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SELF-EMBEDDINGS OF KLEINIAN GROUPS

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ABSTRACT. – We give sufficient conditions for a geometrically finite Kleinian group G acting in the hyperbolic space $\mathbf{H}^n$ to have co-Hopf property, i.e., not to contain non-trivial proper subgroups isomorphic to itself. We provide examples of freely indecomposable geometrically finite non-elementary Kleinian groups which are not co-Hopf if our sufficient condition does not hold. We prove that any topologically tame non-elementary Kleinian group in dimension 3 can not be conjugate by an isometry to its proper subgroup. © Elsevier, Paris

RÉSUMÉ. – Nous donnons des conditions suffisantes pour qu'un groupe kleinien G géométriquement fini agissant sur l'espace hyperbolique $\mathbf{H}^n$ possède la propriété co-Hopf, c'est-à-dire que G ne contienne aucun sous-groupe propre isomorphe à G. Nous fournissons des exemples de groupes géométriquement finis, non-élémentaires, qui ne sont pas produits libres non-triviaux et qui ne sont pas co-Hopf si notre condition suffisante n'est pas vérifiée. Nous prouvons que tout groupe kleinien géométriquement sage non-élémentaire en dimension 2 ne peut pas être conjugué par une isométrie à son sous-groupe propre. © Elsevier, Paris

1. Introduction

The aim of this paper is to study proper monomorphisms $\varphi : G \to G$ where the group G belongs to the class of geometrically finite Kleinian groups acting on n-dimensional sphere $\mathbf{S}^n$ $(n \geq 2)$.

Let us recall that an abstract group G is called cohopfian (or a co-Hopf group) if any monomorphism $\chi : G \to G$ to itself is in fact an isomorphism. There is a dual definition which says that a group G is hopfian if any epimorphism of it to itself is an isomorphism. These two properties are quite different, in particular all finitely generated subgroups of the linear group $GL_n(\mathbf{R})$ are hopfian as being residually finite but there are many of them which are not co-Hopf.

The study of the co-Hopf property was started by Baer [B] in the 40's for surface groups. Such a question was first considered for 3-manifold groups by F. Gonzales-Acuna and W. Whitten [GW] where they completely described the class of cohopfian fundamental groups of Haken manifolds with boundary which is a union of tori. Different criteria of cohopficity were given: for a closed geometric 3-manifolds by S. Wang and Y.-Q. Wu [WW] and for geometric 3-manifolds with boundary by L. Potyagailo and S. Wang [PW]. The problem to describe the class of 3-manifolds having co-Hopf fundamental groups is quite far from being solved.

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The same question remains meaningful in more geometric context. For instance, the question became quite popular since M. Gromov's study of word-hyperbolic groups. He showed that $\pi_1(V)$ is co-Hopf for any compact aspherical pseudomanifold V with a non-elementary hyperbolic π_1 [Gro, 5.4.B]. He also conjectured there that this should be true for any word-hyperbolic group connected at infinity. Z. Sela [Se] proved that a non-elementary, torsion-free, word-hyperbolic group is co-Hopf if and only if it is not a non-trivial free product.

Before stating our results, we recall that a parabolic subgroup H of a discrete (Kleinian) group G of the isometry group $Iso_+\mathbf{H}^n$ of the hyperbolic space $\mathbf{H}^n$ is said to be maximal if there is no any other parabolic subgroup of G containing it and not maximal otherwise. A maximal parabolic subgroup $K \subset G$ is said to have virtual rank r if it contains an abelian subgroup of finite index of rank r. It turns out that co-Hopf property is related to splittings of G over virtually abelian subgroups.

One of our main results is;

THEOREM 1.1. – *Let G be a geometrically finite, non-elementary, torsion-free, discrete (Kleinian) subgroup of the orientation preserving isometry group $Iso_+\mathbf{H}^n$ of the hyperbolic space $\mathbf{H}^n$ $(n \geq 3)$. Then G is co-Hopf if G does not split as HNN-extension or amalgamated free product over a virtually abelian subgroup $H \subset G$ of virtual rank k $(0 \leq k \leq n-1)$.*

Furthermore for any n and $k \in \{1, ..., n-2\}$ there are freely indecomposable non-elementary, geometrically finite, torsion-free Kleinian groups which split over non-maximal parabolic subgroups of rank k and which are not co-Hopf.

