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CRYSTALLINE CONJECTURE VIA K-THEORY

BY WIESŁAWA NIZIOŁ

In Memoriam Robert Thomason

ABSTRACT. – We give a new proof of the Crystalline Conjecture of Fontaine comparing étale and crystalline cohomologies of smooth projective varieties over p-adic fields in the good reduction case. Our proof is based on Thomason's results relating étale cohomology to algebraic K-theory © Elsevier, Paris

RÉSUMÉ. – Nous fournissons une nouvelle démonstration de la conjecture cristalline de Fontaine, qui compare les cohomologies étale et cristalline de variétés projectives lisses sur les corps p-adiques ayant une bonne réduction. Notre preuve repose sur les résultats de Thomason reliant cohomologie étale et K-théorie algébrique. © Elsevier, Paris

1. Introduction

The aim of this article is to show how the Crystalline Conjecture of Fontaine for projective schemes [8] (a theorem of Fontaine-Messing [11] and Faltings [5]) can be derived from Thomason's comparison theorem [24] between algebraic and étale K-theories.

Let us recall the formulation of this conjecture. Let K be a complete discrete valuation field of mixed characteristic $(0,p)$ with ring of integers V and a perfect residue field. Let X be a smooth projective V-scheme. Assume that the relative dimension of X is less than $p-2$ and that V is absolutely unramified. Then the conjecture postulates the existence of a natural period "almost-isomorphism" (a notion which can be made precise)

$$\alpha : H^*(X_{\overline{K}}, \mathbf{Z}/p^n) \otimes B^+_{\mathrm{cr}} \simeq H^*_{dR}(X_n/V_n) \otimes B^+_{\mathrm{cr}},$$

where $\overline{K}$ is an algebraic closure of K, the subscript n denoting reduction mod p^n, and where B^+_{cr} is a certain ring of periods introduced by Fontaine. The ring B^+_{cr} is equipped with a filtration, Galois action and a Frobenius operator, and the period "almost-isomorphism" is expected to preserve these structures. This, in particular, allows one to recover étale cohomology from de Rham cohomology (with all the extra structures) and vice versa (Theorem 4.1).

To prove the conjecture, by a standard argument (see, [11], [5]), it suffices to construct a map

$$(1) \qquad \alpha : H^*(X_{\overline{K}}, \mathbf{Z}/p^n) \to H^*_{dR}(X_n/V_n) \otimes B^+_{\mathrm{cr}}$$

compatible with all the above structures, and, in addition, with Poincaré duality and some cycle classes.

Our idea was to construct the map α via the higher K-theory groups. In the first step of the construction, one represents the right-hand side of (1) as a cohomology group of the scheme $X_{\overline{V}}$, where $\overline{V}$ is the integral closure of V in $\overline{K}$. Namely, using that $B^+_{\mathrm{cr,n}} \simeq H^*_{\mathrm{cr}}(\overline{V}_n/V_n, \mathcal{O}_{\overline{V}_n/V_n})$ [7, p.105], we get that, by the Künneth formula,

$$H^*_{dR}(X_n/V_n) \otimes B^+_{\mathrm{cr}} \simeq H^*_{\mathrm{cr}}(X_{\overline{V},n}/V_n, \mathcal{O}_{X_{\overline{V},n}/V_n}).$$

Hence, we need to construct a well-behaved map

$$\alpha_{ai} : H^a(X_{\overline{K}}, \mathbf{Z}/p^n(i)) \to H^a_{\mathrm{cr}}(X_{\overline{V},n}/V_n, \mathcal{O}_{X_{\overline{V},n}/V_n}(i)),$$

at least for large enough i. Here, the twist in the crystalline cohomology refers to twisting the Hodge filtration and the Frobenius. Our construction is based on the following diagram

$$\begin{tikzcd}
F^i_\gamma/F^{i+1}_\gamma K_j(X_{\overline{V}}; \mathbf{Z}/p^n) \arrow[rr, "\sim", "j*"'] \arrow[d, "\overline{c}^{\mathrm{cr}}_{ij}"] & & F^i_\gamma/F^{i+1}_\gamma K_j(X_{\overline{K}}; \mathbf{Z}/p^n) \arrow[d, "\overline{c}^{\text{ét}}_{ij}"] \\
H^{2i-j}_{\mathrm{cr}}(X_{\overline{V},n}/V_n, \mathcal{O}_{X_{\overline{V},n}/V_n}(i)) & & H^{2i-j}(X_{\overline{K}}, \mathbf{Z}/p^n(i)), \arrow[ll, "\alpha_{2i-j,i}"'] \arrow[ul, "\omega_{ij}"]
\end{tikzcd}$$

where $K_j(\cdot; \mathbf{Z}/p^n)$ is the K-theory with coefficients and $F^i_\gamma K_j(\cdot; \mathbf{Z}/p^n)$ is the γ-filtration. The maps

$$\overline{c}^{\text{ét}}_{ij} : K_j(X_{\overline{K}}; \mathbf{Z}/p^n) \to H^{2i-j}(X_{\overline{K}}, \mathbf{Z}/p^n(i)),$$
$$\overline{c}^{\mathrm{cr}}_{ij} : K_j(X_{\overline{V}}; \mathbf{Z}/p^n) \to H^{2i-j}_{\mathrm{cr}}(X_{\overline{V},n}/V_n, \mathcal{O}_{X_{\overline{V},n}/V_n}(i))$$

are the étale [13] and the crystalline Chern class maps respectively. The arrow ω_{ij} is to be constructed below. Our map $\alpha_{2i-j,i}$ will make this diagram commute.

First, we prove (Lemma 3.1) that the restriction j^* is an isomorphism. Next, using the work of Thomason and Soulé, we show (Proposition 4.1) that, for large j and large p, the étale Chern class map $\overline{c}^{\text{ét}}_{ij}$ is surjective and that the elements in its kernel are annihilated by a power of the Bott element $\beta_n \in K_2(X_{\overline{K}}; \mathbf{Z}/p^n)$. That allows us to construct the arrow ω_{ij} in the above diagram: a map defined only modulo powers of the Bott element β_n, which makes the diagram commute. We set $\alpha_{2i-j,i} = \overline{c}^{\mathrm{cr}}_{ij}\,\omega_{ij}$. Then we prove that the Bott element maps via $\overline{c}^{\mathrm{cr}}_{ij}$ to a non-zero divisor in the image of crystalline Chern class map. This gives that $\alpha_{2i-j,i}$ is a well-defined map.

Our construction of the map $\alpha_{2i-j,i}$ makes it now easy to check its compatibility with Poincaré duality and cycle classes.

The above argument proves the Crystalline Conjecture for primes p larger than, roughly speaking, the cube of the relative dimension of the scheme X. This bound originates from the constant appearing in Thomason's theorem. It is worse than the postulated bound (equal to the relative dimension plus 2).

We also use this argument to prove the rational Crystalline Conjecture (Theorem 5.1). This time, we only need to assume that $p \neq 2$.

This paper grew out of my collaboration with S. Bloch on a related project. It is a pleasure to thank him for many stimulating discussions, his help and his encouragement. W. Gajda helped me to learn étale K-theory, C. Soulé helped me to understand his paper [21] and to simplify the proof of Proposition 4.1, B. Totaro read carefully parts of a preliminary version of this paper. I would like to thank them all for their help.

Throughout the paper, let p be a fixed prime, let $\overline{K}$ denote a chosen algebraic closure of a field K, and, for a scheme X, let $X_n = X \otimes \mathbf{Z}/p^n$.

2. Preliminaries

Let V be a complete discrete valuation ring with its field of fractions K of characteristic 0 and with perfect residue field k of characteristic p. Let $W(k)$ be the ring of Witt vectors with coefficients in k with its field of fractions K_0 . Set $G_K = \mathrm{Gal}(\overline{\mathrm{K}}/\mathrm{K})$, and let σ be the absolute Frobenius on $W(\overline{k})$.

2.1 **The rings of periods**. Consider the ring $R = \mathrm{projlim}\overline{V}/p\overline{V}$, where $\overline{V}$ is the integral closure of V in $\overline{K}$, the maps in the projective system being the p-th power maps. With addition and multiplication defined coordinatewise R, is a ring of characteristic p. Take its ring of Witt vectors $W(R)$; then B^+_{cr} is the p-adic completion of the divided power envelope $D_\xi(W(R))$ of the ideal $\xi W(R)$ in $W(R)$. Here $\xi = [(p)] + p[(-1)]$, where (p), $(-1) \in R$ are reductions mod p of sequences of p-roots of p and -1 respectively (if $p \neq 2$, we may and will choose $(-1) = -1$) and for $x \in R$, $[x] = [x, 0, 0, \ldots] \in W(R)$ is its Teichmüller representative.

The ring B^+_{cr} is a topological $W(k)$-module having the following properties:

1. $W(\overline{k})$ is embedded as a subring of B^+_{cr} and σ extends naturally to a Frobenius ϕ on B^+_{cr};
2. B^+_{cr} is equipped with a decreasing separated filtration $F^n B^+_{\mathrm{cr}}$ such that, for $n < p$, $\phi(F^n B^+_{\mathrm{cr}}) \subset p^n B^+_{\mathrm{cr}}$ (in fact, $F^n B^+_{\mathrm{cr}}$ is the closure of the n-th divided power of the PD ideal of $D_\xi(W(S))$);
3. G_K acts on B^+_{cr}; the action is $W(\overline{k})$-semilinear, continuous, commutes with ϕ and preserves the filtration;
4. there exists an element $t \in F^1 B^+_{\mathrm{cr}}$ such that $\phi(t) = pt$, and G_K acts on t via the cyclotomic character: if we fix $\varepsilon \in R$ – a sequence of nontrivial p-roots of unity, then $t = \log([\varepsilon])$.

B_{cr} is defined as the ring $B^+_{\mathrm{cr}}[p^{-1}, t^{-1}]$ with the induced topology, filtration, Frobenius and the Galois action.

For us, in this paper, it will be essential that the ring B^+_{cr} can be thought of as a cohomology of an 'arithmetic point', namely [7, p.105],

$$B^+_{\mathrm{cr}}/p^n \simeq H^*_{\mathrm{cr}}(\mathrm{Spec}(\overline{\mathrm{V}}/\mathrm{p}^n)/\mathrm{W}_n(\mathrm{k}), \mathcal{O}_{\overline{\mathrm{V}}_n/\mathrm{W}_n(\mathrm{k})}).$$

Let

$$B^+_{dR} = \mathrm{projlim}_r(\mathbf{Q} \otimes \mathrm{projlim}_n B^+_{\mathrm{cr,n}}/F^r B^+_{\mathrm{cr,n}}), \quad B_{dR} = B^+_{dR}[t^{-1}].$$

The ring B^+_{dR} has a discrete valuation given by powers of t. Its quotient field is B_{dR}. We will denote by $F^n B_{dR}$ the filtration induced on B_{dR} by powers of t.

2.2. **Crystalline representations**. Assume that $V = W(k)$. For the integral theory we will need the following abelian categories:

1. $\mathcal{MF}_{big}(V)$ – an object is given by a p-torsion V-module M and a family of p-torsion V-modules $F^i M$ together with V-linear maps $F^i M \to F^{i-1} M$, $F^i M \to M$ and σ-semilinear maps $\phi^i : F^i M \to M$ satisfying certain compatibility conditions (see [10]);
2. $\mathcal{MF}(V)$ – the full subcategory of $\mathcal{MF}_{big}(V)$ with objects finite V-modules M such that $F^i M = 0$ for $i \gg 0$, the maps $F^i(M) \to M$ being injective and $\sum \mathrm{Im}\, \phi^i = M$;
3. $\mathcal{MF}_{[a,b]}(V)$ – the full subcategory of objects M of $\mathcal{MF}(V)$ such that $F^a M = M$ and $F^{b+1} M = 0$.

Consider the category $\mathcal{MF}_{[a,b]}(V)$ with $b-a\leq p-2$. There exists an exact and fully faithfull functor

$$\mathbf{L}(M)=\ker(F^0(M\otimes B^+_{\mathrm{cr}}\{-b\}(-b))\xrightarrow{1-\phi^0}M\otimes B^+_{\mathrm{cr}}(-b)),$$

where $\{-b\}$, $(-b)$ are the $\mathcal{MF}$ and Tate twists respectively, from $\mathcal{MF}_{[a,b]}(V)$ to finite $\mathbf{Z}_p$-Galois representations. Its essential image is called the category of *crystalline representations of weight between* a *and* b. This category is closed under taking tensor products and duals (assuming we stay in the admissible range of the filtration).

We will need the following theorem.

