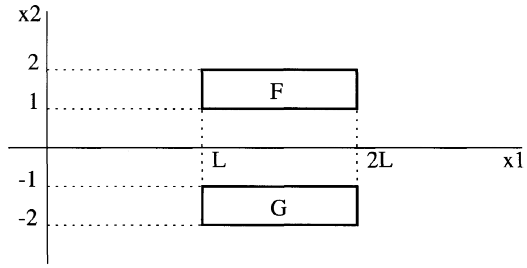

Fig. 2. Example 5.1.

The basic idea behind the examples below is to choose appropriate sets F and G in $\mathbb{R}^n$, take as data $f=\chi_F$ and $g=\chi_G$ their characteristic functions, and then restrict $B_{(++)}$, $B_{(+-)}$, with the representation given by (21), to the largest possible sets in t,x for which the corresponding exponential factors are essentially constant.

Example 5.1 (*Necessity of* (5)). – We first check the estimates (3) by testing the boundedness of $B_{(++)}$ when the data f and g concentrate along the same direction and in the same frequency scale. We consider the function $B=B_{(++)}(\chi_F,\chi_G)$, where F and G are the sets (see Fig. 2):

$$F=\{\eta\colon\ L<\eta_1<2L,\ 1<\eta_2<2,\ |\eta''|<1\},$$
$$G=\{\zeta\colon\ L<\zeta_1<2L,\ -2<\zeta_2<-1,\ |\zeta''|<1\}.$$

Take $\eta\in F$, $\zeta\in G$ and let θ be the angle between η and ζ, then we have

$$|\eta|\approx|\zeta|\approx|\eta+\zeta|\approx|\eta|+|\zeta|\approx L,\qquad \theta\approx L^{-1},$$
$$|\eta|+|\zeta|-|\eta+\zeta|\approx\frac{|\eta||\zeta|}{|\eta|+|\zeta|}\theta^2\approx L^{-1},$$
$$|\eta|-\eta_1\approx\frac{|\eta'|^2}{|\eta|}\approx L^{-1},\qquad |\zeta|-\zeta_1\approx\frac{|\zeta'|^2}{|\zeta|}\approx L^{-1},$$
$$\eta_1+\zeta_1\approx L,\qquad |\eta'+\zeta'|\lesssim 1.$$

From (21) we have

$$B(t,x)=\int\limits_F\int\limits_G e^{i\varphi_+(t,x;\eta,\zeta)}W_+(\eta,\zeta)\,\mathrm{d}\eta\,\mathrm{d}\zeta.$$

The weight W_+ given by (23) is then of order

$$W_+\approx\frac{L^{\beta_0}L^{\beta_+}L^{-\beta_-}}{L^{\alpha_1}L^{\alpha_2}}=L^{\beta_0+\beta_+-\beta_--\alpha_1-\alpha_2}=L^{-2\beta_--\frac{n-1}{2}}.$$

We write the phase function (22) as

$$\begin{aligned}\varphi_+&=t(|\eta|+|\zeta|)+x\cdot(\eta+\zeta)\\&=t(|\eta|-\eta_1+|\zeta|-\zeta_1)+(t+x_1)(\eta_1+\zeta_1)+x'\cdot(\eta'+\zeta')\\&=t\,\mathrm{O}(L^{-1})+(t+x_1)\,\mathrm{O}(L)+x'\cdot\mathrm{O}(1).\end{aligned}$$

It is then possible to choose a region R in $\mathbb{R}^{1+n}$, defined by the conditions

$$|t|\lesssim L,\qquad |t+x_1|\lesssim L^{-1},\qquad |x'|\lesssim 1,$$

such that we have $|\varphi_+| < \pi/3$, when $\eta \in F$, $\zeta \in G$ and $(t,x) \in R$. Therefore we can neglect the oscillating factor, since in that case $|e^{i\varphi_+} - 1| < 1/2$ and $\Re(e^{i\phi_+}) \geqslant 1/2$. Thus we have

$$\left|B(t,x)\right| \geqslant \Re\left(B(t,x)\right) \geqslant \frac{1}{2} \int\limits_F \int\limits_G W_+(\eta,\zeta)\,\mathrm{d}\eta\,\mathrm{d}\zeta \gtrsim L^{-2\beta_- - \frac{n-1}{2}} |F||G|,$$

whenever $(t,x) \in R$. Consequently

$$\frac{\|B\|}{\|\chi_F\|\|\chi_G\|} \gtrsim L^{-2\beta_- - \frac{n-1}{2}} |F|^{1/2}|G|^{1/2}|R|^{1/2} = L^{-2\beta_- - \frac{n-1}{2}} L.$$

We let $L \to \infty$ and therefore, to have the desired inequality we need,

$$-2\beta_- - \frac{n-1}{2} + 1 \leqslant 0,$$

which is equivalent to (5).

Example 5.2 (*Necessity of* (7)). – We still look at the $(++)$ case, but this time we stretch only one of the data along one direction, keeping the other fixed. We consider the function $B = B_{(++)}(\chi_F, \chi_G)$, where now F and G are defined as follow (see Fig. 3):

$$F = \left\{\eta\colon\ L < \eta_1 < 2L,\ 1 < \eta_2 < 2,\ |\eta''| < 1\right\},$$
$$G = \left\{\zeta\colon\ 1 < \zeta_1 < 2,\ -2 < \zeta_2 < -1,\ |\zeta''| < 1\right\}.$$

Take $\eta \in F$, $\zeta \in G$ and let θ be the angle between η and ζ, then we have

$$\begin{aligned}
&|\eta| \approx |\eta + \zeta| \approx |\eta| + |\zeta| \approx L, \qquad |\zeta| \approx 1, \qquad \theta \approx 1,\\
&|\eta| + |\zeta| - |\eta + \zeta| \approx \frac{|\eta||\zeta|}{|\eta| + |\zeta|}\theta^2 \approx 1,\\
&|\eta| - \eta_1 \approx \frac{|\eta'|^2}{|\eta|} \approx L^{-1}, \qquad |\zeta| - \zeta_1 \approx \frac{|\zeta'|^2}{|\zeta|} \approx 1,\\
&\eta_1 + \zeta_1 \approx L, \qquad |\eta' + \zeta'| \lesssim 1.
\end{aligned}$$

The weight W_+ is then of order

$$L^{\beta_0 + \beta_+ - \alpha_1} = L^{\alpha_2 - \beta_- - \frac{n-1}{2}}.$$

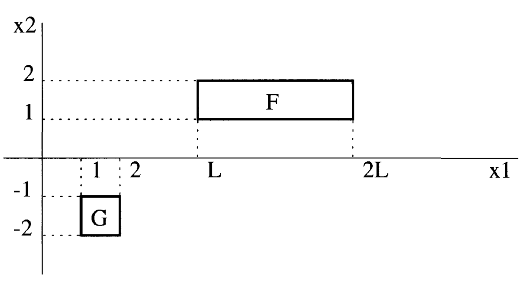

Fig. 3. Example 5.2.

We write the phase function as

$$\begin{aligned}\varphi_+ &= t\big(|\eta|+|\zeta|\big)+x\cdot(\eta+\zeta)\\ &= t\big(|\eta|-\eta_1+|\zeta|-\zeta_1\big)+(t+x_1)(\eta_1+\zeta_1)+x'\cdot(\eta'+\zeta')\\ &= t\,\mathrm{O}(1)+(t+x_1)\,\mathrm{O}(L)+x'\cdot\mathrm{O}(1).\end{aligned}$$

It is then possible to choose a region R in $\mathbb{R}^{1+n}$, defined by

$$|t|\lesssim 1, \qquad |t+x_1|\lesssim L^{-1}, \qquad |x'|\lesssim 1,$$

on which, as in the previous example, we can make the oscillating factor $e^{i\varphi_+}$ as close to 1 as we want. Thus we have

$$\big|B(t,x)\big|\gtrsim L^{\alpha_2-\beta_--\frac{n-1}{2}}|F||G|,$$

whenever $(t,x)\in R$. Consequently

$$\frac{\|B\|}{\|\chi_F\|\|\chi_G\|}\gtrsim L^{\alpha_2-\beta_--\frac{n-1}{2}}|F|^{1/2}|G|^{1/2}|R|^{1/2}\approx L^{\alpha_2-\beta_--\frac{n-1}{2}}.$$

Taking the limit $L\to\infty$, we find the necessary condition,

$$\alpha_2-\beta_--\frac{n-1}{2}\leqslant 0.$$

Similarly, exchanging the role of F and G we get

$$\alpha_1-\beta_--\frac{n-1}{2}\leqslant 0.$$

Hence, we obtain the necessity of condition (7).

Remark 5.3. – Condition (7) follows also by considering a slightly different scaling, with F obtained by a parabolic rescaling (instead of a linear rescaling along one dimension) and G still fixed:

$$\begin{aligned}F&=\big\{\eta\colon\ L<\eta_1<2L,\ \sqrt{L}<\eta_2<2\sqrt{L},\ |\eta''|<1\big\},\\ G&=\big\{\zeta\colon\ 1<\zeta_1<2,\ -2<\zeta_2<-1,\ |\zeta''|<1\big\}.\end{aligned}$$

This example implies the same condition in the contest of L^2 norms, but will provide different information when we shall consider the L^p theory later on.

Example 5.4 (*Necessity of* (6) *with* $\geqslant$). – Again look at the $(++)$ case and this time consider the interaction of data supported in opposite directions. We consider the function $B=B_{(++)}(\chi_F,\chi_G)$, where F and G are the balls of radius $1/4$ centered at $\eta^*=(L,1,0)$ and $\zeta^*=(-L,1,0)$ (see Fig. 4).

Take $\eta\in F$, $\zeta\in G$, we have

$$\begin{aligned}&|\eta|\approx|\zeta|\approx|\eta|+|\zeta|\approx L, \qquad |\eta+\zeta|\approx 1,\\ &|\eta|+|\zeta|-|\eta+\zeta|\approx L,\\ &|\eta|-|\eta^*|\lesssim 1, \qquad |\zeta|-|\zeta^*|\lesssim 1.\end{aligned}$$

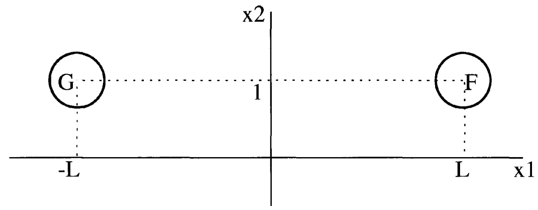

Fig. 4. Examples 5.4 and 5.5.

The weight W_+ is then of order

$$L^{\beta_+ + \beta_- - \alpha_1 - \alpha_2} = L^{-\beta_0 - \frac{n-1}{2}}.$$

We subtract from the phase function an innocuous term that doesn't depend on η or ζ,

$$\begin{aligned}\varphi_+ - t\big(|\zeta^*| + |\zeta^*|\big) &= t\big(|\eta| - |\eta^*| + |\zeta| - |\zeta^*|\big) + x \cdot (\eta + \zeta)\\ &= t\,\mathrm{O}(1) + x \cdot \mathrm{O}(1).\end{aligned}$$

As in the previous examples, it is possible to choose a region R in $\mathbb{R}^{1+n}$, defined by

$$|t| \lesssim 1, \qquad |x| \lesssim 1,$$

on which we can make the oscillating factor $e^{i\varphi_+ - t(|\zeta^*| + |\zeta^*|)}$ as close to 1 as we want. Thus we have

$$\big|B(t,x)\big| \gtrsim L^{-\beta_0 - \frac{n-1}{2}} |F||G|,$$

whenever $(t,x) \in R$. Consequently

$$\frac{\|B\|}{\|\chi_F\|\|\chi_G\|} \gtrsim L^{-\beta_0 - \frac{n-1}{2}} |F|^{1/2} |G|^{1/2} |R|^{1/2} \approx L^{-\beta_0 - \frac{n-1}{2}}.$$

We let $L \to \infty$ and we get the necessary condition,

$$-\beta_0 - \frac{n-1}{2} \leqslant 0,$$

which is a non sharp version of (6).

Example 5.5 ((8)). – This time we look at the $(+-)$ case when the data concentrate along opposite directions. Consider the function $B = B_{(+-)}(\chi_F, \chi_G)$, where F and G are the balls of radius $1/4$ centered at $(L,1,0)$ and $(-L,1,0)$ (see Fig. 4).

Take $\eta \in F$, $\zeta \in G$ and let θ be the angle between η and $-\zeta$, then we have

$$\begin{aligned}&|\eta| \approx |\zeta| \approx L, \qquad |\eta + \zeta| \approx 1, \qquad \theta \approx L^{-1},\\ &|\eta + \zeta| - \big||\eta| - |\zeta|\big| \approx \frac{|\eta||\zeta|}{|\eta + \zeta|}\theta^2 \approx 1,\\ &|\eta| - \eta_1 \approx \frac{|\eta'|^2}{|\eta|} \approx L^{-1}, \qquad |\zeta| + \zeta_1 \approx \frac{|\zeta'|^2}{|\zeta|} \approx L^{-1},\\ &\eta_1 + \zeta_1 \approx 1, \qquad |\eta' + \zeta'| \lesssim 1.\end{aligned}$$

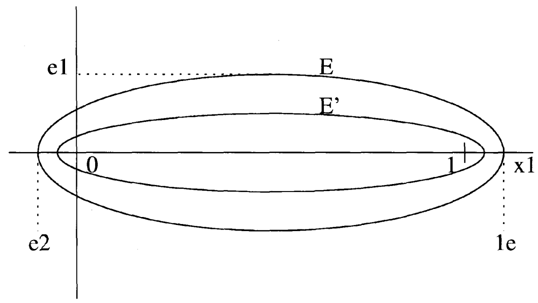

Fig. 5. Example 6.1.

The weight W_- given by (24) is then of order $L^{-\alpha_1-\alpha_2}$. We write the phase function as

$$\begin{aligned}\varphi_- &= t\big(|\eta| - |\zeta|\big) + x\cdot(\eta+\zeta)\\ &= t\big(|\eta| - \eta_1 - |\zeta| - \zeta_1\big) + (t+x_1)(\eta_1+\zeta_1) + x'\cdot(\eta'+\zeta')\\ &= t\,\mathrm{O}\big(L^{-1}\big) + (t+x_1)\,\mathrm{O}(1) + x'\cdot\mathrm{O}(1).\end{aligned}$$

It is then possible to choose a region R in $\mathbb{R}^{1+n}$, defined by

$$|t| \lesssim L, \qquad |t+x_1| \lesssim 1, \qquad |x'| \lesssim 1,$$

on which we can control the phase φ_-. Thus we have

$$\big|B(t,x)\big| \gtrsim L^{-\alpha_1-\alpha_2}|F||G|,$$

whenever $(t,x)\in R$. Consequently

$$\frac{\|B\|}{\|\chi_F\|\|\chi_G\|} \gtrsim L^{-\alpha_1-\alpha_2}|F|^{1/2}|G|^{1/2}|R|^{1/2} = L^{-\alpha_1-\alpha_2+1/2}.$$

We let $L\to\infty$ and we get the necessary condition,

$$-\alpha_1-\alpha_2+\frac{1}{2}\leqslant 0,$$

which is (8).

6. Counterexamples for the critical cases

The counterexamples in the previous section were easy to find in physical space, however for the critical cases (9) and (10) it is essential to take into account the interaction between different frequency scales. This is easier to do in frequency space, where we have the advantage of working with quantities which are, essentially, positive.

The following example is a straightforward generalization of the counterexample given in [6] adapted here to any dimension $n\geqslant 2$.

Example 6.1 (*Necessity of* (9)). – Assume that

(36) $$\alpha_1 = \frac{n+1}{4}, \qquad \alpha_2 = \beta_0+\beta_+, \qquad \beta_- = -\frac{n-3}{4}.$$

Consider the function $B = \tilde{B}_{(++)}(f,g)$, with

$$f(\xi) = \frac{\chi_E(\xi)}{|\xi|^{\frac{n+1}{4}}}, \qquad g(\xi) = \chi_E(\xi),$$

where E is the interior of the ellipsoid $\mathcal{E}(1+2\varepsilon^2, (1,0,\dots,0))$. We can compute the norms of f and g with the aid of Proposition 4.3. Use (29) with $a = (n+1)/2$ and $b = 0$ to get

$$\begin{aligned}\int_E \frac{\mathrm{d}\eta}{|\eta|^{\frac{n+1}{2}}} &= \int_1^{1+2\varepsilon^2} \int \frac{\delta(\tau - |\eta| - |(1,0,\dots,0)-\eta|)}{|\eta|^{\frac{n+1}{2}}} \,\mathrm{d}\eta\,\mathrm{d}\tau \\ &\approx \int_1^{1+2\varepsilon^2} (\tau-1)^{\frac{n-3}{2}} \big|\log(\tau-1)\big| \,\mathrm{d}\tau \approx \varepsilon^{n-1} |\log\varepsilon|;\end{aligned}$$

use (28) with $a = b = 0$ to get

$$\int_E \mathrm{d}\eta = \int_1^{1+2\varepsilon^2} \int \delta\big(\tau - |\eta| - \big|(1,0,\dots,0)-\eta\big|\big)\,\mathrm{d}\eta\,\mathrm{d}\tau \approx \int_1^{1+2\varepsilon^2} (\tau-1)^{\frac{n-3}{2}}\,\mathrm{d}\tau \approx \varepsilon^{n-1}. \tag{37}$$

We obtain

$$\|f\| \approx \varepsilon^{\frac{n-1}{2}} |\log\varepsilon|^{1/2}, \qquad \|g\| \approx \varepsilon^{\frac{n-1}{2}}.$$

Let E' be the interior of the ellipsoid $\mathcal{E}(1+\varepsilon^2, (1,0,\dots,0))$ and consider the region

$$D = \left\{(\tau,\xi)\colon\ \xi \in E',\ |\xi| > \frac{1}{2},\ \frac{\varepsilon^2}{2} < \tau - |\xi| < \varepsilon^2\right\}.$$

We have $\mathcal{E}(\tau,\xi) \subset E$ for every $(\tau,\xi) \in D$. Indeed, if $\eta \in \mathcal{E}(\tau,\xi)$ then

$$\begin{aligned}|\eta| + \big|(1,0,\dots,0)-\eta\big| &\leqslant |\eta| + |\xi-\eta| + \big|(1,0,\dots,0)-\xi\big| \\ &= (\tau - |\xi|) + \big(|\xi| + \big|(1,0,\dots,0)-\xi\big|\big) \\ &< \varepsilon^2 + \big(1+\varepsilon^2\big) = 1 + 2\varepsilon^2.\end{aligned}$$

When $(\tau,\xi) \in D$ we have $|\xi| \approx \tau \approx 1$ and $\tau - |\xi| \approx \varepsilon^2$, hence, using Lemma 4.3 with $a = (n+1)/2 \geqslant b = \beta_0 + \beta_+$, we find

$$B(\tau,\xi) \approx \varepsilon^{-\frac{n-3}{2}} \int \frac{\delta(\tau - |\eta| - |\xi-\eta|)}{|\eta|^{\frac{n+1}{2}} |\xi-\eta|^{\beta_0+\beta_+}}\,\mathrm{d}\eta \approx \varepsilon^{\frac{n-3}{2}} |\log\varepsilon|.$$

By a computation analogous to (37), the measure of D is of order $\varepsilon^2\varepsilon^{n-1}$ and hence

$$\frac{\|B\|}{\|f\|\|g\|} \gtrsim \frac{\varepsilon^{\frac{n-3}{2}} |\log\varepsilon| \varepsilon^{\frac{n+1}{2}}}{\varepsilon^{\frac{n-1}{2}} \varepsilon^{\frac{n-1}{2}} |\log\varepsilon|^{1/2}} = |\log\varepsilon|^{1/2},$$

which becomes unbounded as $\varepsilon \to 0$.

The next example proves the necessity of condition (10). This is, essentially, the example given first in [2] for the case of dimension $n = 3$.

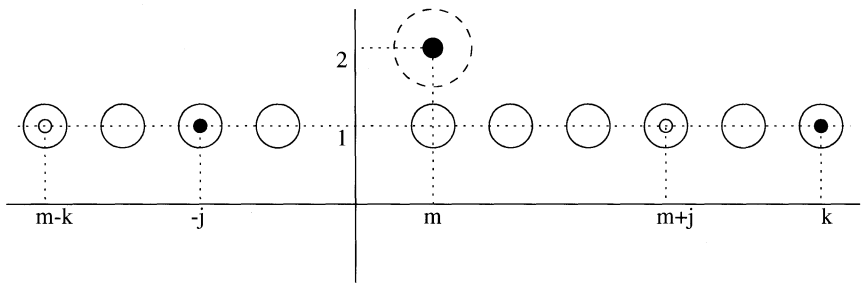

Fig. 6. Example 6.2.

Example 6.2 (*Necessity of* (10)). – Assume that

$$\alpha_1+\alpha_2=\frac{1}{2},\qquad \beta_0+\beta_+=-\frac{n-1}{4},\qquad \beta_-=-\frac{n-3}{4}. \tag{38}$$

Let $B(\tau,\xi)=\tilde{B}_{(+-)}(f,g)(\tau,\xi)$, with $f=g=\sum_{k=-N}^{N}\chi_k$, where, for each integer k, χ_k is the characteristic function of the ball B_k of center $(k,1,0,\dots,0)$ and radius $1/4$. We have $\|f\|=\|g\|\approx N^{1/2}$.

The support of $\tilde{B}_{(+-)}(\chi_k,\chi_j)$ is contained in the ball of center $(k+j,2,0,\dots,0)$ and radius $1/2$. Summing over all such disjoint balls, by orthogonality we have

$$\|B\|^2=\sum_{j,k,m} I(j,k,m),\qquad I(j,k,m)=\big\langle \tilde{B}_{(+-)}(\chi_k,\chi_{m-k}),\tilde{B}_{(+-)}(\chi_{m+j},\chi_{-j})\big\rangle, \tag{39}$$

where the indices in the sum are restricted by the condition

$$-N\leqslant k,k-m,j,j+m\leqslant N. \tag{40}$$

Each addendum in the sum in (39) is positive, hence we get a lower bound if we further restrict the indices to a subset of (40).

Let's look at a single term $I(j,k,m)$ (see Fig. 6). From the definition (20) of $\tilde{B}_{(+-)}$ we have

$$I(j,k,m)=\iiint\limits_{\substack{\eta\in B_k\\ \xi-\eta\in B_{m-k}\\ \xi-\zeta\in B_{m+j}\\ \zeta\in B_{-j}}}\frac{\delta(|\eta|+|\zeta|-|\xi-\eta|-|\xi-\zeta|)\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta}{|\xi|^{\frac{n-1}{2}}(|\xi|-||\eta|-|\xi-\eta||)^{\frac{n-3}{2}}|\eta|^{\alpha_1}|\xi-\eta|^{\alpha_2}|\xi-\zeta|^{\alpha_1}|\zeta|^{\alpha_2}}.$$

If we require that $1<2m<j<k$ then we have the following estimates,

$$|\eta|\approx|\xi-\eta|\approx k,\qquad |\xi-\zeta|\approx|\zeta|\approx j,\qquad |\xi|\approx m,\qquad |\xi|-\big||\eta|-|\xi-\eta|\big|\approx m^{-1}.$$

Also, let $B_{\varepsilon,k}$ be the ball of center $(k,1,0,\dots,0)$ and radius ε, where ε is a positive constant to be fixed later, and B_m^* be the ball of center $(m,2,0,\dots,0)$ and radius $1/8$. If $\varepsilon<1/8$, then

$$\eta\in B_{\varepsilon,k},\ \zeta\in B_{\varepsilon,-j},\ \xi\in B_m^*\Rightarrow \eta\in B_k,\ \xi-\eta\in B_{m-k},\ \xi-\zeta\in B_{m+j},\ \zeta\in B_{-j},$$

and we obtain the lower bound

$$I(j,k,m) \gtrsim \frac{1}{mj^{1/2}k^{1/2}} \iiint\limits_{\substack{\eta\in B_{\varepsilon,k}\\ \zeta\in B_{\varepsilon,-j}\\ \xi\in B_m^*}} \delta\big(|\eta|+|\zeta|-|\xi-\eta|-|\xi-\zeta|\big)\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta.$$

We use now formula (17). In our case $\phi(\xi)=|\eta|+|\zeta|-|\xi-\eta|-|\xi-\zeta|=0$ defines a surface $\mathcal{E}(\eta,\zeta)$, which is the ellipsoid of revolution with foci at η and ζ that contains the origin. We have

$$\begin{aligned}\big|\nabla\phi(\xi)\big| &= \left|\frac{\xi-\eta}{|\xi-\eta|}+\frac{\xi-\zeta}{|\xi-\zeta|}\right| \simeq \left(1+\frac{\xi-\eta}{|\xi-\eta|}\cdot\frac{\xi-\zeta}{|\xi-\zeta|}\right)^{1/2}\\ &\approx \text{angle between } \xi-\eta \text{ and } \zeta-\xi \approx \frac{1}{j}+\frac{1}{k}\approx\frac{1}{j}.\end{aligned}$$

Using (17) we obtain

$$I(j,k,m) \gtrsim \frac{1}{m}\frac{j^{1/2}}{k^{1/2}} \iint\limits_{B_{\varepsilon,k}\times B_{\varepsilon,-j}} \left(\int\limits_{\mathcal{E}(\eta,\zeta)\cap B_m^*} \mathrm{d}S_\xi\right)\mathrm{d}\eta\,\mathrm{d}\zeta.$$

LEMMA 6.3. – *It is possible to choose $\varepsilon,\lambda\in\,]0,1/2]$ and a constant $C>0$, independent of N, such that, if (j,k,m) belong to the set*

$$J_{N,\lambda}=\left\{(j,k,m)\colon\ \frac{1}{\lambda}\leqslant j\leqslant\frac{N}{4},\ \frac{N}{2}\leqslant k\leqslant N,\ 1\leqslant m\leqslant\lambda j\right\}, \tag{41}$$

and $\eta\in B_{\varepsilon,k}$, $\zeta\in B_{\varepsilon,-j}$, then the intersection of the ellipsoid $\mathcal{E}(\eta,\zeta)$ with the ball B_m^* has an area greater than C:

$$\inf_{\substack{(j,k,m)\in J_{N,\lambda}\\ \eta\in B_{\varepsilon,k}\\ \zeta\in B_{\varepsilon,-j}}} \int\limits_{\mathcal{E}(\eta,\zeta)\cap B_m^*} \mathrm{d}S_\xi \geqslant C>0.$$

With this lemma we can fix ε and λ, and for large values of N we have

$$\begin{aligned}\big\|\tilde{B}_{(+-)}(f,g)\big\|^2 &\geqslant \sum_{(j,k,m)\in J_{N,\lambda}} I(j,k,m)\\ &\gtrsim \sum_{1/\lambda\leqslant j\leqslant N/4} j^{1/2} \sum_{N/2\leqslant k\leqslant N} k^{-1/2} \sum_{1\leqslant m\leqslant\lambda j} m^{-1}\\ &\gtrsim N^{1/2} \sum_{1/\lambda\leqslant j\leqslant N/4} j^{1/2}\log(j) \gtrsim N^2\log N.\end{aligned}$$

This shows that

$$\frac{\|B_{(+-)}(f,g)\|}{\|f\|\|g\|} \gtrsim (\log N)^{1/2},$$

and as $N\to\infty$ disproves the estimate for the choice (38).

