ANNALES SCIENTIFIQUES DE L'É.N.S.

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Annales scientifiques de l'É.N.S. 4e série, tome 32, nº 6 (1999), p. 743-767
<http://www.numdam.org/item?id=ASENS_1999_4_32_6_743_0>

Ann. scient. Éc. Norm. Sup.,
4e série, t. 32, 1999, p. 743 à 767.

THE RELATIVE DIXMIER PROPERTY FOR INCLUSIONS OF VON NEUMAN ALGEBRAS OF FINITE INDEX

BY SORIN POPA(*)

"Il ne faut pas toujours intégrer. Il faut aussi désintégrer."
Eugen Ionesco: La leçon

Dedicated to Professor Jacques Dixmier

ABSTRACT. – We prove that if an inclusion of von Neumann algebras $\mathcal{N} \subset \mathcal{M}$ has a conditional expectation $\mathcal{E} : \mathcal{M} \to \mathcal{N}$ satisfying the finite index condition $\mathcal{E}(x) \geq cx, \forall x \in \mathcal{M}_+$, for some $c > 0$, then $\mathcal{N} \subset \mathcal{M}$ satisfies the relative version of Dixmier's property on averaging elements by unitaries in $\mathcal{N}$, i.e., for any $x \in \mathcal{M}$, the norm closure of the convex hull of $\{uxu^* \mid u \text{ unitary element in } \mathcal{N}\}$ contains elements of $\mathcal{N}' \cap \mathcal{M}$. Moreover, in the case $\mathcal{N}, \mathcal{M}$ are factors of type II_1 and $\mathcal{N}$ has separable predual, the finiteness of the index of the inclusion is proved equivalent to the relative Dixmier property and to the property that a normal state on $\mathcal{N}$ has only normal state extensions to $\mathcal{M}$. We give applications of these results. © Elsevier, Paris

RÉSUMÉ. – Nous démontrons que si une inclusion d'algèbres de von Neumann $\mathcal{N} \subset \mathcal{M}$ est d'indice fini, i.e., si elle admet une espérance conditionnelle $\mathcal{E} : \mathcal{M} \to \mathcal{N}$ satisfaisant la condition $\mathcal{E}(x) \geq cx, \forall x \in \mathcal{M}_+$, pour un certain $c > 0$, alors $\mathcal{N} \subset \mathcal{M}$ satisfait la version relative de la propriété de Dixmier, i.e., pour tout $x \in \mathcal{M}$, la fermeture en norme de l'ensemble convexe $\mathrm{co}\{uxu^* \mid u \text{ élément unitaire dans } \mathcal{N}\}$ contient des éléments de $\mathcal{N}' \cap \mathcal{M}$. Si en plus $\mathcal{N}, \mathcal{M}$ sont des facteurs de type II_1 et $\mathcal{N}$ a un prédual séparable, alors on démontre que, réciproquement, si $\mathcal{N} \subset \mathcal{M}$ a la propriété de Dixmier relative alors $\mathcal{N} \subset \mathcal{M}$ est d'indice fini. De même, on démontre que la finitude de l'indice d'une inclusion de facteurs de type II_1, $\mathcal{N} \subset \mathcal{M}$, est équivalente au fait que les états normaux du sous-facteur $\mathcal{N}$ n'ont que des extensions normales à $\mathcal{M}$. D'autres résultats et quelques applications sont aussi donnés. © Elsevier, Paris

We prove in this paper a version for inclusions of von Neumann algebras of finite index of Dixmier's classical result on the norm closure of "averaging" elements by unitaries, as follows:

THEOREM. – *Let $\mathcal{N} \subset \mathcal{M}$ be an inclusion of von Neumann algebras with a conditional expectation $\mathcal{E} : \mathcal{M} \to \mathcal{N}$ of finite index, i.e., $\exists c > 0$ such that $\mathcal{E}(x) \geq cx$, $\forall x \in \mathcal{M}_+$. Then $\mathcal{N} \subset \mathcal{M}$ has the relative Dixmier property, i.e., for any $x \in \mathcal{M}$, we have $\overline{\mathrm{co}}^n\{uxu^* \mid u$ unitary element in $\mathcal{N}\} \cap \mathcal{N}' \cap \mathcal{M} \neq \emptyset$.*

In the last part of the paper we present several applications of this theorem, notably a result showing that for type II_1 subfactors with separable preduals, the finiteness of the Jones' index is in fact equivalent to the relative Dixmier property.

(*) Supported by NSF-Grant 9500882

To prove the above theorem, we treat separately the finite and properly infinite cases. The proof consists then in a series of applications of the Hahn-Banach theorem and of the classical Dixmier theorem for single von Neumann algebras [D1] (including its refined form in [SZ1]), in some analysis specific to finite index situations, in convexity arguments and in reduction-disintegration type techniques. Also, to prove the properly infinite, semifinite case we make use of the general result ([PoR]) on derivations of von Neumann algebras into the ideal generated by the finite projections of a semifinite von Neumann algebra.

For the elements of reduction theory needed in the properly infinite case we refer the reader to [SZ1], [H] and [ApZ]. For basics of finite index analysis, see [PiPo], [Po2,4].

1. Some Preliminaries

We begin by making several considerations that will reduce the proof of the theorem to proving it for selfadjoint elements "orthogonal" to $\mathcal{N} \vee (\mathcal{N}' \cap \mathcal{M})$, with $\mathcal{N}$, $\mathcal{M}$ being both finite or both properly infinite von Neumann algebras.

But first of all, let us note that an expectation $\mathcal{E}$ satisfying the condition $\mathcal{E}(x) \geq cx, \forall x \geq 0$ is automatically normal and faithful (cf. remark 1.1.2, (i) in [Po2]):

1.1 Proposition. *If $\mathcal{M}_0 \subset \mathcal{M}_1$ is an inclusion of von Neumann algebras with a conditional expectation $\mathcal{E}$ of $\mathcal{M}_1$ onto $\mathcal{M}_0$ satisfying $\mathcal{E}(x) \geq cx, \forall x \in \mathcal{M}_{1+}$, for some $c > 0$, then $\mathcal{E}$ is automatically normal and faithful.*

Proof. The faithfulness is clear from the condition.

Let $\mathcal{M}_0^{**} \subset \mathcal{M}_1^{**}$ be the inclusion between the biduals of $\mathcal{M}_0, \mathcal{M}_1$ ([S]) induced by the initial inclusion and $\mathcal{E}^{**} : \mathcal{M}_1^{**} \to \mathcal{M}_0^{**}$ the bidual of $\mathcal{E}$. Thus $\mathcal{E}^{**}$ is a normal conditional expectation onto $\mathcal{M}_0^{**}$, it satisfies the same inequality as $\mathcal{E}$ does and coincides with $\mathcal{E}$ when restricted to $\mathcal{M}_1$.

For each $i = 0, 1$ let $p_i \in \mathcal{Z}(\mathcal{M}_i^{**})$ denote the "normal" projection for $\mathcal{M}_i$, i.e. $p_i \mathcal{M}_i^{**} = p_i \mathcal{M}_i$ and $x \in \mathcal{M}_i, xp_i = 0$ implies $x = 0$ (this latter condition simply means that $\mathcal{M}_i \ni x \mapsto xp_i \in \mathcal{M}_i p_i$ is an isomorphism; note that p_i can also be characterized as the largest projection p in $\mathcal{Z}(\mathcal{M}_i^{**})$ such that $\mathcal{M}_i^{**} p = \mathcal{M}_i p$). Then clearly $p_0 \geq p_1$ and $x \in \mathcal{M}_0, xp_1 = 0$ implies $x = 0$ (both conditions due to $\mathcal{M}_0$ being a von Neumann subalgebra of $\mathcal{M}_1$). In particular it follows that $\mathcal{E}^{**}(p_1)$ lies in $\mathcal{Z}(\mathcal{M}_0)$ and has support p_0.

Let then $\mathcal{E}' : \mathcal{M}_1^{**} p_1 \to \mathcal{M}_0^{**} p_1$ be defined by

$$\mathcal{E}'(Xp_1) = \mathcal{E}^{**}(Xp_1)(\mathcal{E}^{**}(p_1))^{-1} p_1, X \in \mathcal{M}_1^{**}.$$

Since $p_1 \leq p_1(\mathcal{E}^{**}(p_1))^{-1} \leq c^{-1} p_1$ and $\mathcal{E}^{**}$ is normal, it follows that $\mathcal{E}'$ is normal. Also, $\mathcal{E}^{**}(Xp_1) \geq cXp_1, \forall X \in \mathcal{M}_{1+}^{**}$, so that $||\mathcal{E}'(Xp_1)|| \geq ||c\mathcal{E}^{**}(Xp_1)^{-1/2} Xp_1 \mathcal{E}^{*}(Xp_1)^{-1/2}|| \geq ||cXp_1||$. But $\mathcal{M}_1^{**} p_1 \simeq \mathcal{M} p_1$ and $\mathcal{M}_0^{**} p_1 \simeq \mathcal{M}_0$. Thus, by [BaDH] (see also [Po2]), $\mathcal{E}'$ implements a normal conditional expectation of $\mathcal{M}_1$ onto $\mathcal{M}_0$, still denoted $\mathcal{E}'$, satisfying $\mathcal{E}'(x) \geq cx, \forall x \in \mathcal{M}_{1+}$.

Let us show that this implies $p_1 = p_0$. To this end let $\varphi \in (\mathcal{M}_1^{**})_*, \varphi \geq 0$ be such that $s(\varphi) \leq p_0$. Thus $\varphi_{|\mathcal{M}_0}$ is normal (since $s(\varphi_{|\mathcal{M}_0^{**}}) \leq p_0$). But then $\varphi \circ \mathcal{E}'$ is normal on $\mathcal{M}_1$, and since $\mathcal{E}'(x) \geq cx$, we also have $\varphi(x) \leq c^{-1} \varphi \circ \mathcal{E}'(x), \forall x \in \mathcal{M}_{1+}$. Thus φ is normal on $\mathcal{M}_1$, so $s(\varphi) \leq p_1$. This shows that $p_0 = p_1$, so by the formula defining $\mathcal{E}'$ we have $\mathcal{E} = \mathcal{E}'$. Since $\mathcal{E}'$ is normal, it follows that $\mathcal{E}$ is normal.

□

Next we will show that we only need to prove the relative Dixmier property for $x \in \mathcal{M}$ satisfying the condition $x \in \mathcal{Z}(\mathcal{N})' \cap \mathcal{M}$. To see that we may do so we need a technical lemma. Its proof is elementary but, "faute de mieux", it is somewhat long and tedious.

1.2 Lemma. *Let $\mathcal{Z}$ be a commutative von Neumann algebra and $\mathcal{Z}_i \subset \mathcal{Z}, i = 0, 1,$ be von Neumann subalgebras such that $\mathcal{Z}_0 \vee \mathcal{Z}_1 = \mathcal{Z}$ and such that there exist conditional expectations $E_i : \mathcal{Z} \to \mathcal{Z}_i$ with $\mathrm{Ind}E_i < \infty, i = 0, 1$. Then there exists a finite dimensional *-subalgebra $A_0 \subset \mathcal{Z}_0$ such that $A_0 \vee \mathcal{Z}_1 = \Sigma_i p_i \mathcal{Z}_1 = \mathcal{Z}$, where $\{p_i\}_i$ are the minimal projections of A_0.*

Proof. Before any other consideration, let us note that if $\mathcal{A}_0 \subset \mathcal{A}$ is an inclusion of abelian von Neumann algebras with a normal conditional expectation $\mathcal{F} : \mathcal{A} \to \mathcal{A}_0$ such that $\mathrm{Ind}\mathcal{F} \stackrel{\mathrm{def}}{=} (\max\{c \mid \mathcal{F}(x) \geq cx, \forall x \in \mathcal{A}_+\})^{-1}$, then $\mathrm{Ind}\mathcal{F}$ can alternatively be described by $(\mathrm{Ind}\mathcal{F})^{-1} = \inf\{\mathcal{F}(p)(f(s)) \mid s \in \Omega, p \in \mathcal{P}(\mathcal{A}), p(s) \neq 0\}$, where Ω is the spectrum of $\mathcal{A}$, Ω_0 is the spectrum of $\mathcal{A}_0$ and $f : \Omega \to \Omega_0$ is the continuous surjection implementing the inclusion $C(\Omega_0) = \mathcal{A}_0 \subset \mathcal{A} = C(\Omega)$. Also, if we denote $\alpha = (\mathrm{Ind}\mathcal{F})^{-1}$, then for any $p \in \mathcal{P}(\mathcal{A})$ the spectral projection e of $\mathcal{F}(p)$ corresponding to the open interval $(0, 1)$ satisfies $\alpha e \leq \mathcal{F}(p)e \leq (1 - \alpha)e$, in other words if $s_0 \in \Omega_0$ is so that $0 < \mathcal{F}(p)(s_0) < 1$ then $\alpha \leq \mathcal{F}(p)(s_0) \leq 1 - \alpha$. Moreover, note that by ([J], [BDH]) we always have $\mathrm{Ind}\mathcal{F} \geq 1$ and if $\mathrm{Ind}\mathcal{F} < 2$ then $\mathrm{Ind}\mathcal{F} = 1$, i.e., $\mathcal{A}_0 = \mathcal{A}$.

Let then $\mathcal{V}$ denote the set of all isomorphism classes of 5-tuples

$$V = (\mathcal{Z}, \mathcal{Z}_0, \mathcal{Z}_1, \mathcal{E}_0, \mathcal{E}_1)$$

in which $\mathcal{Z}$ is a commutative von Neumann, $\mathcal{Z}_i \subset \mathcal{Z}, i = 0, 1$ are von Neumann subalgebras with $\mathcal{Z}_0 \vee \mathcal{Z}_1 = \mathcal{Z}$ and $E_i : \mathcal{Z} \to \mathcal{Z}_i$ are conditional expectations with $\mathrm{Ind}E_i < \infty, i = 0, 1$.

By the preceding observation, for any $V = (\mathcal{Z}, \mathcal{Z}_0, \mathcal{Z}_1, \mathcal{E}_0, \mathcal{E}_1) \in \mathcal{V}$, any constant $c' > 0$ with $\mathrm{Ind}E_i \leq c'^{-1}, i = 1, 2$, and any $p \in \mathcal{P}(\mathcal{Z}_0)$, the element $E_0(E_1(p))$ has gaps in its spectrum, between 0 and c'^2 and between $1 - c'^2$ and 1, i.e., if $s_0 \in \Omega_0 \stackrel{\mathrm{def}}{=} \mathrm{spectrum}(\mathcal{Z}_0)$ is so that $0 < E_0(E_1(p))(s_0) < 1$ then $c'^2 \leq E_0(E_1(p))(s_0) \leq 1 - c'^2$.

To prove the lemma we need to show:

(*) $\forall V = (\mathcal{Z}, \mathcal{Z}_0, \mathcal{Z}_1, \mathcal{E}_0, \mathcal{E}_1) \in \mathcal{V}$, $\exists A_0 \subset \mathcal{Z}_0$ finite dimensional $*$-subalgebra such that $A_0 \vee \mathcal{Z}_1 = \mathcal{Z}$.

Let $\mathcal{V}_0 = \{V \in \mathcal{V} \mid (*) \text{ doesn't hold true for } V\}$. Assume, on the contrary, that $\mathcal{V}_0 \neq \emptyset$ and let $\beta = \inf\{\mathrm{Ind}V \mid V \in \mathcal{V}_0\}$, where for $V = (\mathcal{Z}, \mathcal{Z}_0, \mathcal{Z}_1, \mathcal{E}_0, \mathcal{E}_1) \in \mathcal{V}$ we set $\mathrm{Ind}V = \mathrm{Ind}E_0 + \mathrm{Ind}E_1$.

Let $V = (\mathcal{Z}, \mathcal{Z}_0, \mathcal{Z}_1, \mathcal{E}_0, \mathcal{E}_1) \in \mathcal{V}_0$ be such that $\mathrm{Ind}V < \beta + \beta^{-2}/2$. Let $\mathcal{P}_0$ be the set of projections q in $\mathcal{P}(\mathcal{Z}_0)$ such that $E_0(E_1(q)) \leq (1 - c^2)1$, where $c = (\mathrm{Ind}V)^{-1}$. By the weak continuity of $E_0 \circ E_1$ it follows that if $\{p_i\}_i \subset \mathcal{P}(\mathcal{Z}_0)$ is an increasing net of projections in $\mathcal{P}_0$ then its limit is still in $\mathcal{P}_0$. Thus $\mathcal{P}_0$ is inductively ordered, with respect to the usual order.

Let $p \in \mathcal{P}_0$ be a maximal element. Let $p_0 \in \mathcal{P}(\mathcal{Z}_0)$ be the support of $E_0(E_1(p))$. Then p_0 belongs to $\mathcal{Z}_0 \cap \mathcal{Z}_1$ and $\mathcal{Z}_0(1 - p_0) = \mathcal{Z}_1(1 - p_0) = \mathcal{Z}(1 - p_0)$.