THEOREM 2.1. – *Let X be a smooth and proper scheme over $V=W(k)$ of pure relative dimension d. For $d\leq p-2$, $H^a_{\mathrm{cr}}(X/V,\mathcal{O}_{X/V}/p^n)$ lies in $\mathcal{MF}_{[0,d]}(V)$.*

It was proved by Kato [17, 2.5] (the projective case), by Fontaine-Messing [11, 2.7], and by Faltings [5, 4.1]. Here,

$$H^a_{\mathrm{cr}}(X/V,\mathcal{O}_{X/V}/p^n)\simeq H^a_{dR}(X_n/V_n):=H^a(X_n,\Omega^{\bullet}_{X_n/V_n})$$

is equipped with the Hodge filtration

$$F^iH^a_{\mathrm{cr}}(X/V,\mathcal{O}_{X/V}/p^n)=\mathrm{Im}(H^a(X_n,\Omega^{\geq i}_{X_n/V_n})\to H^a(X_n,\Omega^{\bullet}_{X_n/V_n})),$$

and the mappings

$$\phi^i=\phi^*/p^i:F^iH^a_{\mathrm{cr}}(X/V,\mathcal{O}_{X/V}/p^n)\to H^a_{\mathrm{cr}}(X/V,\mathcal{O}_{X/V}/p^n),$$

where ϕ denotes the crystalline Frobenius.

2.3. **Syntomic regulators.** Let X be a syntomic, that is, a flat and local complete intersection scheme over $W(k)$. Recall the differential definition [17] of syntomic cohomology of Fontaine-Messing [11]. Assume first that we have an immersion $i:X\hookrightarrow Z$ over $W(k)$ such that Z is a smooth $W(k)$-scheme endowed with a lifting of the Frobenius $F:Z\to Z$. Let $D_n=D_{X_n}(Z_n)$ be the PD-envelope of X_n in Z_n and J_{D_n} be the ideal of X_n in D_n. Consider the following complexes of sheaves on X_{Zar}

$$s_n(r)_X:=\mathrm{Cone}(J^{[r-\cdot]}_{D_n}\otimes\Omega^{\bullet}_{Z_n/W_n(k)}\xrightarrow{1-\phi^r}\mathcal{O}_{D_n}\otimes\Omega^{\bullet}_{Z_n/W_n(k)})[-1],\quad \text{for } 0\leq r\leq p-1,$$
$$s'_n(r)_X:=\mathrm{Cone}(J^{[r-\cdot]}_{D_n}\otimes\Omega^{\bullet}_{Z_n/W_n(k)}\xrightarrow{p^r-\phi}\mathcal{O}_{D_n}\otimes\Omega^{\bullet}_{Z_n/W_n(k)})[-1],$$

where ϕ^r is the "divided by p^r" Frobenius. The complexes $s_n(r)_X$, $s'_n(r)_X$ are, up to canonical quasi-isomorphism, independent of the choice of i and F.

In general, immersions as above exist locally, and one defines $s_n(r)_X\in\mathbf{D}^+(X_{\mathrm{Zar}},\mathbf{Z}/p^n)$ by gluing the local complexes. Finally, one defines $s_n(r)_{X_{\overline{V}}}\in\mathbf{D}^+((X_{\overline{V}})_{\mathrm{Zar}},\mathbf{Z}/p^n)$ as the inductive limit of $s_n(r)_{X_{V'}}$, where V' varies over the integral closures of V in all finite extensions of K in $\overline{K}$. Similarly, we define $s'_n(r)_X$ and $s'_n(r)_{X_{\overline{V}}}$.

One sets

$$H^i(X,s_n(r)):=H^i_{\mathrm{Zar}}(X,s_n(r)_X),\quad H^i(X_{\overline{V}},s_n(r)):=H^i_{\mathrm{Zar}}(X_{\overline{V}},s_n(r)_{X_{\overline{V}}}).$$

We list the following properties:

(1) There is a long exact sequence

$$\to H^i(X_{\overline{V}}, s_n(r)) \to H^i_{\mathrm{cr}}(X_{\overline{V}}/W(k), J_n^{[r]}) \xrightarrow{1-\phi^r} H^i_{\mathrm{cr}}(X_{\overline{V}}/W(k), \mathcal{O}_{X_{\overline{V}}/W(k)}/p^n) \to .$$

(2) There exists a well-behaved product

$$s_n(r)_X \otimes^{\mathbf{L}} s_n(r')_X \to s_n(r+r')_X, \quad r+r' \leq p-1.$$

Since, for X smooth and proper over $W(k)$ of relative dimension less than p,

$$H^i_{\mathrm{cr}}(X_{\overline{V}}/W(k), \mathcal{O}_{X_{\overline{V}}/W(k)}/p^n) \simeq H^i_{\mathrm{cr}}(X/W(k), \mathcal{O}_{X/W(k)}/p^n) \otimes B^+_{\mathrm{cr}},$$
$$H^i_{\mathrm{cr}}(X_{\overline{V}}/W(k), J_n^{[r]}) \simeq F^r(H^i_{\mathrm{cr}}(X/W(k), \mathcal{O}_{X/W(k)}/p^n) \otimes B^+_{\mathrm{cr}}) \qquad [11, 1.5,]$$

Property (1) yields a natural map

$$H^i(X_{\overline{V}}, s_n(r)) \to \mathbf{L}(H^i_{\mathrm{cr}}(X/W(k), \mathcal{O}_{X/W(k)}/p^n)\{-r\}) \quad \text{for} \quad p-2 \geq r \geq \dim X_K.$$

For a scheme X, let $K_*(X)$ be the higher K-theory groups of X as defined by Quillen [19]. Similarly, for a noetherian scheme X, let $K'_*(X)$ be Quillen's K'-theory. The corresponding groups, with coefficients $\mathbf{Z}/l$ [24], will be denoted by $K_i(X;\mathbf{Z}/l)$ and $K'_i(X;\mathbf{Z}/l)$. Unless otherwise stated, we will assume that $l \neq 2,3,4,8$. In that case, $K_*(X;\mathbf{Z}/l)$ is an anticommutative ring.

For a noetherian regular scheme X, set

$$F^j_\gamma K_0(X)$$
$$= \begin{cases} K_0(X) & \text{if } j \leq 0, \\ \langle \gamma_{i_1}(x_1)\cdots\gamma_{i_n}(x_n) | \varepsilon(x_1) = \ldots = \varepsilon(x_n) = 0, i_1 + \cdots + i_n \geq k, \rangle & \text{if } j > 0, \end{cases}$$
$$F^k_\gamma K_q(X;\mathbf{Z}/p^n) = \langle a\gamma_{i_1}(x_1) \cup \cdots \cup \gamma_{i_n}(x_n) | a \in F^{i_0}_\gamma K_0(X), i_0 + i_1 + \cdots + i_n \geq k, \rangle,$$

where ε is the augmentation on $K_0(X)$.

We will use the following theorem of Gros [15]:

THEOREM 2.2. – *For any syntomic $W(k)$-scheme X and any integer $0 \leq i < p-1$, there are functorial Chern classes*

$$c^{\mathrm{syn}}_{ij} : K_j(X) \to H^{2i-j}(X, s_n(i))$$

compatible with the crystalline Chern classes

$$c^{\mathrm{cr}}_{ij} : K_j(X) \to H^{2i-j}_{\mathrm{cr}}(X_n/W_n(k), \mathcal{O}_{X_n/W_n(k)}).$$

Gros' construction follows the method of Gillet [13], which can be extended easily [22] to yield functorial Chern classes

$$\overline{c}^{\mathrm{syn}}_{ij} : K_j(X;\mathbf{Z}/p^n) \to H^{2i-j}(X, s_n(i)) \quad \text{for } j \geq 2 \text{ and } j = 0,$$

compatible with the classes c^{syn}_{ij}.

LEMMA 2.1. – *We record the following properties of syntomic Chern classes:*

1. c_{ij}^{syn}, *for* $j > 0$, *is a group homomorphism.*
2. $\overline{c}_{ij}^{\mathrm{syn}}$, *for* $j \geq 2$ *and* $p \neq 2$, *is a group homomorphism.*
3. $\overline{c}_{ij}^{\mathrm{syn}}$ *are compatible with the reduction maps* $s_n(i) \to s_m(i)$, $n \geq m$.
4. *If* $\alpha \in K_l(X; \mathbf{Z}/p^n)$ *and* $\alpha' \in K_q(X; \mathbf{Z}/p^n)$, *then*

$$\overline{c}_{ij}^{\mathrm{syn}}(\alpha\alpha') = -\sum_{r+s=i} \frac{(i-1)!}{(r-1)!(s-1)!} \overline{c}_{rl}^{\mathrm{syn}}(\alpha)\overline{c}_{sq}^{\mathrm{syn}}(\alpha'),$$

assuming that $l, q \geq 2$, $l + q = j$, $2i \geq j$, $i \geq 0$, $p \neq 2$.

Moreover, if X *is regular, then*

5. *If* $\alpha \in F_\gamma^j K_0(X)$, $j \neq 0$, *and* $\alpha' \in F_\gamma^k K_q(X; \mathbf{Z}/p^n)$, $q \geq 2$, $p \neq 2$, *then*

$$\overline{c}_{j+k,q}^{\mathrm{syn}}(\alpha\alpha') = -\frac{(j+k-1)!}{(j-1)!(k-1)!} c_{j0}^{\mathrm{syn}}(\alpha)\overline{c}_{kq}^{\mathrm{syn}}(\alpha').$$

Similarly, for $c_{j+k,q}^{\mathrm{syn}}$, if $\alpha' \in F_\gamma^k K_0(X)$, $k \neq 0$.

6. *The integral Chern class maps* c_{i0}^{syn} *restrict to zero on* $F_\gamma^{i+1} K_0(X)$, $i \geq 1$.
7. *The Chern class maps* $\overline{c}_{ij}^{\mathrm{syn}}$ *restrict to zero on* $F_\gamma^{i+1} K_j(X; \mathbf{Z}/p^n)$, $j \geq 2$, $p \neq 2$.

Proof. – Recall the construction (due to Gillet [13]) of the higher characteristic classes $\overline{c}_{ij}^{\mathrm{syn}}$. The fastest way to describe it is as follows. One constructs universal classes $C_{i,l}^{\mathrm{syn}} \in H^{2i}(B.GL_l/W(k), s_n(i))$ (de Rham classes of the universal locally free sheaf on $B.GL_{l,n}/W_n(k)$). They yield compatible universal classes $c_{i,l}^{\mathrm{syn}} \in H^{2i}(X, GL_l(\mathcal{O}_X), s_n(i))$, hence a natural map of pointed simplicial sheaves on X, $C_i^{\mathrm{syn}} : B.GL(\mathcal{O}_X) \to \mathcal{K}(2i, \tilde{s}_n(i)_X)$, where $\mathcal{K}$ is the Dold–Puppe functor of $\tau_{\geq 0}\tilde{s}_n(i)_X[2i]$ and $\tilde{s}_n(i)_X$ is an injective resolution of $s_n(i)_X$. The characteristic classes $\overline{c}_{ij}^{\mathrm{syn}}$ are now defined [13, 2.22] as the composition

$$\begin{aligned} K_j(X; \mathbf{Z}/p^n) &\to H^{-j}(X, \mathbf{Z} \times B.GL(\mathcal{O}_X)^+; \mathbf{Z}/p^n) \to H^{-j}(X, B.GL(\mathcal{O}_X)^+; \mathbf{Z}/p^n) \\ &\xrightarrow{H^{-j}(C_i^{\mathrm{syn}})} H^{-j}(X, \mathcal{K}(2i, \tilde{s}_n(i)_X); \mathbf{Z}/p^n) \xrightarrow{f} H^{2i-j}(X, s_n(i)), \end{aligned}$$

where $B.GL(\mathcal{O}_X)^+$ is the (pointed) simplicial sheaf on X associated to the $+$-construction [21, 4.2]. Here, for a (pointed) simplicial sheaf $\mathcal{E}.$ on X, $H^{-j}(X, \mathcal{E}.; \mathbf{Z}/p^n) = \pi_j(R\Gamma(X, \mathcal{E}.); \mathbf{Z}/p^n)$ is the generalized sheaf cohomology of $\mathcal{E}.$ [13, 1.7]: if we let $\mathcal{P}_X^j$ denote the constant sheaf of j-dimensional mod p^n Moore spaces, then $H^{-j}(X, \mathcal{E}.; \mathbf{Z}/p^n) = [\mathcal{P}_X^j, \mathcal{E}.]$, where, for two pointed simplicial sheaves $\mathcal{F}.$, $\mathcal{F}'.$ on X, $[\mathcal{F}., \mathcal{F}'.]$ denotes the morphisms from $\mathcal{F}.$ to $\mathcal{F}'.$ in the homotopy category.