Proof of Lemma 6.3. – Let $(j,k,m)\in J_{N,\lambda}$ and $\eta\in B_{\varepsilon,k}$, $\zeta\in B_{\varepsilon,-j}$. To prove the lemma it is enough to show that, for ε and λ small enough, the ellipsoid $\mathcal{E}(\eta,\zeta)$ intersects the ball of center $(m,2,0,\ldots,0)$ and radius $1/16$.

First, we verify this when $\eta=(k,1,0,\dots,0)$ and $\zeta=(-j,1,0,\dots,0)$. Since $1<2m<j<k$, it follows that the point $(m,2,0,\dots,0)$ lies inside $\mathcal{E}_0=\mathcal{E}(\eta,\zeta)$. The plane tangent to $\mathcal{E}_0$ at the point $(0,2,0,\dots,0)$ makes an angle $\alpha\approx 1/j$ with the ξ_1 direction, hence it intersects the ball B_m^{**} of center $(m,2,0,\dots,0)$ and radius $1/32$, if $m/j\leqslant\lambda$ for some positive constant λ. Thus B_m^{**} contains also points that are outside $\mathcal{E}_0$, so it must intersect $\mathcal{E}_0$.

Then, when $\eta=(k,1,0,\dots,0)+\mathrm{O}(\varepsilon)$ and $\zeta=(-j,1,0,\dots,0)+\mathrm{O}(\varepsilon)$, the distance of each point of $\mathcal{E}_0$ from $\mathcal{E}(\eta,\zeta)$ is at most of order ε, and we can choose ε small enough to make it less than $1/32$. □

Example 6.4. – The following example is a refinement of that given by Example 5.4 and shows that the estimate (3) is in fact false when we have equality in (6). Assume that

$$\beta_0=-\frac{n-1}{2},\qquad \beta_++\beta_-=\alpha_1+\alpha_2.$$

Let $B(\tau,\xi)=\tilde{B}_{(++)}(\chi_F,\chi_G)(\tau,\xi)$, where F and G are the sets

$$F=\{\eta\colon |\eta_1-1|<\varepsilon^2,\ |\eta'|<\varepsilon\},$$
$$G=\{\zeta\colon |\zeta_1+1|<2\varepsilon^2,\ |\zeta'|<2\varepsilon\},$$

with ε a small positive parameter. We have $\|\chi_F\|=|F|^{1/2}\simeq\varepsilon^{\frac{n+1}{2}}$ and $\|\chi_G\|=|G|^{\frac12}\simeq\varepsilon^{\frac{n+1}{2}}$.

The conditions $\eta\in F$, $\xi-\eta\in G$ and $\eta\in\mathcal{E}(\tau,\xi)$ imply that

$$|\eta|\approx|\xi-\eta|\approx\tau\approx\tau-|\xi|\approx 1,\qquad |\xi|\lesssim\varepsilon,$$

and in particular we have

$$B(\tau,\xi)\approx|\xi|^{-\frac{n-1}{2}}\int_{\substack{\eta\in F\\ \xi-\eta\in G}}\delta\big(\tau-|\eta|-|\xi-\eta|\big)\,\mathrm{d}\eta.$$

The support of B contains the region D defined by

$$D=\{(\tau,\xi)\colon |\xi_1|\ll\varepsilon^2,\ |\xi'|\ll\varepsilon,\ |\tau-2|\ll\varepsilon^2\}.$$

If $(\tau,\xi)\in D$ and $\eta\in\mathcal{E}(\tau,\xi)\cap F$ then we have $\zeta=\xi-\eta\in G$, since

$$|\zeta_1+1|\leqslant|\xi_1|+|-\eta_1+1|<2\varepsilon^2,\qquad |\zeta'|\leqslant|\xi'|+|\eta'|<2\varepsilon.$$

Hence, when $(\tau,\xi)\in D$ we have

$$B(\tau,\xi)\approx|\xi|^{-\frac{n-1}{2}}\int_{\eta\in F}\delta\big(\tau-|\eta|-|\xi-\eta|\big)\,\mathrm{d}\eta=|\xi|^{-\frac{n-1}{2}}\int_{\mathcal{E}(\tau,\xi)\cap F}\frac{\mathrm{d}S_\eta}{|\nabla_\eta(|\eta|+|\xi-\eta|)|}.$$

The gradient in the denominator doesn't create any problem, indeed we have

$$\big|\nabla_\eta\big(|\eta|+|\xi-\eta|\big)\big|=\left|\frac{\eta}{|\eta|}-\frac{\xi-\eta}{|\xi-\eta|}\right|\approx\text{angle between }\eta\text{ and }\xi-\eta\approx 1.$$

The intersection $\mathcal{E}(\tau,\xi)\cap F$ is a set of measure $\approx \varepsilon^{n-1}$, indeed the conditions $(\tau,\xi)\in D$, $\eta\in\mathcal{E}(\tau,\xi)$ and $|\eta'|\ll\varepsilon$ imply $\eta\in F$, since we have

$$|\eta|-\eta_1\ll\varepsilon^2,\qquad |\xi-\eta|-\eta_1\ll\varepsilon^2,\qquad \big||\eta|+|\xi-\eta|-2\big|\ll\varepsilon^2\Rightarrow\big|2(\eta_1-1)\big|\ll\varepsilon^2.$$

We obtain that for $(\tau,\xi)\in D$ we have

$$B(\tau,\xi)\gtrsim|\xi|^{-\frac{n-1}{2}}\varepsilon^{n-1},$$

and consequently,

$$\|B\|^2\gtrsim\varepsilon^{2(n-1)}\iint\limits_{D}\frac{\mathrm{d}\xi\,\mathrm{d}\tau}{|\xi|^{n-1}}\gtrsim\varepsilon^{2(n-1)}\int\limits_{2-\varepsilon^2}^{2+\varepsilon^2}\int\limits_{0}^{\varepsilon^2}\int\limits_{|\xi'|\ll\varepsilon}\frac{\mathrm{d}\xi'}{(|\xi'|+\xi_1)^{n-1}}\,\mathrm{d}\xi_1\,\mathrm{d}\tau.$$

We compute the innermost integral with the help of Lemma 4.2,

$$\int\limits_{|\xi'|\ll\varepsilon}\frac{\mathrm{d}\xi'}{(|\xi'|+\xi_1)^{n-1}}\simeq\int\limits_0^{\varepsilon}\frac{r^{n-2}\,\mathrm{d}r}{(r+\xi_1)^{n-1}}=\int\limits_0^1\frac{t^{n-2}\,\mathrm{d}t}{\left(t+\frac{\xi_1}{\varepsilon}\right)^{n-1}}=H^{1-n}_{n-2}\left(\frac{\xi_1}{\varepsilon}\right)\approx\left|\log\left(\frac{\xi_1}{\varepsilon}\right)\right|.$$

Hence, for small values of ε we have

$$\|B\|^2\gtrsim\varepsilon^{2(n-1)}\varepsilon^2\int\limits_0^{\varepsilon^2}\left|\log\left(\frac{\xi_1}{\varepsilon}\right)\right|\mathrm{d}\xi_1\approx\varepsilon^{2(n+1)}|\log\varepsilon|.$$

We finally obtain

$$\frac{\|B\|}{\|\chi_F\|\|\chi_G\|}\gtrsim\frac{\varepsilon^{n+1}|\log\varepsilon|^{1/2}}{\varepsilon^{\frac{n+1}{2}}\varepsilon^{\frac{n+1}{2}}}=|\log\varepsilon|^{1/2},$$

which diverges logarithmically as $\varepsilon\to 0$.

7. Proof for the $(++)$

In this section we prove the estimate for the $(++)$ case. We consider the operator $B_{(++)}$ defined in (19). In this case we don't need to consider the conditions (8) and (10) which, as we have learned from the examples discussed in the previous sections, are relevant only for the $(+-)$ case.

The support of $B_{(++)}$ is contained in the region where $|\xi|\leqslant\tau$, we divide it into two part: the region near the light cone, where $|\xi|\approx\tau$, and the inner region, where $|\xi|\ll\tau$. Looking at the corresponding symbols we see that near the light cone the operator D behaves like D_+, while D_- degenerates along null directions. On the other hand, in the inner region D degenerates along the time direction, while D_+ and D_- behave both as time derivatives.

PROPOSITION 7.1. – *We have the estimate*

$$\big\|B_{(++)}(f,g)\big\|_{L^2(\frac{\tau}{4}\leqslant|\xi|\leqslant\tau)}\lesssim\|f\|_{L^2}\|g\|_{L^2}\tag{42}$$

whenever $\alpha_1,\alpha_2,\beta_0,\beta_+,\beta_-$ *verify conditions* (4), (5), (7) *and* (9).

Proof. – By a simple application of Cauchy–Schwarz relative to the measure $\delta(\tau-|\eta|-|\xi-\eta|)\,\mathrm{d}\eta$ we have

$$|\tilde{B}_{(++)}(f,g)(\tau,\xi)|^2 \lesssim A(\tau,\xi)\int f^2(\eta)g^2(\xi-\eta)\delta(\tau-|\eta|-|\xi-\eta|)\,\mathrm{d}\eta, \tag{43}$$

where

$$A(\tau,\xi)=|\xi|^{2\beta_0}\tau^{2\beta_+}(\tau-|\xi|)^{2\beta_-}\int\frac{\delta(\tau-|\eta|-|\xi-\eta|)}{|\eta|^{2\alpha_1}|\xi-\eta|^{2\alpha_2}}\,\mathrm{d}\eta.$$

Integrating (43) in τ and ξ gives us (42), provided that we can show that the quantity $A(\tau,\xi)$ is uniformly bounded for $\tau/4\leqslant|\xi|\leqslant\tau$. To do so we use Proposition 4.3 with $a=2\alpha_1$ and $b=2\alpha_2$. Since $\tau/4\leqslant|\xi|\leqslant\tau$, if $\max\{\alpha_1,\alpha_2\}<(n+1)/4$, using (28), (4) and (5) we have

$$A(\tau,\xi)\approx\left(\frac{\tau-|\xi|}{\tau}\right)^{2\beta_-+\frac{n-3}{2}}\lesssim 1;$$

if $\max\{\alpha_1,\alpha_2\}>(n+1)/4$, using (28), (4) and (7) we have

$$A(\tau,\xi)\approx\left(\frac{\tau-|\xi|}{\tau}\right)^{-2\max\{\alpha_1,\alpha_2\}+2\beta_-+n-1}\lesssim 1;$$

if $\max\{\alpha_1,\alpha_2\}=(n+1)/4$, using (29), (4), (5) and (9) we have

$$A(\tau,\xi)\approx\left(\frac{\tau-|\xi|}{\tau}\right)^{2\beta_-+\frac{n-3}{2}}\log\left(\frac{\tau}{\tau-|\xi|}\right)\lesssim 1.\quad\square$$

Remark 7.2. – The proof of Proposition 7.1 works fine also for the inner region if $\beta_0\geqslant 0$, since in that case $|\xi|^{\beta_0}$ is not singular and we can use $|\xi|\leqslant\tau$. To obtain sharp results with $\beta_0<0$ we need however a different argument.

PROPOSITION 7.3. – *We have the estimate*

$$\|B_{(++)}(f,g)\|_{L^2(|\xi|\leqslant\frac{\tau}{4})}\lesssim\|f\|_{L^2}\|g\|_{L^2}, \tag{44}$$

whenever $\alpha_1,\alpha_2,\beta_0,\beta_+,\beta_-$ *verify conditions* (4) *and* (6).

Proof. – The conditions $\tau=|\eta|+|\xi-\eta|$ and $|\xi|\leqslant\tau/4$ imply that $|\eta|\approx|\xi-\eta|\approx\tau$. In this region we have

$$\tilde{B}_{(++)}(f,g)(\tau,\xi)\approx|\xi|^{\beta_0}\tau^{\beta_++\beta_--\alpha_1-\alpha_2}\int f(\eta)g(\xi-\eta)\delta(\tau-|\eta|-|\xi-\eta|)\,\mathrm{d}\eta.$$

The vectors η and $\xi-\eta$ are essentially of the same magnitude and their sum is comparatively small, hence they have almost opposite directions, in the sense that the angle θ between η and $-(\xi-\eta)$ has to be less than a small fixed constant, say $\theta\leqslant\theta_0\leqslant\pi/4$. This means that, by decomposing f and g into a finite number of pieces, we reduce to the case of f and g supported in opposite conical regions.

Indeed, decompose the unit sphere into a finite number of disjoint components, $\mathbb{S}^{n-1}=\bigcup_{j=1}^N\Omega_j$, so that the angle between two unit vectors belonging to the same piece Ω_j is always less than θ_0. Then define the cones $\Gamma_j = \{\xi \in \mathbb{R}^n \setminus 0\colon \xi/|\xi| \in \Omega_j\}$ and let χ_j be the characteristic function of Γ_j. Write $f = \sum_{j=1}^N f_j$, where $f_j = \chi_j f$, and similarly for g. By the above observation we have

$$\left\|B_{(++)}(f,g)\right\|_{L^2(|\xi| \leqslant \frac{\tau}{4})} \leqslant \sum_{j,k} \left\|B_{(++)}(f_j,g_k)\right\|_{L^2(|\xi| \leqslant \frac{\tau}{4})},$$

with the sum restricted to the pairs of indices (j,k) for which there exists a cone Γ of aperture $2\theta_0$ such that $\Gamma_j \subset \Gamma$ and $\Gamma_k \subset -\Gamma$. Since it is a finite decomposition, we also have $\sum_{j,k} \|f_j\|\|g_k\| \lesssim \|f\|\|g\|$. Hence, we can assume f supported in a conical region Γ of aperture $2\theta_0$ and g supported in $-\Gamma$.

We now compute the L^2 norm of $B_{(++)}$ using the so called doubling technique. This method, introduced first in [11] for the $(+-)$ case, consists of writing the square of $\tilde{B}_{(++)}$ as a double integral,

$$\begin{aligned}\left|\tilde{B}_{(++)}(f,g)(\tau,\xi)\right|^2 \approx |\xi|^{2\beta_0} \tau^{2(\beta_+ + \beta_- - \alpha_1 - \alpha_2)} \iint f(\eta) g(\xi-\eta) f(\xi-\zeta) g(\zeta) \\ \times \delta\bigl(\tau - |\eta| - |\xi-\eta|\bigr) \delta\bigl(\tau - |\xi-\zeta| - |\zeta|\bigr) \, \mathrm{d}\eta \, \mathrm{d}\zeta,\end{aligned}$$

and then perform the integration with respect to τ and ξ. Recall that by (4) we have $\beta_+ + \beta_- - \alpha_1 - \alpha_2 = -\beta_0 - \frac{n-1}{2}$. We obtain

$$\begin{aligned}&\|B_{(++)}\|^2_{L^2(|\xi| \leqslant \frac{\tau}{4})} \\ &\lesssim \iiint\limits_{\substack{\eta \in \Gamma \\ \zeta \in -\Gamma \\ \xi-\zeta \in \Gamma \\ \xi-\eta \in -\Gamma \\ |\xi| \ll |\eta| \approx |\zeta|}} \frac{f(\eta) g(\zeta) f(\xi-\zeta) g(\xi-\zeta)}{|\xi|^{-2\beta_0} |\eta|^{\beta_0 + \frac{n-1}{2}} |\zeta|^{\beta_0 + \frac{n-1}{2}}} \delta\bigl(|\eta| - |\zeta| + |\xi-\eta| - |\xi-\zeta|\bigr) \, \mathrm{d}\eta \, \mathrm{d}\zeta \, \mathrm{d}\xi.\end{aligned}$$

We apply now Cauchy–Schwarz to separate the pair $f(\eta)g(\zeta)$ from $f(\xi-\zeta)g(\xi-\zeta)$,

$$\|B_{(++)}\|^2_{L^2(|\xi| \leqslant \frac{\tau}{4})} \lesssim \iiint\limits_{\substack{\eta \in \Gamma,\ \zeta \in -\Gamma \\ |\xi| \ll |\eta| \approx |\zeta|}} f^2(\eta) g^2(\zeta) \frac{\delta(|\eta| - |\zeta| + |\xi-\eta| - |\xi-\zeta|)}{|\xi|^{-2\beta_0} |\eta|^{\beta_0 + \frac{n-1}{2}} |\zeta|^{\beta_0 + \frac{n-1}{2}}} \, \mathrm{d}\eta \, \mathrm{d}\zeta \, \mathrm{d}\xi,$$

and we only have to prove the boundedness of the quantity

$$\lambda^{-2\beta_0 - n + 1} \int\limits_{|\xi| \leqslant \lambda} \frac{\delta(|\eta| - |\zeta| + |\xi-\eta| - |\xi-\zeta|)}{|\xi|^{-2\beta_0}} \, \mathrm{d}\xi,$$

uniformly for $|\eta| \approx |\zeta| \approx \lambda$, with $\eta \in \Gamma$ and $\zeta \in -\Gamma$. By a rescaling we can take $\lambda = 1$.

The delta function in the last integral restricts ξ to a hypersurface $\mathcal{H}(\eta,\zeta)$, which is a hyperboloid of revolution around the line through η and ζ with foci at η and ζ, which contains the point $\eta + \zeta$. The fact that η and ζ are almost opposite points implies a uniform upper bound on the curvature of $\mathcal{H}(\eta,\zeta)$ and allows us to treat $\mathcal{H}(\eta,\zeta)$ as if it were a hyperplane. More precisely we have

$$\left|\nabla_\xi(|\xi-\eta| - |\xi-\zeta|)\right| \approx \text{angle between } \xi-\eta \text{ and } \xi-\zeta \approx 1.$$

Moreover, ξ is confined on a bounded region of $\mathcal{H}(\eta,\zeta)$ by the condition $|\xi| \leqslant 1$, so we don't have to worry about divergencies coming from the unboundedness of $\mathcal{H}(\eta,\zeta)$. Parametrizing $\mathcal{H}(\eta,\zeta)$

by a projection on a hyperplane orthogonal to $\eta-\zeta$, we check that

$$\int_{|\xi|\leqslant 1} \frac{\delta(|\eta|-|\zeta|+|\xi-\eta|-|\xi-\zeta|)}{|\xi|^{-2\beta_0}}\,\mathrm{d}\xi \lesssim \int_{\substack{\xi\in\mathcal{H}(\eta,\zeta)\\ |\xi|\leqslant 1}} \frac{\mathrm{d}S_\xi}{|\xi|^{-2\beta_0}} \lesssim \int_{\substack{\xi'\in\mathbb{R}^{n-1}\\ |\xi'|\leqslant 1}} \frac{\mathrm{d}\xi'}{|\xi'|^{-2\beta_0}},$$

which is a bounded quantity whenever $-2\beta_0<n-1$. □

8. Proof for the $(+\,-)$ case

Now we turn our attention to the $(+\,-)$ case. We consider the operator $B_{(+-)}$ defined in (20). Since $\tilde{B}_{(+-)}$ is supported in the region where $|\tau|\leqslant|\xi|$ we have $D\approx D_+$ and we can replace the operator $D^{\beta_0}D_+^{\beta_+}$ by D^β, $\beta=\beta_0+\beta_+$. It is convenient to split the integration over the hyperboloids $\mathcal{H}(\tau,\xi)$ into two parts: $\tilde{B}_{(+-)}=B_L+B_H$, where

$$B_L(f,g)(\tau,\xi)=\int_{|\eta|+|\xi-\eta|\leqslant 2|\xi|} \delta\big(\tau-|\eta|+|\xi-\eta|\big)\frac{|\xi|^\beta(|\xi|-|\tau|)^{\beta_-}}{|\eta|^{\alpha_1}|\xi-\eta|^{\alpha_2}}f(\eta)g(\xi-\eta)\,\mathrm{d}\eta,$$

$$B_H(f,g)(\tau,\xi)=\int_{|\eta|+|\xi-\eta|\geqslant 2|\xi|} \delta\big(\tau-|\eta|+|\xi-\eta|\big)\frac{|\xi|^\beta(|\xi|-|\tau|)^{\beta_-}}{|\eta|^{\alpha_1}|\xi-\eta|^{\alpha_2}}f(\eta)g(\xi-\eta)\,\mathrm{d}\eta.$$

The low frequency part B_L can be treated as in the $(+\,+)$ case, since the integration is restricted to the "elliptic" portion of $\mathcal{H}(\tau,\xi)$.

PROPOSITION 8.1. – *We have the estimate*

$$\big\|B_L(f,g)\big\|\lesssim\|f\|\|g\| \tag{45}$$

whenever $\alpha_1,\alpha_2,\beta_0,\beta_+,\beta_-$ *verify conditions* (4), (5), (7) *and* (9).

Proof. – By a simple application of Cauchy–Schwarz with respect to the measure $\delta(\tau-|\eta|+|\xi-\eta|)\,\mathrm{d}\eta$, we have

$$\big|B_L(f,g)(\tau,\xi)\big|^2\leqslant A(\tau,\xi)\int\big|f(\eta)\big|^2\big|g(\xi-\eta)\big|^2\delta\big(\tau-|\eta|+|\xi-\eta|\big)\,\mathrm{d}\eta, \tag{46}$$

where

$$A(\tau,\xi)=|\xi|^{2\beta}\big(|\xi|-|\tau|\big)^{2\beta_-}\int_{|\eta|+|\xi-\eta|\leqslant 2|\xi|}\frac{\delta(\tau-|\eta|+|\xi-\eta|)}{|\eta|^{2\alpha_1}|\xi-\eta|^{2\alpha_2}}\,\mathrm{d}\eta.$$

Integrating (46) in τ and ξ gives us (45), provided that we can show that the quantity $A(\tau,\xi)$ is uniformly bounded for $|\tau|<|\xi|$. To do so we use Proposition 4.5 with $a=2\alpha_1$ and $b=2\alpha_2$. By symmetry, we can assume $\tau\geqslant 0$. Using (33), (4) and (5) we have

$$A(\tau,\xi)\approx\left(\frac{|\xi|-\tau}{|\xi|}\right)^{2\beta_-+\frac{n-3}{2}}\lesssim 1;$$

if $\alpha_2 > (n+1)/4$, using (33), (4) and (7) we have

$$A(\tau,\xi) \approx \left(\frac{|\xi|-\tau}{|\xi|}\right)^{-2\alpha_1+2\beta_-+n-1} \lesssim 1;$$

if $\alpha_2 = (n+1)/4$, using (34), (4), (5) and (9) we have

$$A(\tau,\xi) \approx \left(\frac{|\xi|-\tau}{|\xi|}\right)^{2\beta_-+\frac{n-3}{2}} \left(1+\log\left(\frac{|\xi|}{|\xi|-\tau}\right)\right) \lesssim 1. \quad \square$$

The high frequency part B_H is more delicate and requires the use of the doubling technique.