For assume on the contrary that there exists a projection $q \in \mathcal{Z}_0(1 - p_0)$ such that $E_0(E_1(q)) \neq q$ and denote $q_0 = q - q_1$, where q_1 is the spectral projection of $E_0(E_1(q))$ corresponding to the set $\{1\}$. Thus q_0 is a non-zero projection in $\mathcal{Z}_0$ and $E_0(E_1(q_0))$ doesn't have 1 as an eigenvector, i.e., $E_0(E_1(q_0)) \leq (1 - c^2)1$. But then the projection $p + q_0 \in \mathcal{Z}_0$

lies in $\mathcal{P}_0$, i.e., $E_0(E_1(p+q_0)) \leq (1-c^2)1$. To see that this is indeed the case, note first that since $q_0 E_0(E_1(p)) = 0$ we have $q_0 E_1(p) = 0$ as well and thus $E_1(q_0)E_1(p) = 0$, showing that $E_1(p)$ and $E_1(q_0)$ have mutually disjoint supports. Thus, if we assume that $p + q_0$ is not in $\mathcal{P}_0$, then there would exist a non-zero projection $q_0' \in \mathcal{Z}_0$ satisfying $q_0' \leq E_0(E_1(p+q_0))$, and thus $q_0' \leq E_1(p+q_0)$ as well. But for any $p' \in \mathcal{Z}$ we have $(1-p')E_1(p')(1-p') = (1-p') - (1-p')E_1(1-p')(1-p') \leq (1-c)(1-p')$. By applying this to $p' = p+q_0$ it follows that if q_0' satisfies the above inequalities then $q_0' \leq p+q_0$. So by replacing if necessary q_0' by either $q_0'p$ or by $q_0'q_0$ we may assume that either $q_0' \leq p$ or that $q_0' \leq q_0$, while still having all the previous properties for q_0'. If $q_0' \leq p$ then the inequality $q_0' \leq E_1(p)+E_1(q_0)$ implies $q_0' = q_0'(1-q_0) \leq (1-q_0)E_1(p)(1-q_0)+(1-q_0)E_1(q_0)(1-q_0)$ with $(1-q_0)E_1(p)(1-q_0) = (1-q_0)E_1(p)$ having support orthogonal to the support of $(1-q_0)E_1(q_0)(1-q_0) = (1-q_0)E_1(q_0)$ and with $(1-q_0)E_1(q_0)(1-q_0) \leq (1-c^2)(1-q_0)$, thus forcing $q_0' \leq E_1(p)$ and consequently $q_0' \leq E_0(E_1(p))$, contradicting $p \in \mathcal{P}_0$. Similarly, if $q_0' \leq q_0$ then $q_0' = q_0'q_0 \leq q_0E_0(E_1(p+q_0))q_0 = q_0E_0(E_1(q_0))q_0$, so that $q_0' \leq E_0(E_1(q_0)) \leq (1-c^2)1$, a contradiction.

These contradictions show that $p + q_0$ must in fact lie in $\mathcal{P}_0$. But since $p + q_0$ is strictly larger than p this contradicts the maximality of p. We conclude that indeed the support p_0 of the maximal element p in $\mathcal{P}_0$ satisfies $p_0 \in \mathcal{Z}_0 \cap \mathcal{Z}_1$ and $\mathcal{Z}_0(1-p_0) = \mathcal{Z}_1(1-p_0) = \mathcal{Z}(1-p_0)$.

Since (*) doesn't hold true for $V = (\mathcal{Z}, \mathcal{Z}_0, \mathcal{Z}_1, \mathcal{E}_0, \mathcal{E}_1)$, this shows that it doesn't hold true for $(\mathcal{Z}p_0, \mathcal{Z}_0p_0, \mathcal{Z}_1p_0, \mathcal{E}_0, \mathcal{E}_1)$ either, with this latter having the same index as V. We may thus assume, without loss of generality, that in fact V is so that the maximal projection $p \in \mathcal{P}_0$ satisfies $\mathrm{supp}E_0(E_1(p)) = 1$.

Let then $p_1 \in \mathcal{Z}_1$ be the spectral projection of $E_1(p)$ corresponding to the set $\{1\}$, $p' \in \mathcal{Z}_1$ the spectral projection of $E_1(p)$ corresponding to the set $[c, 1-c]$, $p_2 = p'p, p_3 = p'(1-p) \in \mathcal{Z}$ and $p_4 \in \mathcal{Z}_1$ the spectral projection of $E_1(p)$ corresponding to the set $\{0\}$. Thus $p_1 + p_2 + p_3 + p_4 = 1$.

For each $1 \leq j \leq 4$ let $\mathcal{Z}^j = \mathcal{Z}p_j, \mathcal{Z}_i^j = \mathcal{Z}_ip_j, i = 0, 1$. Also, for each $1 \leq j \leq 4$ for which $p_j \neq 0$ we define the expectations $E_i^j : \mathcal{Z}^j \to \mathcal{Z}_i^j$ by $E_i^j(x) = E_i(xp_j)(E_i(p_j))^{-1}p_j \in \mathcal{Z}_ip_j = \mathcal{Z}_i^j, i = 0, 1, x \in \mathcal{Z}^j$.

It is then immediate to see that $\mathrm{Ind}E_i^j \leq \mathrm{Ind}E_i$ and that for $j = 1, 4$ we have $\mathrm{Ind}E_0^j \leq (1-c^2)\mathrm{Ind}E_0$ while for $j = 2, 3$ we have $\mathrm{Ind}E_1^j \leq (1-c^2)\mathrm{Ind}E_1$. Thus, if we let $V_j = (\mathcal{Z}^j, \mathcal{Z}_0^j, \mathcal{Z}_1^j, \mathcal{E}_0^j, \mathcal{E}_1^j)$ then by using that $\mathrm{Ind}E_i^j \geq 1$ we get

$$\mathrm{Ind}V_j = \mathrm{Ind}E_0^j + \mathrm{Ind}E_1^j \leq \mathrm{Ind}E_0 + \mathrm{Ind}E_1 - c^2 = \mathrm{Ind}V - c^2 < \beta.$$

Thus (*) does hold true for each $V_j, 1 \leq j \leq 4$, so there exist finite dimensional subalgebras $A^j \subset Z_0^j = \mathcal{Z}_0p_j$ such that $\mathcal{Z}^j = \mathcal{A}^j \vee \mathcal{Z}_1^j, 1 \leq j \leq 4$. Let then $A_0^j \subset \mathcal{Z}_0$ be finite dimensional subalgebras such that $A_0^jp_j = A^j$ and define $A_0 \subset \mathcal{Z}_0$ to be the algebra generated by $p, 1$ and the algebras $A_0^j, 1 \leq j \leq 4$. Then A_0 is clearly a finite dimensional von Neumann subalgebra of $\mathcal{Z}_0$ and it satisfies $A_0 \vee \mathcal{Z}_1 = \mathcal{Z}$, thus contradicting the fact that (*) doesn't hold true for V, i.e., $V \in \mathcal{V}_0$. This final contradiction completes the proof of the lemma.

□

1.3 Corollary. *If $\mathcal{N} \subset \mathcal{M}$ is an inclusion of von Neumann algebras as in the statement of the theorem then there exists a finite dimensional *-subalgebra A_0 of $\mathcal{Z}(\mathcal{N})$ such that*

$A_0 \vee \mathcal{Z}(\mathcal{M}) \supset \mathcal{Z}(\mathcal{N})$. *In particular, there exists a finite number of unitary elements* $v_1, \ldots, v_m$ *in* $\mathcal{Z}(\mathcal{N})$ *such that* $\Sigma_j v_j x v_j^* \in \mathcal{Z}(\mathcal{N})' \cap \mathcal{M}$, $\forall x \in \mathcal{M}$.

Proof. By the previous lemma, it is sufficient to show that if we denote $\mathcal{Z}_0 \stackrel{\text{def}}{=} \mathcal{Z}(\mathcal{N})$, $\mathcal{Z}_1 \stackrel{\text{def}}{=} \mathcal{Z}(\mathcal{M})$ and $\mathcal{Z} \stackrel{\text{def}}{=} \mathcal{Z}(\mathcal{N}) \vee \mathcal{Z}(\mathcal{M}) = \mathcal{Z}_0 \vee \mathcal{Z}_1$ then there exist normal conditional expectations E_i of finite index from $\mathcal{Z}$ onto $\mathcal{Z}_i, i = 0, 1$.

Noting that $\mathcal{Z}$ is included in $\mathcal{N}' \cap \mathcal{M}$ and that $\mathcal{E}(\mathcal{N}' \cap \mathcal{M}) \subset \mathcal{N} \cap \mathcal{N}' = \mathcal{Z}(\mathcal{N}) = \mathcal{Z}_0$, it follows that $\mathcal{E}(\mathcal{Z}) = \mathcal{Z}_0$. Since $\mathrm{Ind}E_0 \leq \mathrm{Ind}\mathcal{E} < \infty$, we may simply take $E_0 \stackrel{\text{def}}{=} \mathcal{E}_{|\mathcal{Z}}$.

To define E_1, note that, since $\mathcal{E}(x) \geq cx, \forall x \in \mathcal{M}_+$, it follows that if $\mathcal{M} \stackrel{\mathcal{E}_1}{\subset} \mathcal{M}_1$ is the basic construction for $\mathcal{N} \stackrel{\mathcal{E}}{\subset} \mathcal{M}$, then $\mathcal{E}_1(x) \geq c^2 x, \forall x \in \mathcal{M}_{1+}$ (see for instance 1.2.1 in [Po2]). Note also that if $x \in \mathcal{M}' \cap \mathcal{M}_1$ then $\mathcal{E}_1(x) \in \mathcal{Z}_1 = \mathcal{Z}(\mathcal{M})$. By canonical conjugation it follows that there exists a conditional expectation E_1 of $\mathcal{N}' \cap \mathcal{M}$ (=the canonical conjugate of $\mathcal{M}' \cap \mathcal{M}_1$) onto $\mathcal{Z}_1$ (=the canonical conjugate of itself) such that $E_1(x) \geq c^2 x, \forall x \in (\mathcal{N}' \cap \mathcal{M})_+$. In particular, if we still denote by E_1 the restriction of E_1 to $\mathcal{Z} = \mathcal{Z}_0 \vee \mathcal{Z}_1$, then $E_1(x) \geq c^2 x, \forall x \in \mathcal{Z}_+$, so $\mathrm{Ind}E_1 < \infty$.

□

1.4 Notation. For $x \in \mathcal{M}$ denote $C_{\mathcal{N}}(x) = \overline{\mathrm{co}}^n\{uxu^* \mid u \in \mathcal{U}(\mathcal{N})\}$. Remark that, with this notation, the relative Dixmier property for the inclusion $\mathcal{N} \subset \mathcal{M}$ can be reformulated as follows: "$C_{\mathcal{N}}(x) \cap \mathcal{N}' \cap \mathcal{M} \neq \emptyset, \forall x \in \mathcal{M}$". If $x \in \mathcal{M}$ is a given element for which $C_{\mathcal{N}}(x) \cap \mathcal{N}' \cap \mathcal{M} \neq \emptyset$, then we will say (occasionally!) that we have the relative Dixmier property for x.

Note that if $y \in C_{\mathcal{N}}(x)$ then $C_{\mathcal{N}}(y) \subset C_{\mathcal{N}}(x)$, so that if $C_{\mathcal{N}}(y) \cap \mathcal{N}' \cap \mathcal{M} \neq \emptyset$ then $C_{\mathcal{N}}(x) \cap \mathcal{N}' \cap \mathcal{M} \neq \emptyset$. By taking $y \in C_{\mathcal{N}}(x)$ of the form $y = \frac{1}{m}\sum_{j=1}^m v_j x v_j^*$, with $v_1, \ldots, v_m$ the unitary elements in $\mathcal{Z}(\mathcal{M})$ given by Corollary 1.3, it thus follows that in order to show that $C_{\mathcal{N}}(x) \cap \mathcal{N}' \cap \mathcal{M} \neq \emptyset, \forall x \in \mathcal{M}$, it is sufficient to show that $C_{\mathcal{N}}(y) \cap \mathcal{N}' \cap \mathcal{M} \neq \emptyset, \forall y \in \mathcal{Z}(\mathcal{N})' \cap \mathcal{M}$.

But this means that we may simply replace the algebra $\mathcal{M}$ by the smaller algebra $\mathcal{Z}(\mathcal{N})' \cap \mathcal{M}$, i.e., we only need to check the relative Dixmier property for the inclusion $\mathcal{N} \subset \mathcal{Z}(\mathcal{N})' \cap \mathcal{M}$. Since the center of the algebra $\mathcal{Z}(\mathcal{N})' \cap \mathcal{M}$ contains $\mathcal{Z}(\mathcal{N})$, this shows that in order to prove the relative Dixmier property for all inclusions of von Neumann algebras with finite index $\mathcal{N} \subset \mathcal{M}$ it is sufficient to prove it for those that satisfy $\mathcal{Z}(\mathcal{N}) \subset \mathcal{Z}(\mathcal{M})$.

Next we will show that we do have the relative Dixmier property for elements $x \in \mathcal{N} \vee \mathcal{N}' \cap \mathcal{M}$. This will enable us to concentrate on x's that are "perpendicular" to $\mathcal{N} \vee \mathcal{N}' \cap \mathcal{M}$. But first note:

1.5 Lemma. *If* $\mathcal{N} \subset \mathcal{M}$ *is an inclusion of von Neumann algebras with a normal faithful conditional expectation* $\mathcal{E}$ *then there exists a unique normal faithful conditional expectation of* $\mathcal{M}$ *onto* $\mathcal{N} \vee \mathcal{N}' \cap \mathcal{M}$, *denoted* $\mathcal{E}_{\mathcal{N} \vee \mathcal{N}' \cap \mathcal{M}}$. *Moreover this expectation satisfies* $\mathcal{E} = \mathcal{E} \circ \mathcal{E}_{\mathcal{N} \vee \mathcal{N}' \cap \mathcal{M}}$.

Proof. Let ϕ_0 be a normal semifinite faithful weight on $\mathcal{N}$. Then clearly $\phi = \phi_0 \circ \mathcal{E}$ is a normal semifinite faithful weight on $\mathcal{M}$ and its restriction to $\mathcal{N} \vee \mathcal{N}' \cap \mathcal{M}$ is also semifinite. Moreover, since the modular automorphism group σ_ϕ associated with ϕ satisfies $\sigma_\phi(\mathcal{N}) = \mathcal{N}$, it follows that $\sigma_\phi(\mathcal{N}' \cap \mathcal{M}) = \mathcal{N}' \cap \mathcal{M}$, so that

$\sigma_\phi(\mathcal{N}\vee\mathcal{N}'\cap\mathcal{M}) = \mathcal{N}\vee\mathcal{N}'\cap\mathcal{M}$. This shows that the conditions in Tkesaki's criterion ([T]) for the existence of a ϕ-preserving normal conditional expectation of $\mathcal{M}$ onto $\mathcal{N}\vee\mathcal{N}'\cap\mathcal{M}$ are met.

Since $(\mathcal{N}\vee\mathcal{N}'\cap\mathcal{M})'\cap\mathcal{M}$ is included in $\mathcal{N}\vee\mathcal{N}'\cap\mathcal{M}$, by a result of Connes ([C1]), the expectation is unique (not only among those expectations that preserve the weight ϕ). If we denote it by $\mathcal{E}_{\mathcal{N}\vee\mathcal{N}'\cap\mathcal{M}}$ then $\phi = \phi\circ\mathcal{E}$ and $\phi = \phi\circ\mathcal{E}_{\mathcal{N}\vee\mathcal{N}'\cap\mathcal{M}}$, thus $\phi = \phi\circ\mathcal{E}\circ\mathcal{E}_{\mathcal{N}\vee\mathcal{N}'\cap\mathcal{M}}$. Thus, since both $\mathcal{E}$ and $\mathcal{E}\circ\mathcal{E}_{\mathcal{N}\vee\mathcal{N}'\cap\mathcal{M}}$ are normal ϕ-preserving conditional expectations onto $\mathcal{N}$, it follows that $\mathcal{E}\circ\mathcal{E}_{\mathcal{N}\vee\mathcal{N}'\cap\mathcal{M}} = \mathcal{E}$.

$\square$

For the proof of the next result it is useful to note that if x is an element in $\mathcal{M}$ for which there exists a sequence $y_0 = x, y_1, y_2, \ldots$ in $\mathcal{M}$ such that $y_{n+1} \in C_{\mathcal{N}}(y_n), \forall n \geq 0$ and $\mathrm{d}(y_n, \mathcal{N}'\cap\mathcal{M}) \to 0$ then $C_{\mathcal{N}}(x)\cap\mathcal{N}'\cap\mathcal{M} \neq \emptyset$. Indeed, because if $\mathrm{d}(y_n, \mathcal{N}'\cap\mathcal{M}) < \varepsilon$ then $\mathrm{d}(y, \mathcal{N}'\cap\mathcal{M}) < \varepsilon$ for all $y \in C_{\mathcal{N}}(y_n)$ (since all unitaries in $\mathcal{N}$ used for averaging y_n commute with $\mathcal{N}'\cap\mathcal{M}$) so that $\mathrm{diam} C_{\mathcal{N}}(y_n) < 2\varepsilon$, and thus the condition $\mathrm{d}(y_n, \mathcal{N}'\cap\mathcal{M}) \to 0$ implies $\cap_n C_{\mathcal{N}}(y_n)$ is a single element set $\{y'\}$ for some y' which lies both in $C_{\mathcal{N}}(x)$ and in $\mathcal{N}'\cap\mathcal{M}$.