The map f is defined as the composition

$$\begin{aligned} &H^{-j}(X, \mathcal{K}(2i, \tilde{s}_n(i)_X); \mathbf{Z}/p^n) \\ &= \pi_j(K(2i, \tilde{s}_n(i)(X)); \mathbf{Z}/p^n)) \xrightarrow{h_j} H_j(K(2i, \tilde{s}_n(i)(X)); \mathbf{Z}/p^n)) \\ &\to H_j(CK(2i, \tilde{s}_n(i)(X))) \xleftarrow{\sim} H_j(NK(2i, \tilde{s}_n(i)(X))) \\ &= H_j(\tilde{s}_n(i)(X)[2i]) = H^{2i-j}(X, s_n(i)) \end{aligned}$$

Here, h_j is the Hurewicz morphism, and, for a simplicial abelian group $\mathcal{A}$, $C\mathcal{A}$ and $N\mathcal{A}$ denote respectively the associated chain complex and the normalized chain complex.

Since, for $p \neq 2$, the Hurewicz morphism is compatible with addition and products [22, II.2.2], so is the map f.

The classes c_{ij}^{syn} are constructed in an entirely analogous manner.

Since the map $\Delta_{\mathcal{P}_X^j} : \mathcal{P}_X^j \to \mathcal{P}_X^j \wedge \mathcal{P}_X^j$ is nullhomotopic for odd p, properties 1 and 2 follow as in [13, 2.26] from the Whitney sum formula for the syntomic classes of representations of sheaves of groups, which in turn follows from the fact that the de Rham cohomology of $B.GL(n,m)/W(k)$ is nontrivial only in even degrees (hence the even-degree syntomic cohomology injects into the de Rham cohomology) and the Whitney sum formula is true in the de Rham cohomology.

Property 3 is easy to check. For the remaining properties, consider the following diagram

$$\begin{array}{ccc}
B.GL(\mathcal{O}_X)^+ \wedge B.GL(\mathcal{O}_X)^+ & \xrightarrow{\mu_0} & B.GL(\mathcal{O}_X)^+ \\
\Big\downarrow {\scriptstyle C^{\mathrm{syn}}_{\cdot} \wedge C^{\mathrm{syn}}_{\cdot}} & & \Big\downarrow {\scriptstyle C^{\mathrm{syn}}_{\cdot}} \\
\prod_{i \geq 1} \mathcal{K}(2i, \tilde{s}_n(i)_X) \wedge \prod_{i \geq 1} \mathcal{K}(2i, \tilde{s}_n(i)_X) & \xrightarrow{*} & \prod_{i \geq 1} \mathcal{K}(2i, \tilde{s}_n(i)_X).
\end{array}$$

Here, $C^{\mathrm{syn}}_{\cdot}$ is the total Chern class, μ_0 is the Loday multiplication, and $*$ is the Grothendieck multiplication [2, 0.3]. We claim that the above diagram commutes. This follows, as in [13, 2.32], from the tensor product formula for the syntomic classes of representations of sheaves of groups, which in turn follows, just like the Whitney sum formula above, from the fact that the de Rham cohomology of $B.(GL_n/W(k) \times GL_m/W(k))$ is nontrivial only in even degrees.

We have [20, 2.3.4]

$$(\sum_{i \geq 1} x_i) * (\sum_{i \geq 1} y_i) = \sum_{l \geq 0} P_l(x_1, \ldots, x_l, y_1, \ldots, y_l),$$

where $P_l(x_1, \ldots, x_l, y_1, \ldots, y_l) = (\sum_{r+s=l} a_{rs} x_r y_s + Z_r(x) T_s(y))$ and $a_{rs} = -(l-1)!/((r-1)!(s-1)!)$. Here, Z_r and T_s are polynomials of weight r and s respectively, and if $r+s=l$ then either Z_r or T_s is decomposable, i.e., can be represented as a sum of products of monomials of lower weight. To prove properties 4 and 5, it suffices to show that the terms $Z_r(x)T_s(y)$ disappear when evaluated on the corresponding K-theory classes.

For any $\eta \in K_j(X; \mathbf{Z}/p^n)$, $j \geq 2$, and any $a, b \geq 1$, the following diagram commutes

$$\begin{array}{ccc}
\mathcal{P}_X^j & \xrightarrow[\Delta_{\mathcal{P}_X^j}]{} & \mathcal{P}_X^j \wedge \mathcal{P}_X^j \\
\Big\downarrow {\scriptstyle \eta} & & \Big\downarrow {\scriptstyle \eta \wedge \eta} \\
B.GL(\mathcal{O}_X)^+ & \xrightarrow[\Delta]{} & B.GL(\mathcal{O}_X)^+ \wedge B.GL(\mathcal{O}_X)^+ \\
\Big\downarrow {\scriptstyle C_a^{\mathrm{syn}} \cdot C_b^{\mathrm{syn}}} & & \Big\downarrow {\scriptstyle C_a^{\mathrm{syn}} \wedge C_b^{\mathrm{syn}}} \\
\mathcal{K}(2(a+b), \tilde{s}_n(a+b)_X) & \xleftarrow{\mu_{a+b}} & \mathcal{K}(2a, \tilde{s}_n(a)_X) \wedge \mathcal{K}(2b, \tilde{s}_n(b)_X).
\end{array}$$

Thus, since $\Delta_{\mathcal{P}_X^j}$ is nullhomotopic, the decomposable terms Z_r or T_s disappear and we get Property 4.

Assume now that we know that the Chern classes in properties 6 and 7 vanish on the monomials $\gamma_k(x)$, $k \geq i+1$. Notice that it suffices to prove properties 5, 6, and 7 for the products of monomials $\gamma_l(x)$: the general case will follow from the Whitney sum formula and property 2. We will thus assume now that all the K-theory classes are products of monomials as above.

We will argue for property 5, the case of $\alpha' \in F_\gamma^k K_0(X)$, $k \neq 0$, and property 6 simultaneously, by induction on the number of monomials $\gamma_k(x)$ in the appearing K-theory classes. If we use $\mathcal{S}_X^0$ (the simplicial version of the 0-sphere) instead of $\mathcal{P}_X^j$ above, since (by induction from property 6) $C_a^{\mathrm{syn}}(\alpha) = 0$ for $a < j$ (and similarly for α'), we get the vanishing of the terms $Z_r T_s$ we wanted. Property 6 follows now by induction from the multiplication formula in property 5.

We will argue for property 5, the case of $\alpha' \in F_\gamma^k K_q(X; \mathbf{Z}/p^n)$, by the same type of induction. By induction from properties 6 and 7, all the terms $Z_r T_s$ disappear. Property 7 follows by induction from the multiplication formula in property 5 and from property 6.

It remains to show that $\overline{c}_{ij}^{\mathrm{syn}} = 0$ on elements of the form $\gamma_k(x)$, $k \geq i+1$, $x \in K_j(X; \mathbf{Z}/p^n)$ (and an analogous fact for the integral K-theory classes). We will reduce to a similar statement for the equivariant Chern classes of group representations.

Recall how the γ-operation γ_k acts on $K_j(X; \mathbf{Z}/p^n)$ [23, 4]. Write $K(GL_N)$ for the Grothendieck group of representations of the group scheme GL_N over $\mathbf{Z}$. There is a morphism of abelian groups

$$r_N : K(GL_N) \to [\mathbf{Z} \times B.GL_N(\mathcal{O}_X)^+, \mathbf{Z} \times B.GL(\mathcal{O}_X)^+]$$

induced by a transformation, which to every representation $\rho : GL_N \to GL_M$ of the group scheme GL_N associates a morphism of (pointed) simplicial sheaves $B.GL_N(\mathcal{O}_X) \to B.GL_M(\mathcal{O}_X)$. Passing first to the morphism induced on generalized cohomology groups, then to the limit over N, we get a morphism of abelian groups

$$r : K(GL) \to \mathrm{Hom}(K_j(X; \mathbf{Z}/p^n) \to K_j(X; \mathbf{Z}/p^n)).$$

Now, the operation γ_k is defined as the image under r of the element $(\gamma_k(\mathrm{Id}_N - N))_N \in K(GL)$, where Id_N, N are the classes of the natural, respectively trivial, representation of GL_N.

From the above description it is clear that it suffices to show that the composition

$$B.GL_N(\mathcal{O}_X) \xrightarrow{\gamma_k} B.GL(\mathcal{O}_X)^+ \xrightarrow{C_i^{\mathrm{syn}}} \mathcal{K}(2i, \tilde{s}_n(i)_X)$$

is homotopically trivial.

Set, after Gillet [13, 2.27],

$$\begin{aligned} H^{2i}(X, \mathbf{Z} \times B.GL_N(\mathcal{O}_X)^+, s_n(i)) &:= [\mathbf{Z} \times B.GL_N(\mathcal{O}_X)^+, \mathcal{K}(2i, \tilde{s}_n(i)_X)]; \\ \widetilde{H}^*(X, \mathbf{Z} \times B.GL_N(\mathcal{O}_X)^+, s_n(*)) &:= H^0(X, \mathbf{Z} \times B.GL_N(\mathcal{O}_X)^+, \mathbf{Z}) \times \{1\} \times \\ &\qquad \prod_{i \geq 1} H^{2i}(X, \mathbf{Z} \times B.GL_N(\mathcal{O}_X)^+, s_n(i)). \end{aligned}$$

$\widetilde{H}^*(X, \mathbf{Z} \times B.GL_N(\mathcal{O}_X)^+, s_n(*))$ is a λ-ring [13, 2.27]. Consider now the augmented total Chern class map

$$\widetilde{C} : K(GL_N) \to \widetilde{H}^*(X, \mathbf{Z} \times B.GL_N(\mathcal{O}_X)^+, s_n(*)); \qquad \rho \mapsto (rank(\rho), C.(\rho)).$$

We claim that this map is a λ-ring homomorphism. Indeed, it is clear that the map is well defined. Compatibility with addition and multiplication follows from the Whitney sum formula and the tensor product formula, which, as we have mentioned above, hold for the syntomic Chern classes.

For the compatibility with λ-operations, it suffices to pass to the correspoding $B.GL_M/W(k)$ and check that $\widetilde{C}(\bigwedge^l \mathcal{E}) = \bigwedge^l \widetilde{C}(\mathcal{E}) \in \widetilde{H}^*(B.GL_M/W(k), s_n(*))$, where $\mathcal{E}$ is the universal locally free sheaf on $B.GL_M/W(k)$. As before, it suffices to check this formula in the de Rham cohomology of $B.GL_M/W(k)$. This can be done easily using the splitting principle.

Returning to the main line of the argument, we have to show that the class of $c_{i,N}^{\mathrm{syn}}(\gamma_k(\mathrm{Id}_N - N)) \in H^{2i}(X, GL_N(\mathcal{O}_X), s_n(i))$ is zero. This now is standard. Set $x_N = \mathrm{Id}_N - N$. The above yields that the Chern polynomial $c_{t,N}^{\mathrm{syn}}$ is a λ-ring homomorphism. Hence we have

$$\begin{aligned} c_{t,N}^{\mathrm{syn}}(\gamma_k(x_N)) &= \gamma_k(c_{t,N}^{\mathrm{syn}}(x_N)) \\ &= \gamma_k(1 + c_{1,N}^{\mathrm{syn}}(x_N)t + c_{2,N}^{\mathrm{syn}}(x_N)t^2 + \ldots) \qquad [2, 0.1.2] \\ &= 1 + (-1)^{k-1}(k-1)! c_{k,N}^{\mathrm{syn}}(x_N)t^k + \ldots \end{aligned}$$

Thus $c_{i,N}^{\mathrm{syn}}(\gamma_k(x_N)) = 0$ for $0 < i < k$, as wanted.

Similarly, by recalling [2, 0.1.2] again, we get that the integral Chern class maps c_{i0}^{syn}, $i \geq 1$, are trivial on $\gamma_k(x)$, $k \geq i+1$, for any $x \in K_0(\cdot)$, $\varepsilon(x) = 0$.

Remark 1. – For any K-scheme Y, one can paraphrase the above proof to show that the étale Chern class maps

$$c_{ij}^{\text{ét}} : K_j(X) \to H^{2i-j}(Y, \mathbf{Z}/p^n(i)), \quad \overline{c}_{ij}^{\text{ét}} : K_j(X; \mathbf{Z}/p^n) \to H_{\mathrm{cr}}^{2i-j}(Y, \mathbf{Z}/p^n(i))$$

have analogous properties to those of the syntomic Chern classes.

Similarly, working with the complex $s_n'(r)$ instead of the complex $s_n(r)$, we get cohomology groups $H^i(X, s_n'(r))$ and $H^i(X_{\overline{V}}, s_n'(r))$, which are targets of higher Chern classes [15, 2.3]. All of the above listed properties of s_n-Chern classes carry over to the s_n'-theory.