We remark that it has been previously shown in dimension $n = 3$ that a finitely generated, non-elementary, torsion-free Kleinian group $G \subset PSL_2\mathbf{C}$ is co-Hopf if G does not split over trivial or cyclic subgroups [PW]. The condition not to be non-trivial free product is that what one obviously needs to impose. In fact if the group $G = G_1 * G_2$ is such a product then it contains a proper subgroup $G =< G_1^{t_2}, G_2^{t_1} >\cong G_1^{t_2} * G_2^{t_1}$ where $t_i \in G_i$ $(i = 1, 2)$. On the other hand, the condition not to split over a cyclic subgroup was hoped for some time to be superfluous and more or less related in [PW] to the method of proving the result. In this paper we provide, however, an unexpectedly quite simple example of a non-elementary, torsion-free, geometrically finite, freely indecomposable Kleinian group acting on $\overline{\mathbf{C}}$ which is not co-Hopf and splits over $\mathbf{Z}$.

The proof of the sufficient condition of 1.1 occupies the section 3 of the paper. In the section 4 we give examples of non-elementary, torsion-free, geometrically finite, freely indecomposable Kleinian groups in all dimensions which split as the central HNN extension over non maximal abelian parabolic subgroups of rank k $(k \in \{1, ..., n-2\})$. For these examples, we shall prove that they are not co-Hopf.

The following is an intriguing question which is from one side a particular case of the co-Hopf property and from the other will be involved in the proof of 1.1.

QUESTION 1.2. – *Let G be a discrete finitely generated non-elementary subgroup in the group $Iso_+\mathbf{H}^n$. Then does the inclusion $\gamma G \gamma^{-1} \subset G$ imply $\gamma G \gamma^{-1} = G$, where $\gamma \in Iso_+\mathbf{H}^n$?*

The above question which we call the "proper conjugation property" for a Fuchsian group G in dimension $n = 2$ was firstly raised by H. Hopf (see [H]). A positive answer is given for a finitely generated group by G. Huber (see [H]), and an example of infinitely generated Fuchsian group of the first kind without this property is given in the paper of T. Jorgensen, A. Marden and C. Pommerenke [JMP].

In [PW] it was proved that the answer to the question is "yes" for any freely indecomposable discrete subgroup of $PSL_2\mathbf{C}$ and afterwards the same answer was given for any geometrically finite subgroup of $Iso_+\mathbf{H}^n$ [WZ].

We now state a theorem whose first part is proved in the section 3 (Lemma 3.4 below) and the second one dealing with topologically tame Kleinian subgroups in dimension 3 is proved in the section 5. A famous still unsolved conjecture of A. Marden [Mar] says that all finitely generated Kleinian groups in $PSL_2\mathbf{C}$ would be topologically tame.

THEOREM 1.3. – *Suppose that $G \subset Iso_+\mathbf{H}^n$ is a non-elementary, torsion-free finitely generated Kleinian group. Then $\alpha G \alpha^{-1} \subset G$ implies $\alpha G \alpha^{-1} = G$ for $\alpha \in Iso_+\mathbf{H}^n$ in one of the following cases (a) (b):*

a) either $n > 3$ and G is isomorphic to a geometrically finite Kleinian group $\Gamma \subset Iso_+\mathbf{H}^n$ which does not split over virtually abelian subgroups (including the trivial one);

b) or $n = 3$ and $G \subset PSL_2\mathbf{C}$ is a topologically tame Kleinian group.

2. Preliminaries

Let us start with the group $SO_+(n,1)$ which is the connected component of the identity in the special Lorentz group $SO(n,1)$. The group $SO_+(n,1)$ is isomorphic to the orientation preserving part of the isometry group $Iso_+\mathbf{H}^n$ of the hyperbolic space $\mathbf{H}^n$. This group will be also denoted throughout the paper by $M(n)$ meaning that this group is the group of conformal transformations of the sphere at infinity $\mathbf{S}^{n-1}_\infty = \partial\mathbf{H}^n$. We use the notation $d_{\mathbf{H}^n}$ for the hyperbolic distance in $\mathbf{H}^n$.

Any discrete subgroup G of $M(n)$ is called Kleinian group. The set of accumulation points of any orbit $G(z)$ (where $z \in \mathbf{H}^n$) is called the limit set and is denoted $\Lambda(G)$. It is well-known that either $Card(\Lambda(G)) \in \{0,1,2\}$ or $\Lambda(G)$ is a perfect closed subset of $\mathbf{S}^{n-1}_\infty$ [Ma]. In the first three cases we call our group G elementary and in the latter non-elementary. The complementary set $\Omega(G) = \mathbf{S}^{n-1}_\infty \setminus \Lambda(G)$ is called the domain of discontinuity of the group G.