1.6 Proposition. *If $\mathcal{N}, \mathcal{M}, \mathcal{E}$ are as in the hypothesis of the Theorem then:*

a). $\mathcal{N}\vee\mathcal{N}'\cap\mathcal{M}$ has a finite orthonormal basis over $\mathcal{N}$ which is contained in $\mathcal{N}'\cap\mathcal{M}$.

b). $C_{\mathcal{N}}(x)\cap\mathcal{N}'\cap\mathcal{M} \neq \emptyset, \forall x \in \mathcal{N}\vee\mathcal{N}'\cap\mathcal{M}$ and if $x \in \mathcal{M}$ is so that $\mathcal{E}_{\mathcal{N}\vee\mathcal{N}'\cap\mathcal{M}}(x) = 0$ then either $C_{\mathcal{N}}(x)\cap\mathcal{N}\vee\mathcal{N}'\cap\mathcal{M} = \{0\} = C_{\mathcal{N}}(x)\cap\mathcal{N}'\cap\mathcal{M}$ or $C_{\mathcal{N}}(x)\cap\mathcal{N}\vee\mathcal{N}'\cap\mathcal{M} = \emptyset$.

c). $C_{\mathcal{N}}(x)\cap\mathcal{N}'\cap\mathcal{M} \neq \emptyset, \forall x \in \mathcal{M}$ if and only if $C_{\mathcal{N}}(x)\cap\mathcal{N}'\cap\mathcal{M} \neq \emptyset, \forall x = x^* \in \mathcal{M}$ satisfying $\mathcal{E}_{\mathcal{N}\vee\mathcal{N}'\cap\mathcal{M}}(x) = 0$.

Proof. a). Note first that $\mathcal{E}(\mathcal{N}'\cap\mathcal{M}) = \mathcal{Z}(\mathcal{N})$ and that any orthonormal basis of $\mathcal{N}'\cap\mathcal{M}$ over $\mathcal{Z}(\mathcal{N})$ (with respect to $\mathcal{E}$) is an orthonormal basis of $\mathcal{N}\vee\mathcal{N}'\cap\mathcal{M}$ over $\mathcal{N}$ (with respect to the same $\mathcal{E}$).

Thus $\mathcal{Z}(\mathcal{N}) \subset \mathcal{N}'\cap\mathcal{M}$ is an inclusion of finite index with $\mathcal{Z}_1 = \mathcal{Z}(\mathcal{N})$ abelian. It follows that $\mathcal{N}'\cap\mathcal{M}$ is of type I_{fin} with the homogeneous type I_{n_i} von Neumann algebras entering in its direct sum decomposition satisfying $\sup_i n_i < \infty$ (see 1.1.2 (iii) in [Po2]). Thus $\mathcal{N}'\cap\mathcal{M}$ is a finitely generated module over $\mathcal{Z} = \mathcal{Z}(\mathcal{N}'\cap\mathcal{M})$.

Since $\mathcal{Z}_1 \subset \mathcal{Z}$ has finite index as well, if we define $\mathcal{Z}_0$ to be equal to $\mathcal{Z}$ and apply Lemma 1.2 to $\mathcal{Z}_0, \mathcal{Z}_1, \mathcal{Z}$, it follows that $\mathcal{Z}$ is finitely generated over $\mathcal{Z}_1$. Altogether this shows that $\mathcal{N}'\cap\mathcal{M}$ is a finitely generated module over $\mathcal{Z}_1 = \mathcal{Z}(\mathcal{N})$. By orthonormalizing a given finite set of generators (see 1.1.7 in [Po2] or 1.1.3 in [Po4]), it follows that $\mathcal{N}'\cap\mathcal{M}$ has a finite orthonormal basis over $\mathcal{Z}(\mathcal{N})$.

b). By a) $\mathcal{N} \subset \mathcal{N}\vee\mathcal{N}'\cap\mathcal{M}$ has an orthonormal basis $\{m_j\}_{1\leq j\leq n} \subset \mathcal{N}'\cap\mathcal{M}$. Thus, for any $x \in \mathcal{N}\vee\mathcal{N}'\cap\mathcal{M}$, we have $x = \sum_{j=1}^{n} m_j y_j'$, for some $y_j' \in \mathcal{N}$. By applying the classical Dixmier theorem to $y_1', y_2', \ldots, y_n' \in \mathcal{N}$, given any $\varepsilon > 0$ we can find unitary elements $\{u_k\}_{1\leq k\leq m} \subset \mathcal{U}(\mathcal{N})$ such that

$$\left\| \frac{1}{m} \sum_{k=1}^{m} u_k y_j' u_k^* - z_j \right\| < \varepsilon/n\|m_j\|,$$

$1 \leq j \leq n$, for some $z_j \in \mathcal{Z}(\mathcal{N})$. Thus

$$\left\| \frac{1}{m} \sum_{k=1}^{m} u_k x u_k^* - \sum_{j=1}^{n} m_j z_j \right\| < \varepsilon.$$

Since $\frac{1}{m}\sum_{k=1}^{m} u_k x u_k^*$ is still in $\mathcal{N} \vee \mathcal{N}' \cap \mathcal{M}$, it follows that we may use the above recursively for $\varepsilon = 1/2^n$ to get a sequence of elements $y_0 = x, y_1, y_2, \ldots$ in $\mathcal{N} \vee \mathcal{N}' \cap \mathcal{M}$ such that $y_{n+1} \in C_{\mathcal{N}}(y_n), \forall n \geq 0$ and $\mathrm{d}(y_n, \mathcal{N}' \cap \mathcal{M}) \to 0$. Thus, by the observation preceding the statement of the Proposition, $C_{\mathcal{N}}(x) \cap \mathcal{N}' \cap \mathcal{M} \neq \emptyset$.

Now if in turn $x \in \mathcal{M}$ is so that $\mathcal{E}_{\mathcal{N} \vee \mathcal{N}' \cap \mathcal{M}}(x) = 0$ then clearly $\mathcal{E}_{\mathcal{N} \vee \mathcal{N}' \cap \mathcal{M}}(y) = 0$ for any $y \in C_{\mathcal{N}}(x)$. Thus if $C_{\mathcal{N}}(x) \cap \mathcal{N} \vee \mathcal{N}' \cap \mathcal{M} \neq \emptyset$ and we take $y \in C_{\mathcal{N}}(x) \cap \mathcal{N} \vee \mathcal{N}' \cap \mathcal{M}$ then we have both that y is in $\mathcal{N} \vee \mathcal{N}' \cap \mathcal{M}$ and $\mathcal{E}_{\mathcal{N} \vee \mathcal{N}' \cap \mathcal{M}}(y) = 0$. But this implies $y = 0$ and so $C_{\mathcal{N}}(x) \cap \mathcal{N} \vee \mathcal{N}' \cap \mathcal{M} = \{0\}$.

c). Assume $C_{\mathcal{N}}(x) \cap \mathcal{N}' \cap \mathcal{M} \neq \emptyset$ (so $= \{0\}$ by part b)) for all $x = x^* \in \mathcal{M}$ satisfying $\mathcal{E}_{\mathcal{N} \vee \mathcal{N}' \cap \mathcal{M}}(x) = 0$. Let us first prove that this implies $C_{\mathcal{N}}(x) \cap \mathcal{N}' \cap \mathcal{M} \neq \emptyset$ for all arbitrary selfadjoint elements x in $\mathcal{M}$. To this end fix $x = x^* \in \mathcal{M}$ and denote $x_0 = \mathcal{E}_{\mathcal{N} \vee \mathcal{N}' \cap \mathcal{M}}(x)$, $x_1 = x - x_0$. By assumption it follows that $\forall \varepsilon > 0, \exists v_1, v_2, \ldots, v_n \in \mathcal{U}(\mathcal{N})$ such that

$$\left\| \frac{1}{n} \sum_{k=1}^{n} v_k x_1 v_k^* \right\| < \varepsilon/2.$$

But then $y = \frac{1}{n}\sum_{k=1}^{n} v_k x_0 v_k^*$ still belongs to $\mathcal{N} \vee \mathcal{N}' \cap \mathcal{M}$ and so by part b) there exist unitary elements $u_j \in \mathcal{U}(\mathcal{N}), 1 \leq j \leq m$ and some $y_1' \in \mathcal{N}' \cap \mathcal{M}$ such that

$$\left\| \frac{1}{m} \sum_{j=1}^{m} u_j y u_j^* - y_1' \right\| < \varepsilon/2.$$

Altogether this shows that we have

$$\begin{aligned} &\left\| \frac{1}{nm} \sum_{j,k} u_j v_k x v_k^* u_j^* - y_1' \right\| \\ &\quad \leq \left\| \frac{1}{nm} \sum_{j,k} u_j v_k x_1 v_k^* u_j^* \right\| + \left\| \frac{1}{nm} \sum_{j,k} u_j v_k x_0 v_k^* u_j^* - y_1' \right\| \\ &\quad \leq \left\| \frac{1}{n} \sum_{k} v_k x_1 v_k^* \right\| + \left\| \frac{1}{m} \sum_{j=1}^{m} u_j y u_j^* - y_1' \right\| < \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

Thus, if we let $y_1 = \frac{1}{nm}\sum_{j,k} u_j v_k x v_k^* u_j^*$ then y_1 belongs to $C_{\mathcal{N}}(x)$ and $\|y_1 - y_1'\| < \varepsilon$.

Thus, by applying all this for $\varepsilon = 1/2^n, n \geq 1$, we recursively obtain a sequence of elements $y_0 = x, y_1, y_2, \ldots$ in $\mathcal{M}$ with $y_{n+1} \in C_{\mathcal{N}}(y_n), n \geq 0$, $\mathrm{d}(y_n, \mathcal{N}' \cap \mathcal{M}) < 1/2^n, n \geq 1$. But by the observation preceding the statement of the Proposition this implies that $C_{\mathcal{N}}(x) \cap \mathcal{N}' \cap \mathcal{M} \neq \emptyset$.

Take now an arbitrary element x in $\mathcal{M}$. By writing $x = \mathrm{Re}(x) + i\mathrm{Im}(x)$ and using the same argument as in the proof of the classical single algebra case (see [D1]) together with the above relative Dixmier property for selfadjoint elements, it readily follows that in fact $C_{\mathcal{N}}(x) \cap \mathcal{N}' \cap \mathcal{M} \neq \emptyset$ as well.

□

At this point let us recall (see e.g., 1.1.2 (iii) in [Po2]) that $\mathrm{Ind}(\mathcal{N} \subset \mathcal{M}) < \infty$ implies there exists a projection p in $\mathcal{Z}(\mathcal{N}) \cap \mathcal{Z}(\mathcal{M})$ such that $p\mathcal{N} \subset p\mathcal{M}$ is an inclusion of finite von Neumann algebras and $(1-p)\mathcal{N} \subset (1-p)\mathcal{M}$ is an inclusion of properly infinite von Neumann algebras. Thus, since obviously $\mathcal{N}_1 \subset \mathcal{M}_1, \mathcal{N}_2 \subset \mathcal{M}_2$ have the relative Dixmier property if and only if $\mathcal{N}_1 \oplus \mathcal{N}_2 \subset \mathcal{M}_1 \oplus \mathcal{M}_2$ has the relative Dixmier property, we may deal separately with the case $\mathcal{N} \subset \mathcal{M}$ is an inclusion of finite von Neumann algebras and respectively the case $\mathcal{N} \subset \mathcal{M}$ is an inclusion of properly infinite von Neumann algebras.

Altogether we have shown in this Section that in order to prove the Theorem it is sufficient to prove the following statement:

1.7. *Let $\mathcal{N} \subset \mathcal{M}$ be an inclusion of von Neumann algebras with a conditional expectation $\mathcal{E} : \mathcal{M} \to \mathcal{N}$ of finite index and satisfying $\mathcal{Z}(\mathcal{N}) \subset \mathcal{Z}(\mathcal{M})$. Assume that either $\mathcal{N}, \mathcal{M}$ are both finite or both properly infinite. Then given any $x = x^* \in \mathcal{M}$ with $\mathcal{E}_{\mathcal{N} \vee \mathcal{N}' \cap \mathcal{M}}(x) = 0$, we have $C_{\mathcal{N}}(x) \cap \mathcal{N}' \cap \mathcal{M} \neq \emptyset$.*

Thus, we will assume throughout Sections 2 and 3 below, which contain the proofs of the finite and respectively properly infinite cases of 1.7, that $\mathcal{N}, \mathcal{M}$ and x are as in 1.7.

2. Proof of the Finite Case

Consider first the case when $\mathcal{N}, \mathcal{M}$ are finite von Neumann algebras. We will proceed by contradiction, using the Hahn-Banach theorem to separate $C_{\mathcal{N}}(x)$ from $\mathcal{N} \vee \mathcal{N}' \cap \mathcal{M}$ with a functional equal to some linear combination of states that can be taken tracial on $\mathcal{N}$. The key observation which will then lead this to a contradiction is that the hypothesis "$\mathcal{E}(x) \geq cx, \forall x \in \mathcal{M}_+$" implies "$\varphi \leq c^{-1}\varphi \circ \mathcal{E}, \forall \varphi \in \mathcal{M}_+^*$", so that, if a state φ on $\mathcal{M}$ is normal on $\mathcal{N}$, it is automatically normal on all $\mathcal{M}$. The argument is carried to an end by using the "weak" version of the realtive Dixmier property in [Po1], showing that $\overline{C_{\mathcal{N}}(x)}^w \cap \mathcal{N}' \cap \mathcal{M} \neq \emptyset, \forall x \in \mathcal{M}$.

With this plan in mind, assume, on the contrary, that there exists an element $x = x^*$ in $\mathcal{M}$, with $\mathcal{E}_{\mathcal{N} \vee \mathcal{N}' \cap \mathcal{M}}(x) = 0$, such that $C_{\mathcal{N}}(x) \cap \mathcal{N}' \cap \mathcal{M} = \emptyset$. By 1.6 this implies $C_{\mathcal{N}}(x) \cap \mathcal{N} \vee \mathcal{N}' \cap \mathcal{M} = \emptyset$. Moreover, it implies that $\inf\{\|y\| \mid y \in C_{\mathcal{N}}(x)\} > 0$ and, in fact, $\inf\{\|y - b\| \mid y \in C_{\mathcal{N}}(x), b \in \mathcal{N} \vee \mathcal{N}' \cap \mathcal{M}\} > 0$ as well. Indeed, the first inequality is clear (or else $0 \in C_{\mathcal{N}}(x)$ so $C_{\mathcal{N}}(x) \cap \mathcal{N}' \cap \mathcal{M} \neq \emptyset$) and for the second one we use the fact that if $y \in C_{\mathcal{N}}(x)$ and $b \in B = \mathcal{N} \vee \mathcal{N}' \cap \mathcal{M}$ then $\mathcal{E}_{\mathcal{N} \vee \mathcal{N}' \cap \mathcal{M}}(y - b) = -b, \forall y \in C_{\mathcal{N}}(x)$ so that $\|b\| = \|\mathcal{E}_B(y - b)\| \leq \|y - b\|$, thus $\|y\| \leq \|y - b\| + \|b\| \leq 2\|y - b\|$, implying that

$$\inf\{\|y - b\| \mid y \in C_{\mathcal{N}}(x), b \in B\} \geq 2^{-1} \inf\{\|y\| \mid y \in C_{\mathcal{N}}(x)\} > 0.$$

This shows that in the Banach quotient space $\mathcal{M}/B$ (which has an isometric "adjoint" operation $*$ inherited from $\mathcal{M}$) the set $C_{\mathcal{N}}(x)/B$ is away from 0. By using the Hahn-Banach theorem in the quotient space $\mathcal{M}/B$ it follows that there exists a selfadjoint functional Φ' on $\mathcal{M}/B$ which separates the self-adjoint set $C_{\mathcal{N}}(x)/B$ from 0. By composing Φ' with the quotient map from $\mathcal{M}$ onto $\mathcal{M}/B$, it follows that there exists a functional $\Phi = \Phi^*$ on $\mathcal{M}$ and $\varepsilon_0 > 0$ such that Φ vanishes on $B = \mathcal{N} \vee \mathcal{N}' \cap \mathcal{M}$ and $\Phi(y) \geq \varepsilon_0, \ \forall y \in C_{\mathcal{N}}(x)$.