For a V-scheme X, let X_0 denote the special fiber of X. Recall that, for X smooth and proper over V, Kato-Messing [18] have constructed the following isomorphisms

$$\begin{aligned} &H_{\mathrm{cr}}^i(X_0/W(k), \mathcal{O}_{X_0/W(k)}) \otimes \mathbf{Q} \otimes B_{\mathrm{cr}}^+ \\ &\quad \xrightarrow[h_{\mathrm{cr}}]{\sim} \mathbf{Q} \otimes \operatorname*{projlim}_n H_{\mathrm{cr}}^i(X_{\overline{V}}/W(k), \mathcal{O}_{X_{\overline{V}}/W(k)}/p^n) \qquad [18, 1.2], \\ &H_{dR}^i(X_K/K) \otimes B_{dR}^+ \\ &\quad \simeq \operatorname*{proj\,lim}_N(\mathbf{Q} \otimes \operatorname*{proj\,lim}_n H_{\mathrm{cr}}^i(X_{\overline{V}}/W(k), \mathcal{O}_{X_{\overline{V},n}/W_n(k)}/J_n^{[N]}) \qquad [18, 1.4], \\ &F^r(H_{dR}^i(X_K/K) \otimes B_{dR}^+) \xrightarrow[h_{dR}]{\sim} \operatorname{proj\,lim}_N(\mathbf{Q} \otimes \operatorname*{proj\,lim}_n H_{\mathrm{cr}}^i(X_{\overline{V}}/W(k), J_n^{[r]}/J_n^{[N]})). \end{aligned}$$

We will need to know that

LEMMA 2.2. – 1. *The following two compositions of maps are equal*

$$\begin{aligned}
&\mathbf{Q}\otimes \operatorname{proj}\lim_n H^i(X_{\overline{V}}, s'_n(r)) \to \operatorname{proj}\lim_N(\mathbf{Q}\otimes \operatorname{proj}\lim_n H^i_{\mathrm{cr}}(X_{\overline{V}}/W(k), J_n^{[r]}/J_n^{[N]}))\\
&\qquad \xleftarrow[\sim]{h_{dR}} F^r(H^i_{dR}(X_K/K)\otimes B^+_{dR}) \simeq F^0(H^i_{dR}(X_K/K)\{-r\}\otimes B^+_{dR})\\
&\qquad \to H^i_{dR}(X_K/K)\otimes B^+_{dR};\\
&\mathbf{Q}\otimes \operatorname{proj}\lim_n H^i(X_{\overline{V}}, s'_n(r)) \to \mathbf{Q}\otimes \operatorname{proj}\lim_n H^i_{\mathrm{cr}}(X_{\overline{V}}/W(k), \mathcal{O}_{X_{\overline{V}}/W(k)}/p^n)\\
&\qquad \xleftarrow[\sim]{h_{\mathrm{cr}}} H^i_{\mathrm{cr}}(X_0/W(k))\otimes_{W(k)}\mathbf{Q}\otimes B^+_{\mathrm{cr}} \xrightarrow{\gamma} H^i_{dR}(X_K/K)\otimes B^+_{dR},
\end{aligned}$$

where δ is induced by the Berthelot-Ogus isomorphism [3, 2.2] $H^i_{\mathrm{cr}}(X_0/W(k))\otimes_{W(k)} K \simeq H^i_{dR}(X_K/K)$.

2. *For any smooth and projective scheme X over V, the Kato-Messing isomorphism*

$$h_{\mathrm{cr}} : \mathbf{Q}\otimes \operatorname{proj}\lim_n H^i_{\mathrm{cr}}(X_{\overline{V}}/W(k), \mathcal{O}_{X_{\overline{V}}/W(k)}/p^n) \simeq H^i_{\mathrm{cr}}(X_0/W(k), \mathcal{O}_{X_0/W(k)})\otimes\mathbf{Q}\otimes B^+_{\mathrm{cr}}$$

is compatible on the image of the crystalline Chern classes

$$c^{\mathrm{cr}}_{i0} : K_0(X) \to \mathbf{Q}\otimes \operatorname{proj}\lim_n H^{2i}_{\mathrm{cr}}(X/W(k), \mathcal{O}_{X/W(k)}/p^n)$$

with the natural map

$$\mathbf{Q}\otimes \operatorname{proj}\lim_n H^i_{\mathrm{cr}}(X/W(k), \mathcal{O}_{X/W(k)}/p^n) \to \mathbf{Q}\otimes H^i_{\mathrm{cr}}(X_0/W(k), \mathcal{O}_{X_0/W(k)}).$$

Proof. – Let $\widetilde{V} = \overline{V}\otimes_{W(k)} V$, $\widetilde{X} = X\otimes_{W(k)}\widetilde{V}$, $\widetilde{B}^+_{\mathrm{cr}} = \operatorname{proj}\lim_n H^0_{\mathrm{cr}}(\widetilde{V}_n/V_n)$. By the syntomic base change theorem [1, 2.3.5] $H^*_{\mathrm{cr}}(\widetilde{V}_n/V_n) \simeq H^*_{\mathrm{cr}}(\overline{V}_n/W_n(k))\otimes_{W(k)} V$. Hence $H^i_{\mathrm{cr}}(\widetilde{V}_n/V_n) = 0$, for $i > 0$, and $\widetilde{B}^+_{\mathrm{cr}} = B^+_{\mathrm{cr}}\otimes_{W(k)} V$. In particular, $\widetilde{B}^+_{\mathrm{cr}}$ is flat over V. Using that and unwinding the definition of the Kato-Messing isomorphism $\hbar_{\mathrm{cr}}$, we easily check that the following compositions of maps are equal

$$\begin{aligned}
&\mathbf{Q}\otimes \operatorname{proj}\lim_n H^i_{\mathrm{cr}}(X_{\overline{V},n}/W_n(k)) \xleftarrow[\sim]{h_{\mathrm{cr}}} H^i_{\mathrm{cr}}(X_0/W(k))\otimes_{W(k)}\mathbf{Q}\otimes B^+_{\mathrm{cr}}\\
&\qquad \to H^i_{\mathrm{cr}}(X_0/W(k))\otimes_{W(k)}\mathbf{Q}\otimes B^+_{\mathrm{cr}}\otimes_{W(k)} V\\
&\qquad \xrightarrow[\lambda_2]{\sim} H^i_{\mathrm{cr}}(X_{0,V}/V)\otimes_V \mathbf{Q}\otimes\widetilde{B}^+_{\mathrm{cr}};\\
&\mathbf{Q}\otimes \operatorname{proj}\lim_n H^i_{\mathrm{cr}}(X_{\overline{V},n}/W_n(k)) \to \mathbf{Q}\otimes \operatorname{proj}\lim_n H^i_{\mathrm{cr}}(X_{\widetilde{V},n}/W_n(k))\\
&\qquad \xleftarrow[\lambda_1]{\sim} \mathbf{Q}\otimes H^i_{\mathrm{cr}}(X_1/V)\otimes_V \operatorname{proj}\lim_n H^0_{\mathrm{cr}}(\widetilde{V}_n/V_n)\\
&\qquad \xrightarrow[\delta]{\sim} H^i_{\mathrm{cr}}(X_{0,V}/V)\otimes_V \mathbf{Q}\otimes\widetilde{B}^+_{\mathrm{cr}}
\end{aligned}$$

Here, $X_{0,V} = X_0\otimes_{W(k)} V$, the map λ_2 is the base change morphism, and the map λ_1 is the Künneth morphism. This yields Property (1) of the lemma. Concerning Property (2), since the morphism

$$\lambda_0 : H^i_{\mathrm{cr}}(X_0/W(k))\otimes\mathbf{Q}\otimes B^+_{\mathrm{cr}} \to H^i_{\mathrm{cr}}(X_0/W(k))\otimes_{W(k)}\mathbf{Q}\otimes B^+_{\mathrm{cr}}\otimes_{W(k)} V$$

is injective, it suffices to show that the composition

$$H^i_{\mathrm{cr}}(X_1/W(k)) \to H^i_{\mathrm{cr}}(X_0/W(k)) \to H^i_{\mathrm{cr}}(X_{0,V}/V) \underset{\sim}{\to} H^i_{\mathrm{cr}}(X_1/V)$$

is the natural map $H^i_{\mathrm{cr}}(X_1/W(k)) \to H^i_{\mathrm{cr}}(X_1/V)$ on the level of Chern classes, and that the composition

$$H^i_{\mathrm{cr}}(X/V) \to \text{proj lim}_n H^i_{\mathrm{cr}}(X_{\widetilde{V},n}/V_n) \underset{\sim}{\xrightarrow{\lambda_1}} H^i_{\mathrm{cr}}(X_1/V) \otimes_V \widetilde{B}^+_{\mathrm{cr}}$$

is compatible with the natural map $H^i_{\mathrm{cr}}(X/V) \to H^i_{\mathrm{cr}}(X_1/V)$. The latter assertion is easy to show, and the former one follows from [3, 3.7].

3. K-theory lemma

Let V be a complete discrete valuation ring with a field of fractions K. Let $\overline{V}$ denote the integral closure of V in $\overline{K}$.

LEMMA 3.1. – *Let X be a smooth V-scheme. For any integer n, the natural morphism $j^* : K_i(X \otimes_V \overline{V}; \mathbf{Z}/n) \to K_i(X \otimes_V \overline{K}; \mathbf{Z}/n)$ is an isomorphism.*

Proof. – We may assume the residue field of V to be separably closed. Let $L/E/K$ be finite field extensions with rings of integers V_L and V_E, respectively. Let e be the ramification index of L/E.

The map $f : X_{V_L} \to X_{V_E}$ being flat, the localization theorem in K'-theory yields the following commutative diagram of maps of exact sequences

$$\begin{array}{ccccccccc} K'_{i+1}(X_L;\mathbf{Z}/n) & \longrightarrow & K'_i(X_{\overline{k}} \otimes_{V_E} V_L;\mathbf{Z}/n) & \longrightarrow & K'_i(X_{V_L};\mathbf{Z}/n) & \longrightarrow & K'_i(X_L;\mathbf{Z}/n) & \longrightarrow \\ {\scriptstyle f^*_{X_E}}\Big\uparrow & & {\scriptstyle f^*_{X_{\overline{k}}}=ei_*}\Big\uparrow & & {\scriptstyle f^*_{X_{V_E}}}\Big\uparrow & & {\scriptstyle f^*_{X_E}}\Big\uparrow & \\ K'_{i+1}(X_E;\mathbf{Z}/n) & \longrightarrow & K'_i(X_{\overline{k}};\mathbf{Z}/n) & \longrightarrow & K'_i(X_{V_E};\mathbf{Z}/n) & \longrightarrow & K'_i(X_E;\mathbf{Z}/n) & \longrightarrow \end{array}$$

The scheme $X_{\overline{k}} \otimes_{V_E} V_L$ is non-reduced, with underlying reduced scheme $X_{\overline{k}}$, and we write i for the inclusion $X_{\overline{k}} \hookrightarrow X_{\overline{k}} \otimes_{V_E} V_L$. We claim that $f^*_{X_{\overline{k}}} = ei_*$. Indeed, let $\mathcal{M}(Z)$ denote the exact category of coherent sheaves on a scheme Z. Then $f^*_{X_{\overline{k}}} = f^K_*$ for the functor $f^K : \mathcal{M}(X_{\overline{k}}) \to \mathcal{M}(X_{\overline{k}} \otimes_{V_E} V_L)$, $f^K(\mathcal{F}) = \mathcal{F} \otimes_{V_E} V_L$. Since $\overline{k} \otimes_{V_E} V_L \simeq \overline{k}[x]/x^e$, filtering f^K, we compute that $f^K_* = \sum_{l=1}^e f^l_*$, where every functor $f^l : \mathcal{M}(X_{\overline{k}}) \to \mathcal{M}(X_{\overline{k}} \otimes_{V_E} V_L)$ is naturally equivalent to i_*.

From that, we easily see that there is a natural isomorphism

$$\underset{L/K}{\text{inj lim}}\, K'_i(X_{V_L};\mathbf{Z}/n) \underset{\sim}{\xrightarrow{j^*_X}} \text{inj lim}_{L/K} K'_i(X_L;\mathbf{Z}/n),$$

where the limit is over finite field extensions of K. Since the original scheme X was smooth, all the schemes X_{V_L} are regular, and we can pass to K-theory in the above isomorphism getting the statement of our lemma.

4. Integral theory

Let the notation be that of section 2. Let X be a smooth quasi-projective V-scheme of relative dimension less than $p-2$, $V = W(k)$. We would like now to construct functorial Galois equivariant morphisms

$$\alpha_{ab} : H^a(X_{\overline{K}}, \mathbf{Z}/p^n(b)) \to \mathbf{L}(H^a_{\mathrm{cr}}(X/V, \mathcal{O}_X/p^n)\{-b\}).$$

We will be able to do it (4) only under certain additional restrictions on the integers a, b and d.