PROPOSITION 8.2. – *We have the estimate*

$$\left\|B_H(f,g)\right\| \lesssim \|f\|\|g\| \tag{47}$$

whenever $\alpha_1, \alpha_2, \beta_0, \beta_+, \beta_-$ *verify conditions* (4), (5), (8) *and* (10).

Proof. – Observe that $\eta \in \mathcal{H}(\tau,\xi)$ and $|\eta| + |\xi-\eta| \geqslant 2|\xi|$ implies $|\xi| \leqslant 2\min\{|\eta|, |\xi-\eta|\}$ and $|\eta| \approx |\xi-\eta|$. In particular we can assume that $\alpha_1 = \alpha_2$, since $|\eta|^{\alpha_1}|\xi-\eta|^{\alpha_2} \approx |\eta|^{\alpha}|\xi-\eta|^{\alpha}$, for $\alpha = (\alpha_1+\alpha_2)/2$, and the conditions (4), (5), (8) and (10) depend only on the sum $\alpha_1+\alpha_2$. Recall also that we may assume, without loss of generality, that f and g are positive functions.

If we proceed by a direct application of the Cauchy–Schwarz inequality, as in the $(++)$ case, we encounter the difficulty that the surface $\mathcal{H}(\tau,\xi)$ is unbounded. We circumvent this difficulty by using the doubling technique, as it was introduced in [11]. This allows us to rewrite the L^2 norm $B_H(f,g)$ in a way which transforms the integration over hyperboloids to one over ellipsoids.

To calculate the L^2 norm of $B_H(f,g)$ we first write $|B_H(f,g)(\tau,\xi)|^2$ as a double integral and then integrate over τ, ξ. After applying the Fubini–Tonelli theorem and rearranging the integrand, we derive:

$$\begin{aligned}\left\|B_H(f,g)\right\|^2 \leqslant \iint \iint_{|\xi|\leqslant 2\min\{|\eta|,|\zeta|\}} & \frac{f(\eta)g(\zeta)}{|\eta|^{\alpha}|\zeta|^{\alpha}} \cdot \frac{f(\xi-\zeta)g(\xi-\eta)}{|\xi-\zeta|^{\alpha}|\xi-\eta|^{\alpha}} \\ & \times |\xi|^{2\beta}\big(|\xi|-|\tau|\big)^{2\beta_-}\delta\big(\tau-|\eta|+|\xi-\eta|\big)\delta\big(\tau-|\xi-\zeta|+|\zeta|\big)\,\mathrm{d}\tau\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta.\end{aligned}$$

Now the important step is to apply Cauchy–Schwarz correctly. As it is suggested by the way we wrote the integrand, we separate the pair $f(\eta)g(\zeta)$ from the pair $f(\xi-\zeta)g(\xi-\eta)$, then apply Cauchy–Schwarz with respect to the measure

$$|\xi|^{2\beta}\big(|\xi|-|\tau|\big)^{2\beta_-}\delta\big(\tau-|\eta|+|\xi-\eta|\big)\delta\big(\tau-|\xi-\zeta|+|\zeta|\big)\,\mathrm{d}\tau\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta.$$

After an obvious change of variables we derive,

$$\begin{aligned}\left\|B_H(f,g)\right\|^2 \leqslant \iint \iint_{|\xi|\leqslant 2\min\{|\eta|,|\zeta|\}} & f^2(\eta)g^2(\zeta)\frac{|\xi|^{2\beta}(|\xi|-|\tau|)^{2\beta_-}}{|\eta|^{2\alpha}|\zeta|^{2\alpha}} \\ & \times \delta\big(\tau-|\eta|+|\xi-\eta|\big)\delta\big(\tau-|\xi-\zeta|+|\zeta|\big)\,\mathrm{d}\tau\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta.\end{aligned}$$

From this we see that (47) will follow, if we can show that the quantity

$$I(\eta,\zeta)=\iint\limits_{|\xi|<2\min\{|\eta|,|\zeta|\}} \frac{|\xi|^{2\beta}\big(|\xi|-|\tau|\big)^{2\beta_-}}{|\eta|^{2\alpha}|\zeta|^{2\alpha}}\delta\big(\tau-|\eta|+|\xi-\eta|\big)\delta\big(\tau-|\xi-\zeta|+|\zeta|\big)\,\mathrm{d}\tau\,\mathrm{d}\xi,$$

is bounded uniformly in η and ζ. The proof of the boundedness of $I(\eta,\zeta)$ requires several steps and it will be treated in the following three sections. □

9. Another parameterization for ellipsoids

The integral defining $I(\eta,\zeta)$ corresponds to an integration on the ellipsoid of revolution

$$\mathcal{E}(\eta,\zeta)=\big\{\xi\colon\ |\eta|+|\zeta|=|\xi-\eta|+|\xi-\zeta|\big\},$$

with foci at η and ζ, that contains the origin.

To study $I(\eta,\zeta)$ we introduce polar coordinates. Write $\xi=r\omega_0$, $\eta=\rho\omega_1$, $\zeta=\sigma\omega_2$, with $|\tau|<r<2\min\{\rho,\sigma\}$ and $\omega_0,\omega_1,\omega_2\in\mathbb{S}^{n-1}$. Also define the cosines $a=\omega_1\cdot\omega_0$, $b=\omega_2\cdot\omega_0$, $c=\omega_1\cdot\omega_2$.

To deal with the delta functions, introduce the auxiliary variables

$$\begin{aligned}X&=\tau-|\eta|+|\xi-\eta|=\tau-\rho+\sqrt{r^2-2r\rho a+\rho^2},\\ Y&=\tau-|\xi-\zeta|+|\zeta|=\tau+\sigma-\sqrt{r^2-2r\sigma b+\sigma^2}.\end{aligned}$$

The system $X=0$, $Y=0$ is equivalent to say that $\eta\in\mathcal{H}(\tau,\xi)$, $\zeta\in\mathcal{H}(-\tau,\xi)$, and, using formula (32) for the parameterization of our hyperboloids, we have

$$\rho=\frac{r^2-\tau^2}{2(-\tau+ra)},\qquad \sigma=\frac{r^2-\tau^2}{2(\tau+rb)}.$$

From these two equations we infer that $(-\tau+ra)\rho=(\tau+rb)\sigma$. Hence

$$\tau=rv, \tag{48}$$

where

$$v=\frac{\rho a-\sigma b}{\rho+\sigma}=\frac{\eta-\zeta}{|\eta|+|\zeta|}\cdot\omega_0. \tag{49}$$

Substituting for τ in the formula for ρ we find that $r=2\rho(-v+a)/(1-v^2)$, which, using (49), simplifies to

$$r=\frac{4\rho\sigma}{\rho+\sigma}\,\frac{u}{1-v^2}, \tag{50}$$

where

$$u=\frac{a+b}{2}=\frac{\omega_1+\omega_2}{2}\cdot\omega_0\geqslant 0. \tag{51}$$

We need also to compute the jacobian of the transformation $(\tau,r)\to(X,Y)$,

$$J=\left|\frac{\partial(X,Y)}{\partial(\tau,r)}\right|_{\substack{X=0\\Y=0}}=\left|\begin{matrix}\dfrac{\partial X}{\partial\tau} & \dfrac{\partial X}{\partial r}\\ \dfrac{\partial Y}{\partial\tau} & \dfrac{\partial Y}{\partial r}\end{matrix}\right|=\left|\begin{matrix}1 & \dfrac{\xi-\eta}{|\xi-\eta|}\cdot\omega_0\\ 1 & -\dfrac{\xi-\zeta}{|\xi-\zeta|}\cdot\omega_0\end{matrix}\right|$$

$$=\left|\frac{r-\rho a}{\rho-\tau}+\frac{r-\sigma b}{\sigma+\tau}\right|=\left|\frac{r(\rho+\sigma)-\tau(\rho a-\sigma b)-\rho\sigma(a+b)}{(\rho-\tau)(\sigma+\tau)}\right|.$$

To simplify this expression observe that by (48), (49), (50) and (51) we have

$$r(\rho+\sigma)=\frac{4\rho\sigma u}{1-v^2},\qquad \tau(\rho a-\sigma b)=\frac{4\rho\sigma u v^2}{1-v^2},\qquad \rho\sigma(a+b)=2\rho\sigma u,$$

hence $r(\rho+\sigma)-\tau(\rho a-\sigma b)-\rho\sigma(a+b)=2\rho\sigma u$ and

$$J=\frac{2\rho\sigma u}{(\rho-\tau)(\sigma+\tau)}.$$

Since we know that in the region of integration $\rho-\tau=|\xi-\eta|\approx|\eta|=\rho$ and $\sigma+\tau=|\xi-\zeta|\approx|\zeta|=\sigma$, the above jacobian simplifies to

$$J=\left|\frac{\partial(X,Y)}{\partial(\tau,r)}\right|_{\substack{X=0\\Y=0}}\approx u. \tag{52}$$

Now we are ready to rewrite $I(\eta,\zeta)$ using polar coordinates and formulas (48), (50), (52) (keep in mind that by (4) we have $2\beta+2\beta_-+n-1=4\alpha$):

$$\begin{aligned}I(\eta,\zeta)&=\int\frac{r^{2\beta}(r-|\tau|)^{2\beta_-}}{\rho^{2\alpha}\sigma^{2\alpha}}r^{n-1}J^{-1}\,\mathrm{d}S_{\omega_0}\approx\int\frac{r^{4\alpha}(1-|v|)^{2\beta_-}}{\rho^{2\alpha}\sigma^{2\alpha}u}\,\mathrm{d}S_{\omega_0}\\&\approx\left(\frac{4\rho\sigma}{(\rho+\sigma)^2}\right)^{2\alpha}\int\frac{(1-|v|)^{2\beta_-}u^{4\alpha-1}}{(1-v^2)^{4\alpha}}\,\mathrm{d}S_{\omega_0}\\&\approx\left(\frac{4\rho\sigma}{(\rho+\sigma)^2}\right)^{2\alpha}\int\frac{u^{4\alpha-1}}{(1-|v|)^{4\alpha-2\beta_-}}\,\mathrm{d}S_{\omega_0},\end{aligned}$$

with the integration always restricted to the region of the unit sphere corresponding to the points ω_0 such that $r=r(\omega_0)\leqslant 2\min\{\rho,\sigma\}$. From identity (50) we see that this condition is equivalent to say that $u\lesssim 1-|v|$. Hence,

$$I(\eta,\zeta)\approx\left(\frac{4\rho\sigma}{(\rho+\sigma)^2}\right)^{2\alpha}\int\limits_{\substack{\omega_0\in\mathbb{S}^{n-1}\\u\lesssim 1-|v|}}\frac{u^{4\alpha-1}}{(1-|v|)^{4\alpha-2\beta_-}}\,\mathrm{d}S_{\omega_0}. \tag{53}$$

10. The case $\beta_->-\frac{n-3}{4}$

We prove the boundedness of $I(\eta,\zeta)$ first in the case where we have strict inequality in (5), which means

$$\beta_->-\frac{n-3}{4},\qquad \alpha\geqslant\frac{1}{4}.$$

Using the fact that $u \lesssim 1-|v|$, we can simplify the numerator with the denominator in the integrand of (53) and we have,

$$I(\eta,\zeta) \lesssim \int \frac{\mathrm{d}S_{\omega_0}}{(1-|v|)^{1-2\beta_-}}.$$

To estimate this last integral we write $v = \lambda w$ where

$$\lambda = \left|\frac{\eta-\zeta}{|\eta|+|\zeta|}\right| = \left(1 - \frac{4\rho\sigma}{(\rho+\sigma)^2}\frac{1+c}{2}\right)^{1/2} \leqslant 1, \qquad w = \frac{\eta-\zeta}{|\eta-\zeta|}\cdot\omega_0,$$

then, using the notation of Lemma 4.2,

$$\int \frac{\mathrm{d}S_{\omega_0}}{(1-|v|)^{1-2\beta_-}} \lesssim \int_{-1}^{1} \frac{(1-w^2)^{\frac{n-3}{2}}}{(1-\lambda|w|)^{1-2\beta_-}}\,\mathrm{d}w \approx \lambda^{2\beta_- -1}\int_0^1 \frac{(1-w)^{\frac{n-3}{2}}}{(\frac{1-\lambda}{\lambda}+(1-w))^{1-2\beta_-}}\,\mathrm{d}w$$

$$= \lambda^{2\beta_- -1} H^{2\beta_- -1}_{\frac{n-3}{2}}\left(\frac{1-\lambda}{\lambda}\right), \tag{54}$$

which is bounded, since $(2\beta_- - 1) + \frac{n-3}{2} + 1 > 0$.

11. The case $\beta_- = -\frac{n-3}{4}$

Now we look at the more delicate cases where we have equality in (5), which by (10) implies an inequality in (8),

$$\beta_- = -\frac{n-3}{4}, \qquad \alpha > \frac{1}{4}, \qquad \beta = 2\alpha - \frac{n+1}{4} > -\frac{n-1}{4}.$$

These cases have already been established for $n=2$ and $n=3$ in [11], where it was taken advantage of the fact that β_- is nonnegative. That method doesn't apply when $\beta_- < 0$ and we need a more precise asymptotic control of the integral defining $I(\eta,\zeta)$ in terms of the angle between the vectors η and $-\zeta$ when this tends to zero.

To simplify the problem, observe that by symmetry and scaling invariance we can assume $|\zeta| \leqslant |\eta| = 1$. As it will become clear later, the quantity $I(\eta,\zeta)$ then depends essentially on two parameters: $\sigma = |\zeta|$, which measures the ratio of the magnitudes of ζ and η, and $\varepsilon = |\omega_1+\omega_2|/2 = \sqrt{(1+c)/2}$, which measures the angle between η and $-\zeta$. Both parameters, ε and σ, are contained in the interval $[0,1]$.

Remark 11.1. – We can also assume that

$$\frac{1}{4} < \alpha \leqslant \frac{1}{2},$$

since otherwise we can use the inequality $u \lesssim 1-|v|$ to reduce simultaneously the powers in the numerator and denominator of the integrand in (53).

We have to prove the boundedness of the quantity

$$I(\sigma,\varepsilon) = \sigma^{2\alpha} \int_{u \lesssim 1-|v|} \frac{u^{4\alpha-1}}{(1-|v|)^{4\alpha+\frac{n-3}{2}}}\,\mathrm{d}S_{\omega_0},$$

where

$$u=\frac{\omega_1+\omega_2}{2}\cdot\omega_0, \qquad v=\frac{\omega_1-\sigma\omega_2}{1+\sigma}\cdot\omega_0.$$

If we follow the same steps as in the previous section, we obtain an upper bound that diverges for $\varepsilon\to 0$. Indeed, with our assumptions we have $v=\lambda w$, with

$$\lambda=\left|\frac{\eta-\zeta}{|\eta|+|\zeta|}\right|=\left(1-\frac{4\sigma\varepsilon}{(1+\sigma)^2}\right)^{1/2}, \quad 1-\lambda\approx\sigma\varepsilon,$$

and we can always assume $1/2\leqslant\lambda\leqslant 1$, otherwise the estimate becomes trivial. By a computation similar to (54), using $u\lesssim 1-|v|$, we find

$$\begin{aligned}I(\sigma,\varepsilon)&\lesssim\sigma^{2\alpha}\int\frac{\mathrm{d}S_{\omega_0}}{(1-|v|)^{\frac{n-1}{2}}}\lesssim\sigma^{2\alpha}\int_{-1}^{1}\frac{(1-w^2)^{\frac{n-3}{2}}}{(1-\lambda|w|)^{\frac{n-1}{2}}}\,\mathrm{d}w\approx\sigma^{2\alpha}\lambda^{-\frac{n-1}{2}}H_{\frac{n-3}{2}}^{-\frac{n-1}{2}}\left(\frac{1-\lambda}{\lambda}\right)\\&\approx\sigma^{2\alpha}\left|\log(1-\lambda)\right|\approx\varepsilon^{-2\alpha}(\sigma\varepsilon)^{2\alpha}\left|\log(\sigma\varepsilon)\right|\lesssim\varepsilon^{-2\alpha},\end{aligned}$$

which shows that $I(\sigma,\varepsilon)$ is bounded uniformly in σ if ε is bigger than a positive constant, but diverges logarithmically as $\varepsilon\to 0$.

It remains to see how to control the integral $I(\sigma,\varepsilon)$ when ε is small.

Assume $n\geqslant 3$. We can always choose a coordinate system so that

$$\begin{aligned}\omega_1&=\left(\sqrt{1-\varepsilon^2},\varepsilon,0,\dots,0\right),\\\omega_2&=\left(-\sqrt{1-\varepsilon^2},\varepsilon,0,\dots,0\right),\\\omega_0&=\left(x,y,\sqrt{1-x^2-y^2}\omega''\right),\quad x^2+y^2\leqslant 1,\quad \omega''\in\mathbb{S}^{n-3}.\end{aligned}$$

It follows that

$$\begin{aligned}u=\varepsilon y,\qquad v&=\sqrt{1-\varepsilon^2}x+\frac{1-\sigma}{1+\sigma}\varepsilon y,\\\mathrm{d}S_{\omega_0}&=(1-x^2-y^2)^{\frac{n-4}{2}}\,\mathrm{d}S_{\omega''}\,\mathrm{d}x\,\mathrm{d}y.\end{aligned}$$

It is convenient to use polar coordinates for the pair (x,y): $x=R\cos\theta$, $y=R\sin\theta$, with $0\leqslant R\leqslant 1$, $0\leqslant\theta\leqslant\pi$. Then

$$u=R\varepsilon\sin\theta, \tag{55}$$

$$v=RQ,\quad Q=\sqrt{1-\varepsilon^2}\cos\theta+\frac{1-\sigma}{1+\sigma}\varepsilon\sin\theta,\quad |Q|\leqslant 1, \tag{56}$$

$$\mathrm{d}S_{\omega_0}=R(1-R^2)^{\frac{n-4}{2}}\,\mathrm{d}S_{\omega''}\,\mathrm{d}R\,\mathrm{d}\theta. \tag{57}$$

With this parameterization, we have

$$\begin{aligned}I(\sigma,\varepsilon)&\lesssim\sigma^{2\alpha}\varepsilon^{4\alpha-1}\int_0^{\pi}\int_0^{1}\frac{R^{4\alpha}(1-R^2)^{\frac{n-4}{2}}(\sin\theta)^{4\alpha-1}}{(1-R|Q|)^{4\alpha+\frac{n-3}{2}}}\,\mathrm{d}R\,\mathrm{d}\theta\\&\lesssim\sigma^{2\alpha}\varepsilon^{4\alpha-1}\int_0^{\pi}\frac{(\sin\theta)^{4\alpha-1}}{|Q|^{4\alpha+\frac{n-3}{2}}}\int_0^{1}\frac{(1-R)^{\frac{n-4}{2}}}{(\frac{1-|Q|}{|Q|}+1-R)^{4\alpha+\frac{n-3}{2}}}\,\mathrm{d}R\,\mathrm{d}\theta\end{aligned}$$

$$\approx \sigma^{2\alpha}\varepsilon^{4\alpha-1}\int_0^{\pi} \frac{(\sin\theta)^{4\alpha-1}}{|Q|^{4\alpha+\frac{n-3}{2}}} H^{-4\alpha-\frac{n-3}{2}}_{\frac{n-4}{2}}\left(\frac{1-|Q|}{|Q|}\right)\mathrm{d}\theta. \tag{58}$$

We use Lemma 4.2, the fact that

$$-4\alpha - \frac{n-3}{2} + \frac{n-4}{2} + 1 < 0 \quad \text{and} \quad \frac{n-4}{2} > -1,$$

to reduce to

$$I(\sigma,\varepsilon) \lesssim \sigma^{2\alpha}\varepsilon^{4\alpha-1}\int_0^{\pi} \frac{(\sin\theta)^{4\alpha-1}}{(1-|Q|)^{4\alpha-\frac{1}{2}}}\,\mathrm{d}\theta. \tag{59}$$

We split the last integral into two pieces: the contribution coming from the region where $|Q| \leqslant \frac{1}{2}$, which is easily bounded, and that where $|Q| > \frac{1}{2}$, which needs more attention.

$$I_1 = \sigma^{2\alpha}\varepsilon^{4\alpha-1}\int_{|Q|\leqslant\frac{1}{2}} \frac{(\sin\theta)^{4\alpha-1}}{(1-|Q|)^{4\alpha-\frac{1}{2}}}\,\mathrm{d}\theta \lesssim \sigma^{2\alpha}\varepsilon^{4\alpha-1} \leqslant 1,$$

$$I_2 = \sigma^{2\alpha}\varepsilon^{4\alpha-1}\int_{|Q|>\frac{1}{2}} \frac{(\sin\theta)^{4\alpha-1}}{(1-|Q|)^{4\alpha-\frac{1}{2}}}\,\mathrm{d}\theta.$$

From its definition

$$Q = Q(\theta) = \sqrt{1-\varepsilon^2}\cos\theta + \frac{1-\sigma}{1+\sigma}\varepsilon\sin\theta. \tag{60}$$

If ε is small enough the condition $|Q| > 1/2$ requires $|\cos\theta|$ to be close to 1. In particular Q and $\cos\theta$ have the same sign, hence

$$I_2 \lesssim \sigma^{2\alpha}\varepsilon^{4\alpha-1}\left(\int_0^{\frac{\pi}{3}} \frac{(\sin\theta)^{4\alpha-1}}{(1-Q)^{4\alpha-\frac{1}{2}}}\,\mathrm{d}\theta + \int_{\pi-\frac{\pi}{3}}^{\pi} \frac{(\sin\theta)^{4\alpha-1}}{(1+Q)^{4\alpha-\frac{1}{2}}}\,\mathrm{d}\theta\right). \tag{61}$$

LEMMA 11.2. – *Let*

$$J_+ = \int_0^{\frac{\pi}{3}} \frac{(\sin\theta)^{4\alpha-1}}{(1-Q)^{4\alpha-\frac{1}{2}}}\,\mathrm{d}\theta, \qquad J_- = \int_{\pi-\frac{\pi}{3}}^{\pi} \frac{(\sin\theta)^{4\alpha-1}}{(1+Q)^{4\alpha-\frac{1}{2}}}\,\mathrm{d}\theta.$$

Then we have $J_+ \lesssim (\sigma\varepsilon)^{-4\alpha+1}$ *and* $J_- \lesssim \varepsilon^{-4\alpha+1}$.

Using this lemma, (61) becomes

$$I_2 \lesssim \sigma^{2\alpha}\varepsilon^{4\alpha-1}\big((\sigma\varepsilon)^{-4\alpha+1} + \varepsilon^{-4\alpha+1}\big) \approx \sigma^{1-2\alpha} \lesssim 1,$$

which is bounded in view of the assumption made in Remark 11.1.