From these properties of Φ and from the definition of $C_{\mathcal{N}}(x)$, it follows that $\Psi(x) \geq \varepsilon_0$, $\forall \Psi \in \mathrm{co}\{\Phi(u \cdot u^*) \mid u \in \mathcal{U}(\mathcal{N})\}$ so that $\Psi(x) \geq \varepsilon_0$, $\forall \Psi \in C_{\mathcal{N}}(\Phi) \stackrel{\text{def}}{=} \overline{\mathrm{co}}^{\sigma(M^*,M)}\{\Phi(u \cdot u^*) \mid u \in \mathcal{U}(\mathcal{N})\}$ and in fact $\Psi(y) \geq \varepsilon_0$, $\forall y \in C_{\mathcal{N}}(x)$ as well. Also, each $\Psi \in C_{\mathcal{N}}(\Phi)$ vanishes on $\mathcal{N} \vee \mathcal{N}' \cap \mathcal{M}$. In order to get a contradiction we will first show that there exists a Ψ in $C_{\mathcal{N}}(\Phi)$ with the property that it can be written as $\Psi = \Psi_1 - \Psi_2$, with $\Psi_{1,2} \in S(\mathcal{M})$ commuting with $\mathcal{N}$ and being factorial traces when restricted to $\mathcal{N}$. By using Sakai's Radon-Nykodim type theorem for $\Psi_{1,2}$ and $\psi = \Psi_{1,2} \circ \mathcal{E}$, we will then obtain positive elements $a_{1,2} \in \mathcal{N}' \cap \mathcal{M}_+$ such that $\psi \circ \mathcal{E}(a_1^{1/2} \cdot a_1^{1/2}) - \psi \circ \mathcal{E}(a_2^{1/2} \cdot a_2^{1/2})$ is close to Ψ, and thus, still separates x from 0. But this will easily be seen to lead to a contradiction by using [Po1].

To prove the existence of $\Psi = \Psi_1 - \Psi_2 \in C_{\mathcal{N}}(\Phi)$ with $\Psi_{1,2}$ states commuting with $\mathcal{N}$ and being factorial on $\mathcal{N}$, we need the following:

2.1. Lemma. *Let $\psi = (\psi_1, \psi_2, \ldots, \psi_n) \in S(\mathcal{M})^n$, and set*

$$C_{\mathcal{N}}(\psi) = \overline{\mathrm{co}}^{\sigma((\mathcal{M}^*)^n,\mathcal{M}^n)}\{(\psi_i(u \cdot u^*))_{1\leq i\leq n} \mid u \in \mathcal{U}(\mathcal{N})\} \subset (S(\mathcal{M}))^n.$$

Then there exists $(\alpha_1, \ldots, \alpha_n) \in C_{\mathcal{N}}(\psi)$ such that $\alpha_{i|\mathcal{Z}(\mathcal{N})} = \psi_{i|\mathcal{Z}(\mathcal{N})}$ and $\mathcal{N}$ is in the centralizer of $\alpha_i, 1 \leq i \leq n$.

Proof. We first show that $C_{\mathcal{N}}(\psi) \cap S_{\mathcal{N},\psi}(\mathcal{M}) \neq \emptyset$, where $S_{\mathcal{N},\psi}(\mathcal{M}) \stackrel{\text{def}}{=} \{(\alpha_1, \ldots, \alpha_n)$ in $S(\mathcal{M})^n \mid \alpha_{i|\mathcal{N}}$ is the unique trace on $\mathcal{N}$ such that $\alpha_{i|\mathcal{Z}(\mathcal{N})} = \psi_{i|\mathcal{Z}(\mathcal{N})},\ 1 \leq i \leq n\}$.

Note that $(\mathcal{M}^n)^* = (\mathcal{M}^*)^n$, with the duality being given by

$$\langle (x_1, \ldots, x_n),\ (\varphi_1, \ldots, \varphi_n)\rangle = \textstyle\sum_{i=1}^n \varphi_i(x_i),$$

then remark that, in the $\sigma((\mathcal{M}^*)^n,\ \mathcal{M}^n)$ topology on $S(\mathcal{M})^n$, both $C_{\mathcal{N}}(\psi)$ and $S_{\mathcal{N},\psi}(\mathcal{M})$ are convex and compact. Assume first that $\mathcal{N} = \mathcal{M}$, in which case $S_{\mathcal{N},\psi}(\mathcal{N}) = \{(\alpha_1^0, \ldots, \alpha_n^0)\}$, where $\alpha_i^0 \in S(\mathcal{N})$ is the unique trace state with $\alpha^0_{i\,|\mathcal{Z}(\mathcal{N})} = \psi_{i|\mathcal{Z}(\mathcal{N})}$, $1 \leq i \leq n$. Assume $(\alpha_1^0, \ldots, \alpha_n^0) \notin C_{\mathcal{N}}(\psi)$. By the Hahn-Banach theorem there exist $(x_1, \ldots x_n) \in \mathcal{N}^n$ and $\varepsilon_0 > 0$ such that $\sum\limits_{i=1}^n \varphi_i(x_i) \geq \sum\limits_{i=1}^n \alpha_i^0(x_i) + \varepsilon_0,\ \forall(\varphi_1, \ldots, \varphi_n) \in C_{\mathcal{N}}(\psi)$.

By the classical Dixmier theorem there exist unitary elements $u_1, \ldots, u_m \in \mathcal{U}(\mathcal{N})$ such that $\|\frac{1}{m} \sum\limits_{j=1}^m u_j x_i u_j^* - \mathrm{ctr}_{\mathcal{N}}\ x_i\| < \varepsilon_0/2n,\ 1 \leq i \leq n$, where $\mathrm{ctr}_{\mathcal{N}}$ denotes the central trace on $\mathcal{N}$. Then $(\varphi_1^0, \ldots, \varphi_n^0) \stackrel{\text{def}}{=} (\frac{1}{m} \sum\limits_{j=1}^m \psi_i(u_j \cdot u_j^*))_{1\leq i\leq n}$ belongs to $C_{\mathcal{N}}(\psi)$ and we have

$$|\varphi_i^0(x_i) - \alpha_i^0(x_i)| < \varepsilon_0/2n, \quad 1 \leq i \leq n.$$

Thus we get

$$\sum_{i=1}^n \varphi_i^0(x_i) \leq \sum_{i=1}^n\ \alpha_i^0(x_i) + \varepsilon_0/2$$

which together with $\sum_{i=1}^{n} \alpha_i^0(x_i) + \varepsilon_0 \leq \sum_{i=1}^{n} \varphi_i^0(x_i)$ (that we have because $(\varphi_i^0)_i \in C_{\mathcal{N}}(\psi)$ and because all elements in $C_{\mathcal{N}}(\psi)$ satisfy this inequality) gives a contradiction.

Now for the general case consider the linear application $E : (S(\mathcal{M}^*))^n \to (S(\mathcal{N}^*))^n$ defined by $E((\phi_i)_{1\leq i\leq n}) = (\phi_i \circ \mathcal{E})_{1\leq i\leq n}$.

Clearly $E(S_{\mathcal{N},\psi}(\mathcal{M})) = \{(\alpha_i^0)_{1\leq i\leq n}\}$ and $E(C_{\mathcal{N}}(\psi)) = C_{\mathcal{N}}(\psi \circ \mathcal{E})$. Note also that $E^{-1}(\{(\alpha_i^0)_{1\leq i\leq n}\}) = S_{\mathcal{N},\psi}(\mathcal{M})$. By the first part we have $E(C_{\mathcal{N}}(\psi)) \ni (\alpha_i^0)_i$. But then any $\phi = (\phi_1, \ldots, \phi_n) \in C_{\mathcal{N}}(\psi)$ with $E(\phi) = (\alpha_i^0)_i$ will belong to $C_{\mathcal{N}}(\psi) \cap S_{\mathcal{N},\psi}(\mathcal{M})$, ending the proof of the fact that $C_{\mathcal{N}}(\psi) \cap S_{\mathcal{N},\psi}(\mathcal{M}) \neq \emptyset$.

Fix now a $(\phi_i)_i$ in $C_{\mathcal{N}}(\psi)\cap S_{\mathcal{N},\psi}(\mathcal{M})$. Let $K = \overline{\text{co}}^n\{(\phi_i(u\cdot u^*))_{1\leq i\leq n} \mid u \in \mathcal{U}(\mathcal{N})\}$, the closure being taken in the usual uniform norm on $(\mathcal{M}^*)^n = (\mathcal{M}^n)^*$. We want to prove that K is compact in the $\sigma((\mathcal{M}^n)^*, \mathcal{M}^n\})$ topology. By Akemann's compactness criteria [A], in order to show this, we need to prove that if $\{e_n\}_n$ is a sequence of mutually orthogonal projections in $\mathcal{M}$, then $\phi'(e_n) \to 0$ uniformly in $\phi' \in \cup_i K_i$ where $K_i = \overline{\text{co}}^n\{\phi_i(u \cdot u^*) \mid u \in \mathcal{U}(\mathcal{N})\}$. It is clearly sufficient to check this uniform convergence for ϕ' of the form $\phi_i(u \cdot u^*)$, $u \in \mathcal{U}(\mathcal{N})$. But $\phi_i(ue_nu^*) \leq c^{-1}\phi_i(u\mathcal{E}(e_n)u^*) = c^{-1}\phi_i(\mathcal{E}(e_n))$ and the sequence $\{\phi_i(\mathcal{E}(e_n))\}_n$ tends to zero (because $\Sigma_n\phi_i(\mathcal{E}(e_n)) \leq 1$) independently on u.

Altogether these show that K, with the affine action $(\phi_i')_i \to (\phi_i'(u \cdot u^*))_i$ of the group $\mathcal{U}(\mathcal{N})$ on it, satisfies the hypothesis of the Ryll-Nardjewski fixed point theorem (see e.g. [SZ2]). Thus there exists $(\alpha_1, ..., \alpha_n) \in K$ with $\alpha_i = \alpha_i(u \cdot u^*), \forall u \in \mathcal{U}(\mathcal{N})$. Since by its definition K is included in $C_{\mathcal{N}}(\psi)$, we are done.

□

For the separating functional $\Phi(= \Phi^*)$ considered before the lemma, let now $\Phi = \Phi_+ - \Phi_-$ be its decomposition into positive and negative parts. Since Φ vanishes on $\mathcal{N} \vee \mathcal{N}' \cap \mathcal{M}$, it follows that Φ_+ and Φ_-coincide when restricted to this algebra, in particular $\Phi_+(1) = \Phi_-(1)$. By replacing Φ with $\Phi_\pm(1)^{-1}\Phi$, we may assume that $\Phi_\pm$ are states. Apply the previous Lemma for $n = 2$, $\psi_1 = \Phi_+$, $\psi_2 = \Phi_- \in S(\mathcal{M})$. It follows that there exists $\Psi \in C_{\mathcal{N}}(\Phi)$ such that $\Psi = \Psi_1 - \Psi_2$ for some $\Psi_{1,2} \in S(\mathcal{M})$ which both have $\mathcal{N}$ in their centralizers. Also $\Psi(x) \geq \varepsilon_0 > 0$ and Ψ vanishes on $\mathcal{N} \vee \mathcal{N}' \cap \mathcal{M}$, meaning that $\Psi_{1,2}$ coincide on this algebra.

Let us prove that in addition to these properties $\Psi_{1,2}$ can be taken to be factorial traces when restricted to $\mathcal{N}$. To this end, let $C = \{(\psi_1, \psi_2) \in (S(\mathcal{M}))^2 \mid (\psi_1 - \psi_2)(x) \geq \varepsilon_0, \psi_{1|\mathcal{N}\vee\mathcal{N}'\cap\mathcal{M}} = \psi_{2|\mathcal{N}\vee\mathcal{N}'\cap\mathcal{M}}$ and $\psi_{1,2}$ commute with $\mathcal{N}\}$. Thus $C \neq \emptyset$ and clearly C is compact in the $\sigma((\mathcal{M}^*)^2, \mathcal{M}^2)$ topology. Since the function $(\psi_1, \psi_2) \to (\psi_1 - \psi_2)(x)$ is continuous in this σ^*-topology, it follows that there exist $(\psi_1', \psi_2') \in C$ such that $(\psi_1' - \psi_2')(x) = \sup\{(\psi_1 - \psi_2)(x) \mid (\psi_1, \psi_2) \in C\} \stackrel{\text{def}}{=} \varepsilon_1$.

Let $C_0 = \{(\psi_1', \psi_2') \in C \mid (\psi_1' - \psi_2')(x) = \varepsilon_1\}$. Thus C_0 is convex, σ^*-compact and non-empty. Let $(\psi_1, \psi_2) \in C_0$ be an extremal point in C_0. We claim that $\psi_{1,2}$ are (equal) factor states when restricted to $\mathcal{N}$. To see this, let $p \in \mathcal{P}(\mathcal{Z}(\mathcal{N}))$ be such that $p, 1-p$ are non-zero and note that both $(t^{-1}\psi_1(\cdot p), t^{-1}\psi_2(\cdot p)), ((1 - t)^{-1}\psi_1(\cdot(1 - p)), (1 - t)^{-1}\psi_2(\cdot(1 - p))$, where $t = \psi_{1,2}(p)$, belong to C_0. (Recall all the way that $\mathcal{Z}(\mathcal{N}) \subset \mathcal{Z}(\mathcal{M})$!) By extremality, it follows that either $\psi_{1,2}(p) = 0$ or $\psi_{1,2}(1-p) = 0$. This shows that $\psi_{1,2|\mathcal{Z}(\mathcal{N})}$ are (equal) characters on $\mathcal{Z}(\mathcal{N})$ and so, since $\psi_{1,2|\mathcal{N}} = \psi_{1,2} \circ \text{ctr}_{\mathcal{N}}$, $\psi_{1,2|\mathcal{N}}$ follow (equal) factor states on $\mathcal{N}$ (as in the proof of 2.1, $\text{ctr}_{\mathcal{N}}$ denotes the central trace on $\mathcal{N}$).

Denote by $\omega = \psi_{1,2|\mathcal{Z}(\mathcal{N})}$ the corresponding element in the spectrum of $\mathcal{Z}(\mathcal{N})$, $\Omega_{\mathcal{N}}$. Also, set $\psi = \omega \circ \mathrm{ctr}_{\mathcal{N}} \circ \mathcal{E}$ and remark that $\psi = \psi_1 \circ \mathcal{E} = \psi_2 \circ \mathcal{E}$. Since $X \leq c^{-1}\mathcal{E}(X), \forall X \in \mathcal{M}_+$, we have $\psi_1(X) \leq c^{-1}\psi_1(\mathcal{E}(X)) = c^{-1}\psi(X)$, so that $\psi_1 \leq c^{-1}\psi$. Similarly, $\psi_2 \leq c^{-1}\psi$. We next want to prove the following:

2.2. Lemma. *$\forall \delta_1 > 0, \exists a_{1,2} \in \mathcal{M}_+$, such that $a_i \leq c^{-1}1$, $\|\psi_i - \psi(a_i^{1/2} \cdot a_i^{1/2})\| < \delta_1$ and $\|u a_i u^* - a_i\|_\psi < \delta_1, \forall u \in \mathcal{U}(\mathcal{N}), i = 1,2$.*

Proof. Let $(\pi_\psi, \xi_\psi, \mathcal{H}_\psi)$ be the GNS representation for $(\mathcal{M}, \psi)$. Let $p \in \overline{\pi_\psi(\mathcal{M})}$ be the support projection of the state implemented by $\langle \cdot \xi_\psi, \xi_\psi \rangle$ on $\overline{\pi_\psi(\mathcal{M})}$. Since $\mathcal{N}$ is in the centralizer of ψ we have $p \in \pi_\psi(\mathcal{N})' \cap \overline{\pi_\psi(\mathcal{M})}$.

Denote $\mathcal{M}^\omega = p\overline{\pi_\psi(\mathcal{M})}p$, $\mathcal{N}^\omega = \overline{\pi_\psi(\mathcal{N})}p$ and still denote by ψ the state $\langle \cdot \xi_\psi, \xi_\psi \rangle$ on $\mathcal{M}^\omega$. Note that ξ_ψ is cyclic and separating for $\mathcal{M}^\omega$, so that the norm $\|X\|_\psi = \|X\xi_\psi\|$ implements the so-topology on the unit ball of $\mathcal{M}^\omega$.

Since $\mathcal{N}^\omega$ is in the centralizer of ψ, there is a unique ψ-preserving conditional expectation $\mathcal{E}^\omega$ of $\mathcal{M}^\omega$ onto $\mathcal{N}^\omega$, which is implemented by the orthogonal projection of $p\mathcal{H}_\psi$ onto $\overline{\pi_\psi(\mathcal{N})\xi_\psi} = \overline{\mathcal{N}^\omega \xi_\psi}$.