Our construction will be based on the following diagram

$$\text{(2)}\qquad \begin{array}{ccc} F^i_\gamma/F^{i+1}_\gamma K_j(X_{\overline{V}}; \mathbf{Z}/p^n) & \xrightarrow[j^*]{\sim} & F^i_\gamma/F^{i+1}_\gamma K_j(X_{\overline{K}}; \mathbf{Z}/p^n) \\ \Big\downarrow{\scriptstyle \overline{c}^{\mathrm{syn}}_{ij}} & & \Big\downarrow{\scriptstyle \overline{c}^{\text{ét}}_{ij}} \\ H^{2i-j}(X_{\overline{V}}, s_n(i)) & & H^{2i-j}(X_{\overline{K}}, \mathbf{Z}/p^n(i)). \end{array}$$

Here $1 \le i < p-1$, $j \ge 3$, $p^n \ge 5$, $p \ne 2$. The Chern class map $\overline{c}^{\mathrm{syn}}_{ij} : F^i_\gamma K_j(X_{\overline{V}}; \mathbf{Z}/p^n) \to H^{2i-j}(X_{\overline{V}}, s_n(i))$ is defined as a limit over finite extensions V'/V of the syntomic Chern class maps $F^i_\gamma K_j(X_{V'}; \mathbf{Z}/p^n) \to H^{2i-j}(X_{V'}, s_n(i))$. Due to Lemma 2.1, the Chern class maps $\overline{c}^{\text{ét}}_{ij}$ and $\overline{c}^{\mathrm{syn}}_{ij}$ factor through F^{i+1}_γ yielding the maps in the above diagram.

We will need to know how Bott elements in K-theory behave under higher Chern class maps. Recall that to every sequence of nontrivial p-roots of unity $\zeta_n \in \overline{\mathbf{Q}}_p$, $\zeta_n^{p^n} = 1$, $\zeta^p_{n+1} = \zeta_n$, one can associate a compatible sequence of Bott classes $\beta_n \in K_2(\overline{\mathbf{Q}}_p; \mathbf{Z}/p^n)$, $n \ge 1$. Namely, β_n is the unique element, which maps to ζ_n in the following exact sequence

$$0 \to K_2(\overline{\mathbf{Q}}_p; \mathbf{Z}/p^n) \xrightarrow{\partial} K_1(\overline{\mathbf{Q}}_p) \xrightarrow{p^n} K_1(\overline{\mathbf{Q}}_p) \to 0.$$

We will denote by $\tilde{\beta}_n$ the unique element in $K_2(\overline{\mathbf{Z}}_p; \mathbf{Z}/p^n)$ mapping to $\beta_n \in K_2(\overline{\mathbf{Q}}_p; \mathbf{Z}/p^n)$ (such an element exists and is unique by Lemma 3.1). We have a commutative diagram

$$\begin{array}{ccccc} K_2(\overline{\mathbf{Q}}_p; \mathbf{Z}/p^n) & \xrightarrow{\partial} & K_1(\overline{\mathbf{Q}}_p) & \xrightarrow{\sim} & \overline{\mathbf{Q}}_p^* \\ \Big\uparrow & & \Big\uparrow & & \Big\uparrow \\ K_2(\overline{\mathbf{Z}}_p; \mathbf{Z}/p^n) & \xrightarrow{\partial} & K_1(\overline{\mathbf{Z}}_p) & \xrightarrow{\sim} & \overline{\mathbf{Z}}_p^* \end{array}$$

where all the vertical maps are injections. Hence $\partial(\tilde{\beta}_n) = \zeta_n$ as well.

LEMMA 4.1. – *We have*

$$\overline{c}^{\mathrm{syn}}_{1,2}(\tilde{\beta}_n) = t \in H^0(\overline{\mathbf{Z}}_p, s_n(1)), \quad \overline{c}^{\mathrm{syn}}_{1,2}(\tilde{\beta}_n) = t \in H^0(\overline{\mathbf{Z}}_p, s'_n(1)),$$
$$\overline{c}^{\text{ét}}_{1,2}(\beta_n) = \zeta_n \in H^0(\overline{\mathbf{Q}}_p, \mathbf{Z}/p^n(1)),$$

where $t \in B^+_{\mathrm{cr}}(\mathbf{Z}_p)$ is the element associated to ζ_n's (cf., section 2.1).

Proof. – By Soulé [22, IV.1.3],

$$\overline{c}^{\text{ét}}_{1,2} : K_2(\overline{\mathbf{Q}}_p; \mathbf{Z}/p^n) \to H^0(\overline{\mathbf{Q}}_p, \mathbf{Z}/p^n(1))$$

is the Bockstein map followed by the determinant map:

$$\overline{c}^{\text{ét}}_{1,2} : K_2(\overline{\mathbf{Q}}_p; \mathbf{Z}/p^n) \xrightarrow{\partial} {}_{p^n}K_1(\overline{\mathbf{Q}}_p) \xrightarrow{\det} \mu_{p^n}(\overline{\mathbf{Q}}_p) = H^0(\overline{\mathbf{Q}}_p, \mathbf{Z}/p^n(1)).$$

Hence $\overline{c}^{\text{ét}}_{1,2}(\beta_n) = \zeta_n$.

For the syntomic class, notice that we have an injection $H^0(\overline{\mathbf{Z}}_p, s_n(1)) \hookrightarrow H^0_{\text{cr}}((\overline{\mathbf{Z}}_p/p^n)/W_n(k), J_n)$. Since the syntomic classes and the crystalline classes are compatible, it suffices to show that $\overline{c}^{\text{cr}}_{1,2}(\tilde{\beta}_n) = t \in H^0_{\text{cr}}((\overline{\mathbf{Z}}_p/p^n)/W_n(k), J_n) \simeq F^1B^+_{\text{cr}}/p^n$. Consider the Kummer exact sequence

$$0 \to \mathbf{Z}/p^n(1) \to \mathbf{G}_m \xrightarrow{p^n} \mathbf{G}_m \to 0$$

on the Zariski site of $\text{Spec}(\overline{\mathbf{Z}}_\text{p})$. Define classes of the natural representations $c_1(\text{Id}_k) \in H^2_{\text{Zar}}(\overline{\mathbf{Z}}_p, GL_k(\overline{\mathbf{Z}}_p), \mathbf{Z}/p^n(1))$ as the image via the boundary map

$$H^1_{\text{Zar}}(\overline{\mathbf{Z}}_p, GL_k(\overline{\mathbf{Z}}_p), \mathbf{G}_m) \xrightarrow{\partial} H^2_{\text{Zar}}(\overline{\mathbf{Z}}_p, GL_k(\overline{\mathbf{Z}}_p), \mathbf{Z}/p^n(1))$$

of the class of the $GL_k(\overline{\mathbf{Z}}_p)$-invertible sheaf associated to $\det(\text{Id}_k)$. Paraphrasing the above mentioned argument of Soulé one can show that, if we induce a map

$$c_{1,2} : K_2(\overline{\mathbf{Z}}_p; \mathbf{Z}/p^n) \to H^0_{\text{Zar}}(\overline{\mathbf{Z}}_p, \mathbf{Z}/p^n(1))$$

via mod p^n Hurewicz morphism and via the Künneth homomorphism

$$\omega : H^2_{\text{Zar}}(\overline{\mathbf{Z}}_p, GL(\overline{\mathbf{Z}}_p), \mathbf{Z}/p^n(1)) \to \bigoplus_{i=0}^{2} \text{Hom}(H_{2-i}(GL(\overline{\mathbf{Z}}_p); \mathbf{Z}/p^n), H^i_{\text{Zar}}(\overline{\mathbf{Z}}_p, \mathbf{Z}/p^n(1)))$$

from the classes $c_1(\text{Id}_k)$, then $c_{1,2}(\tilde{\beta}_n) = \zeta_n$.

Consider now the following diagram

$$\begin{tikzcd} & K_2(\overline{\mathbf{Z}}_p; \mathbf{Z}/p^n) \arrow[dl, "\overline{c}^{\text{cr}}_{1,2}"'] \arrow[dr, "c_{1,2}"] & \\ H^0_{\text{Zar}}(\overline{\mathbf{Z}}_p, u_*J_n) & & H^0_{\text{Zar}}(\overline{\mathbf{Z}}_p, \mathbf{Z}/p^n(1)), \arrow[ll, "\alpha"'] \end{tikzcd}$$

where $u : (\text{Spec}(\overline{\mathbf{Z}}_\text{p}/\text{p}^\text{n})/\text{W}_\text{n}(\text{k}))_{\text{cr}} \to \text{Spec}(\overline{\mathbf{Z}}_\text{p})_{\text{Zar}}$ is the projection from the crystalline to the Zariski topos and the map α is defined by sending ζ_n to t on the open $\text{Spec}(\overline{\mathbf{Z}}_\text{p})$ and ζ_n to 0 on the open $\overline{\mathbf{Q}}_p$. It suffices now to prove that this diagram commutes. Since both maps $c_{1,2}$ and $\overline{c}^{\text{cr}}_{1,2}$ are defined via the same procedure from characteristic classes of representations (to see this for the crystalline classes, adapt the argument of Shekhtman [20, 2.2] to K-theory with coefficients), it suffices to prove that the map

$$\alpha : H^2_{\text{Zar}}(\overline{\mathbf{Z}}_p, GL_k(\overline{\mathbf{Z}}_p), \mathbf{Z}/p^n(1)) \to H^2_{\text{Zar}}(\overline{\mathbf{Z}}_p, GL_k(\overline{\mathbf{Z}}_p), u_*J_n)$$

maps $c_1(\text{Id}_k)$ to $c^{\text{cr}}_1(\text{Id}_k)$. But the last classes come via Bockstein maps

$$\mathbf{G}_m \xrightarrow{\partial} \mathbf{Z}/p^n(1)[-1], \qquad \mathbf{G}_m \to \mathbf{G}_{m,\overline{\mathbf{Z}}/p^n} \xrightarrow{\log \partial^{\text{cr}}} u_*J_n[-1]$$

from the class of the $GL_k(\overline{\mathbf{Z}}_p)$-invertible sheaf associated to $\det(\mathrm{Id}_k)$. Here, ∂, ∂^{cr} are the boundary maps coming from the Kummer sequence and from the exact sequence

$$\begin{array}{ccccccccc} 0 & \to & (1+u_*J_n)(\overline{\mathbf{Z}}_p) & \to & ((u_*\mathcal{O}_n)(\overline{\mathbf{Z}}_p))^* & \to & (\overline{\mathbf{Z}}/p^n)^* & \to & 0 \\ & & \| & & \| & & \| & & \\ 0 & \to & 1+F^1B^+_{\mathrm{cr}}/p^n & \to & (B^+_{\mathrm{cr}}/p^n)^* & \to & (\overline{\mathbf{Z}}/p^n)^* & \to & 0, \end{array}$$

respectively. Hence, it suffices to show that the following diagram commutes (in the derived category)

$$\begin{array}{ccc} \mathbf{G}_m & \xrightarrow{\partial} & \mathbf{Z}/p^n(1)[-1] \\ \downarrow & & \downarrow \alpha \\ \mathbf{G}_{m,\overline{\mathbf{Z}}/p^n} & \xrightarrow{\log\partial^{\mathrm{cr}}} & u_*J_n[-1]. \end{array}$$

It is clear on the open $\mathrm{Spec}(\overline{\mathbf{Q}}_{\mathrm{p}})$. For the commutativity on the open $\mathrm{Spec}(\overline{\mathbf{Z}}_{\mathrm{p}})$, consider the following commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \to & 1+F^1B^+_{\mathrm{cr}}/p^n & \to & (B^+_{\mathrm{cr}}/p^n)^* & \to & (\overline{\mathbf{Z}}/p^n)^* & \to & 0 \\ & & \uparrow & & \gamma\uparrow & & \uparrow & & \\ 0 & \to & \text{proj lim}\ \mu_{p^n}(\overline{\mathbf{Z}}_p^*) & \to & \text{proj lim}\ \overline{\mathbf{Z}}_p^* & \xrightarrow{p} & \overline{\mathbf{Z}}_p^* & \to & 0 \end{array}$$

where the maps in the projective limit are the p-th power maps, and the map γ sends a sequence (a_i) to $[(a_i), 0, \dots]$. This diagram yields the following commutative diagram

$$\begin{array}{ccccc} (\overline{\mathbf{Z}}/p^n)^* & \xrightarrow{\partial^{\mathrm{cr}}} & (1+F^1B^+_{\mathrm{cr}}/p^n)[-1] & \xrightarrow{\log} & F^1B^+_{\mathrm{cr}}/p^n[-1] \\ v\uparrow & & \uparrow & & \| \\ \overline{\mathbf{Z}}_p^* & \xrightarrow{\partial} & \text{proj lim}\mu_{p^n}(\overline{\mathbf{Z}}_p^*)[-1] & & F^1B^+_{\mathrm{cr}}/p^n[-1] \\ \| & & \downarrow & & \| \\ \overline{\mathbf{Z}}_p^* & \xrightarrow{\partial} & \mu_{p^n}(\overline{\mathbf{Z}}_p^*)[-1] & \xrightarrow{\alpha} & F^1B^+_{\mathrm{cr}}/p^n[-1], \end{array}$$

which completes our argument.