The convex hull $N_G \in \mathbf{H}^n$ of the limit set $\Lambda(G)$ is called Nielsen hull. It is a minimal convex subset of $\mathbf{H}^n$ invariant under the G-action. We say that a finitely generated Kleinian group G is geometrically finite if $Vol_{\mathbf{H}^n}(M_G = N_G/G) < +\infty$.

By Margulis' lemma, it is known that there is a positive constant ϵ_0 such that for any Kleinian group $G \subset M(n)$ and $\epsilon \leq \epsilon_0$, the part of $\mathbf{H}^n/G$, where the injectivity radius is less than ϵ is a disjoint union of tubular neighbourhoods of closed geodesics whose lengths are less than 2ϵ and cusp neighbourhoods. We denote by $(\mathbf{H}^n/G)_0$ the complement of cusp neighbourhoods corresponding to some ϵ less than the Margulis constant ϵ_0 above. If G is geometrically finite, then $(\mathbf{H}^n/G)_0$ is compact.

A sequence of discrete faithful representations $\rho_m : F \to M(n)$ of an abstract finitely generated group F converges algebraically to a representation ρ_∞ if $\rho_m(f_i)$ converges to $\rho_\infty(f_i)$ in the compact-open topology of $\mathbf{S}^{n-1}_\infty$ for a fixed finite generator system $\{f_1, ..., f_l\}$ of F. Denote $Def(F, M(n))$ the space of discrete faithful representations of F into $M(n)$ modulo conjugacy which we endow respectively with the quotient topology.

In dimension $n = 3$, we call a Kleinian group $G \subset PSL_2\mathbf{C}$ topologically tame if the manifold $M(G) = \mathbf{H}^3/G$ is homeomorphic to the interior of a compact 3-manifold. By Scott's core theorem ([Sc]), it is known that in general for any 3-dimensional Kleinian group G, the quotient $\mathbf{H}^3/G$ has a compact 3-submanifold which is homotopy equivalent to $\mathbf{H}^3/G$ by the inclusion. Such a submanifold is called a core of $\mathbf{H}^3/G$. By the relative core theorem due to McCullough ([Mc]), we can choose a core C so that C is contained in $(\mathbf{H}^3/G)_0$ and $(C, C \cap \partial(\mathbf{H}^3/G)_0)$ is relatively homotopy equivalent to $((\mathbf{H}^3/G)_0, \partial(\mathbf{H}^3/G)_0)$ as pairs by the inclusion. We call such a core a relative core of $(\mathbf{H}^3/G)_0$.

An end of $(\mathbf{H}^3/G)_0$ is said to be geometrically finite if it has a neighbourhood which intersects no closed geodesics, and otherwise geometrically infinite. The components of the frontier of a relative core correspond one-to-one to the ends of $(\mathbf{H}^3/G)_0$ by taking a frontier component to an end such that a component of the complement of the core whose closure contains the frontier component is its neighbourhood. We call the corresponding end "the end facing the frontier component". Conversely we also say that the frontier component is facing the end.

When G is topologically tame, for an end e facing a frontier component Σ of a relative core, the end e has a neighbourhood homeomorphic to $\Sigma \times \mathbf{R}$.

Convention. – We will be mostly concerned with torsion-free case and suppose, unless mentioned otherwise, that all Kleinian groups are torsion free. Recall that the famous Selberg lemma says that one can always find a torsion-free subgroup of finite index in any finitely generated subgroup of the linear group $GL_n(\mathbf{R})$.

3. Sufficient conditions

In this section we shall prove the following.

PROPOSITION 3.1. – *Let $G \subset M(n)$ be a geometrically finite, non-elementary, torsion-free Kleinian group. Suppose that G does not split over a virtually abelian subgroup of virtual rank k such that $0 \leq k \leq n-1$. Then G is co-Hopf.*

Let F be a non-virtually abelian finitely generated group and $\{f_1, ..., f_s\}$ a finite generator system of F. We shall first study some pre-compact sequences in the deformation space $Def(F, M(n))$.

LEMMA 3.2. – *Suppose that $\rho_m : F \to M(n)$ ($m \in \mathbf{N}$) is a sequence of discrete faithful representations of the group F representing a converging sequence in $Def(F, M(n))$. Suppose also that for any $m \in \mathbf{N}$, the group $\rho_m(F)$ is a subgroup of the same geometrically finite Kleinian group $G \subset M(n)$. Then there exist a point $x_0 \in \mathbf{H}^n$ and a sequence of elements $(\alpha_k) \subset G$ such that*

$$\max_{1 \leq i \leq s} d_{\mathbf{H}^n}(x_0, \alpha_k \rho_{m_k}(f_i) \alpha_k^{-1}(x_0)) \leq R < +\infty$$

for some subsequence $\{\rho_{m_k}\}$ of $\{\rho_m\}$ and a constant R.