Moreover, if $X \in \mathcal{M}$ is such that $\mathcal{E}(X) = 0$, then we have $\mathcal{E}(Y^*X) = 0, \forall Y \in \mathcal{N}$, so that $\langle p\pi_\psi(X)p\xi_\psi, \pi_\psi(Y)\xi_\psi \rangle = \langle \pi_\psi(X)\xi_\psi, \pi_\psi(Y)\xi_\psi \rangle = \psi(Y^*X) = 0, \forall Y \in \mathcal{N}$, implying that $\mathcal{E}^\omega(p\pi_\psi(X)p) = 0$. Thus $\mathcal{E}^\omega(p\pi_\psi(X')p) = \pi_\psi(\mathcal{E}(X'))p, \forall X' \in \mathcal{M}$. In particular, since $\mathcal{E}(X') \geq cX', \forall X' \in \mathcal{M}_+$, we get $\mathcal{E}^\omega(p\pi_\psi(X')p) \geq cp\pi_\psi(X')p, \forall X' \in \mathcal{M}_+$. By density, $\mathcal{E}^\omega(x') \geq cx', \forall x' \in \mathcal{M}^\omega_+$.

Note that since ψ is a normal faithful trace on $\mathcal{N}^\omega$, we have that $\mathcal{N}^\omega$ and $\mathcal{M}^\omega$ are finite von Neumann algebras (cf. 1.1.3 in [Po2]). In fact, note that $\mathcal{N}^\omega$ is isomorphic to the type II_1 factor $\mathcal{N}/[\omega]$, where $[\omega]$ denotes the (maximal) ideal in $\mathcal{N}$ generated by ω (cf. [Sa]).

Moreover, note that the inequality $\psi_i \leq c^{-1}\psi$ on $\mathcal{M}$ implies that ψ_i implements a state on $\mathcal{M}^\omega$, still denoted by ψ_i, satisfying $\psi_i(p\pi_\psi(X)p) = \psi_i(X), \forall X \in \mathcal{M}$, and $\psi_i \leq c^{-1}\psi$, $i = 1,2$ on $\mathcal{M}^\omega$. Also, as states on $\mathcal{M}^\omega$, ψ_i have $\mathcal{N}^\omega$ in their centralizers.

By the Radon-Nykodim theorem, it follows that there exist $A_i \in \mathcal{N}^{\omega'} \cap \mathcal{M}^\omega$, $0 \leq A_i \leq c^{-1}1$, such that $\psi_i = \psi(A_i^{1/2} \cdot A_i^{1/2})$ on $\mathcal{M}^\omega$, $i = 1,2$.

By the density of $\pi_\psi(\mathcal{M})$ in $\overline{\pi_\psi(\mathcal{M})}$ and Kaplansky's theorem, it follows that $\forall \delta > 0$ there exists $a_i \in \mathcal{M}$ such that $0 \leq a_i \leq c^{-1}1$ and $\|\pi_\psi(a_i^{1/2})\xi_\psi - A_i^{1/2}\xi_\psi\| < \delta$. Since $[u, A_i^{1/2}] = 0, [u, \xi_\psi] = 0, \forall u \in \mathcal{U}(\mathcal{N})$, this implies that $\|\pi_\psi(u a_i^{1/2} u^* - A_i^{1/2})\xi_\psi\| < \delta$, so that $\|u a_i^{1/2} u^* - a_i^{1/2}\|_\psi < 2\delta$ on $\mathcal{M}$.

Moreover, we have the estimate $\|\psi(a_i^{1/2} \cdot a_i^{1/2}) - \psi_i\| = \|\psi(a_i^{1/2} \cdot a_i^{1/2}) - \psi(A_i^{1/2} \cdot A_i^{1/2})\| < 2c^{-1}\delta$. By taking, from the beginning, δ to be less than $c\delta_1/4$, this completes the proof. $\square$

We next want to prove that the positive elements $a_i \in \mathcal{M}$ in the previous lemma are small perturbations of elements in $\mathcal{N}' \cap \mathcal{M}$.

2.3. Lemma. *If $b \in \mathcal{M}$ is such that $\|ubu^* - b\|_\psi \leq \delta, \forall u \in \mathcal{U}(\mathcal{N})$, then there exists $b' \in \mathcal{N}' \cap \mathcal{M}$ such that $\|b - b'\|_\psi \leq \delta$.*

Proof. Let $\mathcal{E}_{\mathcal{N}' \cap \mathcal{M}}$ be the unique normal conditional expectation onto $\mathcal{N}' \cap \mathcal{M}$ that preserves $\mathrm{ctr}_{\mathcal{N}} \circ \mathcal{E}$, i.e., such that $\mathrm{ctr}_{\mathcal{N}} \circ \mathcal{E} \circ \mathcal{E}_{\mathcal{N}' \cap \mathcal{M}} = \mathrm{ctr}_{\mathcal{N}} \circ \mathcal{E}$. By replacing if necessary b by $b - \mathcal{E}_{\mathcal{N}' \cap \mathcal{M}}(b)$ (which clearly still satisfies the inequality in the hypothesis) we may assume $\mathcal{E}_{\mathcal{N}'\cap\mathcal{M}}(b) = 0$. We will prove that this and the commutation relations in the hypothesis of the lemma imply $\|b\|_\Psi \leq \delta$.

To this end, let us show that $\exists u \in \mathcal{U}(\mathcal{N})$ such that Re $(\text{ctr}_{\mathcal{N}}(\mathcal{E}(b^*ubu^*))) \leq (\delta^2/2)1$ as operators in $\mathcal{Z}(\mathcal{N})$. Indeed, let $\{(q_i, u_i)\}_{i\in I}$ be a maximal family of pairs of elements with q_i mutually orthogonal projections in $\mathcal{P}(\mathcal{Z}(\mathcal{N}))$ and $u_i \in \mathcal{U}(\mathcal{N}q_i)$ such that Re $(\text{ctr}_{\mathcal{N}}(\mathcal{E}(b^*u_ibu_i^*))) \leq (\delta^2/2)q_i, \forall i \in I$. If $q = 1 - \Sigma_{i\in I}q_i \neq 0$ then apply (2.3 in [Po1]) to the element $bq \in \mathcal{M}q$, with $\mathcal{E}_{(\mathcal{N}q)'\cap\mathcal{M}q}(bq) = 0$, to obtain a projection $q_0 \in \mathcal{Z}(\mathcal{N}), 0 \neq q_0 \leq q$, and a unitary element $u_0 \in \mathcal{N}q_0$ such that Re $(\text{ctr}_{\mathcal{N}}(\mathcal{E}(b^*u_0bu_0^*))) \leq (\delta^2/2)q_0$, contradicting the maximality of $\{(q_i, u_i)\}_{i\in I}$. Thus $\Sigma_i q_i = 1$ and so $u = \Sigma_i u_i$ is a unitary element in $\mathcal{N}$. We then have:

$$
\begin{aligned}
\delta^2 &\geq \|ubu^* - b\|_\psi^2 = 2\|b\|_\psi^2 - 2\text{Re } \psi(b^*ubu^*) \\
&= 2\|b\|_\psi^2 - 2\text{Re } \omega(\text{ctr}_{\mathcal{N}}(\mathcal{E}(b^*ubu^*))) \geq 2\|b\|_\psi^2 - 2\delta^2/2.
\end{aligned}
$$

This shows that $\|b\|_\psi \leq \delta$.

$\square$

With these technical results in hand, let us now proceed to end the proof of the finite case of the theorem.

Thus, let $x = x^* \in \mathcal{M}$ satisfy $\mathcal{E}_{\mathcal{N}\vee\mathcal{N}'\cap\mathcal{M}}(x) = 0, \varepsilon_1 = (\psi_1 - \psi_2)(x) > 0$, with $\psi_{1,2}$ states on $\mathcal{M}$ that have $\mathcal{N}$ in their centralizers and are factor states on $\mathcal{N}$, as before.

By applying Lemmas 2.2, 2.3 above to ψ_1, ψ_2 and $\psi(= \psi_1 \circ \mathcal{E} = \psi_2 \circ \mathcal{E})$, it follows that $\forall\delta > 0, \delta < \varepsilon_1/4, \exists a_1, a_2 \in \mathcal{N}' \cap \mathcal{M}, 0 \leq a_i \leq c^{-1}1$ such that $\|\psi_i - \psi(a_i^{1/2} \cdot a_i^{1/2})\| < \delta$. Then we get:

$$
\begin{aligned}
&\psi(a_1^{1/2}xa_1^{1/2}) - \psi(a_2^{1/2}xa_2^{1/2}) \geq \\
&\qquad \psi_1(x) - \psi_2(x) - \|\psi_1 - \psi(a_1^{1/2} \cdot a_1^{1/2})\| - \|\psi_2 - \psi(a_2^{1/2} \cdot a_2^{1/2})\| \geq \varepsilon_1 - 2\delta.
\end{aligned}
$$

But $\mathcal{E} = \mathcal{E} \circ \mathcal{E}_{\mathcal{N}\vee\mathcal{N}'\cap\mathcal{M}}$, so that $\psi = \psi \circ \mathcal{E}_{\mathcal{N}\vee\mathcal{N}'\cap\mathcal{M}}$, and since $\mathcal{E}_{\mathcal{N}\vee\mathcal{N}'\cap\mathcal{M}}(a_i^{1/2}xa_i^{1/2}) = a_i^{1/2}\mathcal{E}_{\mathcal{N}\vee\mathcal{N}'\cap\mathcal{M}}(x)a_i^{1/2} = 0$, we get

$$
0 \geq \varepsilon_1 - 2\delta \geq \varepsilon_1 - \varepsilon_1/2 = \varepsilon_1/2 > 0
$$

a contradiction which completes the proof of the finite case.

2.4. *Remarks.* 1°. The proof of the finite factorial case of the Theorem can be made considerably shorter. Thus, 1.1-1.7, 2.2 and 2.3 are not needed and the Radon-Nykodim argument at the end of the proof can be applied immediately after lemma 2.1, by using that "$\mathcal{E}(x) \geq cx, \forall x \in \mathcal{M}_+$" implies "$\varphi \leq c^{-1}\varphi \circ \mathcal{E}, \forall\varphi \in \mathcal{M}_+^*$" (see A.1 in [Po5] for an elementary and short proof of this factorial type II_1 case). The more general case when $\mathcal{E}$ preserves a normal faithful trace on $\mathcal{M}$ can also be given a shorter proof. Most of the difficulties encountered for the general case stem from the case when there exist no trace preserving conditional expectations of finite index, yet there do exist expectations of finite index. For an example of such an inclusion consider the locally trivial Jones subfactors $N_k \subset M_k$ of index $(1/k)^{-1} + (1 - 1/k)^{-1} = k^2/(k-1)$ and $E_k : M_k \to N_k$ be the expectation of minimal index, $\text{Ind}E_k = 4$. Then $\mathcal{N} = \oplus_k N_k \subset \oplus_k M_k = \mathcal{M}$, and the expectation $\mathcal{E} = \oplus_k E_k$ has finite index but the unique trace preserving expectation E of $\mathcal{M}$ onto $\mathcal{N}$ has infinite index.

2°. Note that once we have the Dixmier property for all inclusions of finite von Neumann algebras with conditional expectations having index majorized by a fixed constant c^{-1}, it follows that for any $\varepsilon > 0$, there exists $n = n(c, \varepsilon) \geq 1$, such that given any inclusion $\mathcal{N} \subset \mathcal{M}$ of finite von Neumann algebras, with a normal faithful conditional expectation $\mathcal{E} : \mathcal{M} \to \mathcal{N}$ satisfying $\mathrm{Ind}\mathcal{E} \leq c^{-1}$, we have that $\forall x \in \mathcal{M}, \|x\| \leq 1, \exists u_1, u_2, ..., u_n \in \mathcal{U}(\mathcal{N})$ such that $\mathrm{d}(\frac{1}{n}\sum_{k=1}^{n} u_k x u_k^*, \mathcal{N}' \cap \mathcal{M}) < \varepsilon$. For assume this is not the case. Then $\exists \varepsilon_0 > 0, \exists c_0 < \infty$ such that $\forall n \geq 1, \exists \mathcal{N}_n \subset \mathcal{M}_n$, with $\mathcal{E}_n : \mathcal{M}_n \to \mathcal{N}_n$ satisfying $\mathcal{E}_n(X) \geq c_0 X, \forall X \in \mathcal{M}_+$, and $\exists x_n \in \mathcal{M}_n, \|x_n\| \leq 1$, such that $\mathrm{d}(\frac{1}{n}\sum_{k=1}^{n} u_k x_n u_k^*, \mathcal{N}_n' \cap \mathcal{M}_n) \geq \varepsilon_0, \forall u_1, u_2, ..., u_n \in \mathcal{U}(\mathcal{N}_n)$. But then applying the relative Dixmier property for $\mathcal{N} = \oplus_n \mathcal{N}_n \subset \oplus_n \mathcal{M}_n = \mathcal{M}$, $\mathcal{E} = \oplus_n \mathcal{E}_n$ and $X = \oplus_n x_n$ we get some $V_1 = \oplus_n v_{1,n}, ..., V_k = \oplus_n v_{k,n} \in \mathcal{U}(\mathcal{N})$ such that

$$\varepsilon_0/2 > \mathrm{d}\left(\frac{1}{k}\sum_{j=1}^{k} V_j X V_j^*, \mathcal{N}' \cap \mathcal{M}\right) = \sup_n \mathrm{d}\left(\frac{1}{k}\sum_{j=1}^{k} v_{j,n} x_n v_{j,n}^*, \mathcal{N}_n' \cap \mathcal{M}_n\right)$$

This gives a contradiction if $n > k$.

3. Proof of the Properly Infinite Case

Unlike the finite case, where we used a contradiction argument and the Hahn-Banach theorem, the proof of the properly infinite case will be direct and more or less constructive.

Thus, in this section we will assume that $\mathcal{N} \subset \mathcal{M}$ is an inclusion of properly infinite von Neumann algebras, with $\mathcal{Z}(\mathcal{N}) \subset \mathcal{Z}(\mathcal{M})$ as always.

We first prove the theorem under the additional assumption that $\mathcal{N}$ (and thus $\mathcal{M}$) is countably decomposable, or equivalently σ-finite, i.e., $\mathcal{N}$ has normal faithful states. This assumption will be needed for the reduction theory considerations below.

For $X \in \mathcal{M}$ and $u = (u_1, u_2, ..., u_n) \subset \mathcal{U}(\mathcal{N})$, it will be convenient to denote $T_u(X) = \frac{1}{n}\sum_{j=1}^{n} u_j X u_j^*$. Note that $\mathcal{M} \ni x \mapsto T_u(x) \in \mathcal{M}$ is completely positive and unital, thus $T_u(X)^* T_u(X) \leq T_u(X^*X)$, by Kadison's inequality.

Let further $\mathcal{J} \subset \mathcal{M}$ denote the norm closed ideal generated by the finite projections of $\mathcal{M}$. Note that $\mathcal{E}(\mathcal{J}) = \mathcal{J} \cap \mathcal{N}$ is the ideal generated by finite projections in $\mathcal{N}$ and that $\mathcal{E}$ implements a conditional expectation, still denoted by $\mathcal{E}$, of $\mathcal{M}/\mathcal{J}$ onto $\mathcal{N}/\mathcal{J}$. Note also that if $\mathcal{N}, \mathcal{M}$ are purely infinite then $\mathcal{J} = 0$.

Then for $K \in \mathcal{J}, K \geq 0$ we have $\mathcal{E}(K) \geq cK$ and by ([H] or [SZ1]) it follows that for any $\varepsilon > 0$ there exists $u = (u_1, ..., u_n) \subset \mathcal{U}(\mathcal{N})$ such that $\|T_u(\mathcal{E}(K))\| < \varepsilon$, thus $\|T_u K\| < \varepsilon c^{-1}$. Thus $0 \in C_{\mathcal{N}}(K)$.

If $X \in \mathcal{M}$ then we denote by $\|X\|_e$ the norm of X in the "Calkin" algebra $\mathcal{M}/\mathcal{J}$. Thus $\|X\|_e = \|X\|$ if $\mathcal{M}$ is purely infinite. Note that if $X \in \mathcal{N}$ then $\|X/\mathcal{J} \cap \mathcal{N}\| = \|X/\mathcal{J}\| (= \|X\|_e)$.

For $t \in \Omega_{\mathcal{N}}$ denote by I_t the maximal ideal in $\mathcal{N}$ generated by t. Similarly, if $s \in \Omega_{\mathcal{M}}$ then denote by I_s' the maximal ideal in $\mathcal{M}$ generated by s. Recall ([SZ1]) that if $y \in \mathcal{N}$ (resp. $x \in \mathcal{M}$) then the function $t \to \|y/I_t\|$ (resp. $s \to \|x/I_s'\|$) is continuous on $\Omega_{\mathcal{N}}$ (resp. on $\Omega_{\mathcal{M}}$). Also, with the above notations, we have $\sup\{\|y/I_t\| \mid t \in \Omega_{\mathcal{N}}\} = \|y\|_e$, $\sup\{\|x/I_s'\| \mid s \in \Omega_{\mathcal{M}}\} = \|x\|_e$ (cf. [SZ1], since $\mathcal{N}, \mathcal{M}$ are countably decomposable).