Proof of Lemma 11.2. – Let's look first at J_-, which is easier. When $\theta \in [\frac{2}{3}\pi, \pi]$ we have

$$\begin{aligned} 1+Q &\geqslant 1 + \sqrt{1-\varepsilon^2}\cos\theta \approx 1 - \big(1-\varepsilon^2\big)(\cos\theta)^2 \\ &= \varepsilon^2 + \big(1-\varepsilon^2\big)(\sin\theta)^2 \approx (\varepsilon + \sin\theta)^2. \end{aligned} \tag{62}$$

Hence,

$$J_- \lesssim \int_0^{\frac{\pi}{3}} \frac{(\sin\theta)^{4\alpha-1}\,\mathrm{d}\theta}{(\varepsilon+\sin\theta)^{8\alpha-1}} \lesssim \int_0^1 \frac{x^{4\alpha-1}\,\mathrm{d}x}{(\varepsilon+x)^{8\alpha-1}} \approx H_{4\alpha-1}^{-8\alpha+1}(\varepsilon) \approx \varepsilon^{-4\alpha+1}.$$

When $\theta \in [0, \pi/3]$ we have

$$\begin{aligned} 1-Q &= \left(1-\varepsilon\sin\theta-\sqrt{1-\varepsilon^2}\cos\theta\right)+\frac{2\sigma}{1+\sigma}\varepsilon\sin\theta \\ &\approx \left((1-\varepsilon\sin\theta)^2-\left(1-\varepsilon^2\right)(\cos\theta)^2\right)+\sigma\varepsilon\sin\theta=(\varepsilon-\sin\theta)^2+\sigma\varepsilon\sin\theta. \end{aligned} \tag{63}$$

Hence

$$J_+ \lesssim \int_0^{\frac{\pi}{3}} \frac{(\sin\theta)^{4\alpha-1}\mathrm{d}\theta}{((\varepsilon-\sin\theta)^2+\sigma\varepsilon\sin\theta)^{4\alpha-\frac{1}{2}}} \lesssim \int_0^1 \frac{x^{4\alpha-1}\mathrm{d}x}{((\varepsilon-x)^2+\sigma\varepsilon x)^{4\alpha-\frac{1}{2}}}.$$

To understand which term is dominant in the denominator of the last integral, observe that $(\varepsilon-x)^2<\sigma\varepsilon x$ if and only if $x_-<x<x_+$, where

$$x_\pm=\frac{\varepsilon}{2}\left(2+\sigma\pm\sqrt{4\sigma+\sigma^2}\right)\approx\varepsilon,$$

and notice that

$$x_+-x_-\approx\sigma^{1/2}\varepsilon, \qquad |x_\pm-\varepsilon|\approx\sigma^{1/2}\varepsilon.$$

We have

$$\int_0^1 \frac{x^{4\alpha-1}\,\mathrm{d}x}{((\varepsilon-x)^2+\sigma\varepsilon x)^{4\alpha-\frac{1}{2}}} \lesssim \int_{x_-}^{x_+} \frac{x^{4\alpha-1}\,\mathrm{d}x}{(\sigma\varepsilon x)^{4\alpha-\frac{1}{2}}} + \int_0^{x_-} \frac{x^{4\alpha-1}\,\mathrm{d}x}{(\varepsilon-x)^{8\alpha-1}} + \int_{x_+}^1 \frac{x^{4\alpha-1}\,\mathrm{d}x}{(x-\varepsilon)^{8\alpha-1}}.$$

The first integral in the right hand side is easily estimated,

$$\int_{x_-}^{x_+} \frac{x^{4\alpha-1}\mathrm{d}x}{(\sigma\varepsilon x)^{4\alpha-\frac{1}{2}}} \approx \frac{\varepsilon^{4\alpha-1}(\sigma^{1/2}\varepsilon)}{(\sigma\varepsilon^2)^{4\alpha-\frac{1}{2}}}=(\sigma\varepsilon)^{-4\alpha+1}.$$

For the second, since $8\alpha-1>1$, we have

$$\begin{aligned} \int_0^{x_-} \frac{x^{4\alpha-1}\,\mathrm{d}x}{(\varepsilon-x)^{8\alpha-1}} &\lesssim \int_0^{x_-/2} \frac{x^{4\alpha-1}\,\mathrm{d}x}{\varepsilon^{8\alpha-1}} + \int_{x_-/2}^{x_-} \frac{\varepsilon^{4\alpha-1}\,\mathrm{d}x}{((\varepsilon-x_-)+(x_--x))^{8\alpha-1}} \\ &\lesssim \varepsilon^{1-4\alpha}+\varepsilon^{4\alpha-1}\int_0^{\varepsilon} \frac{\mathrm{d}y}{(\sigma^{1/2}\varepsilon+y)^{8\alpha-1}} \\ &= \varepsilon^{1-4\alpha}+\varepsilon^{1-4\alpha}\int_0^1 \frac{\mathrm{d}t}{(\sigma^{1/2}+t)^{8\alpha-1}} \approx \varepsilon^{1-4\alpha}\left(1+\sigma^{\frac{1}{2}(1-8\alpha+1)}\right)\approx(\sigma\varepsilon)^{-4\alpha+1}. \end{aligned}$$

The third is treated in a similar way,

$$\int_0^{x_-} \frac{x^{4\alpha-1}\,\mathrm{d}x}{(x-\varepsilon)^{8\alpha-1}} \lesssim \int_{2x_+}^{1} \frac{x^{4\alpha-1}\mathrm{d}x}{x^{8\alpha-1}} + \int_{x_+}^{2x_+} \frac{\varepsilon^{4\alpha-1}\mathrm{d}x}{((x-x_+)+(x_+-\varepsilon))^{8\alpha-1}}$$

$$\lesssim \varepsilon^{1-4\alpha} + \varepsilon^{4\alpha-1} \int_0^{\varepsilon} \frac{\mathrm{d}y}{(y+\sigma^{1/2}\varepsilon)^{8\alpha-1}} \approx (\sigma\varepsilon)^{-4\alpha+1}. \quad \square$$

This completes the proof when $n \geqslant 3$.

When $n=2$ the above procedure can be simplified. We can always choose a coordinate system so that

$$\omega_1 = \left(\sqrt{1-\varepsilon^2}, \varepsilon\right), \qquad \omega_2 = \left(-\sqrt{1-\varepsilon^2}, \varepsilon, 0\right), \qquad \omega_0 = (\cos\theta, \sin\theta), \quad \theta \in [0,\pi].$$

We have

$$u = \varepsilon \sin\theta, \qquad v = Q = \sqrt{1-\varepsilon^2}\cos\theta + \frac{1-\sigma}{1+\sigma}\varepsilon\sin\theta, \qquad \mathrm{d}S_{\omega_0} = \mathrm{d}\theta.$$

We obtain

$$I(\sigma,\varepsilon) = \sigma^{2\alpha} \int_{u \lesssim 1-|v|} \frac{u^{4\alpha-1}}{(1-|v|)^{4\alpha-\frac{1}{2}}}\,\mathrm{d}S_{\omega_0} \leqslant \sigma^{2\alpha}\varepsilon^{4\alpha-1} \int \frac{(\sin\theta)^{4\alpha-1}}{(1-|Q|)^{4\alpha-\frac{1}{2}}}\,\mathrm{d}\theta,$$

and we already know from the analysis above that the last integral is bounded.

Remark 11.3. – If we compare the formulas, we can see that the above calculations, in practice, reduce the problem in dimension $n \geqslant 3$ to that of dimension $n=2$.

12. Frequency localized estimates

There is another way of proving Theorem 1.1, with the exception of some limiting cases, by using dyadic decompositions. This is done by following the steps below:

(1) Decompose the functions, and their product, into dyadic pieces relative to the frequency space.
(2) Obtain sharp estimates for the product of solutions corresponding to data supported in different dyadic regions.
(3) Sum the pieces together exploiting the orthogonality properties of the convolution structure.

Step 1: decomposition. Since we deal with L^2 theory we don't need any refined Littlewood–Paley theory, we can simply cut the frequency space into disjoint dyadic shells. Let χ_λ be the characteristic function of the region $\{\lambda \leqslant |\xi| < 2\lambda\}$ and define the operator $S_\lambda : f \mapsto f_\lambda$, where

$$\hat{f}_\lambda(\xi) = (S_\lambda f)\hat{}(\xi) = \chi_\lambda(\xi)\hat{f}(\xi).$$

Each function can be decomposed into dyadic pieces, $f = \sum_{\lambda \in 2^{\mathbb{Z}}} f_\lambda$. Throughout this section, sums in λ and μ will always be taken over dyadic values, $\sum_\lambda = \sum_{\lambda \in 2^{\mathbb{Z}}}$.

If u and v are solutions of the homogeneous wave equation, $\Box u = \Box v = 0$, with initial data [7] $u(0) = f$, $\partial_t u(0) = 0$ and $v(0) = g$, $\partial_t v(0) = 0$, then we can write the product uv as the sum [8] of three pieces, $uv = \Sigma_1 + \Sigma_2 + \Sigma_3$, where

$$\Sigma_1 = \sum_{\substack{\lambda,\mu\in 2^{\mathbb{Z}}\\ \mu\ll\lambda}} u_\lambda v_\mu, \qquad \Sigma_2 = \sum_{\substack{\lambda,\mu\in 2^{\mathbb{Z}}\\ \mu\approx\lambda}} u_\lambda v_\mu, \qquad \Sigma_3 = \sum_{\substack{\lambda,\mu\in 2^{\mathbb{Z}}\\ \mu\gg\lambda}} u_\lambda v_\mu.$$

The sums Σ_1 and Σ_3 contain the interactions between high and low frequencies and, as it was made clear by Example 5.2, their control is more critical when α_1 or α_2 are close to their maximum values allowed by (7). The sum Σ_2, instead, takes into account of the interactions between same frequency levels and, as it was made clear by Examples 5.4 and 5.5, its control is more critical when β_0 or $\alpha_1 + \alpha_2$ are close to their minimum values allowed by (8) and (6).

Step 2: dyadic estimates. This is really the main step. We prove the following:

THEOREM 12.1. – *Let $0 < \mu \lesssim \lambda$ and $\gamma \geqslant -(n-3)/4$. Take f and g to be functions whose Fourier transforms have compact supports and let u and v be the solutions of the homogeneous wave equation, $\Box u = \Box v = 0$, with initial data $u(0) = f$, $\partial_t u(0) = 0$ and $v(0) = g$, $\partial_t v(0) = 0$. We have the following dyadic estimates*:

- *suppose* $\operatorname{supp}\hat f \subset \{|\xi| \approx \lambda\}$ *and* $\operatorname{supp}\hat g \subset \{|\xi| \lesssim \mu\}$, *then*

$$\big\|D_-^\gamma(uv)\big\| \lesssim \mu^{\gamma+\frac{n-1}{2}}\|f\|\|g\|; \tag{64}$$

- *suppose* $\operatorname{supp}\hat f, \operatorname{supp}\hat g \subset \{|\xi| \approx \lambda\}$, *then*

$$\big\|D_-^\gamma S_\mu(u^+v^+)\big\| \lesssim \lambda^\gamma \mu^{\frac{n-1}{2}}\|f\|\|g\|, \tag{65}$$

$$\big\|D_-^\gamma S_\mu(u^+v^-)\big\| \lesssim \lambda^{\frac12} \mu^{\gamma+\frac{n-2}{2}}\|f\|\|g\|. \tag{66}$$

Proof of (64). – We have to prove that

$$\left\| \big||\tau| - |\xi|\big|^\gamma \int_{|\xi-\eta|\lesssim\mu} \delta\big(\tau - |\eta| \mp |\xi-\eta|\big) f(\eta) g(\xi-\eta)\,\mathrm{d}\eta \right\|_{L^2_{\tau,\xi}} \lesssim \mu^{\gamma+\frac{n-1}{2}}\|f\|\|g\|.$$

Applying Cauchy–Schwarz, as in the proof of Proposition 7.1, this reduces to show that

$$\sup_{\tau,\xi}\big||\tau| - |\xi|\big|^{2\gamma} \int_{|\xi-\eta|\lesssim\mu} \delta\big(\tau - |\eta| - \mp|\xi-\eta|\big)\,\mathrm{d}\eta \lesssim \mu^{2\gamma+n-1}.$$

The case $\mu \approx \lambda$ is easy and can be treated precisely as in the proof of Proposition 7.1 or Proposition 8.1, by using Lemma 4.3 or Lemma 4.5 with $a = b = 0$. Assume instead that $\mu \ll \lambda$, then $|\eta| \approx \lambda$, $|\xi - \eta| \lesssim \mu$ imply $|\tau| \approx |\xi| \approx \lambda$ and $\big||\tau| - |\xi|\big| \lesssim \mu$. What we need is then the following lemma, together with the condition $2\gamma + ((n-3)/2) \geqslant 0$. $\Box$

[7] For simplicity and without loss of generality we take $\phi_0 = f$, $\phi_1 = 0$ and $\psi_0 = g$, $\psi_1 = 0$.

[8] This way of decomposing a product is essentially the main ingredient used in paradifferential calculus.

LEMMA 12.2. – *Let* $|\xi| \gg \mu$. *Then*

$$\int\limits_{|\xi-\eta|\lesssim\mu} \delta\big(\tau-|\eta|\mp|\xi-\eta|\big)\,\mathrm{d}\eta \lesssim \big||\tau|-|\xi|\big|^{\frac{n-3}{2}}\mu^{\frac{n+1}{2}}.$$

Proof. – Consider the case of integration over the ellipsoid. We make use of formula (25) in Lemma 4.1 with $F(|\eta|,|\xi-\eta|)=\chi(|\xi-\eta|\lesssim\mu)$,

$$\begin{aligned} I(\tau,\xi) &= \int\limits_{|\xi-\eta|\lesssim\mu} \delta\big(\tau-|\eta|-|\xi-\eta|\big)\,\mathrm{d}\eta \\ &= \big(\tau^2-|\xi|^2\big)^{\frac{n-3}{2}} \int\limits_{\substack{-1<x<1\\ \tau-|\xi|x\lesssim\mu}} \big(\tau^2-|\xi|^2x^2\big)\big(1-x^2\big)^{\frac{n-3}{2}}\,\mathrm{d}x. \end{aligned}$$

We are in a situation where $\tau\approx|\xi|$ and

$$0<1-x\leqslant\frac{\tau-|\xi|x}{|\xi|}\lesssim\frac{\mu}{|\xi|}\ll 1,$$

hence

$$\begin{aligned} I &\lesssim \big(\tau-|\xi|\big)^{\frac{n-3}{2}}\tau^{\frac{n-1}{2}}\mu \int\limits_{0<1-x\lesssim\mu/|\xi|} (1-x)^{\frac{n-3}{2}}\,\mathrm{d}x \\ &\approx \left(\frac{\tau}{|\xi|}\right)^{\frac{n-1}{2}}\big(\tau-|\xi|\big)^{\frac{n-3}{2}}\mu^{\frac{n+1}{2}} \lesssim \big(\tau-|\xi|\big)^{\frac{n-3}{2}}\mu^{\frac{n+1}{2}}. \end{aligned}$$

The case of integration over the hyperboloid is done in a similar way using Lemma 4.4. □

Proof of (65). – The case $\mu\approx\lambda$ is already contained in (64), hence we can assume $\mu\ll\lambda$. If $|\eta|\approx|\xi-\eta|\approx\lambda\gg\mu\approx|\xi|$, then $\tau=|\eta|+|\xi-\eta|\gg|\xi|$ and the D_-^γ becomes essentially λ^γ and doesn't create any problem (in this case γ can take any value). More over, in this case the vectors η and $\xi-\eta$ must occupy almost opposite positions and, since the interaction ξ takes place on a set of size μ, we can reduce to the case where $f=f_{(Q)}$ and $g=g_{(-Q)}$ have supports contained in opposite cubes Q and $-Q$ of size $\approx\mu$ at a distance $\approx\lambda$ from the origin.

Indeed, let $\{Q\}_{Q\in\mathcal{Q}}$ be a family of cubes of size μ that covers the supports of f and g. Decompose $f=\sum_Q f_Q$ and $g=\sum_Q g_Q$, where $f_Q=\chi_Q f$ with χ_Q the characteristic function of the cube Q. Let u_Q and v_Q be the solutions of the homogeneous wave equation corresponding to initial data f_Q and g_Q. Then $S_\mu(uv)=\sum_{Q_1,Q_2}S_\mu(u_{Q_1}v_{Q_2})$ and the sum can be taken over just the pairs of cubes $Q_1,Q_2\in\mathcal{Q}$ such that Q_1+Q_2 is contained in the region $\{|\xi|\leqslant 5\mu\}$. For each fixed $Q_1\in\mathcal{Q}$ there is at most a finite number, depending only on n, of cubes $Q_2\in\mathcal{Q}$ so that the pair (Q_1,Q_2) has that property. This implies that

$$\sum_{\substack{Q_1,Q_2\in\mathcal{Q}\\ Q_1+Q_2\subset\{|\xi|\leqslant5\mu\}}} \|f_{Q_1}\|\|g_{Q_2}\|\lesssim\|f\|\|g\|.$$

Moreover, for each of these pairs of cubes is always possible to find a cube Q of size 10μ such that $Q_1\in Q$ and $Q_2\in -Q$. Hence, we just have to prove

$$\big\|u^+_{(Q)}v^+_{(-Q)}\big\|_{L^2_{\tau,\xi}(|\xi|\approx\mu)}\lesssim\mu^{\frac{n-1}{2}}\|f_{(Q)}\|\|g_{(-Q)}\|.$$

If we apply directly Cauchy–Schwarz we reduce to show that

$$\sup_{\substack{\tau\approx\lambda\\|\xi|\approx\mu}} \int\limits_{|\eta|\in Q} \delta\big(\tau-|\eta|-|\xi-\eta|\big)\,\mathrm{d}\eta \lesssim \mu^{n-1},$$

which is a consequence of the following lemma. □

LEMMA 12.3. – *Let $|\xi|\ll\tau$ and Q be a cube of size μ. Then*

$$\int\limits_{\eta\in Q} \delta\big(\tau-|\eta|-|\xi-\eta|\big)\,\mathrm{d}\eta \lesssim \mu^{n-1}.$$

Proof. – We have

$$\int\limits_{\eta\in Q} \delta\big(\tau-|\eta|-|\xi-\eta|\big)\,\mathrm{d}\eta = \int\limits_{\mathcal{E}(\tau,\xi)\cap Q} \frac{\mathrm{d}S_\eta}{|\nabla_\eta(|\eta|+|\xi-\eta|)|}.$$

The vectors η and $\xi-\eta$ are almost opposite, hence

$$\big|\nabla_\eta\big(|\eta|+|\xi-\eta|\big)\big| = \left|\frac{\eta}{|\eta|}-\frac{\xi-\eta}{|\xi-\eta|}\right| \approx 1.$$

The condition $|\xi|\ll\tau$ forces the shape of the ellipsoid to be close to a ball of radius much bigger than μ, hence $|\mathcal{E}(\tau,\xi)\cap Q|\lesssim\mu^{n-1}$. □

Proof of (66). – As for the proof of (65), we can suppose that $\mu\ll\lambda$ and $f=f_{(Q)}$ and $g=g_{(-Q)}$ have supports contained in opposite cubes Q and $-Q$ of size μ at a distance λ from the origin.

To treat the resulting expression we have to use again the doubling method discussed in Proposition 8.2. This procedure works in any dimension.[9] However, as it was recently observed by Tataru in [21], there is a somewhat simpler proof of the dyadic estimate (66) in dimension $n=2$, which we sketch below for completeness.

The observation is that, in this case, we can follow the same pattern of proof as in the previous estimates. We can in fact apply directly Cauchy–Schwarz and reduce to show that

$$\big(|\xi|-|\tau|\big)^{2\gamma}\int\limits_{Q} \delta\big(\tau-|\eta|+|\xi-\eta|\big)\,\mathrm{d}\eta \lesssim \lambda\mu^{2\gamma}. \tag{67}$$

Compute the gradient of the quantity inside the delta function,

$$\nabla_\eta\big(\tau-|\eta|+|\xi-\eta|\big) = -\frac{\eta}{|\eta|}-\frac{\xi-\eta}{|\xi-\eta|}.$$

From the equation $\tau=|\eta|-|\xi-\eta|$ we infer that

$$\left|\frac{\eta}{|\eta|}+\frac{\xi-\eta}{|\xi-\eta|}\right|^2 = \frac{|\xi|^2-\tau^2}{|\eta||\xi-\eta|}.$$

[9] For a discussion of the doubling method in the particular case of $n=2$ see [11] as well as [14].

Hence, using formula (17) and the fact that we are integrating on a piece of a curve of length $|Q\cap\mathcal{H}(\tau,\xi)|\lesssim\mu$,

$$\int_{Q}\delta\big(\tau-|\eta|+|\xi-\eta|\big)\,\mathrm{d}\eta=\big(|\xi|^2-\tau^2\big)^{-1/2}\int_{Q\cap\mathcal{H}(\tau,\xi)}|\eta|^{1/2}|\xi-\eta|^{1/2}\,\mathrm{d}S_\eta$$
$$\lesssim\big(|\xi|-|\tau|\big)^{-1/2}\mu^{-1/2}\lambda^{1/2}\lambda^{1/2}\mu,$$

and, since $|\xi|-|\tau|\lesssim\mu$, we obtain the desired bound in (67) when $2\gamma\geqslant 1/2$.

Now assume $n\geqslant 3$. This time we perform a Cauchy–Schwarz inequality only after the doubling of the integral, and reduce to prove that

$$\iint_{|\xi|\approx\mu}\big(|\xi|-|\tau|\big)^{2\gamma}\delta\big(\tau-|\eta|+|\xi-\eta|\big)\delta\big(\tau+|\zeta|-|\xi-\zeta|\big)\,\mathrm{d}\tau\,\mathrm{d}\xi\lesssim\lambda\mu^{2\gamma+n-2}.$$

Using the fact that $|\xi|-|\tau|\lesssim\mu$, it is enough to look at the most critical case $\gamma=-(n-3)/4$. By scaling invariance we can also fix $\mu=1$. We have to show that

$$I=\frac{1}{\lambda}\iint_{|\xi|\approx 1}\frac{\delta(\tau-|\eta|+|\xi-\eta|)\delta(\tau+|\zeta|-|\xi-\zeta|)}{(|\xi|-|\tau|)^{\frac{n-3}{2}}}\,\mathrm{d}\tau\,\mathrm{d}\xi\lesssim 1. \tag{68}$$

This is an integral restricted to the ellipsoid $\mathcal{E}=\mathcal{E}(\eta,\zeta)$, of a type that we have already encountered.

We first observe that this integral is much simpler to treat in dimension $n=3$, since the denominator in (68) disappears, and therefore we have

$$I=\frac{1}{\lambda}\int_{|\xi|\approx 1}\delta\big(|\eta|+|\zeta|-|\xi-\eta|-|\xi-\zeta|\big)\,\mathrm{d}\xi=\frac{1}{\lambda}\int_{\substack{\xi\in\mathcal{E}(\eta,\zeta)\\|\xi|\approx 1}}\frac{\mathrm{d}S_\xi}{|\nabla_\xi(|\xi-\eta|+|\xi-\zeta|)|}.$$

Now, from the equation $|\eta|+|\zeta|=|\xi-\eta|+|\xi-\zeta|$ it follows that

$$|\eta||\zeta|+\eta\cdot\zeta=|\xi-\eta||\xi-\zeta|+(\xi-\eta)\cdot(\xi-\zeta)$$

and we have

$$\big|\nabla_\xi\big(|\xi-\eta|+|\xi-\zeta|\big)\big|^2=\frac{2(|\eta||\zeta|+\eta\cdot\zeta)}{|\xi-\eta||\xi-\zeta|}\approx\frac{|\eta||\zeta|\theta^2}{|\xi-\eta||\xi-\zeta|}\approx\frac{1}{\lambda^2}$$

where $\theta\approx\lambda^{-1}$ is the angle between η and $-\zeta$. Hence, since the region of integration has diameter comparable to 1, we have

$$I\approx\int_{\substack{\xi\in\mathcal{E}(\eta,\zeta)\\|\xi|\approx 1}}\mathrm{d}S_\xi\lesssim 1.$$

In general for $n>3$ it is not so easy to adapt this geometric argument, we opt instead for a more analytic approach, with the help of the parameterization for $\mathcal{E}(\eta,\zeta)$ used in Section 11. Choose a coordinate system so that

$$\frac{\eta}{|\eta|} = \big(\sqrt{1-\varepsilon^2}, \varepsilon, 0, \dots, 0\big),$$
$$\frac{\zeta}{|\zeta|} = \big(-\sqrt{1-\varepsilon^2}, \varepsilon, 0, \dots, 0\big),$$
$$\frac{\xi}{|\xi|} = \big(R\cos\theta, R\sin\theta, \sqrt{1-R^2}\omega''\big), \quad 0 \leqslant R \leqslant 1,\ 0 \leqslant \theta \leqslant \pi,\ \omega'' \in \mathbb{S}^{n-3}.$$

By the same calculations which led to (58), taking $\alpha = 1/4$, we have

$$I \lesssim \iint_D \frac{R(1-R)^{\frac{n-4}{2}}\,\mathrm{d}R\,\mathrm{d}\theta}{(1-R|Q|)^{\frac{n-1}{2}}},$$

with the quantity Q still defined by (60) with

$$\varepsilon = \frac{1}{2}\Big|\frac{\eta}{|\eta|} + \frac{\zeta}{|\zeta|}\Big| \ll 1, \qquad \sigma = \frac{|\zeta|}{|\eta|} \approx 1.$$

The integration is restricted to the region $D \subset \{0 \leqslant R \leqslant 1, 0 \leqslant \theta \leqslant \pi\}$ corresponding to $\mathcal{E}(\eta,\zeta) \cap \{|\xi| \approx 1\}$. Using formulas (50), (55) and (56) the condition $|\xi| \approx 1$ becomes equivalent to

$$1 - R|Q| \approx \lambda\varepsilon R\sin\theta, \tag{69}$$

which is the key to proving the desired estimate.