For the s'-class, since the syntomic classes and the crystalline classes are compatible, it suffices to show that $\overline{c}^{\mathrm{cr}}_{1,2}(\tilde{\beta}_n) = t \in H^0_{\mathrm{cr}}((\overline{\mathbf{Z}}_p/p^n)/W_n(k), J_n) \simeq F^1B^+_{\mathrm{cr}}/p^n$. This was done above.

In what follows, we will fix a generator $\zeta = (\zeta_n)$ of $\mathbf{Z}_p(1)$ and denote by β_n, $\tilde{\beta}_n$, and t the elements corresponding to ζ. For a scheme Y over $\overline{\mathbf{Z}}_p$, let $\tilde{\beta}_n \in K_2(Y; \mathbf{Z}/p^n)$ denote the pullback of the Bott element $\tilde{\beta}_n \in K_2(\overline{\mathbf{Z}}_p; \mathbf{Z}/p^n)$. We will denote by β_n the image of $\tilde{\beta}_n$ in $K_2(Y_{\overline{\mathbf{Q}}_p}; \mathbf{Z}/p^n)$. For a positive integer d, let $N(d)$ denote the constant defined by Thomason in [24]: $N(d) = (2/3)d(d+1)(d+2)$.

PROPOSITION 4.1. – *Let Y be a smooth quasi-projective scheme over $\overline{K}$. Let $p \neq 2$ and n be such that $p^n \geq 5$. Let i be any positive integer. There exists a constant $T = T(d, i, j)$ depending only on the dimension d of Y and i, j such that, for $j \geq 2N(d)$,*

1. *the cokernel of $\overline{c}_{ij}^{\text{ét}} : F_{\gamma}^{i}/F_{\gamma}^{i+1}K_j(Y;\mathbf{Z}/p^n) \to H^{2i-j}(Y,\mathbf{Z}/p^n(i))$ is annihilated by T;*
2. *if $[x]$ is in the kernel of $\overline{c}_{ij}^{\text{ét}}$, then $T[x] = [z]$ for $z \in F_{\gamma}^{i}K_j(Y;\mathbf{Z}/p^n)$ annihilated by $\beta_n^{N(d)}$.*

Moreover, for $p \geq \max\{i+1, (d+j+3)/2\}$, we can take $T = 1$.

Proof. – The proposition follows from Thomason's comparison theorem [24] between algebraic K-theory and étale K-theory, and from the degeneration of the Dwyer-Friedlander spectral sequence modulo torsion [21]. We get a uniform constant T as in the theorem through a careful bookkeeping of the appearing torsion - most of the work being done by Soulé in [21].

Soulé [21, 4.1.4] has shown that the algebraic Chern class $c_{ij}^{\text{ét}}$ is equal to the composition of the topological Chern class

$$c_{ij} : K_j^{\text{ét}}(Y;\mathbf{Z}/p^n) \to H^{2i-j}(Y,\mathbf{Z}/p^n(i)), \quad i \geq 1,\ j \geq 2,$$

where $K_j^{\text{ét}}(Y;\mathbf{Z}/p^n)$ is the étale K-theory of Dwyer and Friedlander [4], [12], with the natural map

$$\rho_j : K_j(Y;\mathbf{Z}/p^n) \to K_j^{\text{ét}}(Y;\mathbf{Z}/p^n)$$

from algebraic to étale K-theory.

Recall that we have a Dwyer-Friedlander spectral sequence

$$E_2^{s,q} = \begin{cases} H^s(Y,\mathbf{Z}/p^n(i)) & \text{if } 0 \leq s \leq q = 2i, \\ 0 & \text{otherwise} \end{cases}$$

converging to $K_{q-s}^{\text{ét}}(Y;\mathbf{Z}/p^n)$, $q - s \geq 3$. Let $F^sK_j^{\text{ét}}(Y;\mathbf{Z}/p^n)$ denote the filtration on $K_j^{\text{ét}}(Y;\mathbf{Z}/p^n)$ defined by this spectral sequence. Consider also the following γ-filtration $\widetilde{F}_{\gamma}^{i} = \langle \gamma^k(x) | k \geq i \rangle$. Since the λ-ring multiplication on $K_j^{\text{ét}}(Y;\mathbf{Z}/p^n)$ is trivial, this version of γ-filtration satisfies many of the formal properties of the filtration F_{γ}^{i}. In particular, we can prove, as in Soulé [21, 3.4], that

$$M(d,i,j)F^{2i-j}K_j^{\text{ét}}(Y;\mathbf{Z}/p^n) \subset \widetilde{F}_{\gamma}^{i}K_j^{\text{ét}}(Y;\mathbf{Z}/p^n) \subset F^{2i-j}K_j^{\text{ét}}(Y;\mathbf{Z}/p^n), \tag{3}$$

where the constant $M(d,i,j)$ is to be defined below.

Now, we know [21, 4.2] that c_{ij} restricts to zero on $F^{2i-j+1}K_j^{\text{ét}}(Y;\mathbf{Z}/p^n)$. Hence, it induces a map

$$\overline{c}_{ij} : \widetilde{F}_{\gamma}^{i}/\widetilde{F}_{\gamma}^{i+1}K_j^{\text{ét}}(Y;\mathbf{Z}/p^n) \xrightarrow{f} F^{2i-j}/F^{2i-j+2}K_j^{\text{ét}}(Y;\mathbf{Z}/p^n) \xrightarrow{g} H^{2i-j}(Y,\mathbf{Z}/p^n(i)).$$

We will now study the maps f, g. Let's first recall some constants from [21, 3.4]. Let l be a positive integer, and let w_l be the greatest common divisor of the set of integers $k^N(k^l - 1)$, as k runs over the positive integers and N is large enough with respect to l. Let $M(k)$ be the product of the w_l's for $2l < k$. We will also need the integer $M(k,m,n) = \prod_{2m \leq 2l \leq 2m-n+k+1} M(2l)$.

By the inclusions (3), the map f has kernel and cokernel annihilated by $M(d,i+1,j)$, respectively $M(d,i,j)$.

Concerning the map g, notice that, by Soulé [21, 4.2], the image of c_{ij} in $H^{2i-j}(Y,\mathbf{Z}/p^n(i)) = E_2^{2i-j,2i}$ lies in the kernel $K^{2i-j,2i}$ of all higher differentials d_r, $r \geq 2$, in the Dwyer-Friedlander spectral sequence. Hence, we have a factorization

$$g : F^{2i-j}/F^{2i-j+2}K_j^{\text{ét}}(Y;\mathbf{Z}/p^n) \to K^{2i-j,2i} \hookrightarrow H^{2i-j}(Y,\mathbf{Z}/p^n(i)).$$

Since [21, 3.3.2] $M(d)d_r = 0$ for any $r \geq 2$, the cokernel of the inclusion $K^{2i-j,2i} \hookrightarrow H^{2i-j}(Y,\mathbf{Z}/p^n(i))$ is annihilated by $M(d)$. Consider now the composition

$$\begin{aligned} E_\infty^{2i-j,2i} =& F^{2i-j}/F^{2i-j+2}K_j^{\text{ét}}(Y;\mathbf{Z}/p^n) \to K^{2i-j,2i} \\ &\xrightarrow{\phi_{ij}} F^{2i-j}/F^{2i-j+2}K_j^{\text{ét}}(Y;\mathbf{Z}/p^n) = E_\infty^{2i-j,2i}, \end{aligned}$$

where ϕ_{ij} is the natural projection. This composition is proved in [21, 4.2] to be equal to multiplication by $(-1)^{i-1}(i-1)!$. Hence the kernel of g is annihilated by $(i-1)!$. Also, since $M(d)d_r = 0$ for any $r \geq 2$, the kernel and cokernel of ϕ_{ij} are annihilated by $M(d)$. Hence the cokernel of g is annihilated by $M(d)^2(i-1)!$.

The above yields that the kernel of $\overline{c}_{ij}$ is annihilated by $M(d,i+1,j)(i-1)!$ and its cokernel by $M(d,i,j)M(d)^2(i-1)!$.

Consider now the map

$$\tilde{\rho}_j : \widetilde{F}_\gamma^i/\widetilde{F}_\gamma^{i+1}K_j(Y;\mathbf{Z}/p^n) \xrightarrow{\tilde{\rho}_j} \widetilde{F}_\gamma^i/\widetilde{F}_\gamma^{i+1}K_j^{\text{ét}}(Y;\mathbf{Z}/p^n).$$

We know from Thomason [24], [25, 3.2], that $\tilde{\rho}_j$ is surjective for $j \geq 2N(d)$. For its kernel, let $x \in \widetilde{F}_\gamma^i K_j(Y;\mathbf{Z}/p^n)$ be such that $\rho_j(x) \in \widetilde{F}_\gamma^{i+1}K_j^{\text{ét}}(Y;\mathbf{Z}/p^n)$. If $j \geq 2N(d)$, there exists a $y \in \widetilde{F}_\gamma^{i+1}K_j(Y;\mathbf{Z}/p^n)$ such that $\rho_j(x) = \rho_j(y)$, or $\rho_j(x-y) = 0$. Hence, if we set $z = x - y$, then $y = x - z \in \widetilde{F}_\gamma^{i+1}K_j(Y;\mathbf{Z}/p^n)$ and z is annihilated by $\beta_n^{N(d)}$ (again by Thomason).

Combining all of the above, we get that, for $j \geq 2N(d)$, the cokernel of $\overline{c}_{ij}^{\text{ét}}$ is annihilated by $M(d,i,j)M(d)^2(i-1)!$, and its kernel, restricted to $\widetilde{F}_\gamma^i K_j(Y;\mathbf{Z}/p^n)/\widetilde{F}_\gamma^{i+1}K_j(Y;\mathbf{Z}/p^n)$, is, modulo $M(d,i+1,j)(i-1)!$, annihilated by $\beta_n^{N(d)}$. Since $M(2i)F_\gamma^i K_j(Y;\mathbf{Z}/p^n) \subset \widetilde{F}_\gamma^i K_j(Y;\mathbf{Z}/p^n) + F_\gamma^{i+1}K_j(Y;\mathbf{Z}/p^n)$ (argue as in [21, 3.4]), we can take $T = T(d,i,j) = (i-1)!M(2i)M(d,i,j)M(d,i+1,j)M(d)^2$.

Since an odd prime p divides $M(d,i,j)$ if and only if $p < (j+d+3)/2$, and divides $M(l)$ if and only if $p < (l/2)+1$, we get the last statement of the proposition. □

Remark 2. – The results of Soulé needed in the above proof are stated in [21] for the p-adic topological K-theory. Working with Friedlander's definition of étale K-theory mod p^n [12], i.e., using systematically $P^2(p^n) \wedge X_{\text{ét}}$, where $P^2(p^n)$ is the simplicial Moore space obtained from the mapping cone of multiplication by p^n on the circle and where, for a scheme X, $X_{\text{ét}}$ is its étale topological type, instead of $X_{\text{ét}}$, one sees that Soulé's results hold as well for the p-torsion K-theory (with the usual restriction on the degree of the K-groups) and his proofs carry over to that case almost verbatim.