3.1. Lemma. *For $t_0 \in \Omega_{\mathcal{N}}$ let $J_{t_0} = \{x \in \mathcal{M} \mid \mathcal{E}(x^*x) \in I_{t_0}\}$. Then J_{t_0} is a closed bilateral ideal in $\mathcal{M}$ and $J_{t_0+} = \{x \in \mathcal{M}_+ \mid \mathcal{E}(x) \in I_{t_0}\}$. Moreover, if $\{s_k\}_{1 \leq k \leq n} \subset \Omega_{\mathcal{M}}$ is the preimage of t_0 under the surjection $f : \Omega_{\mathcal{M}} \to \Omega_{\mathcal{N}}$, implemented by the inclusion $C(\Omega_{\mathcal{N}}) = \mathcal{Z}(\mathcal{N}) \subset \mathcal{Z}(\mathcal{M}) = C(\Omega_{\mathcal{M}})$, then $J_{t_0} = \cap_k I'_{s_k}$. Also, if $X \in \mathcal{M}$ then the function $t \to \|X/J_t\|$ is continuous on $\Omega_{\mathcal{N}}$ and its supremum equals $\|X\|_e$.*

Proof. By the way it is defined, J_{t_0} is clearly a closed left ideal in $\mathcal{M}$. To prove that it is a right ideal as well, note that if $x \in J_{t_0}$ then the function $t \to \|\mathcal{E}(x^*x)/I_t\|$ has a zero at t_0. Thus $\forall \varepsilon > 0$ there exists an open-closed neighborhood V_0 of t_0 in $\Omega_{\mathcal{N}}$ such that $\|\mathcal{E}(x^*x)/I_t\| < \varepsilon, \forall t \in V_0$. Equivalently, if p_{V_0} denotes the projection in $\mathcal{Z}(\mathcal{N})$ corresponding to the set V_0, we have $\|\mathcal{E}(x^*x)p_{V_0}\|_e < \varepsilon$. Thus, if $y \in \mathcal{M}$, then, by using that $\mathcal{Z}(\mathcal{N}) \subset \mathcal{Z}(\mathcal{M})$ (so in particular $p_{V_0} \in \mathcal{Z}(\mathcal{M})$) we get:

$$\|\mathcal{E}(y^*x^*xy)p_{V_0}\|_e \leq \|y^*x^*xyp_{V_0}\|_e \leq \|y\|^2\|x^*xp_{V_0}\|_e$$

$$\leq c^{-1}\|y\|^2\|\mathcal{E}(x^*x)p_{V_0}\|_e < \varepsilon c^{-1}\|y\|^2.$$

It thus follows that $\|\mathcal{E}(y^*x^*xy)/I_{t_0}\| < \varepsilon c^{-1}\|y\|^2$. Since ε was arbitrary, this shows that $\|\mathcal{E}(y^*x^*xy)/I_{t_0}\| = 0$, thus $\mathcal{E}(y^*x^*xy) \in I_{t_0}$, proving that $xy \in J_{t_0}$.

If we now take $x \in J_{t_0+}$, then $\mathcal{E}(x^2) \in I_{t_0}$ and since $\mathcal{E}(x^2) \geq \mathcal{E}(x)^2$ it follows that $\mathcal{E}(x)^2 \in I_{t_0}$ so that $\mathcal{E}(x) \in I_{t_0}$ as well. Conversely, if $x \geq 0$ and $\mathcal{E}(x) \in I_{t_0}$ then $x^{1/2} \in J_{t_0}$ (by the definition of J_{t_0}) so that $x \in J_{t_0}$.

To prove the last part of the statement, let $x \in J_{t_0}$. Since the continuous function $t \mapsto \|\mathcal{E}(x^*x)\|$ has a zero at t_0, for each $\varepsilon > 0$, there exists $p \in \mathcal{P}(\mathcal{Z}(\mathcal{N}))$ such that $p(t_0) = 1$ and $\|\mathcal{E}(x^*x)p\|_e < \varepsilon$. By regarding p as an element in $C(\Omega_{\mathcal{M}}) = \mathcal{Z}(\mathcal{M}) \supset \mathcal{Z}(\mathcal{N}) = C(\Omega_{\mathcal{N}})$, we have $p(s_k) = 1, \forall k$. But $\mathcal{E}(x^*x)p \geq cx^*xp$, so that $\|x^*xp\|_e < c^{-1}\varepsilon$, implying that $\|x^*x/I'_{s_k}\| < c^{-1}\varepsilon$. Since ε was arbitrary, this implies $\|x/I'_{s_k}\| = 0$ so that $x \in \cap_k I'_{s_k}$.

Conversely, if $x \in \cap_k I'_{s_k}$, then, there exists $p_k \in \mathcal{P}(\mathcal{Z}(\mathcal{M}))$ such that $p_k(s_k) = 1$ and $\|xp_k\|_e < \varepsilon, \forall k$. If we denote $p = \vee_k p_k$, it follows that $p(s_k) = 1, \forall k$, and $\|xp\|_e < \varepsilon$. But by the proof of Lemma 1.2 there exists $q \in \mathcal{P}(\mathcal{Z}(\mathcal{N}))$ such that $q \leq p$ and $q(t_0) = 1$. Thus $\|xq\|_e < \varepsilon$, so that $\|\mathcal{E}(x^*x)q\|_e < \varepsilon^2$. Thus $\|\mathcal{E}(x^*x)/I_{t_0}\| < \varepsilon$ and since $\varepsilon > 0$ was arbitrary, this implies $\mathcal{E}(x^*x) \in I_{t_0}$, i.e., $x \in J_{t_0}$.

The very last assertion in the statement of the lemma follows now by first noting that for $X \in \mathcal{M}$ and $t \in \Omega_{\mathcal{N}}$ we have

$$\|X/J_t\| = \|X/\bigcap_{s \in f^{-1}(t)} I'_s\| = \|\bigoplus_{s \in f^{-1}(t)} X/I'_s\| = \max_{s \in f^{-1}(t)} \|X/I'_s\|.$$

And since by lemma 1.2 there exist continous functions $g_1, g_2, ..., g_n : \Omega_{\mathcal{N}} \to \Omega_{\mathcal{M}}$ such that $f \circ g_k = id_{\Omega_{\mathcal{N}}}, \forall k$ (i.e., g_k are right inverses for f) and such that

$$f^{-1}(t) \subset \{g_1(t), g_2(t), ..., g_n(t)\}, \forall t \in \Omega_{\mathcal{N}},$$

it follows that the function $\Omega_{\mathcal{N}} \ni t \mapsto \|X/J_t\|$ is the maximum of the continuous functions $\Omega_{\mathcal{N}} \ni \mapsto \|X/I'_{g_i(t)}\|, i = 1, ..., n$ and thus it is itself continuous. Also, since

$\bigcup_{s\in f^{-1}(t)} f^{-1}(t) = \Omega_{\mathcal{M}}$, we have:

$$\sup\{\|X/J_t\| \mid t \in \Omega_{\mathcal{N}}\}$$
$$= \sup\{\|X/I'_s\| \mid s \in f^{-1}(t), t \in \Omega_{\mathcal{N}}\}$$
$$= \sup\{\|X/I'_s\| \mid s \in \Omega_{\mathcal{M}}\} = \|X\|_e.$$

□

The next technical lemma, based on a convexity argument, can be regarded as the crucial point in proving the properly infinite case of the theorem.

3.2. Lemma. *Let $x = x^* \in \mathcal{M}$ and $t \in \Omega_{\mathcal{N}}$. Denote $a = \inf\{\mathrm{sp}(\mathcal{E}(y^2)/I_t) \mid y \in C_{\mathcal{N}}(x)\}$. Then we have:*

a) There exists $y \in C_{\mathcal{N}}(x)$ such that $\|\mathcal{E}(y^2)/I_t - a1\| < \varepsilon$; moreover, given any element $y \in C_{\mathcal{N}}(x)$ satisfying $\|\mathcal{E}(y^2)/I_t - a1\| < \varepsilon$, we have

$$\|\mathcal{E}((T_v y)^2)/I_t - a1\| \leq \varepsilon, \forall v = (v_1, ..., v_m) \subset \mathcal{U}(\mathcal{N})$$

b) If $y \in C_{\mathcal{N}}(x)$ satisfies $\|\mathcal{E}(y^2)/I_t - a1\| < \varepsilon$ then

$$\|(y - v_0 y v_0^*)/J_t\| \leq (8\varepsilon c^{-1})^{1/2}, \forall v_0 \in \mathcal{U}(\mathcal{N}).$$

c) If $y \in C_{\mathcal{N}}(x)$ satisfies $\|\mathcal{E}(y^2)/I_t - a1\| < \varepsilon$ and $y' \in C_{\mathcal{N}}(y)$ then $\|(y - y')/J_t\| \leq (8\varepsilon c^{-1})^{1/2}$

Proof a) Let $y \in C_{\mathcal{N}}(x)$ be such that $\inf\{\mathrm{sp}(\mathcal{E}(y^2)/I_t) \leq a + \varepsilon\}$. By [SZ1] there exists $u = (u_1, ..., u_n) \subset \mathcal{U}(\mathcal{N})$ such that

$$\|(T_u(\mathcal{E}(y^2)) - a1)/I_t\| < \varepsilon.$$

Since $T_u(\mathcal{E}(y^2)) = \mathcal{E}(T_u(y^2))$ we get, by Kadison's inequality applied as above,

$$\mathcal{E}((T_u y)^2) \leq \mathcal{E}(T_u(y^2)) = T_u(\mathcal{E}(y^2)) \leq a + \varepsilon, \text{ modulo } I_t.$$

Since $T_u y \in C_{\mathcal{N}}(x)$, by the definition of a we also get $\mathcal{E}((T_u y)^2) \geq a1$ modulo I_t (i.e., in the quotient C*-algebra $\mathcal{N}/I_t$). Thus $\|(\mathcal{E}((T_u y)^2) - a1)/I_t\| \leq \varepsilon$. Since $T_u y \in C_{\mathcal{N}}(x)$, taking $T_u y$ for y we get the first part of a). Next, if y satisfies $\|(\mathcal{E}(y^2) - a1)/I_t\| \leq \varepsilon$ and $v = (v_1, ..., v_m) \subset \mathcal{U}(\mathcal{N})$ then again modulo I_t we have

$$a \leq \mathcal{E}((T_v y)^2) \leq T_v(\mathcal{E}(y^2)) \leq T_v((a + \varepsilon)1) = a + \varepsilon.$$

b) Since by a) we have $\|(\mathcal{E}((T_v y)^2) - a1)/I_t\| \leq \varepsilon$ for any $v = (v_1, ..., v_m) \subset \mathcal{U}(\mathcal{N})$, it follows that modulo I_t we have

$$0 \leq \mathcal{E}(T_v(y^2) - (T_v y)^2) = T_v(\mathcal{E}(y^2)) - \mathcal{E}((T_v y)^2) \leq \varepsilon$$

Since $\mathcal{E}$ has finite index we get:

$$\|(T_v(y^2) - (T_v y)^2)/J_t\| \leq c^{-1}\|(\mathcal{E}(T_v(y^2)) - \mathcal{E}((T_v y)^2))/I_t\| \leq 2\varepsilon c^{-1}$$

Take then $v_0 \in \mathcal{U}(\mathcal{N})$ and note that $(y - v_0 y v_0^*)^2 = 4(T_v(y^2) - (T_v y)^2)$, where $v = (1, v_0)$. From the above we then get $\|(y - v_0 y v_0^*)/J_t\|^2 \leq 8\varepsilon c^{-1}$.

c) is now trivial from b).

$\square$

Let us now show that from the estimates in lemma 3.2 at a point t we can get similar estimates in a neighborhood of t:

3.3. Lemma. *Let $t_0 \in \Omega_{\mathcal{N}}$ be fixed and assume $y \in \mathcal{M}$ satisfies $\|(uyu^* - y)/J_{t_0}\| < \varepsilon_0$, $\forall u \in \mathcal{U}(\mathcal{N})$. Then there exists an open-closed neighborhood V of t_0 such that $\|(uyu^* - y)p_V\|_e < 2\varepsilon_0$, $\forall u \in \mathcal{U}(\mathcal{N})$.*

Proof. Assume, on the contrary, that for all neighborhood V of t_0, $\exists u \in \mathcal{U}(\mathcal{N})$ satisfying $\|(uyu^* - y)p_V\|_e \geq 2\varepsilon_0$.

Let $\{(p_i, u_i)\}_{i \in I}$ be a maximal family of pairs of elements with p_i mutually orthogonal projections in $\mathcal{Z}(\mathcal{N})$ and $u_i \in \mathcal{U}(\mathcal{N}p_i)$ satisfying $\|(u_i y u_i^* - y)p_i/J_t\| \geq 3\varepsilon_o/2, \forall t \in V_{p_i}$, where V_{p_i} is the open closed set in $\Omega_{\mathcal{N}}$ corresponding to $p_i \in \mathcal{Z}(\mathcal{N})$. Let $p = \Sigma_i p_i$ and $V_p \subset \Omega_{\mathcal{N}}$ its associated open-closed set.

There are two possibilities: either $t_0 \in V_p$ or $t_0 \in \Omega_{\mathcal{N}} \setminus V_p$.

If $t_0 \in V_p$ and we put $u = \Sigma_i u_i + (1 - p)$ then u is a unitary element in $\mathcal{N}$ and thus $\|(uyu^* - y)/J_{t_0}\| < \varepsilon_0$, by hypothesis. Thus, there exists an open-closed subset $U \subset V_p$ such that $t_0 \in U$ and $\|(uyu^* - y)p_U\|_e < \varepsilon_0$. Since $p_U \neq 0$ and $p_U \leq p = \Sigma_i p_i$, it follows that there exists i such that $p_U p_i \neq 0$. But then we get

$$\varepsilon_0 > \|(uyu^* - y)p_U p_i\|_e = \sup\{\|(uyu^* - y)/J_t\| \mid t \in V_{p_i} \cap U\} \geq 3\varepsilon_0/2,$$

a contradiction.

If $t_0 \in \Omega_{\mathcal{N}} \setminus V_p$, then by the initial contradiction assumption applied to $V = \Omega_{\mathcal{N}} \setminus V_p$ it follows that there exists $u \in \mathcal{U}(\mathcal{N})$ satisfying $\|(uyu^* - y)p_V\|_e \geq 2\varepsilon_0$.

It then follows that there exists a non-empty open-closed set $V_0 \subset V$ such that if $p_0 = p_{V_0}$ then

$$\|(uyu^* - y)p_0/J_t\| > 3\varepsilon_0/2, \forall t \in V_0.$$

But then $\{(p_i, u_i)\}_{i \in I} \cup \{(p_0, up_0)\}$ contradicts the maximality of $\{(p_i, u_i)\}_{i \in I}$. This final contradiction completes the proof of the lemma.

$\square$

We will now use 3.2, 3.3 and a compactness argument to obtain an element in $C_{\mathcal{N}}(x)$ that globally satisfies the commutation estimates with unitaries in $\mathcal{N}$:

3.4. Lemma. *Let $x = x^* \in \mathcal{M}$. Given any $\varepsilon > 0$ there exists $y \in C_{\mathcal{N}}(x)$ such that $\|uyu^* - y\|_e \leq \varepsilon, \forall u \in \mathcal{U}(\mathcal{N})$. Moreover, if y' is an arbitrary element in $C_{\mathcal{N}}(y)$ then $\|y - y'\|_e \leq \varepsilon$.*

Proof. By 3.2, $\forall t \in \Omega_{\mathcal{N}}, \exists y_t \in C_{\mathcal{N}}(x)$ such that $\|(uy_tu^* - y_t)/J_t\| \leq \varepsilon/2, \forall u \in \mathcal{U}(\mathcal{N})$. By 3.3 it follows that there exists an open-closed neighborhood V_t of t in $\Omega_{\mathcal{N}}$ such that $\|(uy_tu^* - y_t)p_{V_t}\|_e < \varepsilon, \forall u \in \mathcal{U}(\mathcal{N})$.

Since $\Omega_{\mathcal{N}}$ is compact, there exists $t_1, ..., t_n \in \Omega_{\mathcal{N}}$ such that $\cup_{i=1}^n V_{t_i} = \Omega_{\mathcal{N}}$.

Take $p_1 = p_{V_1}$ and $p_k = p_{V_k} - p_{V_k}(\Sigma_{i=1}^{k-1} p_i)$ if $k \geq 2$, which will thus be mutually orthogonal projections in $\mathcal{Z}(\mathcal{N})$, with sum equal to 1. Let also $y = \Sigma_{k=1}^n y_{t_k} p_k$ and note that $y \in C_{\mathcal{N}}(x)$. Indeed, this is because if $y_1, ..., y_m \in C_{\mathcal{N}}(x)$ and $p_1, ..., p_m$ is a finite partition of the identity with projections in $\mathcal{Z}(\mathcal{N})$ (which is included in $\mathcal{Z}(\mathcal{M})$) then $\Sigma_j y_j p_j \in C_{\mathcal{N}}(x)$, a fact that can be easily seen by the same argument as in the single algebra case (see e.g. [H] or [SZ1]). Since

$$\|uyu^* - y\|_e = \max_k \|up_k y u^* p_k - y p_k\|_e = \max_k \|up_k y_{t_k} u^* p_k - y_{t_k} p_k\|_e \leq \varepsilon$$

it follows that y, defined this way, satisfies all the required conditions.