In the region D_1 where $R \leqslant 1/2$ or $|Q| \leqslant 1/2$ there is no singularity in the integrand and

$$I_1 = \iint_{D_1} \frac{R(1-R)^{\frac{n-4}{2}}\,\mathrm{d}R\,\mathrm{d}\theta}{(1-R|Q|)^{\frac{n-1}{2}}} \lesssim 1.$$

Consider the region D_2 where $R > 1/2$, $|Q| > 1/2$. From (69) we have

$$\max\{1-R, 1-|Q|\} \leqslant 1 - R|Q| \approx \lambda\varepsilon\sin\theta$$

and it follows that we must have $\sin\theta \lesssim \lambda\varepsilon$, since, repeating the argument of (62) or (63), we have

$$(\varepsilon \pm \sin\theta)^2 \lesssim 1 - |Q| \lesssim \lambda\varepsilon\sin\theta.$$

We compute first the integral with respect to R over $0 \leqslant 1 - R \lesssim \lambda\varepsilon\theta$ and then estimate the integration with respect to θ over $0 \leqslant \sin\theta \lesssim \min\{\lambda\varepsilon, 1\}$,

$$I_2 = \iint_{D_2} \frac{(1-R)^{\frac{n-4}{2}}\,\mathrm{d}R\,\mathrm{d}\theta}{(1-R|Q|)^{\frac{n-1}{2}}} \lesssim \int_{\sin\theta \lesssim \lambda\varepsilon} \frac{(\lambda\varepsilon\sin\theta)^{\frac{n-2}{2}}}{(\lambda\varepsilon\sin\theta)^{\frac{n-1}{2}}}\,\mathrm{d}\theta \simeq \frac{\min\{\lambda\varepsilon, 1\}^{1/2}}{(\lambda\varepsilon)^{1/2}} \lesssim 1. \quad \square$$

Step 3: summing up. We now use the estimates proved in Step 2 to sum the pieces together. We start with estimating Σ_1. Since the Fourier transform of the piece $p_\lambda = \sum_{\mu \ll \lambda} u_\lambda v_\mu$ has support in a region where $|\xi| \approx \lambda$, we infer that $\{p_\lambda\}_{\lambda \in 2^{\mathbb{Z}}}$ is a sequence of almost orthogonal functions, therefore

$$\big\|D^{\beta_0} D_+^{\beta_+} D_-^{\beta_-} \Sigma_1\big\|^2 \approx \sum_\lambda \lambda^{2\beta_0}\lambda^{2\beta_+}\|D_-^{\beta_-} p_\lambda\|^2.$$

Using the dyadic estimate (64) of Theorem 12.1 with $\gamma = \beta_-$ we have

$$\|D_-^{\beta_-} p_\lambda\| \leqslant \sum_{\mu \ll \lambda} \|u_\lambda v_\mu\| \lesssim \sum_{\mu \ll \lambda} \mu^{\beta_- + \frac{n-1}{2}} \lambda^{-\alpha_1} \mu^{-\alpha_2} \|D^{\alpha_1} f_\lambda\| \|D^{\alpha_2} g_\mu\|.$$

By (4) and (7) we have that

$$\varepsilon = -(\beta_0 + \beta_+ - \alpha_1) = \beta_- + \frac{n-1}{2} - \alpha_2 \geqslant 0,$$

hence

$$\|D^{\beta_0} D_+^{\beta_+} D_-^{\beta_-} \Sigma_1\|^2 \lesssim \sum_\lambda \|D^{\alpha_1} f_\lambda\|^2 \left(\sum_{\mu \ll \lambda} \left(\frac{\mu}{\lambda}\right)^\varepsilon \|D^{\alpha_2} g_\mu\| \right)^2$$
$$\lesssim \sum_\lambda \|D^{\alpha_1} f_\lambda\|^2 \sum_{\mu \ll \lambda} \|D^{\alpha_2} g_\mu\|^2 \sum_{\mu' \ll \lambda} \left(\frac{\mu'}{\lambda}\right)^{2\varepsilon}.$$

To ensure the boundedness of the last geometric series we need $\varepsilon > 0$, which requires a strict inequality in (7). In that case,

$$\|D^{\beta_0} D_+^{\beta_+} D_-^{\beta_-} \Sigma_1\|^2 \lesssim \sum_\lambda \|D^{\alpha_1} f_\lambda\|^2 \sum_\mu \|D^{\alpha_2} g_\mu\|^2 = \|D^{\alpha_1} f\|^2 \|D^{\alpha_2} g\|^2.$$

The term Σ_3 can be estimated exactly in the same way.

Now consider Σ_2. First decompose Σ_2 into orthogonal dyadic pieces

$$\|D^{\beta_0} D_+^{\beta_+} D_-^{\beta_-} \Sigma_2\|^2 \approx \sum_\mu \mu^{2\beta_0} \|D_+^{\beta_+} D_-^{\beta_-} S_\mu \Sigma_2\|^2 \lesssim \sum_\mu \mu^{2\beta_0} \left(\sum_{\lambda \gtrsim \mu} \|D_+^{\beta_+} D_-^{\beta_-} S_\mu (u_\lambda v_\lambda)\| \right)^2.$$

To be rigorous, we should have written $u_\lambda \sum_{\lambda' \approx \lambda} v_{\lambda'}$ instead of $u_\lambda v_\lambda$, but we can forget about this detail since the operators $S'_\lambda = \sum_{\lambda' \approx \lambda} S_{\lambda'}$ behave essentially like the S_λ's, indeed we have

$$\sum_\lambda \|S'_\lambda f\|^2 = \sum_\lambda \sum_{\lambda' \approx \lambda} \|S_{\lambda'} f\|^2 \simeq \sum_\lambda \|S_\lambda f\|^2 = \|f\|^2.$$

We have to treat the $(++)$ and $(+-)$ cases differently because of the different behavior of the operator D_+. In the $(++)$ case, D_+ corresponds to $\tau \approx \lambda$ and using (65) we find

$$\|D_+^{\beta_+} D_-^{\beta_-} S_\mu (u_\lambda v_\lambda)\| \lesssim \lambda^{\beta_+} \lambda^{\beta_-} \mu^{\frac{n-1}{2}} \lambda^{-\alpha_1 - \alpha_2} \|D^{\alpha_1} f_\lambda\| \|D^{\alpha_2} g_\lambda\|.$$

In the $(+-)$ case, D_+ corresponds to $|\xi| \approx \mu$ and using (66) we find

$$\|D_+^{\beta_+} D_-^{\beta_-} S_\mu (u_\lambda v_\lambda)\| \lesssim \mu^{\beta_+} \lambda^{\frac{1}{2}} \mu^{\beta_- + \frac{n-2}{2}} \lambda^{-\alpha_1 - \alpha_2} \|D^{\alpha_1} f_\lambda\| \|D^{\alpha_2} g_\lambda\|.$$

In both cases, using the scaling condition (4) and setting $\varepsilon = \beta_0 + \frac{n-1}{2}$ for the $(++)$ case and $\varepsilon = \alpha_1 + \alpha_2 - \frac{1}{2}$ for the $(+-)$ case, we reduce to

$$\|D^{\beta_0} D_+^{\beta_+} D_-^{\beta_-} \Sigma_2\|^2 \lesssim \sum_\mu \left(\sum_{\lambda \gtrsim \mu} \left(\frac{\mu}{\lambda}\right)^\varepsilon \|D^{\alpha_1} f_\lambda\| \|D^{\alpha_2} g_\lambda\| \right)^2.$$

To estimate an expression like this we write the square of the sum over λ doubling the sum, and interchange the order of summation,

$$\sum_{\mu}\left(\sum_{\lambda\gtrsim\mu}\left(\frac{\mu}{\lambda}\right)^{\varepsilon}a_{\lambda}b_{\lambda}\right)^{2}=\sum_{\lambda_1,\lambda_2}\left(\sum_{\mu\lesssim\min\{\lambda_1,\lambda_2\}}\frac{\mu^{2\varepsilon}}{\lambda_1^{\varepsilon}\lambda_2^{\varepsilon}}\right)a_{\lambda_1}b_{\lambda_1}a_{\lambda_2}b_{\lambda_2}$$

$$\simeq\sum_{\lambda_1,\lambda_2}\left(\frac{\min\{\lambda_1,\lambda_2\}}{\max\{\lambda_1\lambda_2\}}\right)^{\varepsilon}a_{\lambda_1}b_{\lambda_1}a_{\lambda_2}b_{\lambda_2}$$

$$\lesssim\left(\sum_{\lambda}a_{\lambda}b_{\lambda}\right)^{2}\lesssim\left(\sum_{\lambda}a_{\lambda}^{2}\right)\left(\sum_{\lambda}b_{\lambda}^{2}\right).$$

Here to ensure the convergence of the geometric series in μ we had to require that $\varepsilon>0$; this is equivalent to (6) in the $(++)$ case, while it requires a strict inequality in (8) for the $(+-)$ case. With this assumption we finally obtain

$$\left\|D^{\beta_0}D_+^{\beta_+}D_-^{\beta_-}\Sigma_2\right\|^2\lesssim\sum_{\lambda}\|D^{\alpha_1}f_{\lambda}\|^2\sum_{\mu}\|D^{\alpha_2}g_{\mu}\|^2=\|D^{\alpha_1}f\|^2\|D^{\alpha_2}g\|^2.$$

Remark 12.4. – Using the dyadic estimates of Theorem 12.1 we have been able to reprove Theorem 1.1 except for the cases which correspond to equality in (7) and (8). These correspond to the boundary of the triangle $A_0A_1A_2$ of Fig. 1 and we know that they are allowed, in view of our previous proof, for $\beta_->-(n-3)/4$.

On the other hand the estimates (64), (65), (66) of Theorem 12.1 show, roughly, that for frequency localized data the estimate (3) is true even in the exceptional cases (9), (10) and also for $\beta_0=-(n-1)/2$. This explains why the examples of Section 6 had to take into consideration the interactions among several different frequency levels to produce a weak logarithmic divergence.

13. Null form estimates

Theorem 1.1 has its roots in the study of the improved regularity properties for solutions of nonlinear wave equations with a special null structure. Typically this structure can be described in terms of null bilinear forms. In general a null quadratic form is an expression of the form $Q(\phi,\psi)$ whose symbol $q(\tau,\xi;\lambda,\eta)$ vanishes whenever the space-time vectors $(\tau,\xi),(\lambda,\eta)$ are null and parallel. We give below some of the main examples of bilinear null forms which have appeared so far in the study of Lagrangean field theories, such as the Wave maps and Yang–Mills equations.

Consider first a generic bilinear form $Q(\phi,\psi)$ whose symbol is given by $q(\tau,\xi;\lambda,\eta)$. In other words,

$$\widetilde{Q(\phi,\psi)}(\tau,\xi)=\int q(\tau-\lambda,\xi-\eta;\lambda,\eta)\tilde{\phi}(\tau-\lambda,\xi-\eta)\tilde{\psi}(\lambda,\eta)\,\mathrm{d}\lambda\,\mathrm{d}\eta.$$

Under the assumption that ϕ and ψ are solutions of the homogeneous wave equation, using the notation of Section 3, we can take

$$\phi=\phi^+,\qquad\psi=\psi^\pm.$$

Then the space-time Fourier transform of Q can be written in the form

$$\widetilde{Q(\phi^+,\psi^\pm)}(\tau,\xi)\simeq\int q^\pm(\eta,\xi-\eta)\delta\big(\tau-|\eta|\mp|\xi-\eta|\big)\widehat{\phi_0}(\eta)\widehat{\psi_0}(\xi-\eta)\,\mathrm{d}\eta,$$

where $q^{\pm}(\eta,\xi-\eta)=q(|\xi-\eta|,\xi-\eta;\pm|\eta|,\eta)$ will be called the reduced symbol of the bilinear form Q.

DEFINITION 13.1. – *We define the null forms Q_0, Q_{ij}, Q_{0j} by*

$$\begin{aligned}
Q_0(\phi,\psi)&=-\partial_t\phi\partial_t\psi+\nabla_x\phi\cdot\nabla_x\psi,\\
Q_{ij}(\phi,\psi)&=\partial_i\phi\partial_j\psi-\partial_j\phi\partial_i\psi,\quad 1\leqslant i<j\leqslant n,\\
Q_{0j}(\phi,\psi)&=\partial_t\phi\partial_j\psi-\partial_j\phi\partial_t\psi,\quad 1\leqslant j\leqslant n.
\end{aligned}$$

Their correspondent reduced bilinear symbols are

$$\begin{aligned}
q_0^{\pm}(\eta,\zeta)&=\pm|\eta||\zeta|-\eta\cdot\zeta,\\
q_{ij}^{\pm}(\eta,\zeta)&=-\eta_i\zeta_j+\eta_j\zeta_i,\\
q_{0j}^{\pm}(\eta,\zeta)&=-|\eta|\zeta_j\pm\eta_j|\zeta|.
\end{aligned}$$

These are the null forms that appear in the nonlinear structure of important lagrangean field theories, like Wave Maps and Yang–Mills equations. Indeed, a good model problem for the equations of wave maps is given by the equation (see [6,14])

$$\Box u=Q_0(u,u),$$

and the Yang–Mills equations, with an appropriate choice of a gauge condition, have a nonlinear structure with a quadratic part that is well described by the model system of equations (see [8, 15])

$$\Box u^I=c^I_{JK}D^{-1}Q_{ij}(u^J,u^K)+c^I_{JK}Q_{ij}(D^{-1}u^J,u^K).$$

These equations have better regularity properties than a semilinear wave equation with a generic quadratic nonlinearity, like

$$\Box u=|Du|^2.$$

The main cancellation properties of the above null quadratic forms are summarized in the following lemma.

LEMMA 13.2. – *Let η and ζ be two vectors in $\mathbb{R}^n$. Then we have*:
(1) *for Q_0,*

$$|\eta||\zeta|-\eta\cdot\zeta\approx\big(|\eta|+|\zeta|\big)\big(|\eta|+|\zeta|-|\eta+\zeta|\big),\tag{70}$$

$$|\eta||\zeta|+\eta\cdot\zeta\approx|\eta+\zeta|\big(|\eta+\zeta|-\big||\eta|-|\zeta|\big|\big);\tag{71}$$

(2) *for Q_{ij},*

$$|\eta\wedge\zeta|\lesssim|\eta|^{1/2}|\zeta|^{1/2}|\eta+\zeta|^{1/2}\big(|\eta|+|\zeta|-|\eta+\zeta|\big)^{1/2},\tag{72}$$

$$|\eta\wedge\zeta|\lesssim|\eta|^{1/2}|\zeta|^{1/2}|\eta+\zeta|^{\frac12}\big(|\eta+\zeta|-\big||\eta|-|\zeta|\big|\big)^{1/2};\tag{73}$$

(3) *for Q_{0j},*

$$\big||\eta|\zeta-|\zeta||\eta|\big|\approx|\eta|^{1/2}|\zeta|^{1/2}\big(|\eta|+|\zeta|\big)^{1/2}\big(|\eta|+|\zeta|-|\eta+\zeta|\big)^{1/2},\tag{74}$$

$$\big||\eta|\zeta+|\zeta||\eta|\big|\approx|\eta|^{1/2}|\zeta|^{1/2}|\eta+\zeta|^{1/2}\big(|\eta+\zeta|-\big||\eta|-|\zeta|\big|\big)^{1/2}.\tag{75}$$

Proof. – Inequalities (70) and (71) follow immediately from the identities

$$2\big(|\eta||\zeta| - \eta\cdot\zeta\big) = \big(|\eta| + |\zeta| + |\eta+\zeta|\big)\big(|\eta| + |\zeta| - |\eta+\zeta|\big),$$
$$2\big(|\eta||\zeta| + \eta\cdot\zeta\big) = \big(|\eta+\zeta| + |\eta| - |\zeta|\big)\big(|\eta+\zeta| - |\eta| + |\zeta|\big).$$

Inequalities (72) and (73) follow from the identity

$$\begin{aligned}|\eta\wedge\zeta|^2 &= \big(|\eta||\zeta| + \eta\cdot\zeta\big)\big(|\eta||\zeta| - \eta\cdot\zeta\big)\\ &= \frac{1}{4}\big(|\eta+\zeta| + |\eta| - |\zeta|\big)\big(|\eta+\zeta| - |\eta| + |\zeta|\big)\big(|\eta| + |\zeta| + |\eta+\zeta|\big)\big(|\eta| + |\zeta| - |\eta+\zeta|\big),\end{aligned}$$

indeed, assume $|\eta| \geqslant |\zeta|$, then we just have to notice that

$$\big(|\eta+\zeta| + |\eta| - |\zeta|\big) \approx |\eta+\zeta|, \qquad \big(|\eta+\zeta| - |\eta| + |\zeta|\big) \lesssim |\zeta|,$$
$$\big(|\eta| + |\zeta| + |\eta+\zeta|\big) \approx |\eta|, \qquad \big(|\eta| + |\zeta| - |\eta+\zeta|\big) \lesssim |\zeta|.$$

Inequalities (74) and (75) follow in a similar way from the identities

$$\big||\eta|\zeta - |\zeta|\eta|\big|^2 = |\eta||\zeta|\big(|\eta| + |\zeta| + |\eta+\zeta|\big)\big(|\eta| + |\zeta| - |\eta+\zeta|\big),$$
$$\big||\eta|\zeta + |\zeta|\eta|\big|^2 = |\eta||\zeta|(|\eta+\zeta| + |\eta| - |\zeta|)(|\eta+\zeta| - |\eta| + |\zeta|). \quad \square$$

Let $n \geqslant 2$. Let ϕ, ψ be the solutions of (1), (2), and for simplicity take $\phi_1 = \psi_1 = 0$. We have the following corollaries of Theorem 1.1.

COROLLARY 13.3. – *The estimate*

$$\big\|D^{\beta_0}D_+^{\beta_+}D_-^{\beta_-}Q_0(\phi,\psi)\big\|_{L^2(\mathbb{R}^{1+n})} \lesssim \|D^{\alpha_1}\phi_0\|_{L^2(\mathbb{R}^n)}\|D^{\alpha_2}\psi_0\|_{L^2(\mathbb{R}^n)},$$

holds when $\alpha_1, \alpha_2, \beta_0, \beta_+, \beta_-$ satisfy the following conditions:

$$\begin{aligned}&\beta_0 + \beta_+ + \beta_- = \alpha_1 + \alpha_2 - \frac{n+3}{2},\\ &\beta_- \geqslant -\frac{n+1}{4}, \qquad \beta_0 > -\frac{n-1}{2},\\ &\alpha_1 + \alpha_2 \geqslant \frac{1}{2}, \qquad \alpha_i \leqslant \beta_- + \frac{n+1}{2},\\ &(\alpha_1+\alpha_2, \beta_-) \neq \left(\frac{1}{2}, -\frac{n+1}{4}\right), \qquad (\alpha_i, \beta_-) \neq \left(\frac{n+1}{4}, -\frac{n+1}{4}\right).\end{aligned}$$

Proof. – We could use (70) and (71) to argue at the level of symbols, but in this case we can just use the identity

$$2Q_0(\phi,\psi) = \square(\phi\psi) - (\square\phi)\psi - \phi(\square\psi),$$

to deduce that $Q_0(\phi,\psi)$ behaves like $D_+D_-(\phi\psi)$. More precisely, let ϕ' and ψ' be the solutions of the homogeneous wave equation with data ϕ_0' and ψ_0', where

$$\hat{\phi}_0'(\xi) = \big|\hat{\phi}_0(\xi)\big|, \qquad \hat{\psi}_0'(\xi) = \big|\hat{\psi}_0(\xi)\big|,$$

then $|\tilde{Q}_0(\phi,\psi)| \lesssim (D_+D_-(\phi'\psi'))\tilde{}$ and

$$\big\|D^{\beta_0}D_+^{\beta_+}D_-^{\beta_-}Q_0(\phi,\psi)\big\| \lesssim \big\|D^{\beta_0}D_+^{\beta_++1}D_-^{\beta_-+1}(\phi'\psi')\big\|.$$

Also $\|D^{\alpha_1}\phi_0'\| = \|D^{\alpha_1}\phi_0\|$ and $\|D^{\alpha_2}\psi_0'\| = \|D^{\alpha_2}\psi_0\|$. The result then follows from Theorem 1.1 with β_+ and β_- replaced by $\beta_+ + 1$ and $\beta_- + 1$.

COROLLARY 13.4. – *The estimate*

$$\left\|D^{\beta_0} D_+^{\beta_+} D_-^{\beta_-} Q_{ij}(\phi,\psi)\right\|_{L^2(\mathbb{R}^{1+n})} \lesssim \|D^{\alpha_1}\phi_0\|_{L^2(\mathbb{R}^n)} \|D^{\alpha_2}\psi_0\|_{L^2(\mathbb{R}^n)},$$

holds when $\alpha_1, \alpha_2, \beta_0, \beta_+, \beta_-$ satisfy the following conditions:

$$\begin{aligned}
&\beta_0 + \beta_+ + \beta_- = \alpha_1 + \alpha_2 - \frac{n+3}{2},\\
&\beta_0 > -\frac{n+1}{2}, \qquad \beta_- \geqslant -\frac{n-1}{4},\\
&\alpha_1 + \alpha_2 \geqslant \frac{3}{2}, \qquad \alpha_i \leqslant \beta_- + \frac{n+1}{2}, \quad i = 1,2,\\
&(\alpha_1 + \alpha_2, \beta_-) \neq \left(\frac{3}{2}, -\frac{n-1}{4}\right), \qquad (\alpha_i, \beta_-) \neq \left(\frac{n+3}{4}, -\frac{n-1}{4}\right).
\end{aligned}$$

Proof. – The inequalities (72) for the $(++)$ case and (73) for the $(+-)$, on the level of symbols, allow us to replace $Q_{ij}(\phi,\psi)$ by $D^{1/2}D_-^{1/2}(D^{1/2}\phi D^{1/2}\psi)$, and then we can apply Theorem 1.1 with β_0, β_-, α_1 and α_2 replaced by

$$\beta_0 + \frac{1}{2}, \quad \beta_- + \frac{1}{2}, \quad \alpha_1 - \frac{1}{2} \quad \text{and} \quad \alpha_2 - \frac{1}{2}.$$

Doing in this way we would obtain the estimate valid in the range above except for the condition on β_0, for which we would have only $\beta_0 > -n/2$. But recall that the condition on β_0 becomes relevant only in the $(++)$ case when we are in the region where $|\xi| \ll \tau$. (See Example 5.4 and Proposition 7.3.) There we can use the fact that $|\eta \wedge \zeta| \lesssim |\eta|^{1/2}|\zeta|^{1/2}|\eta + \zeta|$, which is a consequence of (73), to replace $Q_{ij}(\phi,\psi)$ by $D(D^{1/2}\phi D^{1/2}\psi)$ and therefore gaining another half power for β_0. □

COROLLARY 13.5. – *The estimate*

$$\left\|D^{\beta_0} D_+^{\beta_+} D_-^{\beta_-} Q_{0j}(\phi,\psi)\right\|_{L^2(\mathbb{R}^{1+n})} \lesssim \|D^{\alpha_1}\phi_0\|_{L^2(\mathbb{R}^n)} \|D^{\alpha_2}\psi_0\|_{L^2(\mathbb{R}^n)},$$

holds when $\alpha_1, \alpha_2, \beta_0, \beta_+, \beta_-$ satisfy the following conditions:

$$\begin{aligned}
&\beta_0 + \beta_+ + \beta_- = \alpha_1 + \alpha_2 - \frac{n+3}{2},\\
&\beta_0 > -\frac{n-1}{2}, \qquad \beta_- \geqslant -\frac{n-1}{4},\\
&\alpha_1 + \alpha_2 \geqslant \frac{3}{2}, \qquad \alpha_i \leqslant \beta_- + \frac{n+1}{2},\\
&(\alpha_1 + \alpha_2, \beta_-) \neq \left(\frac{3}{2}, -\frac{n-1}{4}\right), \qquad (\alpha_i, \beta_-) \neq \left(\frac{n+3}{4}, -\frac{n-1}{4}\right).
\end{aligned}$$

Proof. – Using inequalities (74) for the $(++)$ case and (75) for the $(+-)$, on the level of symbols, we can treat $Q_{0j}(\phi,\psi)$ as if it were $D_+^{1/2}D_-^{1/2}(D^{1/2}\phi D^{1/2}\psi)$, and then we apply Theorem 1.1 with β_+, β_-, α replaced by $\beta_+ + 1/2$, $\beta_- + 1/2$ and $\alpha_2 - 1/2$. □

Remark 13.6. – The conditions on the exponents $\beta_0, \beta_+, \beta_-, \alpha_1, \alpha_2$ in Corollaries 13.3, 13.4 and 13.5 are not only sufficient but also necessary for the validity of the estimates. One can see it using the same examples given in Section 5 and 6 and applying Lemma 13.2 to the symbols of the null forms. The only thing to notice is that for the null forms Q_{ij} our examples correspond to the region where we can use (72) and (73) with $\lesssim$ replaced by $\approx$.