Let now p, a, b, d satisfy the following condition

(4) $\quad b \geq d,\ p-2 \geq \max\{b+N(d), (1/2)(d+2b-a-1)\},\ 2b-a \geq 2N(d).$

Define the morphisms

$$\alpha_{ab} : H^a(X_{\overline{K}}, \mathbf{Z}/p^n(b)) \to \mathbf{L}(H^a_{\mathrm{cr}}(X/V, \mathcal{O}_{X/V}/p^n)\{-b\})$$

as the composition $\alpha_{ab} := \psi_n \overline{c}^{\mathrm{syn}}_{b,2b-a}(j^*)^{-1}(\overline{c}^{\mathrm{\acute{e}t}}_{b,2b-a})^{-1}$, where ψ_n is the natural map $H^a(X_{\overline{V}}, s_n(b)) \to \mathbf{L}(H^a_{\mathrm{cr}}(X/V, \mathcal{O}_{X/V}/p^n)\{-b\})$. Here $(\overline{c}^{\mathrm{\acute{e}t}}_{b,2b-a})^{-1}(x)$ is defined by taking any element in the preimage of x. We do not know whether this is well-defined, but we do know that its composition with $\psi_n \overline{c}^{\mathrm{syn}}_{b,2b-a}(j^*)^{-1}$ is. Indeed, by the above proposition, an ambiguity in the definition of $(\overline{c}^{\mathrm{\acute{e}t}}_{b,2b-a})^{-1}$ may only come from a class of $z \in K_{2b-a}(X_{\overline{K}}; \mathbf{Z}/p^n)$ such that $\beta_n^{N(d)} z = 0$. So $\tilde{\beta}_n^{N(d)}(j^*)^{-1}(z) = 0$. Hence

$$\begin{aligned} 0 &= \psi_n[\overline{c}^{\mathrm{syn}}_{b+N(d),2b+2N(d)-a}(\tilde{\beta}_n^{N(d)}(j^*)^{-1}(z))] \\ &= (-1)^{N(d)}(N(d)+b-1)!/(b-1)!\psi_n[\overline{c}^{\mathrm{syn}}_{1,2}(\tilde{\beta}_n^{N(d)})\overline{c}^{\mathrm{syn}}_{b,2b-a}((j^*)^{-1}(z))] \\ &= (-1)^{N(d)}(N(d)+b-1)!/(b-1)!t^{N(d)}\psi_n \overline{c}^{\mathrm{syn}}_{b,2b-a}((j^*)^{-1}(z)). \end{aligned}$$

To conclude that $\psi_n \overline{c}^{\mathrm{syn}}_{b,2b-a}((j^*)^{-1}(z)) = 0$, as wanted, it suffices now to show that, for $b+1 \leq p-2$, multiplication by t

$$\mathbf{L}(H^a_{\mathrm{cr}}(X/V, \mathcal{O}_{X/V}/p^n)\{-b\}) \xrightarrow{t} \mathbf{L}(H^a_{\mathrm{cr}}(X/V, \mathcal{O}_{X/V}/p^n)\{-b-1\})$$

is an isomorphism. But the multiplication map

$$H^a_{\mathrm{cr}}(X/V, \mathcal{O}_{X/V}/p^n)\{-b\} \otimes B^+_{\mathrm{cr}} \otimes V/p^n\{-1\} \otimes B^+_{\mathrm{cr}} \to H^a_{\mathrm{cr}}(X/V, \mathcal{O}_{X/V}/p^n)\{-b-1\} \otimes B^+_{\mathrm{cr}}$$

induces a map

$$\mathbf{L}(H^a_{\mathrm{cr}}(X/V, \mathcal{O}_{X/V}/p^n)\{-b\}) \otimes \mathbf{L}(V/p^n\{-1\}) \to \mathbf{L}(H^a_{\mathrm{cr}}(X/V, \mathcal{O}_{X/V}/p^n)\{-b-1\}),$$

which, since $H^a_{\mathrm{cr}}(X/V, \mathcal{O}_{X/V}/p^n)\{-b-1\} \in \mathcal{MF}_{[-b-1,0]}(V)$ (Theorem 2.1), is an isomorphism. Since multiplication by t yields an isomorphism $\mathbf{Z}/p^n \xrightarrow[\sim]{t} \mathbf{L}(V/p^n\{-1\})$ [9, 5.3.6], hence an isomorphism

$$\mathbf{L}(H^a_{\mathrm{cr}}(X/V, \mathcal{O}_{X/V}/p^n)\{-b\}) \xrightarrow{t} \mathbf{L}(H^a_{\mathrm{cr}}(X/V, \mathcal{O}_{X/V}/p^n)\{-b\}) \otimes \mathbf{L}(V/p^n\{-1\}),$$

we are done.

Assume now that X is projective of pure dimension d. We will show that in that case the maps α_{ab} are isomorphisms. First we prove

LEMMA 4.2. – 1. *The maps α_{ab} commute with products.*

2. *The maps α_{ab} are compatible with some cycle class maps, i.e., if Z is an irreducible closed subscheme of codimension j in X, smooth over V, then*

$$\alpha_{2j,b}(\mathrm{cl}^{\mathrm{\acute{e}t}}(Z_K)\zeta^{b-j}) = \mathrm{cl}^{\mathrm{cr}}(Z_k)t^{b-j}.$$

Here $\mathrm{cl}^{\mathrm{cr}}(Z_k) \in H^{2j}_{\mathrm{cr}}(X_1/V)$ *is the one defined by Gros in* [16].

3. *For irreducible $X_{\overline{K}}$, the following diagram commutes*

$$\begin{array}{ccc} H^{2d}(X_{\overline{K}}, \mathbf{Z}/p^n(b)) & \xrightarrow[\sim]{\mathrm{tr}^{\text{ét}}} & \mathbf{Z}/p^n(b-d) \\ \alpha'_{2d,b}\Big\downarrow & & t^{b-d}\Big\downarrow \\ H^{2d}_{\mathrm{cr}}(X/V, \mathcal{O}_{X/V}/p^n)\{-b\} \otimes B^+_{\mathrm{cr}} & \xrightarrow[\sim]{\mathrm{tr}^{\mathrm{cr}}} & B^+_{\mathrm{cr}}\{d-b\}, \end{array}$$

Proof. – Property 1 follows from the multiplication formula in Lemma 2.1.5.

For property 2, let $[\mathcal{O}_Z]$ and $[\mathcal{O}_{Z_K}]$ denote the class of $\mathcal{O}_Z$ and $\mathcal{O}_{Z_K}$ in $K_0(X)$ and $K_0(X_K)$ respectively. From the hypotheses, both Z_K and Z_k are integral and of codimension j in their respective fibers. We will need to do some preliminary computations.

Let $s(d) = \prod_{k=j-1}^{j+d-1} k!$. First, we claim that $s(d)[\mathcal{O}_Z] \in F^j_\gamma K_0(X)$ (it follows then that $s(d)[\mathcal{O}_{Z_K}] \in F^j_\gamma K_0(X_K)$). By the Integral Riemann-Roch [23, 4.6], for any $z \in F^l_\gamma K_0(Z)$ $\gamma_{j+l}(i_*(z)) = i_*((-1)^{j+l-1}(j+l-1)!z + y)$, where $i_* : K_0(Z) \to K_0(X)$ and $y \in F^{l+1}_\gamma K_0(Z)$. Since $F^{d+1}K_0(Z) = 0$, this gives our claim.

Next, we claim that, for $0 < i < j$, the Chern classes $c^{\text{ét}}_{i,0}([\mathcal{O}_{Z_K}])$ and $c^{\mathrm{syn}}_{i,0}([\mathcal{O}_Z])$ are zero. For the étale Chern class, since for an open U in X_K such that the codimension of $X_K \setminus U$ is strictly greater than i, the restriction $H^{2i}_{\text{ét}}(X_K, \mathbf{Z}/p^n) \to H^{2i}_{\text{ét}}(U, \mathbf{Z}/p^n)$ is an injection, we may assume that Z_K is the zero-scheme of a regular section s of a locally free sheaf $\mathcal{E}$ of rank j on X_K. The Koszul complex gives a resolution

$$0 \to \bigwedge^r(\check{\mathcal{E}}) \to \bigwedge^{r-1}(\check{\mathcal{E}}) \to \cdots \to \check{\mathcal{E}} \xrightarrow{\check{s}} \mathcal{O}_{X_K} \to \mathcal{O}_{Z_K} \to 0$$

Hence $[\mathcal{O}_{Z_K}] = \sum(-1)^i[\bigwedge^i(\check{\mathcal{E}})]$ in $K_0(X_K)$. Our claim now follows from [2, 0.1.2].

For the syntomic Chern class, notice that $\psi_n c^{\mathrm{syn}}_{i,0}([\mathcal{O}_Z]) = c^{\mathrm{cr}}_{i,0}([\mathcal{O}_{Z_k}])$. Recall that the crystalline Chern classes factor through the logarithmic de Rham-Witt classes $\tilde{c}^{\mathrm{cr}}_{i,0}([\mathcal{O}_{Z_k}]) \in H^i(X_k, W_n\Omega^i_{X_k,\log})$. Hence it suffices to show that $\tilde{c}^{\mathrm{cr}}_{i,0}([\mathcal{O}_{Z_k}]) = 0$. This can be shown exactly as in the étale case (for the injection of the restriction map $H^i(X_k, W_n\Omega^i_{X_k,\log}) \to H^i(U, W_n\Omega^i_{U,\log})$ refer to the purity result in the logarithmic de Rham-Witt cohomology [16, II.4.1.1]).

Finally, from the Whitney sum formula we have

$$c^{\text{ét}}_{j,0}(s(d)[\mathcal{O}_{Z_K}]) = s(d)c^{\text{ét}}_{j,0}([\mathcal{O}_{Z_K}]), \qquad \psi_n c^{\mathrm{syn}}_{j,0}(s(d)[\mathcal{O}_Z]) = s(d)\psi_n c^{\mathrm{syn}}_{j,0}([\mathcal{O}_Z]).$$

We are now ready to prove property 2. Since $\overline{c}^{\text{ét}}_{1,2}(\beta_n) = \zeta_n$ (β_n corresponding to ζ_n) and $c^{\text{ét}}_{j,0}([\mathcal{O}_{Z_K}]) = (-1)^{j-1}(j-1)!\mathrm{cl}^{\text{ét}}(Z_K)$, we get from the multiplication formula in Lemma 2.1.5, we get that

$$\overline{c}^{\text{ét}}_{b,2(b-j)}(s(d)[\mathcal{O}_{Z_K}]\beta_n^{b-j}) = (-1)^{b-1}(b-1)!s(d)\mathrm{cl}^{\text{ét}}(Z_K)\zeta_n^{b-j}.$$

On the other hand, $\psi_n\overline{c}^{\mathrm{syn}}_{1,2}(\tilde{\beta}_n) = t$ and $\psi_n c^{\mathrm{syn}}_{j,0}([\mathcal{O}_Z]) = c^{\mathrm{cr}}_{j,0}([\mathcal{O}_{Z_k}]) = \mathrm{cl}^{\mathrm{cr}}(Z_k)$ [16, IV.3.1.2]. Hence

$$\psi_n\overline{c}^{\mathrm{syn}}_{b,2(b-j)}(s(d)[\mathcal{O}_Z]\tilde{\beta}_n^{b-j}) = (-1)^{b-1}(b-1)!s(d)\mathrm{cl}^{\mathrm{cr}}(Z_k)t^{b-j}.$$

By the definition of the map $\alpha_{2j,b}$, since our $p \geq \max\{d+j-1, b-1\}$, this gives property 2.

Concerning property 3, we can assume that the residue field of V is algebraically closed. Let P be a rational point of X over V. Recall that $\mathrm{tr}^{\text{ét}}$ and $\mathrm{tr}^{\mathrm{cr}}$ are characterized by the mapping $\mathrm{cl}^{\text{ét}}(P_K)\zeta_n^{b-d}$, respectively $\mathrm{cl}^{\mathrm{cr}}(P_k)$, to 1. It suffices thus to show that

$$\alpha'_{2d,b}(\mathrm{cl}^{\text{ét}}(P_K)\zeta_n^{b-d}) = \mathrm{cl}^{\mathrm{cr}}(P_k)t^{b-d},$$

which is a special case of property 2. □

THEOREM 4.1. – *For any projective, smooth V-scheme X of pure relative dimension d, the Galois equivariant morphism*

$$\alpha_{ab} : H^a(X_{\overline{K}}, \mathbf{Z}/p^n(b)) \to \mathbf{L}(H^a_{\mathrm{cr}}(X/V, \mathcal{O}_{X/V}/p^n)\{-b\})$$

is an isomorphism, if the numbers p, b, d satisfy $b \geq 2d + N(2d)$, $p - 2 \geq 2b + N(2d)$.