The last part is trivial, since any $y' \in C_{\mathcal{N}}(y)$ is a norm limit of elements of the form $T_u y$, and for all such elements we have $\|y - T_u y\|_e \leq \varepsilon$.

□

Let us now proceed with the proof of the properly infinite case ($\mathcal{N}, \mathcal{M}$ are still assumed countably decomposable for the moment). As in the finite case we take $x = x^* \in \mathcal{M}$ such that $\mathcal{E}_{\mathcal{N} \vee \mathcal{N}' \cap \mathcal{M}}(x) = 0$. By Lemma 3.4, there exists $y_1 \in C_{\mathcal{N}}(x)$ such that $\|y_1 - vy_1v^*\|_e < 1/2, \forall v \in \mathcal{U}(\mathcal{N})$ and $\|y_1 - y'\|_e < 1/2, \forall y' \in C_{\mathcal{N}}(y_1)$. More generally, by applying recursively Lemma 3.4 we can construct recursively a sequence of elements $\{y_n\}_n \subset C_{\mathcal{N}}(x)$ such that:(i)$y_n \in C_{\mathcal{N}}(y_{n-1})$; (ii) the diameter of $C_{\mathcal{N}}(y_n)$ in the "Calkin" algebra $\mathcal{M}/\mathcal{J}$ is $< 1/2^n$; (iii) $\|vy_n - y_n v\|_e < 1/2^n, \forall v \in \mathcal{U}(\mathcal{N})$. But then y_n is Cauchy in $\mathcal{M}/\mathcal{J}$ so there exists $y = y^* \in \mathcal{M}$ such that $\|y - y_n\|_e$ tends to 0. By (iii) we also have that y commutes with $\mathcal{N}$ modulo $\mathcal{J}$.

Now in the purely infinite case $\mathcal{J} = 0, \| \|_e = \| \|$ so we get $y \in C_{\mathcal{N}}(x) \cap \mathcal{N}' \cap \mathcal{M}$ and we are done.

When $\mathcal{N}, \mathcal{M}$ are properly infinite but semifinite, by (I.1 in [PoR]) it follows that $y = y' + K$ for some $y' \in \mathcal{N}' \cap \mathcal{M}$ and $K \in \mathcal{J}$. Thus $\forall n \geq 1, \exists K_n \in \mathcal{J}$ such that $\|y' - (y_n + K_n)\| < 1/2^n$. By the remark before Lemma 3.1, there exists $u = (u_1, ..., u_m) \subset \mathcal{U}(\mathcal{N})$ such that $\|T_u(K_n)\| < 1/2^n$. Since $T_u(y_n) \in C_{\mathcal{N}}(x)$ and $T_u(y') = y'$ it follows that y' is at distance $< 1/2^{n-1}$ from $C_{\mathcal{N}}(x), \forall n \geq 1$. Thus $y' \in C_{\mathcal{N}}(x)$. But then, as we've noticed in 1.6 b), $y' = 0$.

This completes the proof of the relative Dixmier property for inclusions of properly infinite, countably decomposable von Neumann algebras with finite index.

To settle the case when $\mathcal{N}, \mathcal{M}$ are properly infinite but not necessarily countably decomposable, let us first prove the following:

3.5. Lemma. *Given any normal element $u \in \mathcal{M}$ there exists a family of countably decomposable projections $\{p_i\}_{i \in I} \subset \mathcal{N}$ (i.e., each p_i is the support of a normal state on $\mathcal{M}$) such that $\Sigma_i p_i = 1$ and $\{p_i\}_{i \in I} \subset \{u\}' \cap \mathcal{N}$.*

Proof. Since any normal element in $\mathcal{M}$ is contained in a von Neumann algebra generated by a unitary element, it is sufficient to prove the statement for $u \in \mathcal{M}$ unitary.

Begin by noting that if $q \in \mathcal{M}$ is a projection of countable type then $Tq \stackrel{\text{def}}{=} \vee_{n \in \mathbb{Z}} u^n q u^{-n}$ and $Sq \stackrel{\text{def}}{=} s(\mathcal{E}(q))$ are also projections of countable type. Indeed, because if q is the support of a normal state ψ on $\mathcal{M}$ then Tq is the support projection of $\Sigma_{n \in \mathbb{Z}} 2^{-|n|} \psi(u^n \cdot u^{-n})$ and Sq is the support projection of $\psi \circ \mathcal{E}$. Also, note that if $\{q_n\}_n$ are of countable type then $\vee_n q_n$ is of countable type.

Let then $\{p_i\}_{i \in I}$ be a maximal family of mutually orthogonal, countably decomposable projections in $\{u\}' \cap \mathcal{N}$ and assume $p = 1 - \Sigma_i p_i \neq 0$. Let $0 \neq q_0 \leq p$ be the support projection of a normal state on $p\mathcal{M}p$. From the above remark it follows that

$p_0 \overset{\text{def}}{=} \vee_{n \geq 0}(ST)^n(q_0)$ is of countable type as well. Note that $q_0 \leq Tq_0 \leq STq_0 \leq TSTq_0 \leq \ldots \nearrow p_0$. Also, since $(ST)^n(q_0) \in \mathcal{N}, \forall n$, we have $p_0 \in \mathcal{N}$. Moreover, for each fixed $k \in \mathbb{Z}$ we have $u^k((ST)^n q_0)u^{-k} \leq T(ST)^n q_0 \leq p_0$ so that $u^k p_0 u^{-k} \leq p_0$. Taking $-k$ for k we get $u^{-k} p_0 u^k \leq p_0$ as well, so that $u^k p_0 u^{-k} = p_0, \forall k \in \mathbb{Z}$. Thus p_0 belongs to $\{u\}' \cap \mathcal{N}$. But then $\{p_i\}_{i \in I} \cup \{p_0\}$ contradicts the maximality of $\{p_i\}_{i \in I}$. Thus, $\Sigma_{i \in I} p_i = 1$. □

By the preceding lemma and by using the fact that if $x = x^* \in \mathcal{M}$ is such that $\mathcal{E}_{\mathcal{N} \vee \mathcal{N}' \cap \mathcal{M}}(x) = 0$ then $\mathcal{E}_{\mathcal{N} \vee \mathcal{N}' \cap \mathcal{M}}(p_i x p_i) = 0, \forall i$, where p_i are as in 3.5, it follows that in order to have the relative Dixmier property for inclusions of arbitrary properly infinite von Neumann algebras with finite index, it is sufficient to have it in the countable decomposable case, provided we can prove that the (minimal) number of averaging unitaries needed depends only on $\|x\|$ and ε. Thus, all we need in order to conclude the proof of the properly infinite case of the Theorem is the following lemma, whose proof is identical to the proof of Remark 2.4.2°, but which we include here in details, for convenience:

3.6. Lemma. *For any $\varepsilon > 0$, there exists $n = n(c, \varepsilon) \geq 1$, such that given any inclusion $\mathcal{N} \subset \mathcal{M}$ of countably decomposable properly infinite von Neumann algebras, with a normal faithful conditional expectation $\mathcal{E} : \mathcal{M} \to \mathcal{N}$ satisfying Ind$\mathcal{E} \leq c^{-1}$, we have that $\forall x \in \mathcal{M}, \|x\| \leq 1, \exists u_1, u_2, \ldots, u_n \in \mathcal{U}(\mathcal{N})$ such that $d(\frac{1}{n}\sum_{k=1}^n u_k x u_k^*, \mathcal{N}' \cap \mathcal{M}) < \varepsilon$.*

Proof. Assume this is not the case. Then $\exists \varepsilon_0 > 0, \exists c_0 < \infty$ such that $\forall n \geq 1, \exists \mathcal{N}_n \subset \mathcal{M}_n$, with $\mathcal{E}_n : \mathcal{M}_n \to \mathcal{N}_n$ satisfying $\mathcal{E}_n(X) \geq c_0 X, \forall X \in \mathcal{M}_+$, and $\exists x_n \in \mathcal{M}_n, \|x_n\| \leq 1$, such that $\mathrm{d}(\frac{1}{n}\sum_{k=1}^n u_k x_n u_k^*, \mathcal{N}_n' \cap \mathcal{M}_n) \geq \varepsilon_0, \forall u_1, u_2, \ldots, u_n \in \mathcal{U}(\mathcal{N}_n)$. But then applying the relative Dixmier property for the inclusion of countably decomposable, properly infinite von Neumann algebras $\mathcal{N} = \oplus_n \mathcal{N}_n \subset \oplus_n \mathcal{M}_n = \mathcal{M}$, $\mathcal{E} = \oplus_n \mathcal{E}_n$ (N.B. $\mathcal{N}, \mathcal{M}$ are countably decomposable as being direct sums of countably many such von Neumann algebras!) and to the element $X = \oplus_n x_n$ we get some $V_1 = \oplus_n v_{1,n}, \ldots, V_k = \oplus_n v_{k,n} \in \mathcal{U}(\mathcal{N})$ such that

$$\varepsilon_0/2 > \mathrm{d}\left(\frac{1}{k}\sum_{j=1}^k V_j X V_j^*, \mathcal{N}' \cap \mathcal{M}\right) = \sup_n \mathrm{d}\left(\frac{1}{k}\sum_{j=1}^k v_{j,n} x_n v_{j,n}^*, \mathcal{N}_n' \cap \mathcal{M}_n\right)$$

This gives a contradiction if $n > k$. □

3.7. *Remark.* 1°. Note that the above proof of the properly infinite case of the Theorem becomes much simpler if we assume the algebras $\mathcal{N}, \mathcal{M}$ are properly infinite factors of countable type. Indeed, in this case no reduction theory argument is needed and so, Lemma 3.2 being stated for the factors $\mathcal{N}, \mathcal{M}$ and the ideal $\mathcal{J}$ (instead of $\mathcal{N}_t, \mathcal{M}_t, J_t$), the argument following 3.4 can be applied directly (Lemmas 3.3 - 3.6 are not needed).

2°. Let us also present a very simple argument proving the Theorem in the separable factorial type III (purely infinite) case. So assume $\mathcal{N} \subset \mathcal{M}$ are purely infinite factors with separable preduals and let φ be a normal faithful state on $\mathcal{M}$ such that $\varphi \circ \mathcal{E} = \varphi$. If $x = x^* \in \mathcal{M}$ is so that $\mathcal{E}_{\mathcal{N} \vee \mathcal{N}' \cap \mathcal{M}}(x) = 0$ and we denote by $C_{\mathcal{N}}^w(x)$ the weak closure of co$\{uxu^* \mid u \in \mathcal{U}(\mathcal{N})\}$ (and thus of $C_{\mathcal{N}}(x)$), then $C_{\mathcal{N}}^w(x) \cap \mathcal{N}' \cap \mathcal{M} \neq \emptyset$ (for instance by 2.1 in [ILPo]). But $\mathcal{E}(x) = 0$ implies $\mathcal{E}(y) = 0, \forall y \in C_{\mathcal{N}}^w(x)$, showing that $C_{\mathcal{N}}^w(x) \cap \mathcal{N}' \cap \mathcal{M} = \{0\}$. By the inferior semicontinuity of the norm $\|y\|_\varphi$ with respect to the weak topology, it follows that $\forall n \geq 1, \exists y_n \in C_{\mathcal{N}}(x)$ such that $\|y_n\|_\phi < 1/n^2$.

Thus, if e_n is the spectral projection of $|y_n|$ corresponding to the interval $[1/n, \|x\|]$ then $\varphi(\mathcal{E}(e_n)) = \varphi(e_n) < 1/n$. Since $\mathcal{E}(e_n) \geq 0$, this implies that $\forall \varepsilon > 0, \exists n \geq 1$ such that $e_{[0,\alpha]}(\mathcal{E}(e_n)) \neq 0$, where $\alpha = c\varepsilon/4\|x\|$. By the well known properties of the Dixmier sets for elements in purely infinite factors (cf e.g. [SZ1]), applied here for the factor $\mathcal{N}$ and the element $\mathcal{E}(e_n) \in \mathcal{N}$, it follows that there exist $u_1, u_2, ..., u_m \in \mathcal{U}(\mathcal{N})$ such that

$$\left\| \frac{1}{m} \sum_{j=1}^{m} u_j \mathcal{E}(e_n) u_j^* \right\| \leq 2\alpha$$

Since $\mathcal{E}(e_n) \geq ce_n$, it follows that

$$2\alpha \geq \frac{1}{m} \sum_{j=1}^{m} u_j \mathcal{E}(e_n) u_j^* \geq \frac{1}{m} \sum_{j=1}^{m} u_j e_n u_j^*.$$

Thus we get

$$\left\| \frac{1}{m} \sum_{j=1}^{m} u_j y_n u_j^* \right\| \leq \|x\| \left\| \frac{1}{m} \sum_{j=1}^{m} u_j e_n u_j^* \right\| + 1/n \leq \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Since $y_n \in C_{\mathcal{N}}(x)$ and ε was arbitrary, it follows that $0 \in C_{\mathcal{N}}(x)$, thus proving the result in this separable, factorial type III case.

4. Applications

We will now mention some consequences of the Theorem. Our first result shows that for inclusions of type II_1 factors $N \subset M$, the finiteness of the index not only implies the relative Dixmier property, but it is actually equivalent to it. Along the lines, we also prove that the finite index condition on $N \subset M$ is equivalent to the property that states on M that are normal when restricted to N follow normal on all M.

4.1. Corollary. *Let $N \subset M$ be an inclusion of type II_1 factors.*

(i). $N \subset M$ has finite Jones index, $[M : N] < \infty$, if and only if the only states on M that are normal when restricted to N are the normal states of M.

(ii). If $N \subset M$ has finite Jones index then $N \subset M$ has the relative Dixmier property. If in addition N has separable predual then, conversely, if $N \subset M$ has the relative Dixmier property then $[M : N] < \infty$.

(iii). If N is isomorphic to the hyperfinite type II_1 factor R, then $[M : N] < \infty$ if and only if any conditional expectation of M onto N is normal (equivalently, if and only if any state on M having N in its centralizer is normal on M).

Proof. If $[M : N] < \infty$, then, by [PiPo], we have $E_N(x) \geq [M : N]^{-1}x, \forall x \in M_+$, where E_N denotes the unique expectation of M onto N that preserves the trace τ of M. Thus $N \subset M$ has the relative Dixmier property by the theorem, proving the implication $\Rightarrow$ in (ii).

Also, we have already mentioned in Section 2 that the implication $\Rightarrow$ in (i) holds true for all inclusions of von Neumann algebras $\mathcal{N} \subset \mathcal{M}$ with conditional expectations

$\mathcal{E}:\mathcal{M}\to\mathcal{N}$ satisfying the finite index condition "$\mathcal{E}(x)\geq cx, \forall x\in\mathcal{M}_+$". Indeed, for if this condition is satisfied then for any state φ on $\mathcal{M}$ we have $\varphi\leq c^{-1}\varphi\circ\mathcal{E}$, so if $\varphi_{|\mathcal{N}}$ is normal, then φ is bounded above by the normal state $\varphi\circ\mathcal{E}$ on $\mathcal{M}$ and thus, by Sakai's Radon-Nykodim theorem ([Sa]), φ is normal itself.

To prove the converse in (i) and (ii), let us first deal with the case when $N'\cap M$ is infinite dimensional. One can actually prove the following more general:

4.2. Lemma. *Let $\mathcal{N}\subset\mathcal{M}$ be an inclusion of von Neumann algebras and assume there exists a sequence of mutually orthogonal non-zero projections $\{p_n\}_n$ in $\mathcal{M}$ such that for each m one has the implication "$x\in\mathcal{N}, xp_m=0\Rightarrow x=0$". Then each normal state on $\mathcal{N}$ can be extended to a singular state on $\mathcal{M}$. If in addition $\mathcal{N}$ has a separable predual, and an infinite dimensional irreducible representation then $\mathcal{N}\subset\mathcal{M}$ doesn't have the relative Dixmier property.*

Proof. Let ψ be a normal state on $\mathcal{N}$. Since $\mathcal{N}$ is isomorphic to $\mathcal{N}p_n$ we can view ψ as a state ψ_n on $\mathcal{N}p_n$. Let φ_n be an arbitrary state on $p_n\mathcal{M}p_n$ extending ψ_n and still denote by φ_n the state on $\mathcal{M}$ defined by compression to $p_n\mathcal{M}p_n$, $\varphi_n(x)=\varphi_n(p_nxp_n), x\in\mathcal{M}$. Let $\phi\in\mathcal{M}^*$ be a weak limit of $\{\varphi_n\}_n\in\mathcal{M}^*$. Thus φ is a state on $\mathcal{M}$, $\varphi_{|\mathcal{N}}=\psi$ and φ is singular on $\mathcal{M}$ (because $\varphi(1-\Sigma_{n\geq m}p_n)=0, \forall m$ and $(1-\Sigma_{n\geq m}p_n)\to 1$ as $m\to\infty$).