14. Conjectures for $L^q L^r$ estimates

Now that the L^2 theory is completely understood, it makes sense to investigate the possible generalization of these bilinear estimates to the L^p context. The starting point in this respect are the classical Strichartz inequalities which we can cast in a bilinear form as follows (see [3] and [5]):

$$\|\phi\psi\|_{L_t^q L_x^r} \lesssim \left(\|D^\alpha \phi_0\|_{L^2(\mathbb{R}^n)} + \|D^{\alpha-1}\phi_1\|_{L^2(\mathbb{R}^n)}\right)\left(\|D^\alpha \psi_0\|_{L^2(\mathbb{R}^n)} + \|D^{\alpha-1}\psi_1\|_{L^2(\mathbb{R}^n)}\right). \tag{76}$$

They hold whenever q, r, α satisfy the conditions

$$\frac{1}{q} \leqslant \frac{n-1}{2}\left(1-\frac{1}{r}\right), \qquad 2\alpha = n\left(1-\frac{1}{r}\right) - \frac{1}{q}, \qquad (q,r,n) \neq (\infty, 1, 3).$$

In view of the L^2 bilinear estimates presented above, it makes sense to consider generalizations of the type

$$\begin{aligned} &\left\|D^{\beta_0} D_+^{\beta_+} D_-^{\beta_-}(\phi\psi)\right\|_{L_t^q L_x^r} \\ &\quad \lesssim \left(\|D^{\alpha_1}\phi_0\|_{L^2(\mathbb{R}^n)} + \|D^{\alpha_1-1}\phi_1\|_{L^2(\mathbb{R}^n)}\right)\left(\|D^{\alpha_2}\psi_0\|_{L^2(\mathbb{R}^n)} + \|D^{\alpha_2-1}\psi_1\|_{L^2(\mathbb{R}^n)}\right), \end{aligned} \tag{77}$$

where by $L_t^q L_x^r$ we denote the Lebesque space with mixed exponents defined through the norm

$$\|F\|_{L_t^q L_x^r} = \left(\int_{-\infty}^{+\infty}\left(\int_{\mathbb{R}^n} \left|F(t,x)\right|^r \mathrm{d}x\right)^{q/r} \mathrm{d}t\right)^{1/q}.$$

Some estimates of this type, with $\beta_+ = \beta_- = 0$ and $\beta_0 < 0$, have been recently proved, and made use of, in [15].

THEOREM 14.1 ([15]). – *The estimate*

$$\left\|D^{\beta}(\phi\psi)\right\|_{L_t^q L_x^r} \lesssim \left(\|D^\alpha \phi_0\|_{L^2(\mathbb{R}^n)} + \|D^{\alpha-1}\phi_1\|_{L^2(\mathbb{R}^n)}\right)\left(\|D^\alpha \psi_0\|_{L^2(\mathbb{R}^n)} + \|D^{\alpha-1}\psi_1\|_{L^2(\mathbb{R}^n)}\right)$$

hold when

$$\frac{1}{q} = \frac{n-1}{2}\left(1-\frac{1}{r}\right), \qquad \beta = 2\alpha + \frac{1}{q} - n\left(1-\frac{1}{r}\right), \qquad \alpha > \frac{1}{2q}, \quad \beta \leqslant 0.$$

The proof is based on the Littlewood–Paley decomposition and some sharp dyadic version of the classical Strichartz inequalities.

Remark 14.2. – The investigation concerning the $L_t^q L_x^r$ bilinear estimates was motivated by the hope that such estimates could give further insight into the problem of optimal regularity for nonlinear wave equations. As an example, consider a system of equations of the form

$$\Box u = Q(u,u),$$

where Q stands for an arbitrary null form (see Definition 13.1). In [6] one has used $L_t^2 L_x^2$ bilinear estimates

$$\|Q(\phi,\psi)\|_{L^2(\mathbb{R}^{1+n})} \lesssim \|\phi_0\|_{\dot{H}^{\frac{n+1}{2}}(\mathbb{R}^n)} \|\psi_0\|_{\dot{H}^1(\mathbb{R}^n)},$$

for two solutions, ϕ and ψ, of the homogeneous wave equation with data $\phi(0)=\phi_0$, $\psi(0)=\psi_0$, $\partial_t\phi(0)=\partial_t\psi(0)=0$, together with the standard Duhamel's principle, to prove $H^{(n+1)/2}$ well-posedness results. Duhamel's principle suggets, however, that an estimate of the type

$$\|Q(\phi,\psi)\|_{L_t^1 L_x^2} \lesssim \|\phi_0\|_{\dot{H}^{\alpha_1}(\mathbb{R}^n)} \|\psi_0\|_{\dot{H}^{\alpha_2}(\mathbb{R}^n)},$$

with optimal choices of α_1 and α_2 might imply a better result. Unfortunately, Example 14.14 shows that this approach fails. Nevertheless, we suspect that the general form of bilinear $L_t^q L_x^r$ estimates may turn out to be very useful.

Let's discuss now the estimates (77). The following condition follows easily by a straightforward scaling argument:

$$\beta_0+\beta_++\beta_- - \frac{1}{q} - \frac{n}{r} = \alpha_1+\alpha_2-n. \tag{78}$$

Using the techniques of Section 5 we want to find other necessary conditions for the exponents $q, r, \beta_0, \beta_+, \beta_-, \alpha_1, \alpha_2$, involved in (77).

The algorithm is still the same:

(1) We have to check the boundedness of the operators

$$B_{(++)}, B_{(+-)} : L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \to L_t^q L_x^r,$$

defined in (21), (22), (23), (24).

(2) We test it relative to the characteristic functions of two sets F and G which may depend on a small parameter ε.

(3) We choose F and G in an appropriate way, corresponding to the geometry of the singularity of the weight $W_\pm$. We want also to be able to estimate the order of magnitude of the weight $W_\pm$ in terms of parameter ε when $\eta \in F$ and $\zeta \in G$, let's say $W_\pm(\eta,\zeta) \approx W_*$ for some constant $W_* = W_*(\varepsilon)$.

(4) To each choice of F and G there corresponds a region R in physical space, depending on ε, so that, when $\eta \in F$, $\zeta \in G$ and $(t,x) \in R$ the phase $\varphi_\pm(t,x;\eta,\zeta)$ is essentially constant (or we can subtract from it a function of (t,x) to make it essentially constant).

(5) This set R will span a time interval of length $T_* = T_*(\varepsilon)$ and all its sections at fixed t will have the same measure, say equal to $X_* = X_*(\varepsilon)$.

(6) Putting these ingredients together we obtain that

$$\frac{\|B_{(+\pm)}\|_{L_t^q L_x^r}}{\|\chi_F\|_{L^2}\|\chi_G\|_{L^2}} \gtrsim W_* |F|^{1/2} |G|^{1/2} T_*^{1/q} X_*^{1/r}.$$

(7) All the quantities on the right hand side will be estimated in terms of power of our parameter ε. The resulting combination will be of order ε^d for some exponent d which depends on the quantities $q, r, \alpha., \beta.$.

(8) For the estimate (77) to be true, ε^d has to be uniformly bounded as $\varepsilon \to 0$, therefore $d \geqslant 0$. This implies a necessary condition on the parameters $q, r, \alpha_1, \alpha_2, \beta_0, \beta_+, \beta_-$.

We classify the examples into two categories, the first contains those with data supported on the same dyadic shell in frequency space, the second contains those with interaction between high and low frequencies.

Interactions of the data at the same frequency level

Example 14.3. – Consider the $(++)$ case. Take

$$\begin{aligned} F &= \{\eta\colon\ 1<\eta_1<2,\ \varepsilon<\eta_2<2\varepsilon,\ |\eta''|<\varepsilon\}, \quad |F|\simeq\varepsilon^{n-1},\\ G &= \{\zeta\colon\ 1<\zeta_1<2,\ -2\varepsilon<\zeta_2<-\varepsilon,\ |\zeta''|<\varepsilon\}, \quad |G|\simeq\varepsilon^{n-1}. \end{aligned}$$

Looking back at (23) and estimating each of the factors $|\xi|^{\beta_0}$, τ_+^{β}, $(\tau-|\xi|)^{\beta_-}$, $|\eta|^{\alpha_1}$ and $|\zeta|^{\alpha_2}$, we have

$$W_+\approx W_*=\frac{1^{\beta_0}1^{\beta_+}(\varepsilon^2)^{\beta_-}}{1^{\alpha_1}1^{\alpha_2}}=\varepsilon^{2\beta_-}.$$

Now write the phase as follows

$$\begin{aligned} \varphi_+ &= t\big(|\eta|-\eta_1+|\zeta|-\zeta_1\big)+(t+x_1)(\eta_1+\zeta_1)+x'\cdot(\eta'+\zeta')\\ &= t\,\mathrm{O}\big(\varepsilon^2\big)+(t+x_1)\,\mathrm{O}(1)+x'\cdot\mathrm{O}(\varepsilon). \end{aligned}$$

Hence, we can choose

$$R=\{(t,x)\colon\ t=\mathrm{O}\big(\varepsilon^{-2}\big),\ x_1=-t+\mathrm{O}(1),\ x'=\mathrm{O}\big(\varepsilon^{-1}\big)\}, \qquad T_*\simeq\varepsilon^{-2}, \qquad X_*\simeq\varepsilon^{-(n-1)}.$$

Then

$$W_*|F|^{\frac12}|G|^{\frac12}T_*^{\frac1q}X_*^{\frac1r}\simeq\varepsilon^{2\beta_-}\varepsilon^{\frac{n-1}{2}}\varepsilon^{\frac{n-1}{2}}\varepsilon^{-\frac2q}\varepsilon^{-\frac{n-1}{r}}=\varepsilon^d,$$

and we obtain the condition

$$d=2\beta_--\frac2q+(n-1)\left(1-\frac1r\right)\geqslant0. \tag{79}$$

Example 14.4. – Consider the $(++)$ case. Take

$$\begin{aligned} F &= \{\eta\colon\ |\eta_1-1|<\varepsilon,\ \varepsilon<\eta_2<2\varepsilon,\ |\eta''|<\varepsilon\}, \quad |F|\simeq\varepsilon^{n},\\ G &= \{\zeta\colon\ |\zeta_1+1|<\varepsilon,\ -2\varepsilon<\zeta_2<-\varepsilon,\ |\zeta''|<\varepsilon\}, \quad |G|\simeq\varepsilon^{n}. \end{aligned}$$

We have

$$\begin{aligned} &W_+\approx W_*=\frac{\varepsilon^{\beta_0}1^{\beta_+}1^{\beta_-}}{1^{\alpha_1}1^{\alpha_2}}=\varepsilon^{\beta_0},\\ &\varphi_+-2t=t\big(|\eta|-1+|\zeta|-1\big)+x_1(\eta_1-1+\zeta_1+1)+x'\cdot(\eta'+\zeta')=t\,\mathrm{O}(\varepsilon)+x\cdot\mathrm{O}(\varepsilon). \end{aligned}$$

Hence, we can choose

$$R=\{(t,x)\colon\ t=\mathrm{O}\big(\varepsilon^{-1}\big),\ x=\mathrm{O}\big(\varepsilon^{-1}\big)\}, \qquad T_*\simeq\varepsilon^{-1}, \qquad X_*\simeq\varepsilon^{-n}.$$

Then

$$W_*|F|^{\frac12}|G|^{\frac12}T_*^{\frac1q}X_*^{\frac1r}\simeq\varepsilon^{\beta_0}\varepsilon^{\frac n2}\varepsilon^{\frac n2}\varepsilon^{-\frac1q}\varepsilon^{-\frac nr}=\varepsilon^d,$$

and we obtain the condition

$$d=\beta_0-\frac{1}{q}+n\left(1-\frac{1}{r}\right)\geqslant 0. \tag{80}$$

Example 14.5. – Consider the $(++)$ case. Take

$$F=\{\eta\colon\ |\eta_1-1|<\varepsilon^2,\ \varepsilon<\eta_2<2\varepsilon,\ |\eta''|<\varepsilon\},\quad |F|\simeq\varepsilon^{n+1},$$
$$G=\{\zeta\colon\ |\zeta_1+1|<\varepsilon^2,\ -2\varepsilon<\zeta_2<-\varepsilon,\ |\zeta''|<\varepsilon\},\quad |G|\simeq\varepsilon^{n+1}.$$

We have

$$W_+\approx W_*=\frac{\varepsilon^{\beta_0}1^{\beta_+}1^{\beta_-}}{1^{\alpha_1}1^{\alpha_2}}=\varepsilon^{\beta_0}.$$

$$\varphi_+-2t=t\big(|\eta|-1+|\zeta|-1\big)+x_1(\eta_1-1+\zeta_1+1)+x'\cdot(\eta'+\zeta')=t\,\mathrm{O}\big(\varepsilon^2\big)+x_1\,\mathrm{O}\big(\varepsilon^2\big)+x'\cdot\mathrm{O}(\varepsilon).$$

Hence, we can choose

$$R=\{(t,x)\colon\ t=\mathrm{O}\big(\varepsilon^{-2}\big),\ x_1=\mathrm{O}\big(\varepsilon^{-2}\big),\ x'=\mathrm{O}\big(\varepsilon^{-1}\big)\},\qquad T_*\simeq\varepsilon^{-2},\qquad X_*\simeq\varepsilon^{-(n+1)}.$$

Then

$$W_*|F|^{\frac12}|G|^{\frac12}T_*^{\frac1q}X_*^{\frac1r}\simeq\varepsilon^{\beta_0}\varepsilon^{\frac{n+1}{2}}\varepsilon^{\frac{n+1}{2}}\varepsilon^{-\frac2q}\varepsilon^{-\frac{n+1}{r}}=\varepsilon^d,$$

and we obtain the condition

$$d=\beta_0-\frac{2}{q}+(n+1)\left(1-\frac{1}{r}\right)\geqslant 0. \tag{81}$$

Example 14.6. – Consider the $(++)$ case. Take

$$F=\{\eta\colon\ |\eta_1-1|<\varepsilon^2,\ |\eta_2|<\varepsilon^2,\ |\eta''|<\varepsilon\},\quad |F|\simeq\varepsilon^{n+2},$$
$$G=\{\zeta\colon\ |\zeta_1|<\varepsilon^2,\ |\zeta_2-1|<\varepsilon^2,\ |\zeta''|<\varepsilon\},\quad |G|\simeq\varepsilon^{n+2}.$$

We have

$$W_+\approx W_*=\frac{1^{\beta_0}1^{\beta_+}1^{\beta_-}}{1^{\alpha_1}1^{\alpha_2}}=1,$$
$$\begin{aligned}\varphi_--2t-x_1-x_2&=t\big(|\eta|-1+|\zeta|-1\big)+x_1(\eta_1-1+\zeta_1)+x_2(\eta_2+\zeta_2-1)+x''\cdot(\eta''+\zeta'')\\&=t\,\mathrm{O}\big(\varepsilon^2\big)+x_1\,\mathrm{O}\big(\varepsilon^2\big)+x_2\,\mathrm{O}\big(\varepsilon^2\big)+x''\cdot\mathrm{O}(\varepsilon).\end{aligned}$$

Hence, we can choose

$$R=\{(t,x)\colon\ t=\mathrm{O}\big(\varepsilon^{-2}\big),\ x_1=\mathrm{O}\big(\varepsilon^{-2}\big),\ x_2=\mathrm{O}\big(\varepsilon^{-2}\big),\ x''=\mathrm{O}\big(\varepsilon^{-1}\big)\},$$
$$T_*\simeq\varepsilon^{-2},\qquad X_*\simeq\varepsilon^{-(n+2)}.$$

Then

$$W_*|F|^{\frac12}|G|^{\frac12}T_*^{\frac1q}X_*^{\frac1r}\simeq\varepsilon^{\frac{n+2}{2}}\varepsilon^{\frac{n+2}{2}}\varepsilon^{-\frac2q}\varepsilon^{-\frac{n+2}{r}}=\varepsilon^d,$$

and we obtain the condition

$$d=-\frac{2}{q}+(n+2)\left(1-\frac{1}{r}\right)\geqslant 0. \tag{82}$$

However, condition (82) is not sharp and can be improved with a more refined choice of the sets F and G. The following example was communicated to us by T. Tao.

Example 14.7 ([19]). – Consider the $(++)$ case. Take

$$F=\{\eta\colon\ |\eta_1-1|<\varepsilon,\ |\eta_2-1|<\varepsilon^2,\ |\eta''|<\varepsilon\},\quad |F|\simeq\varepsilon^{n+1},$$
$$G=\{\zeta\colon\ |\zeta_1-1|<\varepsilon,\ |\zeta_2+1|<\varepsilon^2,\ |\zeta''|<\varepsilon\},\quad |G|\simeq\varepsilon^{n+1}.$$

When $\eta\in F$ and $\zeta\in G$ we have:

$$|\eta|^2=\big(1+(\eta_1-1)\big)^2+\big(1+(\eta_2-1)\big)^2+|\eta''|^2=2+2(\eta_1-1)+\mathrm{O}(\varepsilon^2),$$
$$|\zeta|^2=\big(1+(\zeta_1-1)\big)^2+\big(-1+(\zeta_2+1)\big)^2+|\zeta''|^2=2+2(\zeta_1-1)+\mathrm{O}(\varepsilon^2).$$

Taking square roots we find

$$|\eta|+|\zeta|=2\sqrt{2}+\frac{1}{\sqrt{2}}(\eta_1-1+\zeta_1-1)+\mathrm{O}(\varepsilon^2).$$

Hence

$$\begin{aligned}&\varphi_--2\sqrt{2}t-2x_1\\&=t\,\mathrm{O}(\varepsilon^2)+\left(\frac{1}{\sqrt{2}}t+x_1\right)(\eta_1-1+\zeta_1-1)+x_2(\eta_2-1+\zeta_2+1)+x''\cdot(\eta''+\zeta'')\\&=t\,\mathrm{O}(\varepsilon^2)+x_1\,\mathrm{O}(\varepsilon)+x_2\,\mathrm{O}(\varepsilon^2)+x''\cdot\mathrm{O}(\varepsilon)\end{aligned}$$

and of course we have

$$W_+\approx W_*=\frac{1^{\beta_0}1^{\beta_+}1^{\beta_-}}{1^{\alpha_1}1^{\alpha_2}}=1.$$

We can choose the region R according to our decomposition of the phase,

$$R=\{(t,x)\colon\ t=\mathrm{O}(\varepsilon^{-2}),\ x_1=\mathrm{O}(\varepsilon^{-1}),\ x_2=\mathrm{O}(\varepsilon^{-2}),\ x''=\mathrm{O}(\varepsilon^{-1})\},$$
$$T_*\simeq\varepsilon^{-2},\qquad X_*\simeq\varepsilon^{-(n+1)}.$$

Then

$$W_*|F|^{\frac12}|G|^{\frac12}T_*^{\frac1q}X_*^{\frac1r}\simeq\varepsilon^{\frac{n+1}{2}}\varepsilon^{\frac{n+1}{2}}\varepsilon^{-\frac2q}\varepsilon^{-\frac{n+1}{r}}=\varepsilon^d,$$

and we obtain the condition

$$d=-\frac{2}{q}+(n+1)\left(1-\frac{1}{r}\right)\geqslant 0. \tag{83}$$

Observe that in the above example all weights corresponding to the differential operators are essentially like 1. Hence condition (83) is due solely to the geometric properties of cones and to the bilinear structure of the product. There is an analogous example in the simpler contest of bilinear restriction problems for the sphere, which we will discuss in Section 17 (see Example 17.5).

Example 14.8. – Consider the $(+-)$ case. Take

$$F=\{\eta\colon\ |\eta_1-1|<\varepsilon,\ \varepsilon<\eta_2<2\varepsilon,\ |\eta''|<\varepsilon\},\quad |F|\simeq\varepsilon^n,$$
$$G=\{\zeta\colon\ |\zeta_1+1|<\varepsilon,\ -2\varepsilon<\zeta_2<-\varepsilon,\ |\zeta''|<\varepsilon\},\quad |G|\simeq\varepsilon^n.$$

We have

$$W_- \approx W_* = \frac{\varepsilon^{\beta_0+\beta_+}\varepsilon^{\beta_-}}{1^{\alpha_1}1^{\alpha_2}} = \varepsilon^{\beta_0+\beta_++\beta_-},$$

$$\begin{aligned}\varphi_- &= t\big(|\eta| - \eta_1 - |\zeta| - \zeta_1\big) + (t+x_1)(\eta_1+\zeta_1) + x'\cdot(\eta'+\zeta')\\ &= t\,\mathrm{O}\big(\varepsilon^2\big) + (t+x_1)\,\mathrm{O}(\varepsilon) + x'\cdot \mathrm{O}(\varepsilon).\end{aligned}$$

Hence, we can choose

$$R = \big\{(t,x)\colon\ t = \mathrm{O}\big(\varepsilon^{-2}\big),\ x_1 = -t + \mathrm{O}\big(\varepsilon^{-1}\big),\ x' = \mathrm{O}\big(\varepsilon^{-1}\big)\big\}, \qquad T_* \simeq \varepsilon^{-2}, \qquad X_* \simeq \varepsilon^{-n}.$$

Then

$$W_*|F|^{\frac12}|G|^{\frac12}T_*^{\frac1q}X_*^{\frac1r} \simeq \varepsilon^{\beta_0+\beta_++\beta_-}\varepsilon^{\frac n2}\varepsilon^{\frac n2}\varepsilon^{-\frac2q}\varepsilon^{-\frac nr} = \varepsilon^d,$$

and we obtain the condition

$$d = \beta_0 + \beta_+ + \beta_- - \frac{2}{q} + n\left(1 - \frac{1}{r}\right) \geqslant 0. \tag{84}$$

There is another example for the $(+-)$ interaction at the same frequency level which is associated to an unusual quadratic-cubic scaling and, for some values of q and r is sharper than Example 14.8.

Example 14.9 ([19]). – Consider the $(+-)$ case. Take

$$\begin{aligned}F &= \big\{\eta\colon\ |\eta_1 - 1| < \varepsilon,\ |\eta_2 - \varepsilon| < \varepsilon^3,\ |\eta''| < \varepsilon^2\big\}, \quad |F| \simeq \varepsilon^{2n},\\ G &= \big\{\zeta\colon\ |\zeta_1 + 1| < \varepsilon,\ |\zeta_2 - \varepsilon| < \varepsilon^3,\ |\zeta''| < \varepsilon^2\big\}, \quad |G| \simeq \varepsilon^{2n}.\end{aligned}$$

We have

$$W_- \approx W_* = \frac{\varepsilon^{\beta_0+\beta_+}\varepsilon^{\beta_-}}{1^{\alpha_1}1^{\alpha_2}} = \varepsilon^{\beta_0+\beta_++\beta_-}.$$

Before estimating the phase, observe that for $\eta \in F$ we have

$$\begin{aligned}|\eta| &= \eta_1\left(1 + \frac{\eta_2^2}{\eta_1^2} + \frac{|\eta''|^2}{\eta_1^2}\right)^{\frac12} = \eta_1 + \frac{\eta_2^2}{2\eta_1} + \mathrm{O}\big(\varepsilon^4\big) = \eta_1 + \frac{\varepsilon^2}{2}\frac{1}{\eta_1} + \mathrm{O}\big(\varepsilon^4\big)\\ &= \eta_1 + \frac{\varepsilon^2}{2} - (\eta_1 - 1)\frac{\varepsilon^2}{2} + \mathrm{O}\big(\varepsilon^4\big),\end{aligned}$$

and similarly, for $\zeta \in G$,

$$|\zeta| = -\zeta_1 + \frac{\varepsilon^2}{2} + (\eta_1 + 1)\frac{\varepsilon^2}{2} + \mathrm{O}\big(\varepsilon^4\big).$$

This implies that

$$|\eta| - |\zeta| = (\eta_1 + \zeta_1)\left(1 - \frac{\varepsilon^2}{2}\right) + \mathrm{O}\big(\varepsilon^4\big).$$

The phase can then be written as

$$\varphi_- - 2\varepsilon x_2 = t\,\mathrm{O}\big(\varepsilon^4\big) + \left[t\left(1 - \frac{\varepsilon^2}{2}\right) + x_1\right]\mathrm{O}(\varepsilon) + x_2\,\mathrm{O}\big(\varepsilon^3\big) + x''\cdot\mathrm{O}\big(\varepsilon^2\big).$$

Hence, we can choose

$$R=\left\{(t,x)\colon\ t=\mathrm{O}\big(\varepsilon^{-4}\big),\ x_1=-t\left(1-\frac{\varepsilon^2}{2}\right)+\mathrm{O}\big(\varepsilon^{-1}\big),\ x_2=\mathrm{O}\big(\varepsilon^{-3}\big),\ x''=\mathrm{O}\big(\varepsilon^{-2}\big)\right\},$$

which corresponds to $T_*\simeq\varepsilon^{-4}$ and $X_*\simeq\varepsilon^{-2n}$. Then

$$W_*|F|^{\frac12}|G|^{\frac12}T_*^{\frac1q}X_*^{\frac1r}\simeq\varepsilon^{\beta_0+\beta_++\beta_-}\varepsilon^n\varepsilon^n\varepsilon^{-\frac4q}\varepsilon^{-\frac{2n}{r}}=\varepsilon^d,$$

and we obtain the condition

$$d=\beta_0+\beta_++\beta_--\frac4q+2n\left(1-\frac1r\right)\geqslant 0. \tag{85}$$

Condition (85) is sharper than (84) when q and r belong to the region where

$$\frac1q>\frac n2\left(1-\frac1r\right).$$

The last example for the $(+-)$ interaction at the same frequency level is an adaptation of an argument of Selberg [16].