Proof. – We may assume that k is algebraically closed and that X is irreducible. The line of the argument is standard [11], [6, 2.4]. Both the target and the domain satisfy Poincaré duality:

$$\begin{aligned}
&H^a(X_{\overline{K}}, \mathbf{Z}/p^n(b)) \otimes H^{2d-a}(X_{\overline{K}}, \mathbf{Z}/p^n(b)) \\
&\qquad \xrightarrow{\cup} H^{2d}(X_{\overline{K}}, \mathbf{Z}/p^n(2b)) \xrightarrow{\mathrm{tr}^{\text{ét}}} \mathbf{Z}/p^n(2b-d), \\
&\mathbf{L}(H^a_{\mathrm{cr}}(X/V, \mathcal{O}_{X/V}/p^n)\{-b\}) \otimes \mathbf{L}(H^{2d-a}_{\mathrm{cr}}(X/V, \mathcal{O}_{X/V}/p^n)\{-b\}) \\
&\qquad \xrightarrow{\cup} \mathbf{L}(H^{2d}_{\mathrm{cr}}(X/V, \mathcal{O}_{X/V}/p^n)\{-2b\}) \xrightarrow{\cup^{\mathrm{cr}}} \mathbf{L}(V\{-2b+d\}) \xrightarrow{\sim} \mathbf{Z}/p^n(2b-d)
\end{aligned}$$

By Lemma 4.2, α_{ab} has a left inverse α_{ab}^{-1}. To show that α_{ab}^{-1} is a right inverse as well, it would suffice to show that it commutes with products. Since it commutes with Künneth products (because α_{ab} is functorial and commutes with cup products), it suffices to show that it commutes with Δ^*, $\Delta : X \hookrightarrow X \times X$ being the diagonal embedding, or equivalently, that α_{ab} commutes with Δ_*. That, in turn, would follow, if we would show that α_{ab} respects class of the diagonal and that Δ^* is surjective. But this holds, since $p_1\Delta = \mathrm{Id}$, where $p_1 : X \times X \to X$ is the projection onto the first factor, and we have Lemma 4.2. □

Remark 3. – The lower bound on the prime p appearing here is of order d^3, while Faltings in [5] gets an order of d. While it is perhaps possible to decrease our lower bound slightly by a more careful argument, the main obstruction comes from the constant $N(d)$ of Thomason. It seems reasonnable to believe that the optimal $N(d)$ should be of order d as well.

5. Rational theory

Throughout this section, let the prime p be odd. Let X be a smooth quasi-projective V-scheme, where now the ring V is possibly ramified over $W(k)$. For large b and $p \neq 2$, we will construct Galois equivariant functorial morphisms

$$\alpha_{ab} : H^a(X_{\overline{K}}, \mathbf{Q}_p(b)) \to H^a_{\mathrm{cr}}(X_0/W(k), \mathcal{O}_{X_0/W(k)}) \otimes \mathbf{Q} \otimes B^+_{\mathrm{cr}}.$$

Most of the work was already done in section 4. For $j \geq 3$, $p^n \geq 5$, $p \neq 2$, since Lemma 2.1 holds in this context as well, we get the following diagram

$$\begin{array}{ccc} F^i_\gamma/F^{i+1}_\gamma K_j(X_{\overline{V}}; \mathbf{Z}/p^n) & \xrightarrow[j^*]{\sim} & F^i_\gamma/F^{i+1}_\gamma K_j(X_{\overline{K}}; \mathbf{Z}/p^n) \\ \downarrow{\scriptstyle \overline{c}^{\mathrm{syn}}_{ij}} & & \downarrow{\scriptstyle \overline{c}^{\mathrm{\acute{e}t}}_{ij}} \\ H^{2i-j}(X_{\overline{V}}, s'_n(i)) & & H^{2i-j}(X_{\overline{K}}, \mathbf{Z}/p^n(i)). \end{array}$$

For $b \geq d + N(d)$, define the morphisms

$$\alpha^n_{ab} : H^a(X_{\overline{K}}, \mathbf{Z}/p^n(b)) \to H^a_{\mathrm{cr}}(X_{\overline{V}}/W(k), \mathcal{O}_{X_{\overline{V}}/W(k)}/p^n)\{-b - N(d)\}$$

as the composition

$$\alpha^n_{ab} := l(b,d)t^{N(d)}\psi_n \overline{c}^{\mathrm{syn}}_{b,2b-a}(j^*)^{-1}T(d,b,2b-a)(\overline{c}^{\mathrm{\acute{e}t}}_{b,2b-a})^{-1},$$

where ψ_n is the natural projection

$$\psi_n : H^a(X_{\overline{V}}, s'_n(b)) \to H^a_{\mathrm{cr}}(X_{\overline{V}}/W(k), \mathcal{O}_{X_{\overline{V}}/W(k)}/p^n),$$

and we set $l(b,d) = (-1)^{N(d)}(N(d) + b - 1)!/(b-1)!$. Here $(\overline{c}^{\mathrm{\acute{e}t}}_{b,2b-a})^{-1}(x)$ is defined by taking any element in the preimage of $T(d,b,2b-a)x$ (by Proposition 4.1, $T(d,b,2b-a)x$ lies in the image of $\overline{c}^{\mathrm{\acute{e}t}}_{b,2b-a}$). Again, by Proposition 4.1, any ambiguity in that definition comes from a class of y such that $T(d,b,2b-a)[y] = [z]$ and $z \in K_{2b-a}(X_{\overline{K}}; \mathbf{Z}/p^n)$ is annihilated by $\beta_n^{N(d)}$. Hence

$$\begin{aligned} l(b,d)t^{N(d)}\psi_n \overline{c}^{\mathrm{syn}}_{b,2b-a}&((j^*)^{-1}(T(d,b,2b-a)[y])) \\ &= l(b,d)\psi_n[\overline{c}^{\mathrm{syn}}_{1,2}(\tilde{\beta}_n^{N(d)})\overline{c}^{\mathrm{syn}}_{b,2b-a}((j^*)^{-1}[z])] \\ &= \psi_n[\overline{c}^{\mathrm{syn}}_{b+N(d),2b+2N(d)-a}(\tilde{\beta}_n^{N(d)}(j^*)^{-1}[z])] = 0 \end{aligned}$$

and the map α^n_{ab} is well defined.

Let now $p \neq 2$ and $b \geq d + N(d)$. Define the morphism

$$\alpha_{ab} : H^a(X_{\overline{K}}, \mathbf{Q}_p(b)) \to H^a_{\mathrm{cr}}(X_0/W(k), \mathcal{O}_{X_0/W(k)}) \otimes_{W(k)} B_{\mathrm{cr}}\{-b\}$$

as the composition of $\mathbf{Q} \otimes \operatorname{proj}\lim_n \alpha^n_{ab}$ with the Kato-Messing isomorphism

$$\begin{aligned} h_{\mathrm{cr}} : \mathbf{Q} \otimes \operatorname{proj}\lim_n H^a_{\mathrm{cr}}(X_{\overline{V}}/W(k), \mathcal{O}_{X_{\overline{V}}/W(k)}/p^n) \\ \simeq H^a_{\mathrm{cr}}(X_0/W(k), \mathcal{O}_{X_0/W(k)}) \otimes_{W(k)} B^+_{\mathrm{cr}} \otimes \mathbf{Q} \end{aligned}$$

and the division by $l(b,d)T(d,b,2b-a)^2 t^{N(d)}$.

THEOREM 5.1. – *Let $p \neq 2$ and let X be any projective smooth V-scheme of pure relative dimension d. Then, assuming $b \geq 2d + N(2d)$, the morphism*

$$\alpha_{ab} : H^a(X_{\overline{K}}, \mathbf{Q}_p(b)) \otimes_{\mathbf{Q}_p} B_{\mathrm{cr}} \to H^a_{\mathrm{cr}}(X_0/W(k), \mathcal{O}_{X_0/W(k)}) \otimes_{W(k)} B_{\mathrm{cr}}\{-b\}$$

is an isomorphism.

Moreover, the map α_{ab} *preserves the Frobenius and the action of* $\mathrm{Gal}(\overline{\mathrm{K}}/\mathrm{K})$, *and, after extension to* B_{dR}, *induces an isomorphism of filtrations.*

Proof. – We have the following analogue of Lemma 4.2.

LEMMA 5.1. – 1. *Let* $x \in H^a(X_{\overline{K}}, \mathbf{Z}/p^n(b))$ *and* $y \in H^c(X_{\overline{K}}, \mathbf{Z}/p^n(e))$. *Then*

$$\begin{aligned} &l(b,d)l(e,d)K(b,e)t^{N(d)}T(d,b,2b-a)^2T(d,e,2e-c)^2\alpha^n_{a+c,b+e}(x \cup y)\\ &\qquad = l(b+e,d)K(b,e)T(d,b+e,2b+2e-a-c)^2\alpha^n_{ab}(x) \cup \alpha^n_{ce}(y), \end{aligned}$$

where $K(b,e) = (b+e-1)!/(b-1)!(e-1)!$, *assuming that all the indices are in the valid range.*

2. *The maps* α_{ab} *are compatible with some cycle class maps, i.e., if* Z *is an irreducible closed subscheme of codimension* j *in* X, *smooth over* V, *and* $b \geq j$, *then*

$$\alpha_{2j,b}(\mathrm{cl}^{\text{ét}}(Z_K)\zeta^{b-j}) = \mathrm{cl}^{\mathrm{cr}}(Z_k)t^{b-j}.$$

Here $\mathrm{cl}^{\mathrm{cr}}(Z_k) \in H^{2j}_{\mathrm{cr}}(X_0/V)$ *is the one defined by Gros in* [16].

3. *For* $b \geq d$ *and irreducible* $X_{\overline{K}}$, *the following diagram commutes*

$$\begin{array}{ccc} H^{2d}(X_{\overline{K}}, \mathbf{Q}_p(2b)) & \xrightarrow[\sim]{\mathrm{tr}^{\text{ét}}} & \mathbf{Q}_p(2b-d) \\ {\scriptstyle \alpha_{2d,2b}}\downarrow & & \downarrow{\scriptstyle t^{2b-d}} \\ H^{2d}_{\mathrm{cr}}(X_0/W(k), \mathcal{O}_{X_0/W(k)})\{-2b\} \otimes B_{\mathrm{cr}} & \xrightarrow[\sim]{\mathrm{tr}^{\mathrm{cr}}} & B_{\mathrm{cr}}\{d-2b\}, \end{array}$$

Proof. – For properties (1) and (2) we evoke Lemma 4.2 and argue as in the proof of Lemma 2.2.2. For property (3), take a rational point P of X over V. Since $tr^{\text{ét}}$ and $\mathrm{tr}^{\mathrm{cr}}$ are characterized by the mapping $\mathrm{cl}^{\text{ét}}(P_K)\zeta^{2b-d}$, respectively $\mathrm{cl}^{\mathrm{cr}}(P_k)$, to 1, it suffices to show that

$$\alpha_{2d,2b}(\mathrm{cl}^{\text{ét}}(P_K)\zeta^{2b-d}) = \mathrm{cl}^{\mathrm{cr}}(P_k)t^{2b-d}.$$

But that follows from property (2). □

Assume that k is algebraically closed and that X is irreducible. By construction α_{ab} is a Galois equivariant map compatible with the Frobenius action. Since both sides have the same rank over B_{cr}, it suffices to show that the morphism α_{ab} is injective. This follows from the fact that both the domain and the target satisfy Poincaré duality, and the map α_{ab} commutes with products and traces (by the above lemma).

The proof that α_{ab} induces an isomorphism on filtrations is now standard (see, [18, 3]). By Lemma 2.2.1 extension of α_{ab} to B_{dR} is compatible with the filtrations. One passes to the associated grading and reduces to showing that the induced map

$$\overline{\alpha}_{ab} : \mathbf{C}_p(l) \otimes_{\mathbf{Q}_p} H^a(X_{\overline{K}}, \mathbf{Q}_p(b)) \to \bigoplus_{j \in \mathbf{Z}} \mathbf{C}_p(b+l-j) \otimes_K H^{a-j}(X_K, \Omega^j_{X_K/K}), \quad l \in \mathbf{Z},$$

is an isomorphism (this is the statement of the Hodge-Tate Conjecture). The dimensions of both sides being equal, it suffices to show that $\overline{\alpha}_{ab}$ is injective. Since both the target and the domain of $\overline{\alpha}_{ab}$ satisfy Poincaré duality and $\overline{\alpha}_{ab}$ is compatible with products (Lemma 5.1.1), for $\overline{\alpha}_{ab}$ to have a left inverse, it is sufficient that it be compatible with traces. But, $\overline{\alpha}_{2d,2b}$ is obtained from the composition of the map

$$\alpha'_{2d,2b} : H^{2d}(X_{\overline{K}}, \mathbf{Q}_p(2b)) \to \mathbf{Q} \otimes \text{proj } \lim_n H^{2d}_{\text{cr}}(X_{\overline{V}}/W(k), J_n^{[2b]}/J_n^{[2b+1]})$$

induced by grading the maps $\alpha^n_{2d,2b}$, and the natural isomorphism [18, 1.4]

$$\sigma_{\text{gr}} : \mathbf{Q} \otimes \text{proj } \lim_n H^{2d}_{\text{cr}}(X_{\overline{V}}/W(k), J_n^{[2b]}/J_n^{[2b+1]}) \simeq \mathbf{C}_p(2b-d) \otimes H^d(X_K, \Omega^d_{X_K/K}).$$

Since the Hodge trace is compatible with the de Rham trace, we can use Lemma 2.2.1, Lemma 5.1.3, and the compatibility of the Berthelot-Ogus isomorphism $H^i_{\text{cr}}(X_0/W(k)) \otimes_{W(k)} K \simeq H^i_{dR}(X_K/K)$ with traces [18, B.3.3] to conclude our proof. $\square$

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