If in addition $\mathcal{N}$ has separable predual then let $\{x_k\}_k\in\mathcal{N}$ be a sequence of elements dense in the unit ball of $\mathcal{N}$ in the weak operator topology. If we assume, on the contrary, that $\mathcal{N}\subset\mathcal{M}$ does have the relative Dixmier property, then in particular the element $X=\Sigma_n x_n p_n$ in $\mathcal{M}$ satisfies $C_{\mathcal{N}}(X)\cap\mathcal{N}'\cap\mathcal{M}\neq\emptyset$. But for each n one has $C_{\mathcal{N}}(X)p_n=C_{\mathcal{N}}(x_n)p_n\subset\mathcal{N}p_n$ and since $\mathcal{N}'\cap\mathcal{M}\cap\mathcal{N}p_n=\mathcal{N}'\cap\mathcal{N}p_n=\mathcal{Z}(\mathcal{N})p_n$, it follows that $C_{\mathcal{N}}(X)\cap\mathcal{N}'\cap\mathcal{M}\subset\Sigma_n\mathcal{Z}(\mathcal{N})p_n$. Consequently, $\forall\varepsilon>0$ there exist finitely many unitary elements $u_1,u_2,...,u_m\in\mathcal{U}(\mathcal{N})$ and some central elements $\{z_n\}_n$ in the unit ball of $\mathcal{Z}(\mathcal{N})$ such that

$$\left\|\frac{1}{m}\sum_{k=1}^{m} u_kXu_k^*-\Sigma_n z_np_n\right\|<\varepsilon.$$

Since $Xp_n=x_np_n$ and u_k commute with p_n, $\forall k,n$, we have:

$$\left\|\frac{1}{m}\sum_{k=1}^{m} u_kXu_k^*-\Sigma_n z_np_n\right\|=\left\|\Sigma_n\left(\frac{1}{m}\sum_{k=1}^{m} u_kx_nu_k^*-z_n\right)p_n\right\|$$

$$=\sup_n\left\|\frac{1}{m}\sum_{k=1}^{m} u_kx_nu_k^*-z_n\right\|\geq\left\|\frac{1}{m}\sum_{k=1}^{m} u_kx_nu_k^*-z_n\right\|, \forall n.$$

Thus, $u_1,u_2,...,u_m\in\mathcal{U}(\mathcal{N})$ have the property that

$$\left\|\frac{1}{m}\sum_{k=1}^{m} u_kx_nu_k^*-z_n\right\|<\varepsilon, \forall n$$

But then, since $\{x_k\}_k$ is dense in the unit ball of $\mathcal{N}$ in the weak operator topology, by the inferior semicontinuity of the uniform norm with respect to the wo-topology it follows that $\forall x$ in the unit ball of $\mathcal{N}$ $\exists z$ in the unit ball of $\mathcal{Z}(\mathcal{N})$ such that

$$\left\|\frac{1}{m}\sum_{k=1}^{m} u_kxu_k^*-z\right\|\leq\varepsilon.$$

Moreover, if $\pi : \mathcal{N} \to \mathcal{B}(\mathcal{H})$ is an irreducible representation of N then, by using again the inferior semicontinuity of the uniform norm with respect to the wo topology, the wo-density of the unit ball of $\pi(\mathcal{N})$ in the unit ball of $\mathcal{B}(\mathcal{H})$ and the fact that, for $z \in \mathcal{Z}(\mathcal{N})$, $\pi(z)$ is a scalar multiple of identity in $\mathcal{B}(\mathcal{H})$, it follows that $\forall Y \in \mathcal{B}(\mathcal{H}), \exists \alpha \in \mathbb{C}$ such that

$$\left\| \frac{1}{m} \sum_{k=1}^{m} u_k Y u_k^* - \alpha 1 \right\| \leq \varepsilon.$$

But if π is an infinite dimensional irreducible representation of the von Neumann algebra $\mathcal{N}$, then the corresponding Hilbert space $\mathcal{H}$ is non-separable, and since $u_1, ..., u_m$ is a finite set, there exists an (non-zero) invariant separable Hilbert subspace $\mathcal{H}_0 \subset \mathcal{H}$ which is reducing for all the elements $u_1, ..., u_m$. Thus, if p_0 denotes the orthogonal projection of $\mathcal{H}$ onto $\mathcal{H}_0$ then $u_i p_0 u_i^* = p_0, \forall i$. Taking $Y = p_0$ it follows that for some scalar α we have

$$\|p_0 - \alpha 1\| = \left\| \frac{1}{m} \sum_{k=1}^{m} u_k p_0 u_k^* - \alpha 1 \right\| \leq \varepsilon,$$

which is a contradiction if $\varepsilon < 1/2$, no matter what α one takes. This completes the proof of 4.2.

□

Let us now prove the implication $\Leftarrow$ in (i) in the case $N' \cap M$ is finite dimensional. Note that if the normal states on a von Neumann algebra $\mathcal{N}$ have only normal state extensions to the von Neumann algebra $\mathcal{M}$ ($\supset \mathcal{N}$) and $p \in \mathcal{N}' \cap \mathcal{M}$ is a projection then the normal states on $\mathcal{N}p$ have only normal state extensions to $p\mathcal{M}p$. Note also that if $\dim N' \cap M < \infty$ and $[M : N] = \infty$ then there exists a non-zero minimal projection $p \in N' \cap M$ such that $[pMp : Np] = \infty$ (cf. [J]).

This shows that in order to prove the implication $\Leftarrow$ in (i) it is sufficient to prove that if an inclusion of type II_1 factors $N \subset M$ satisfies $[M : N] = \infty$ and $N' \cap M = \mathbb{C}1$ then there exist singular states on M which restrict to normal states on N.

To see this, recall that by [PiPo] for any $\varepsilon > 0$ there exists $q \in \mathcal{P}(M)$ such that $E_N(q) \leq \varepsilon 1$. Let q_ε be a maximal projection in M such that $E_N(q_\varepsilon) \leq \varepsilon$. Then clearly $E_N(q_\varepsilon)$ has support 1, because if p would be a non-zero projection in N such that $pE_N(q_\varepsilon) = 0$ then again by [PiPo] we can find $q_1 \in \mathcal{P}(pMp), 0 \neq q_1 \leq p$ such that $E_{pNp}(q_1) \leq \varepsilon p$ and so $q_\varepsilon + q_1$ would contradict the maximality of q_ε. Now let $b_\varepsilon = E_N(q_\varepsilon)^{-1/2} q_\varepsilon E_N(q_\varepsilon)^{-1/2}$ and note that $E_N(b_\varepsilon) = 1$ and $\tau(b_\varepsilon) = 1$. Thus $b_\varepsilon \in L^1(M, \tau)_+$ and $\tau(\cdot\, b_\varepsilon)$ defines a normal state on M. Also, since $E_N(b_\varepsilon) = 1$ we have $\tau(yb_\varepsilon) = \tau(y), \forall y \in N$. Furthermore, b_ε satisfies $\tau(s(b_\varepsilon)) = \tau(q_\varepsilon) \leq \varepsilon$.

For each $\varepsilon = 2^{-n}$ take a $a_n = b_{2^{-n}}$ this way constructed. Since $\tau(s(a_n)) \leq 2^{-n}$, if for each m we denote $p_m = \vee_{n \geq m} s(a_n)$ then $\{p_m\}_m$ is dicreasing to 0. Thus, if ψ is a limit point of $\{\tau(\cdot\, b_n)\}_n \in M^*$ in the $\sigma(M^*, M)$-topology, then $\psi(1 - p_n) = 0, \forall n$ and $1 - p_n \to 1$ so that ψ is a singular state on M. Thus M has a singular state which restricted to N is equal to the trace, thus finishing the proof of (i).

The implication $\Leftarrow$ in (ii) in the case $\dim N' \cap M < \infty$ follows now immediately from (i) and the result in [P] showing that if M has singular states that restrict to the trace on N and N has separable predual then $N \subset M$ doesn't have the relative Dixmier property.

Finally, to prove (iii) assume N is isomorphic to R. If $[M : N] < \infty$ and E is a conditional expectation from M onto N and φ is a normal state on N then $\varphi \circ E$ is normal on M by (i). Thus, if $\{x_i\}_i$ is a weakly convergent net in the unit ball of M then $\varphi(E(x_i)) \to 0$, showing that $\{E(x_i)\}_i$ is weakly convergent to 0 as well, thus E is normal.

Conversely, if $[M : N] = \infty$ then by (i) it follows that there exists a singular state φ on M such that $\varphi_{|N} = \tau_N$. Let $\mathcal{U}_0$ be a countable discrete amenable subgroup of $\mathcal{U}(N)$ such that $\mathcal{U}_0'' = N$ and put $\psi = \int \varphi(u \cdot u^*)\mathrm{d}\mu(u)$, where μ is the invariant mean on the discrete amenable group $\mathcal{U}_0$ and the integral is in the usual, Banach limit sense (see e.g., [S]). Since ψ belongs to the closure in the $\sigma(M^*, M)$-topology of a countable set of singular states on M, by [A] it follows that ψ is singular on M. Also, by construction we have $\psi_{|N} = \tau_N$ and $\psi(u \cdot u^*) = \psi, \forall u \in \mathcal{U}_0$. By Connes' lemma in [C3], it thus follows that $\psi(u \cdot u^*) = \psi, \forall u \in \mathcal{U}(N)$. But then ψ is a singular N-hypertrace on M so by the usual construction (see e.g. [C4]) it defines a singular conditional expectation of M onto N. This completes the proof of the Corollary 4.1.

□

4.3. *Remarks*. 1°. Note that in the above Corollary the separability condition on N is essential: indeed, it has been proved in [Po3] that if $N \subset M$ is an irreducible inclusion of type II_1 factors and ω is a free ultrafilter then $N^\omega \subset M^\omega$ does have the relative Dixmier property, even when $[M : N](= [M^\omega : N^\omega])$ is infinite.

2°. The hyperfiniteness assumption on N in the last part of the Corollary is also essential. Thus, if the free group on two generators $\mathbb{F}_2$ is embedded in the usual way in the free group on three generators $\mathbb{F}_3$ and $N \subset M$ denotes the corresponding inclusion of von Neumann factors $L(\mathbb{F}_2) \subset L(\mathbb{F}_3)$, then it is easy to see that any conditional expectation of M onto N must coincide with the trace preserving one, while $[M : N] = \infty$. In fact, by using Day's trick one can show that if $N \subset M$ is an inclusion of type II_1 factors then there exists a singular conditional expectation of M onto N iff $\forall \varepsilon > 0, \forall u_1, ..., u_n \in \mathcal{U}(N), \exists b \in L^1(M, \tau)_+$ such that: $E_N(b) = 1$, $\|u_i b u_i^* - b\|_1 < \varepsilon$ and $\tau(s(b)) < \varepsilon$. In turn, this last condition clearly doesn't hold true for $L(\mathbb{F}_2) \subset L(\mathbb{F}_3)$, for instance by ([MvN] or 2.1 in [Po7]).

Other examples of inclusions $N \subset M$ of infinite index for which any expectation of M onto N is normal are obtained by taking N to have the Connes-Kazhdan property T and M an arbitrary type II_1 factor containing N such that $N' \cap M = \mathbb{C}1$ and $[M : N] = \infty$. For many similar examples, see e.g. [Po3] or [Po6]. For instance, if N has the property T, and if P is an arbitrary finite von Neumann algebra with a faithful normal trace and no minimal projections, then there exists a unique conditional expectation of $N * P$ onto $N(=N * \mathbb{C})$, while there always exist singular conditional expectations of $R * P$ onto R !

3°. The proof of 4.1 obviously works beyond the factorial case, for rather general classes of inclusions of finite von Neumann algebras. Let us mention though that if one drops the finiteness condition, even just for M, the problem may become much more difficult to solve. For instance, if one denotes by $\mathcal{D}$ the algebra of diagonal operators on the infinite dimensional separable Hilbert space $\mathcal{H}$, then it has been noted by J. Anderson in [An] that the relative Dixmier property for $\mathcal{D} \subset \mathcal{B}(\mathcal{H})$ is equivalent to the positive solution to an old standing problem of R.V. Kadison and I. Singer ([KSi]), asking whether each pure state on $\mathcal{D}$ extends to a unique pure state on $\mathcal{B}(\mathcal{H})$. The answer to this is still not known, despite considerable efforts (see e.g., [An], [BHKW], [BoT]). While all the results in these papers are about proving that for certain large classes of operators $X \in \mathcal{B}(\mathcal{H})$ the set $C_{\mathcal{D}}(X)$ contains $\mathcal{E}(X)$, $\mathcal{E}$ being the unique normal conditional expectation of $\mathcal{B}(\mathcal{H})$ onto

$\mathcal{D}$, thus pointing towards a positive answer to the Kadison-Singer problem, the opinion among experts is that the answer to the problem might as well turn out to be negative.

In connection with this, let us point out that despite Ind$\mathcal{E} = \infty$ in this case, the inclusion $\mathcal{D} \subset \mathcal{B}(\mathcal{H})$ does satisfy that a state on $\mathcal{B}(\mathcal{H})$ which is normal on $\mathcal{D}$ is normal on all $\mathcal{B}(\mathcal{H})$ and that $\mathcal{E}$ is the only possible conditional expectation of $\mathcal{B}(\mathcal{H})$ onto $\mathcal{D}$. Thus, the extrapolation of the statement 4.1 (ii) to more general inclusions suggests that the Kadison-Siger conjecture might have a negative answer, i.e., that $\mathcal{D} \subset \mathcal{B}(\mathcal{H})$ does not have the relative Dixmier property, while part (i) of 4.1 suggests that the conjecture might have a positive answer. On the other hand, it is interesting to note that if there does exist a pure state on $\mathcal{D}$ which extends to a pure state φ on $\mathcal{B}(\mathcal{H})$ such that $\varphi \neq \varphi \circ \mathcal{E}$, then φ is still $\mathcal{D}$-linear, in fact even $\mathcal{D}$-multiplicative (cf. [An]; one can also prove this by using the same argument as in 2.1 and in the proof following it). For certain fast growing free ultrafilters ω on $\mathbb{N}$ it is easy to see that if $\varphi_{|\mathcal{D}} = \omega$ then this last condition forces φ to factor through $\mathcal{E}$ thus leading to a contradiction (see [Re]). For general ultrafilters though, it is not clear that this would still be the case.

The next corollary gives a somewhat surprising consequence of the theorem, which, while related in spirit to Connes' well known characterization of proper outerness of automorphisms in [C2], may be interesting on its own:

4.4. Corollary. *Let $\mathcal{Q}$ be an arbitrary von Neumann algebra and $\sigma_1, ..., \sigma_n$ a finite set of properly outer automorphisms of $\mathcal{Q}$ ([C1]). Then*

$$(0, ..., 0) \in \mathrm{co}^n\{(u\sigma_i(u^*))_{1 \leq i \leq n} \mid u \in \mathcal{U}(\mathcal{Q})\}.$$

Proof. This is a trivial consequence of the theorem, applied to the locally trivial inclusion of von Neumann algebras $\mathcal{N} \stackrel{\text{def}}{=} \{x \oplus \sigma_1(x) \oplus ... \oplus \sigma_n(x) \mid x \in \mathcal{Q}\} \subset \mathcal{M} \stackrel{\text{def}}{=} M_{(n+1)\times(n+1)}(\mathcal{Q})$, with its obvious expectation $\mathcal{E}$ of index $(n+1)^2$, and to the elements $\{e_{0j}\}_{1 \leq j \leq n} \subset M$ (see [Po2] for the latter notations).

□

Note that the above result can be applied to the case when $\mathcal{Q}$ is the diagonal algebra $\mathcal{D} \simeq l^2(\mathbb{N})$ mentionned in Remark 4.2.3°, in relation to the Kadison-Singer problem. Thus one obtains that if $X \in \mathcal{B}(\mathcal{H})$ belongs to the C*-algebra generated in $\mathcal{B}(\mathcal{H})$ by $\mathcal{D}$ and the unitaries in the normalizer of $\mathcal{D}$ in $\mathcal{B}(\mathcal{H})$ then $C_{\mathcal{D}}(X) \cap \mathcal{D} = \{\mathcal{E}(X)\}$. Thus we recover some of the results in [An], [BHKW], [BoT].

4.5. Corollary. *Let Q be a type II_1 factor and $\sigma : G \to AutQ$ a properly outer action of a discrete group G on Q. Let $M = Q \times_\sigma G$ be the associated von Neumann cross-product algebra (thus M is a type II_1 factor) and denote by M_0 the reduced C*-cross product algebra, i.e., the C*-algebra generated in M by Q and the unitaries $\{u_g\}_{g \in G} \subset M$ implementing σ. Then $Q \subset M_0$ has the relative Dixmier property but if G is infinite and Q has separable predual then $Q \subset M$ doesn't have this property.*

Proof. The first part is an immediate consequence of Corollary 4.4 while the second part is a trivial consequence of Corollary 4.1 (ii).

□

Finally, let us mention that another significant application of the Theorem is given in [Po5].

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(Manuscript received November 14, 1997.)

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