Example 14.10. – Consider the $(+-)$ case. Take

$$\begin{aligned}&F=\big\{\eta\colon\ 1<\eta_1<2,\ |\eta'|<\varepsilon\big\},\quad |F|\simeq\varepsilon^{n-1},\\ &G=\big\{\zeta\colon\ 1<|\zeta|<2,\ \big||\zeta|+\zeta_1-1\big|<\varepsilon^2\big\},\quad |G|\simeq\varepsilon^2.\end{aligned}$$

Since the angle between $\eta\in F$ and $-\zeta\in -G$ remains large, we have

$$W_-\approx W_*=\frac{1^{\beta_0+\beta_+}1^{\beta_-}}{1^{\alpha_1}1^{\alpha_2}}=1.$$

Observe that for $\eta\in F$ and $\zeta\in G$ we have

$$\big(|\eta|-|\zeta|\big)-(\eta_1+\zeta_1)+1=\big(|\eta|-\eta_1\big)-\big(|\zeta|+\zeta_1-1\big)=\mathrm{O}\big(\varepsilon^2\big).$$

Hence we can write the phase as

$$\varphi_-+t=t\,\mathrm{O}\big(\varepsilon^2\big)+(t+x_1)\,\mathrm{O}(1)+x'\cdot\mathrm{O}(1),$$

which corresponds to $T_*\simeq\varepsilon^{-2}$ and $X_*\simeq 1$. Then

$$W_*|F|^{\frac12}|G|^{\frac12}T_*^{\frac1q}X_*^{\frac1r}\simeq 1\varepsilon^{\frac{n-1}{2}}\varepsilon^1\varepsilon^{-\frac2q}1=\varepsilon^d,$$

and we obtain the condition

$$d=\frac{n+1}{2}-\frac2q\geqslant 0. \tag{86}$$

Condition (86) is relevant only when $n=2$.

Interactions of the data at different frequency levels

Example 14.11 ([19]). – Consider the $(++)$ case. Let $L=\varepsilon^{-1}$ and define

$$\begin{aligned}F&=\{\eta\colon\ L<\eta_1<L+1,\ 1<\eta_2<2,\ |\eta''|<1\},\quad |F|\simeq 1,\\ G&=\{\zeta\colon\ |\zeta_1|<1,\ -2<\zeta_2<-1,\ |\zeta''|<1\},\quad |G|\simeq 1.\end{aligned}$$

We have

$$\begin{aligned}W_+\approx W_*&=\frac{L^{\beta_0}L^{\beta_+}1^{\beta_-}}{L^{\alpha_1}1^{\alpha_2}}=L^{\beta_0+\beta_+-\alpha_1},\\ \varphi_+-L(t+x_1)&=t\big(|\eta|-L+|\zeta|\big)+x_1(\eta_1-L+\zeta_1)+x'\cdot(\eta'+\zeta')\\ &=t\,\mathrm{O}(1)+x_1\,\mathrm{O}(1)+x'\cdot\mathrm{O}(1).\end{aligned}$$

Hence, we can choose

$$R=\{(t,x)\colon\ t=\mathrm{O}(1),\ x=\mathrm{O}(1)\},\qquad T_*\simeq 1,\qquad X_*\simeq 1.$$

Then

$$W_*|F|^{\frac12}|G|^{\frac12}T_*^{\frac1q}X_*^{\frac1r}\simeq L^{\beta_0+\beta_+-\alpha_1}=L^{-d},$$

and, in the limit $L\to\infty$, we obtain the condition

$$d=-\beta_0-\beta_++\alpha_1\geqslant 0. \tag{87}$$

Example 14.12. – Consider the $(++)$ case. Define

$$\begin{aligned}F&=\{\eta\colon\ L<\eta_1<2L,\ \sqrt{L}<\eta_2<2\sqrt{L},\ |\eta''|<\sqrt{L}\},\quad |F|\simeq L^{\frac{n+1}{2}},\\ G&=\{\zeta\colon\ |\zeta_1|<1,\ -2<\zeta_2<-1,\ |\zeta''|<1\},\quad |G|\simeq 1.\end{aligned}$$

We have

$$\begin{aligned}W_+\approx W_*&=\frac{L^{\beta_0}L^{\beta_+}1^{\beta_-}}{L^{\alpha_1}1^{\alpha_2}}=L^{\beta_0+\beta_+-\alpha_1},\\ \varphi_+&=t\big(|\eta|-\eta_1+|\zeta|-\zeta_1\big)+(t+x_1)(\eta_1+\zeta_1)+x'\cdot(\eta'+\zeta')\\ &=t\,\mathrm{O}(1)+(t+x_1)\,\mathrm{O}(L)+x'\cdot\mathrm{O}\big(L^{1/2}\big).\end{aligned}$$

Hence, we can choose

$$R=\{(t,x)\colon\ t=\mathrm{O}(1),\ x_1=-t+\mathrm{O}\big(L^{-1}\big),\ x'=\mathrm{O}\big(L^{-1/2}\big)\},\qquad T_*\simeq 1,\qquad X_*\simeq L^{-\frac{n+1}{2}}.$$

Then

$$W_*|F|^{\frac12}|G|^{\frac12}T_*^{\frac1q}X_*^{\frac1r}\simeq L^{\beta_0+\beta_+-\alpha_1}L^{\frac{n+1}{4}}L^{-\frac{n+1}{2r}}=L^{-d},$$

and we obtain the condition

$$d=-\beta_0-\beta_++\alpha_1-\frac{n+1}{2}\left(\frac12-\frac1r\right)\geqslant 0. \tag{88}$$

Example 14.13. – Consider the $(++)$ case. Let $L=\varepsilon^{-1}$ and define

$$F=\{\eta\colon\ L<\eta_1<2L,\ L<\eta_2<2L,\ |\eta''|<L\},\quad |F|\simeq L^n,$$
$$G=\{\zeta\colon\ |\zeta_1|<1,\ -2<\zeta_2<-1,\ |\zeta''|<1\},\quad |G|\simeq 1.$$

We have

$$W_+\approx W_*=\frac{L^{\beta_0}L^{\beta_+}1^{\beta_-}}{L^{\alpha_1}1^{\alpha_2}}=L^{\beta_0+\beta_+-\alpha_1},$$
$$\varphi_+=t\big(|\eta|+|\zeta|\big)+x\cdot(\eta+\zeta)=t\,\mathrm{O}(L)+x\cdot\mathrm{O}(L).$$

Hence, we can choose

$$R=\{(t,x)\colon\ t=\mathrm{O}(1),\ x=\mathrm{O}\big(L^{-1}\big)\},\qquad T_*\simeq 1,\qquad X_*\simeq L^{-n}.$$

Then

$$W_*|F|^{\frac12}|G|^{\frac12}T_*^{\frac1q}X_*^{\frac1r}\simeq L^{\beta_0+\beta_+-\alpha_1}L^{\frac n2}L^{-\frac1q}L^{-\frac nr}=L^{-d},$$

and we obtain the condition

$$d=-\beta_0-\beta_++\alpha_1+\frac1q-n\left(\frac12-\frac1r\right)\geqslant 0. \tag{89}$$

The last example that we present was also communicated to us by T. Tao.

Example 14.14 ([19]). – Consider the $(++)$ case. Let ε be a small positive number and define

$$F=\{\eta\colon\ \big||\eta|+\eta_1-2\varepsilon\big|<\varepsilon^2,\ |\eta_1|<\varepsilon/2\},\quad |F|\approx\varepsilon^{n+1},$$
$$G=\{\zeta\colon\ |\zeta_1+1|<\varepsilon,\ |\zeta'|<\varepsilon\},\quad |G|\simeq\varepsilon^n.$$

We have

$$W_+\approx W_*=\frac{1^{\beta_0}1^{\beta_+}\varepsilon^{\beta_-}}{\varepsilon^{\alpha_1}1^{\alpha_2}}=L^{\beta_--\alpha_1},$$

and for the phase

$$\begin{aligned}\varphi_++(1-2\varepsilon)t-x_1&=t\big(|\eta|+\eta_1-2\varepsilon+|\zeta|+\zeta_1\big)+(x_1-t)(\eta_1+\zeta_1-1)+x'\cdot(\eta'+\zeta')\\&=t\,\mathrm{O}\big(\varepsilon^2\big)+(x_1-t)\,\mathrm{O}(\varepsilon)+x'\cdot\mathrm{O}(\varepsilon).\end{aligned}$$

Hence, we can choose

$$R=\{(t,x)\colon\ t=\mathrm{O}\big(\varepsilon^{-2}\big),\ x=\mathrm{O}\big(\varepsilon^{-1}\big)\},\qquad T_*\simeq\varepsilon^{-2},\qquad X_*\simeq\varepsilon^{-n}.$$

Then

$$W_*|F|^{\frac12}|G|^{\frac12}T_*^{\frac1q}X_*^{\frac1r}\simeq\varepsilon^{\beta_--\alpha_1+\frac{n+1}{2}+\frac n2-\frac2q-\frac nr}=\varepsilon^d,$$

and, in the limit $\varepsilon\to 0$, we obtain the condition

$$d=-\alpha_1+\beta_-+\frac{n+1}{2}-\frac2q+n\left(\frac12-\frac1r\right)\geqslant 0. \tag{90}$$

Making use of the scaling condition (78), we rearrange and summarize the various conditions that we have found so far in the following proposition.

PROPOSITION 14.15. – *Let $n \geqslant 2$ and $1 \leqslant q, r \leqslant \infty$. If the estimate* (77) *is true then the parameters $q, r, \alpha_1, \alpha_2, \beta_0, \beta_+, \beta_-$ must verify the following conditions*:

- *Scaling invariance*:

$$\beta_0 + \beta_+ + \beta_- = \alpha_1 + \alpha_2 + \frac{1}{q} - n\left(1 - \frac{1}{r}\right);$$

- *Geometry of cones*:

$$\frac{1}{q} \leqslant \frac{n+1}{2}\left(1 - \frac{1}{r}\right), \qquad \frac{1}{q} \leqslant \frac{n+1}{4};$$

- *Concentration along null directions*:

$$\beta_- \geqslant \frac{1}{q} - \frac{n-1}{2}\left(1 - \frac{1}{r}\right);$$

- *Low frequencies in* $(++)$ *interaction*:

$$\beta_0 \geqslant \frac{1}{q} - n\left(1 - \frac{1}{r}\right),$$

$$\beta_0 \geqslant \frac{2}{q} - (n+1)\left(1 - \frac{1}{r}\right);$$

- *Low frequencies in* $(+-)$ *interaction*:

$$\alpha_1 + \alpha_2 \geqslant \frac{1}{q},$$

$$\alpha_1 + \alpha_2 \geqslant \frac{3}{q} - n\left(1 - \frac{1}{r}\right);$$

- *Interaction between high and low frequencies*:

$$\alpha_i \leqslant \beta_- + \frac{n}{2},$$

$$\alpha_i \leqslant \beta_- + \frac{n}{2} - \frac{1}{q} + \frac{n-1}{2}\left(\frac{1}{2} - \frac{1}{r}\right),$$

$$\alpha_i \leqslant \beta_- + \frac{n}{2} - \frac{1}{q} + n\left(\frac{1}{2} - \frac{1}{r}\right),$$

$$\alpha_i \leqslant \beta_- + \frac{n}{2} - \frac{1}{q} + n\left(\frac{1}{2} - \frac{1}{r}\right) + \left(\frac{1}{2} - \frac{1}{q}\right).$$

CONJECTURE 14.16. – *The estimate* (77) *is true* [10] *when the parameters $q, r, \alpha_1, \alpha_2, \beta_0, \beta_+, \beta_-$ satisfy the conditions of Proposition* 14.15, *with the exception of some borderline cases corresponding to equality in the above conditions.* [11]

[10] This and the following conjectures have more the flavor of open questions. They reflect our current understanding of the problem. It is conceivable that the list of counterexamples and necessary conditions presented above is not complete. In particular, in the region where q and r are very different our conjecture seems far to be optimal. For $q = r$ we believe our conjecture are sharp up to borderline cases.

[11] Very recently we found out that T. Wolff has proved a significant part of this conjecture for the case $q = r$, see [24]. He claims that when $\widetilde{\phi}$ and $\widetilde{\psi}$ are supported on disjoint and transversal subsets of the positive null cone at the same frequency level, then estimate (77) holds in the sharp range for $q = r > (n+3)/(n+1)$, leaving open only the question of the endpoint.

We extrapolate from this conjecture also some conjectures about null forms. Again, for simplicity we set $\phi_1 = \psi_1 = 0$.

CONJECTURE 14.17. – *Let $n \geqslant 2$ and $1 \leqslant q, r \leqslant \infty$. Let Q be any of the null forms Q_0 or Q_{ij} introduced in Section 13. The estimate*

$$\big\|Q(\phi,\psi)\big\|_{L_t^q L_x^r} \lesssim \|D^\alpha \phi_0\|_{L^2} \|D^\alpha \psi_0\|_{L^2},$$

is verified whenever q, r, α verify the following conditions:

$$\alpha = 1 - \frac{1}{2q} + \frac{n}{2}\left(1 - \frac{1}{r}\right),$$
$$\frac{1}{q} \leqslant \min\left\{\frac{n+1}{2}\left(1 - \frac{1}{r}\right), \frac{n+1}{4}, \frac{n-1}{2} + \frac{1}{r}, \frac{1}{3} + \frac{n}{3}\left(1 - \frac{1}{r}\right)\right\}.$$

In the particular case where $q = r = p$ the above conjectures reduce to

$$\big\|Q(\phi,\psi)\big\|_{L^p(\mathbb{R}^{1+n})} \lesssim \|D^\alpha \phi_0\|_{L^2} \|D^\alpha \psi_0\|_{L^2}, \tag{91}$$

for

$$p \geqslant \frac{n+3}{n+1} \quad \text{and} \quad \alpha = \frac{1}{2} + \frac{n+1}{2}\left(1 - \frac{1}{p}\right).$$

Estimates of the type (91) were first considered in [6]. There it was shown, using essentially the Example 14.6, that the exponents $p = (n+1)/n$, $\alpha = 1$ are not admissible for dimensions $n = 2, 3$.

CONJECTURE 14.18. – *Let $n \geqslant 2$ and $1 \leqslant q, r \leqslant \infty$. The estimate*

$$\big\|D^{-1} Q_{ij}(\phi,\psi)\big\|_{L_t^q L_x^r} + \big\|Q_{ij}(D^{-1}\phi,\psi)\big\|_{L_t^q L_x^r} \lesssim \|D^\alpha \phi_0\|_{L^2} \|D^\alpha \psi_0\|_{L^2},$$

is verified whenever q, r, α verify the following conditions:

$$\alpha = \frac{1}{2}\left(1 - \frac{1}{q}\right) + \frac{n}{2}\left(1 - \frac{1}{r}\right),$$
$$1 - \frac{n}{r} \leqslant \frac{1}{q} \leqslant \min\left\{\frac{n}{3}\left(1 - \frac{1}{r}\right), n - 1 - \frac{n}{r}, \frac{n-3}{2} + \frac{1}{r}, \frac{n+1}{4}\right\}.$$

In the particular case where $q = r = p$ the above conjecture reduces to

$$\big\|D^{-1} Q_{ij}(\phi,\psi)\big\|_{L^p(\mathbb{R}^{1+n})} + \big\|Q_{ij}(D^{-1}\phi,\psi)\big\|_{L^p(\mathbb{R}^{1+n})} \lesssim \|D^\alpha \phi_0\|_{L^2} \|D^\alpha \psi_0\|_{L^2},$$

for

$$\alpha = \frac{n+1}{2}\left(1 - \frac{1}{p}\right), \quad \frac{n+1}{n-1} \leqslant p \leqslant n+1$$

when $n \geqslant 3$.

15. The case $q = \infty$, $r = 2$

Another situation where it is possible to completely verify Conjecture 14.16 is for energy-type norms, when $q = \infty$ and $r = 2$.

THEOREM 15.1. – *Let $n \geqslant 2$. Let ϕ, ψ be the solutions of* (1), (2). *Then the estimate*

$$\|D^{\beta_0} D_+^{\beta_+} D_-^{\beta_-}(\phi\psi)\|_{L_t^\infty L_x^2}$$
$$\lesssim \left(\|D^{\alpha_1}\phi_0\|_{L^2(\mathbb{R}^n)} + \|D^{\alpha_1-1}\phi_1\|_{L^2(\mathbb{R}^n)}\right)\left(\|D^{\alpha_2}\psi_0\|_{L^2(\mathbb{R}^n)} + \|D^{\alpha_2-1}\psi_1\|_{L^2(\mathbb{R}^n)}\right), \tag{92}$$

holds if and only if $\alpha_1, \alpha_2, \beta_0, \beta_+, \beta_-$ satisfy the following conditions:

$$\beta_0 + \beta_+ + \beta_- = \alpha_1 + \alpha_2 - \frac{n}{2}, \tag{93}$$

$$\beta_- \geqslant -\frac{n-1}{4}, \tag{94}$$

$$\beta_0 > -\frac{n}{2}, \tag{95}$$

$$\alpha_i < \beta_- + \frac{n}{2}, \quad i = 1, 2, \tag{96}$$

$$\alpha_1 + \alpha_2 > 0. \tag{97}$$

Proof of the sufficient part. – By the usual decomposition of ϕ and ψ into their $+$ and $-$ parts it is enough to prove the estimate for $\phi_+\psi_+$ and $\phi_+\psi_-$. Observe that we have

$$D^{\beta_0} D_+^{\beta_+} D_-^{\beta_-}(\phi_+\psi_\pm)(t,x) = D^{\beta_0} D_+^{\beta_+} D_-^{\beta_-}\left(\phi_+^t\psi_\pm^t\right)(0,x), \tag{98}$$

where ϕ^t and ψ^t are solutions of the homogeneous wave equation with initial data given by

$$\widehat{\phi_j^t}(\eta) = e^{it|\eta|}\widehat{\phi_j}(\eta), \qquad \widehat{\psi_j^t}(\zeta) = e^{\pm it|\zeta|}\widehat{\psi_j}(\zeta), \qquad j = 0, 1.$$

Since the exponential factors don't change the norm of the initial data, once we have a bound for the L^2 norm in x of the expression (98) at time $t = 0$, then we automatically get a uniform bound for all times. Hence the estimate (92) is equivalent to the boundedness of the bilinear operators

$$B_\pm : L^2 \times L^2 \to L^2,$$

defined by

$$B_\pm(f,g)(\xi) = \int W_\pm(\eta, \xi - \eta) f(\eta) g(\xi - \eta)\, d\eta,$$

where $W_\pm$ are still given by (23) and (24).

More over, as we did for the proofs of Theorems 1.1 and 12.1, it is convenient to look separately at the contribution coming from the interaction of comparable frequency levels and the contribution coming from the interaction of different frequency levels. Thus, we decompose the operators $B_\pm$ as the sum of

$$B_\pm^0(f,g)(\xi) = \int_{|\eta| \leqslant |\xi-\eta| \approx |\xi|} W_\pm(\eta, \xi - \eta) f(\eta) g(\xi - \eta)\, d\eta, \tag{99}$$

$$B_\pm^1(f,g)(\xi) = \int_{|\eta| \approx |\xi-\eta| \gg |\xi|} W_\pm(\eta, \xi - \eta) f(\eta) g(\xi - \eta)\, d\eta, \tag{100}$$

$$B_\pm^2(f,g)(\xi) = \int_{|\eta| \gg |\xi-\eta| \approx |\xi|} W_\pm(\eta, \xi - \eta) f(\eta) g(\xi - \eta)\, d\eta. \tag{101}$$

Obviously the properties of $B^2_\pm$ are analogous to those of $B^0_\pm$, because of the symmetry $f \leftrightarrow g$, $\alpha_1 \leftrightarrow \alpha_2$. Also we can always assume that f and g are non-negative functions.

We assume first that we have a strict inequality in (94):

$$\beta_- > -\frac{n-1}{4}. \tag{102}$$

The cases with

$$\beta_- = -\frac{n-1}{4}$$

will be considered in Section 16.

Estimates for $B^0_\pm$. Consider B^0_+. If we apply Cauchy–Schwarz directly, we reduce to prove the boundedness of the quantity

$$\sup_\xi \int\limits_{|\eta|\leqslant|\xi-\eta|\approx|\xi|} W^2_\pm(\eta,\xi-\eta)\,\mathrm{d}\eta.$$

In the region where $|\eta| \leqslant |\xi-\eta| \approx |\xi|$ we have

$$|\eta| + |\xi-\eta| - |\xi| \approx \frac{|\xi-\eta|^2 - (|\xi|-|\eta|)^2}{|\xi|} \approx |\eta|\theta^2,$$

where θ is the angle between η and $-\xi$. Hence,

$$W_+(\eta,\xi-\eta) \approx |\xi|^{\beta_0+\beta_+-\alpha_2}|\eta|^{\beta_--\alpha_1}\theta^{2\beta_-},$$

and we have

$$\int\limits_{|\eta|\leqslant|\xi-\eta|\approx|\xi|} W^2_\pm(\eta,\xi-\eta)\,\mathrm{d}\eta \lesssim |\xi|^{2(\beta_0+\beta_+-\alpha_2)} \int\limits_{|\eta|\leqslant|\xi|} |\eta|^{2(\beta_--\alpha_1)}\theta^{4\beta_-}\,\mathrm{d}\eta$$

$$\lesssim |\xi|^{2(\beta_0+\beta_+-\alpha_2)} \int_0^{|\xi|} \rho^{2(\beta_--\alpha_1)+n-1}\,\mathrm{d}\rho \int_0^{\pi} \theta^{4\beta_-+n-2}\,\mathrm{d}\theta.$$

The last two integrals are bounded when $4\beta_- + n - 2 > -1$ and $2(\beta_- - \alpha_1) + n - 1 > -1$, which are ensured by (102) and (96). The scaling condition (93) then ensures that the result is uniformly bounded in ξ.

The estimate for B^0_- is obtained in a similar manner.

Estimates for B^1_+. In the region where $|\eta| \approx |\xi-\eta| \gg |\xi|$ we have $|\eta| + |\xi-\eta| - |\xi| \approx |\eta|$ and, by (93),

$$W_+(\eta,\xi-\eta) \approx |\xi|^{\beta_0}|\eta|^{\beta_++\beta_--\alpha_1-\alpha_2} = |\xi|^{\beta_0}|\eta|^{-\beta_0-\frac{n}{2}}.$$

By the now familiar doubling technique and Cauchy–Schwarz inequality, we find

$$\|B^1_+\| \lesssim \iiint\limits_{|\xi|\ll|\eta|} \frac{|\xi|^{2\beta_0}}{|\eta|^{2\beta_0+n}} f^2(\eta)g^2(\zeta)\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta.$$

The integral in ξ is bounded if $2\beta_0 > -n$, which is ensured by (95), and we have

$$\int\limits_{|\xi|\ll|\eta|} \frac{|\xi|^{2\beta_0}\mathrm{d}\xi}{|\eta|^{2\beta_0+n}} \lesssim 1.$$

Observe that here the proof of the boundedness of B_+^1 works also for the case $\beta_- = -(n-1)/4$, which is not critical in this region.

Estimates for B_-^1. In the region where $|\eta| \approx |\xi-\eta| \gg |\xi|$ we have

$$|\xi| - \big||\eta| - |\xi-\eta|\big| \approx \frac{|\eta||\xi-\eta|}{|\xi|}\varphi^2 \approx |\xi|(\sin\theta)^2,$$

where φ is the angle between η and $\eta-\xi$ and θ is the angle between ξ and η. Indeed, from $|\xi| \ll |\eta|$ it follows that φ is small, in particular $\varphi \approx \sin\varphi$, and by geometric considerations we have

$$\frac{\sin\varphi}{|\xi|} = \frac{\sin\theta}{|\xi-\eta|}.$$

Hence,

$$W_-(\eta,\xi-\eta) \approx \frac{|\xi|^{\beta_0+\beta_++\beta_-}}{|\eta|^{\alpha_1+\alpha_2}}(\sin\theta)^{2\beta_-} = \frac{|\xi|^{\alpha_1+\alpha_2-\frac{n}{2}}}{|\eta|^{\alpha_1+\alpha_2}}(\sin\theta)^{2\beta_-}.$$

We then perform the doubling and apply Cauchy–Schwarz. We obtain

$$\|B_-^1\| \lesssim \iiint\limits_{|\xi|\ll|\eta|} \frac{|\xi|^{2(\alpha_1+\alpha_2)-n}(\sin\theta)^{4\beta_-}}{|\eta|^{2(\alpha_1+\alpha_2)}} f^2(\eta)g^2(\zeta)\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta.$$

The integral in ξ is bounded if $2(\alpha_1+\alpha_2) - n > -n$ and $4\beta_- > -n+1$, which are ensured by (97) and (102), and we have

$$\int\limits_{|\xi|\ll|\eta|} \frac{|\xi|^{2(\alpha_1+\alpha_2)-n}(\sin\theta)^{4\beta_-}}{|\eta|^{2(\alpha_1+\alpha_2)}}\,\mathrm{d}\xi \simeq \frac{1}{|\eta|^{2(\alpha_1+\alpha_2)}}\int\limits_0^{|\eta|}\rho^{2(\alpha_1+\alpha_2)-1}\,\mathrm{d}\rho\int\limits_0^{\pi}(\sin\theta)^{4\beta_-+n-2}\,\mathrm{d}\theta \simeq 1.$$

The proof of the sufficient part of Theorem 15.1, under the assumption (102), is now complete. □

The necessity of conditions (95), (96) and (97) with non-strict inequalities, follows from the examples in Section 14. It remains to show counterexamples to exclude the cases of equality in those conditions.