Proof of the Lemma. – The fact that the sequence (ρ_m) represents a sequence converging algebraically in $Def(F, M(n))$ implies that there exists a sequence $(\gamma_m) \subset M(n)$ such that:

$\lim_{m\to\infty} \gamma_m \rho_m(f_i)\gamma_m^{-1} = \rho(f_i)$, where $\rho : F \to M(n)$ is a discrete faithful representation $(i \in \{1, ..., s\})$ [Ot]. It follows that

$$\max_{1\le i\le s} d_{\mathbf{H}^n}(O, \gamma_m\rho_m(f_i)\gamma_m^{-1}(O)) \le R_1 < +\infty, \tag{1}$$

where O is some point in $\mathbf{H}^n$. Let $D(G) \subset \mathbf{H}^n$ be a convex fundamental polyhedron for the action of G in $\mathbf{H}^n$. We can pick $\alpha_m \in G$ such that $\alpha_m(\gamma_m^{-1}(O)) = x_m \in D(G) \in \mathbf{H}^n$ and rewrite (1) :

$$\max_{1\le i\le s} d_{\mathbf{H}^n}(x_m, \alpha_m\rho_m(f_i)\alpha_m^{-1}(x_m)) \le R_1 < +\infty \tag{2}$$

By passing to a subsequence and keeping the notations we may assume that $x_m \to x_0 \in \overline{\mathbf{H}^n} = \mathbf{H}^n \cup \mathbf{S}_\infty^{n-1}$. We need to show the following:

Claim. – $x_0 \in \mathbf{H}^n$.

Proof of the Claim. – Suppose to the contrary that $x_0 \in \mathbf{S}_\infty^{n-1}$. By denoting the closure of $D(G)$ in $\overline{\mathbf{H}^n}$ by $\overline{D(G)}$, we see that x_0 is contained in $\overline{D(G)} \cap \mathbf{S}_\infty^{n-1}$. Since the group G is geometrically finite there are two possibilities for x_0:

either 1) $x_0 \in \overline{D(G)} \cap \Omega(G)$ or 2) x_0 is a cusped parabolic fixed point of G ([Ma, p.118]).

Case 1. – $x_0 \in \overline{D(G)} \cap \Omega(G)$.

Put $\beta_{m,i} = \alpha_m\rho_m(f_i)\alpha_m^{-1}$ and $y_{m,i} = \beta_{m,i}(x_m)$ $(m \in \mathbf{N}, i \in \{1, ..., s\})$. By the hypothesis, we have that $\rho_m(f_i) \in G$, and since $\alpha_m \in G$, it follows that $\beta_{m,i} \in G$. We have by (2) that $\max_{1\le i\le s} d_{\mathbf{H}^n}(y_{m,i}, x_m) \le R_1$ so $\lim_{m\to\infty} y_{m,i} = \lim_{m\to\infty} x_m = x_0$.

Since $x_0 \in \Omega(G)$, there exists m_0 such that $\forall m > m_0$ $y_{m,i} = \beta_{m,i}(x_m) = x_m$ $(1 \le i \le s)$. This means that the group $\alpha_m\rho_m(F)\alpha_m^{-1}$ is an elementary elliptic group fixing the point $x_m \in \mathbf{H}^n$, hence so is $\rho_m(F)$. This is a contradiction.

Case 2. – $x_0 \in \Lambda(G) \cap \overline{D(G)}$ is a cusped parabolic fixed point.

Consider the upper-half space model for $\mathbf{H}^n$ and assume that $x_0 = \infty$. Then there is $t \in \mathbf{R}_+$ such that the horoball $V = V(t) = \{(x^1, ..., x^{n-1}, x^n) \in \mathbf{H}^n : x^n > t\}$ is precisely invariant in G under its stabiliser subgroup $H = Stab(V, G) = \{h \in G : h(V) = V\}$, which means that $\forall g \in G \setminus H$ $g(V) \cap V = \emptyset$.