Example 15.2. – Assume we have equality in (95), $\beta_0 = -n/2$. Take $f = \chi_F$ and $g = \chi_G$, where F is the ball of center $(1,0,\dots,0)$ and radius $1/2$ and G is the ball of center $(-1,0,\dots,0)$ and radius $1/2$. When $|\xi| \leqslant 1/4$ we have

$$B_+(f,g)(\xi) \approx |\xi|^{-n/2}\int\limits_{\substack{\eta\in F\\ \xi-\eta\in G}}\mathrm{d}\eta \gtrsim |\xi|^{-n/2}.$$

Hence, $B_+(f,g)$ is not a L^2 function.

Example 15.3. – Assume we have equality in (96), $\alpha_1 = \beta_- + n/2$. Take

$$f(\eta) = \frac{\chi_F(\eta)}{|\eta|^{n/2}|\log|\eta||^a}, \qquad g(\zeta) = \chi_G(\zeta),$$

where F is the ball of center 0 and radius $1/2$ and G is the ball of center $(1,0,\dots,0)$ and radius $1/2$. The function f is in L^2 when $a > 1/2$.

When $|\xi - (1,0,\dots,0)| \leqslant 1/4$ we have

$$B_+(f,g)(\xi) \approx \int\limits_{\substack{\eta\in F\\ \xi-\eta\in G}} \frac{(|\eta| + |\xi-\eta| - |\xi|)^{\beta_-}}{|\eta|^{\alpha_1+\frac{n}{2}}|\log|\eta||^a}\,\mathrm{d}\eta \gtrsim \int\limits_{|\eta|\leqslant 1/4} \frac{\theta^{2\beta_-}}{|\eta|^{\alpha_1+\frac{n}{2}-\beta_-}|\log|\eta||^a}\,\mathrm{d}\eta$$

$$\simeq \int\limits_{|\eta|\leqslant 1/4} \frac{\mathrm{d}\eta}{|\eta|^n|\log|\eta||^a}.$$

As before, θ is the angle between η and $-\xi$. The last integral is bounded only if $a > 1$, hence, we obtain a counterexample by choosing $1/2 < a \leqslant 1$.

Example 15.4. – Assume we have equality in (97), $\alpha_1 + \alpha_2 = 0$ and consequently $\beta_0 + \beta_+ + \beta_- = -n/2$. Take $f = \chi_F$ and $g = \chi_G$, where F is the ball of center $(1,0,\dots,0)$ and radius $1/2$ and G is the ball of center $(-1,0,\dots,0)$ and radius $1/2$. When $|\xi| \leqslant 1/4$ we have

$$B_-(f,g)(\xi) \approx |\xi|^{\beta_0+\beta_+} \int\limits_{\substack{\eta\in F\\ \xi-\eta\in G}} \big(|\xi| - \big||\eta| - |\xi-\eta|\big|\big)^{\beta_-}\,\mathrm{d}\eta$$

$$\gtrsim |\xi|^{\beta_0+\beta_++\beta_-} \int\limits_{|\eta-(1,0,\dots,0)|<1/4} \theta^{2\beta_-}\,\mathrm{d}\eta \gtrsim |\xi|^{-n/2}.$$

Here θ is the angle between η and ξ. This shows that $B_-(f,g)$ is not a L^2 function.

16. Removal of a logarithmic divergence

To complete the proof of Theorem 15.1, it remains to show the boundedness of the operators $B^0_\pm$ and B^1_-, defined in (99) and (100), under the assumption that

$$\beta_- = -\frac{n-1}{4}.$$

To prove this critical case it is convenient to perform a dyadic decomposition of the data in frequency space. Using the summation techniques already seen in Section 12, the estimates can then be obtained from their frequency localized versions as stated in the following proposition.

PROPOSITION 16.1. – *Let $0 < \mu \lesssim \lambda$. Take f and g to be functions with compact support. We have the following dyadic estimates*:

- *suppose* $\operatorname{supp} f \subset \{|\xi| \approx \mu\}$ *and* $\operatorname{supp} g \subset \{|\xi| \lesssim \lambda\}$, *then*

$$\big\|B^0_\pm(f,g)\big\| \lesssim \left(\frac{\mu}{\lambda}\right)^{\frac{n+1}{4}-\alpha_1}\left(1 + \left|\log\left(\frac{\mu}{\lambda}\right)\right|\right)\|f\|\|g\|; \tag{103}$$

- *suppose* $\operatorname{supp} f, \operatorname{supp} g \subset \{|\xi| \approx \lambda\}$, *then*

$$\|S_\mu B^1_-(f,g)\| \lesssim \left(\frac{\mu}{\lambda}\right)^{\alpha_1+\alpha_2}\left(1+\left|\log\left(\frac{\mu}{\lambda}\right)\right|\right)\|f\|\|g\|. \tag{104}$$

If we apply the procedures of the previous section we would obtain logarithmic divergencies in the integration of the angular variables. A refined application of the doubling technique will allow us to transfer the logarithmic loss from the angular variables to the radial ones, for which, in view of the strict inequalities in (96) and (97), we may allow a loss of ε in the exponent.

Proof of (104). – Because of the scaling invariance condition (93), we may assume that $\mu = 1 \ll \lambda$. Moreover, by the same procedure described in the proof of (65), we can assume that f and g are supported in opposite cubes, Q and $-Q$, of size ≈ 1 at a distance $\approx \lambda$ from the origin. In the region where $|\eta| \approx |\xi-\eta| \gg |\xi|$ we have

$$|\xi| - \big||\eta| - |\xi-\eta|\big| \approx \frac{|\xi||\eta|}{|\xi|+|\eta|}\theta^2(\xi,-\eta),$$

where $\theta(\xi,-\eta)$ is the angle between the vectors ξ and $-\eta$. Under our hypotheses the operator to be estimated is then

$$S_1 B^1_-(f_{(Q)}, g_{(-Q)})(\xi) \approx \lambda^{-\alpha_1-\alpha_2} \int\limits_{\substack{\eta,\eta-\xi\in Q\\ |\eta|\leqslant|\xi-\eta|}} \frac{f(\eta)g(\xi-\eta)}{\theta^{\frac{n-1}{2}}(\xi,-\eta)}\,\mathrm{d}\eta,$$

in the region where $|\xi| \approx 1$. By doubling the integral we need to show that

$$I = \iiint\limits_{\Omega} \frac{f(\eta)g(\xi-\eta)f(\eta')g(\xi-\eta')}{\theta^{\frac{n-1}{2}}(\xi,-\eta)\theta^{\frac{n-1}{2}}(\xi,-\eta')}\,\mathrm{d}\eta\,\mathrm{d}\eta'\,\mathrm{d}\xi \lesssim (1+\log\lambda)\|f\|^2\|g\|^2,$$

where the region of integration is

$$\Omega = \big\{(\eta,\eta',\xi)\colon \eta,\eta-\xi,\eta',\eta'-\xi \in Q,\ |\xi|\approx 1\big\}.$$

We split the integral $I = I_1 + I_2$ by decomposing the region Ω into two parts. The first is the subset Ω_1 defined by the conditions

$$\min\big\{\theta(\xi,-\eta),\theta(\xi,-\eta')\big\} \lesssim \lambda\theta(\eta,\eta'),$$
$$\min\big\{\theta(\xi,\xi-\eta),\theta(\xi,\xi-\eta')\big\} \lesssim \lambda\theta(\xi-\eta,\xi-\eta');$$

the second is the subset Ω_2 on which we have

$$\lambda\theta(\eta,\eta') \ll \min\big\{\theta(\xi,-\eta),\theta(\xi,-\eta')\big\},$$
$$\lambda\theta(\xi-\eta,\xi-\eta') \ll \min\big\{\theta(\xi,\xi-\eta),\theta(\xi,\xi-\eta')\big\}.$$

(It is possible to adjust the constant in the above inequalities so that $\Omega \subset \Omega_1 \cup \Omega_2$.)

For I_1 we apply Cauchy–Schwarz pairing $f(\eta)$ with $f(\eta')$ and $g(\xi-\eta)$ with $g(\xi-\eta')$. We find

$$I_1^2 \leqslant \iint\limits_{\Omega_1} \frac{f^2(\eta)f^2(\eta')}{\theta^{\frac{n-1}{2}}(\xi,-\eta)\theta^{\frac{n-1}{2}}(\xi,-\eta')}\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\eta' \cdot \iint\limits_{\Omega_1} \frac{g^2(\xi-\eta)g^2(\xi-\eta')}{\theta^{\frac{n-1}{2}}(\xi,-\eta)\theta^{\frac{n-1}{2}}(\xi,-\eta')}\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\eta'.$$

In the right hand side, the second integral becomes similar to the first if we perform the change of variables $(\eta,\eta',\xi)\to(\zeta,\zeta',\xi')=(\xi-\eta,\xi-\eta',-\xi)$. Since we have that

$$\int\limits_{\substack{\theta(\xi,-\eta)\lesssim\lambda\theta(\eta,\eta')\\ \theta(\xi,-\eta')\lesssim\lambda\theta(\eta,\eta')\\ |\xi|\approx 1}} \frac{\mathrm{d}\xi}{\theta^{\frac{n-1}{2}}(\xi,-\eta)\theta^{\frac{n-1}{2}}(\xi,-\eta')}\lesssim 1+\log\lambda,$$

it follows that

$$I_1\lesssim(1+\log\lambda)\|f\|\|g\|.$$

For I_2 we apply Cauchy–Schwarz pairing $f(\eta)$ with $g(\xi-\eta)$ and $f(\eta')$ with $g(\xi-\eta')$, We find

$$I_2^2\leqslant\iint\limits_{\Omega_2}\frac{f^2(\eta)g^2(\xi-\eta)}{\theta^{n-1}(\xi,-\eta)}\,\mathrm{d}\eta'\,\mathrm{d}\eta\,\mathrm{d}\xi\cdot\iint\limits_{\Omega_2}\frac{f^2(\eta')g^2(\xi-\eta')}{\theta^{n-1}(\xi,-\eta')}\,\mathrm{d}\eta\,\mathrm{d}\eta'\,\mathrm{d}\xi.$$

For fixed η the intersection of Q with the set of points η' such that $\theta(\eta,\eta')\leqslant A$ has a volume which is $\lesssim(\lambda A)^{n-1}$. Thus

$$\frac{1}{\theta^{n-1}(\xi,-\eta)}\int\limits_{\substack{\theta(\eta,\eta')\ll\frac{\theta(\xi,-\eta)}{\lambda}\\ \eta'\in Q}}\mathrm{d}\eta'\lesssim 1,$$

which implies

$$I_2\lesssim\|f\|\|g\|.\quad\square$$

Proof of (103). – As before we can assume $\mu=1\leqslant\lambda$. Let's look at the $(++)$ case. In the region where $|\eta|\leqslant|\xi-\eta|\approx|\xi|$ we have

$$|\eta|+|\xi-\eta|-|\xi|\approx|\eta|\theta^2(\eta,\xi-\eta).$$

The operator to be estimated is

$$B_+^0(f,g)(\xi)\approx\lambda^{\alpha_2-\frac{n+1}{4}}\int\limits_{\substack{|\eta|\approx 1\\ |\xi-\eta|\approx|\xi|}}\frac{f(\eta)g(\xi-\eta)}{\theta^{\frac{n-1}{2}}(\eta,\xi-\eta)}\,\mathrm{d}\eta,$$

in the region where $|\xi|\approx\lambda$. Now the situation is different from before since f and g are supported on regions at different frequency scales. This fact will cause us some problems if we try to proceed using the doubling technique at this stage. Instead, we will first observe that, by duality, is enough to prove that

$$J(f,g,h)\lesssim\lambda^{\alpha_2-\frac{n+1}{4}}(1+\log\lambda)\|f\|\|g\|\|h\|,\tag{105}$$

where J is the trilinear expression

$$J(f,g,h)=\int B_+^0(f,g)(\xi)h(\xi)\,\mathrm{d}\xi\approx\lambda^{\alpha_2-\frac{n+1}{4}}\iint\frac{f(\eta)g(\zeta)h(\eta+\zeta)}{\theta^{\frac{n-1}{2}}(\eta,\zeta)}\,\mathrm{d}\eta\,\mathrm{d}\zeta.$$

But we can view (105) also as the duality formulation of the L^2 boundedness for the bilinear operator

$$B'(g,h)(\eta)=\lambda^{\alpha_2-\frac{n+1}{4}}\int\limits_{|\zeta|\approx|\eta+\zeta|\approx\lambda}\frac{g(\zeta)h(\eta+\zeta)}{\theta^{\frac{n-1}{2}}(\eta,\zeta)}\,\mathrm{d}\zeta,$$

on the region $|\eta|\approx 1$. More over g and h can be restricted to be supported in the same cube Q of size 1 at frequency level λ. The proof for the boundedness of B' now can follow the same line as in the proof of (104). (Changing $\zeta\to-\zeta$ and $g(\zeta)\to g(-\zeta)$ we get to estimate precisely the same operator.)

The proof in the $(+\,-)$ case for B^0_-, is analogous to that of the $(+\,+)$ case for B^0_+. The only change is to replace η by $-\eta$ and $f(\eta)$ by $f(-\eta)$. $\square$

17. Bilinear restriction conjecture

In trying to solve Conjecture 14.16, which we expect to be hard whenever $q<2$ or $r<2$, it helps to consider a simpler version of it which makes sense in the classical context of the restriction theorem for the unit sphere $\mathbb{S}=\mathbb{S}^{n-1}$ in $\mathbb{R}^n$.

Define the Stein operator to be the adjoint of the Fourier restriction operator $Rf=\hat{f}|_{\mathbb{S}}$,

$$Sf(x)=R^*f(x)=\int\limits_{\mathbb{S}}e^{ix\cdot\xi}f(\xi)\,\mathrm{d}S_\xi\simeq(f\,\mathrm{d}S)^\vee(x).$$

Recall that the classical Stein–Tomas restriction theorem (see [23,17]) can be formulated as follows.

THEOREM 17.1. – *Let $f\in L^2(\mathbb{S})$ and*

$$\frac{2(n+1)}{n-1}\leqslant p\leqslant\infty.$$

Then $Sf\in L^p(\mathbb{R}^n)$ and

$$\|Sf\|_{L^p(\mathbb{R}^n)}\lesssim\|f\|_{L^2(\mathbb{S})}. \tag{106}$$

The Strichartz inequalities (76) are proved by using the same methods which are used to prove Theorem 17.1. As the bilinear estimates (77) are a generalization of (76), it makes sense to ask for some version of a bilinear extension of the Stein–Tomas theorem. In particular we may consider the following bilinear forms,

$$B(f,g)=Sf\cdot Sg,$$

$$Q(f,g)=Q_{ij}(Sf,Sg)=\partial_iSf\cdot\partial_jSg-\partial_jSf\cdot\partial_iSg.$$

Observe that the symbol of $Q(f,g)$, $\sigma(\xi,\eta)=\xi\wedge\eta$ vanishes precisely when ξ,η are aligned, this corresponds to the worst behavior of the product $Sf\cdot Sg$. It is reasonable, then, to ask what happens with $Sf\cdot Sg$ if the supports of f and g are not aligned, in other words they have disjoint projectivised supports. This leads to the following:

CONJECTURE 17.2. – *Let Ω_1,Ω_2 two disjoint subsets of $\mathbb{S}$ such that*

$$\operatorname{dist}(\Omega_1,\Omega_2)>0,\qquad\operatorname{dist}(\Omega_1,-\Omega_2)>0.$$

Then

$$\big\|B(f,g)\big\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{L^2(\Omega_1)}\|g\|_{L^2(\Omega_2)}, \tag{107}$$

for all f supported in Ω_1 and g supported in Ω_2, whenever

$$p \geqslant \frac{n+2}{n}. \tag{108}$$

This is intimately connected to the following:

CONJECTURE 17.3. – *For arbitrary $f,g \in L^2(\mathbb{S})$, $p \geqslant (n+2)/n$ and*

$$Q(f,g) = Q_{ij}(Sf,Sg) = \partial_i Sf \cdot \partial_j Sg - \partial_j Sf \cdot \partial_i Sg$$

we have,

$$\big\|Q(f,g)\big\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{L^2(\mathbb{S})}\|g\|_{L^2(\mathbb{S})}. \tag{109}$$

Remark 17.4. – The case $p \geqslant (n+1)/(n-1)$ is already contained in Theorem 17.1, since applying Hölder we immediately have

$$\big\|B(f,g)\big\|_{L^p(\mathbb{R}^n)} \leqslant \|Sf\|_{L^{2p}(\mathbb{R}^n)}\|Sg\|_{L^{2p}(\mathbb{R}^n)} \lesssim \|f\|_{L^2(\mathbb{S}^{n-1})}\|g\|_{L^2(\mathbb{S}^{n-1})},$$

for all $p \geqslant (n+1)/(n-1)$.

The Knapp example shows that this is sharp. Consider in fact $f = g = \chi_{C_\varepsilon}$, where C_ε is the cap defined as the intersection of the sphere $\mathbb{S}$ and the rectangular box

$$D_\varepsilon = \big\{\xi \in R^n\colon\ |\xi_1 - 1| \leqslant \varepsilon^2,\ |\xi'| \leqslant \varepsilon\big\},$$

for a small $\varepsilon > 0$. We write

$$Sf(x) = e^{ix_1} \int\limits_{C_\varepsilon} e^{ix_1(\xi_1 - 1)} e^{x'\cdot\xi'}\,\mathrm{d}S_\xi,$$

and observe that it is possible to choose a region R_ε defined by

$$R_\varepsilon = \big\{x \in \mathbb{R}^n\colon\ |x_1| \lesssim \varepsilon^{-2},\ |x'| \leqslant \varepsilon^{-1}\big\},$$

such that $|Sf(x)| \gtrsim |C_\varepsilon|$ when $x \in R_\varepsilon$. Therefore

$$\frac{\|B(f,g)\|_{L^p}}{\|f\|_{L^2}\|g\|_{L^2}} = \frac{\|Sf\|^2_{L^{2p}}}{\|f\|^2_{L^2}} \gtrsim \frac{|C_\varepsilon|^2|R_\varepsilon|^{2/2p}}{|C_\varepsilon|} = |C_\varepsilon||R_\varepsilon|^{1/p} \simeq \varepsilon^{n-1-\frac{n+1}{p}}.$$

In the limit $\varepsilon \to 0$ an inequality like (107) implies $p \geqslant (n+1)/(n-1)$.

Clearly the Knapp example is not relevant to the case of projective disjoint supports. If we try to consider a modification of it, where we take f and g to be the characteristic functions of two disjoint spherical caps, then we are lead to the condition $p \geqslant n/(n-1)$. However, this is not a sharp result, as it shown in the next example, which is one of the main justifications for Conjecture 17.2.

Example 17.5. – Define, for some small $\varepsilon > 0$, two rectangular regions in phase space as

$$D_1 = \{\xi \in \mathbb{R}^n\colon\ |\xi_1 - 1| < \varepsilon^2,\ |\xi_2| < \varepsilon^2,\ |\xi''| < \varepsilon\},$$
$$D_2 = \{\xi \in \mathbb{R}^n\colon\ |\xi_1| < \varepsilon^2,\ |\xi_2 - 1| < \varepsilon^2,\ |\xi''| < \varepsilon\}.$$

Let now $f = \chi_{\mathbb{S}\cap D_1}$ and $g = \chi_{\mathbb{S}\cap D_2}$, then

$$\|f\|_{L^2(\mathbb{S})} = \|g\|_{L^2(\mathbb{S})} = |\mathbb{S}\cap D_1|^{1/2} \approx \varepsilon^{n/2}.$$

It is then possible to fix a region R in physical space, defined by

$$|x_1| \lesssim \varepsilon^{-2}, \qquad |x_2| \lesssim \varepsilon^{-2}, \qquad |x''| \lesssim \varepsilon^{-1},$$

such that for $x \in R$ we have

$$|Sf(x)| \geqslant \frac{1}{2}|\mathbb{S}\cap D_1|, \qquad |Sg(x)| \geqslant \frac{1}{2}|\mathbb{S}\cap D_2|.$$

This implies that

$$\frac{\|B(f,g)\|_{L^p}}{\|f\|_{L^2}\|g\|_{L^2}} \gtrsim |\mathbb{S}\cap D_1|^{1/2}|\mathbb{S}\cap D_2|^{1/2}|R|^{1/p} \approx \varepsilon^{n-\frac{(n+2)}{p}}.$$

For small values of ε, an estimate like (107) will necessarily require $n - (n+2)/p \geqslant 0$, which is possible only if $p \geqslant (n+2)/n$.

The Conjectures 17.2 and 17.3 can be easily proved when $n = 2$. This is due to the fact that, in that case, the optimal exponent is $p = (n+2)/n = 2$. This fact allows us to apply Plancherel's theorem and make use of the simple convolution structure of the Fourier transform of $B(f,g)$.

Indeed,

$$\hat{B}(f,g)(\xi) \simeq (f\,\mathrm{d}S) * (g\,\mathrm{d}S)(\xi) = \int_{\mathbb{R}^2} \delta\big(1 - |\xi-\eta|\big)\delta\big(1-|\eta|\big)f(\xi-\eta)g(\eta)\,\mathrm{d}\eta,$$

and applying Cauchy–Schwarz with respect to the measure $\delta(1 - |\xi-\eta|)\delta(1-|\eta|)\,\mathrm{d}\eta$ we find

$$\big|\hat{B}(f,g)(\xi)\big|^2 \leqslant \hat{B}(1,1)(\xi)\hat{B}\big(|f|^2,|g|^2\big)(\xi). \tag{110}$$

It is not difficult to verify that

$$\hat{B}(1,1)(\xi) \simeq |\xi|^{-1}\big(4 - |\xi|^2\big)_+^{-1/2}.$$

When we integrate (110) with respect to ξ we obtain

$$\big\|B(f,g)\big\|^2_{L^2(\mathbb{R}^2)} \lesssim \iint \frac{\delta(1-|\xi-\eta|)\delta(1-|\eta|)}{|\xi|(4-|\xi|^2)^{1/2}}\big|f(\xi-\eta)\big|^2\big|g(\eta)\big|^2\,\mathrm{d}\eta\,\mathrm{d}\xi.$$

Change variable, $\xi \to \zeta = \xi - \eta$, and observe that when $|\eta| = |\zeta| = 1$ we have

$$|\xi| = |\eta+\zeta| \simeq (1+\eta\cdot\zeta)^{1/2},$$
$$\big(4 - |\xi|^2\big)^{1/2} = \big(4 - |\eta+\zeta|^2\big)^{1/2} \simeq (1-\eta\cdot\zeta)^{1/2},$$

hence

$$\|B(f,g)\|^2_{L^2(\mathbb{R}^2)} \lesssim \iint\limits_{\mathbb{S}^1\times\mathbb{S}^1} \frac{|f(\zeta)|^2|g(\eta)|^2}{(1-(\eta\cdot\zeta)^2)^{1/2}}\,\mathrm{d}S_\eta\,\mathrm{d}S_\zeta.$$

This is an interesting formula. Observe that if the supports of f and g on $\mathbb{S}^1$ are projectionally disjoints, i.e. don't contain points in the same direction, then the quantity $1-(\eta\cdot\zeta)^2$ is bounded below by a positive constant and in this case we obtain the bilinear restriction estimate

$$\|B(f,g)\|_{L^2(\mathbb{R}^2)} \lesssim \|f\|_{L^2(\mathbb{S}^1)}\|g\|_{L^2(\mathbb{S}^1)}.$$

In the same way, using the structure of the symbol $\sigma(\xi,\eta)=\xi\wedge\eta$, we prove,

$$\|Q(f,g)\|_{L^2(\mathbb{R}^2)} \lesssim \|f\|_{L^2(\mathbb{S}^1)}\|g\|_{L^2(\mathbb{S}^1)}.$$

As was mentioned in the introduction the Conjectures 17.2 as well 17.3 have surfaced in discussions between Machedon and Klainerman many years back. They were motivated by their interest in bilinear null estimates for solutions to homogeneous wave equation. The conjectures, as well as their simple proof for $n=2$, were discussed in a conference at MSRI in July 97.

Recently, Tao, Vargas and Vega [20] were able to make some progress on Conjecture 17.2 in dimension $n=3$. In particular, for $n=3$ they could prove (107) for $p>2-\frac{5}{69}$.

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