Let us show that for all but finitely many $m \in \mathbf{N}$ one has $\{x_m, y_{m,i}\} \subset V(t)$ $(i \in \{1, ..., s\})$. In fact obviously $x_m \in V(t)$ since $x_m \in D(G) \cap V(t)$. If to the contrary we get a subsequence of $\{y_{m,i}\}$ (for which we retain the same notation) such that $y_{m,i} \notin V(t)$ then by using the expression for the distance $d_{\mathbf{H}^n}$ for the upper-half space model we obtain:

$$d_{\mathbf{H}^n}(y_{(m,i)}, x_m) \ge \log|\frac{(y_{m,i})^n}{(x_m)^n}| \to \infty,$$

where $(y)^n$ is the n-th coordinate of $y \in \mathbf{H}^n$. This is impossible by (2).

By using the same notation $\beta_{m,i}$ as in the Case 1 we have $y_{m,i} = \beta_{m,i}(x_m) \in V(t)$ and also $x_m \in V(t)$ $(m > m_0, i \in \{1, ..., s\})$. Then since $V(t)$ is precisely invariant under H we obtain $\beta_{m,i} \in H$. This implies that $\alpha_m\rho_m(F)\alpha_m^{-1}$ is an elementary parabolic subgroup of G which is prohibited by the hypothesis. Thus the Claim is proved.

We obtain therefore $x_0 \in \mathbf{H}^n$ and so that $d_{\mathbf{H}^n}(x_0, x_m) \leq R_2 < +\infty$ for all $m \in \mathbf{N}$, hence:

$$d_{\mathbf{H}^n}(x_0, \alpha_m \rho_m(f_i)\alpha_m^{-1}(x_0)) \leq 2 \cdot d_{\mathbf{H}^n}(x_m, x_0) + d_{\mathbf{H}^n}(x_m, \beta_{m,i}(x_m)) \leq 2R_2 + R_1 = R < +\infty.$$

Thus Lemma 3.2 is proved. □

LEMMA 3.3. – *Let $G \subset M(n)$ be a geometrically finite non-elementary Kleinian group. Then the following assertions are valid:*

a) Let $F \subset G$ be a finitely presented subgroup which does not split over its virtually abelian subgroups (including the trivial one). Then there are only finitely many conjugacy classes of subgroups of G isomorphic to F.

b) If in addition, G itself does not split over a virtually abelian subgroup of virtual rank $k (0 \leq k \leq n-1)$ then the group of outer automorphisms $Out(G)$ is finite.

Remark. – A statement analogous to a) was first proved by W. Thurston in the case when $G = \pi_1(M^3)$ where M^3 is an acylindrical hyperbolic 3-manifold and $F = \pi_1(S_g)$ is surface group $(g > 2)$ [Th1, Corollary 8.8.6].

Both statements are also well-known in the theory of word-hyperbolic groups. The statement a) in this context is first proved by M. Gromov [Gro, Thm 5.3.C] and in a more general form by T. Delzant [De]. The statement b) for word-hyperbolic groups is due to F. Paulin.

We shall prove the statements in the above form. Whereas the proof of F. Paulin for the statement b) works for relative hyperbolic groups (see for the definition [Gro]), and consequently for geometrically finite Kleinian groups in general; as for the statement a) one should take some care for groups with accidental parabolics (see [Th1, Remark after 8.8.6] or our example 2 in the next section) for which a similar statement does not hold.

Proof. – Our approach is inspired essentially by the methods of [Be] and [Se].

a) Suppose that we have an infinite sequence of non-conjugate embeddings $\phi_m : F \to G \subset M(n)$. Take any finite generating system $f_1, ..., f_s$ of F and consider the displacement function:

$$d_m = \min_{x \in \mathbf{H}^n} \max_{1 \leq i \leq s} d_{\mathbf{H}^n}(x, \phi_m(f_i)(x)). \tag{1}$$

Note that it is proved that the minimum as above is attained by a point in $\mathbf{H}^n$ in [Be]. Let us first consider the case when d_m does not converge to ∞. Then the standard argument shows that there is a subsequence (denoted by the same symbol) $\phi_m : F \to M(n)$ converging to a discrete faithful representation $\phi : F \to M(n)$ in $Def(F, M(n))$, (see e.g. [Be]). We also know that $\phi_m(F) \subset G$, so that we can apply Lemma 3.2 to obtain a point $x_0 \in \mathbf{H}^n$, and a sequence $\alpha_m \subset G$ such that

$$d_m = \max_{1 \leq i \leq s} d_{\mathbf{H}^n}(x_0, \ \alpha_m \phi_m(f_i)\alpha_m^{-1}(x_0)) \leq R < +\infty$$

after taking a further subsequence, still denoted by ϕ_m. Since $\alpha_m \phi_m(f_i)\alpha_m^{-1} \in G$ $(m \in \mathbf{N}, i \in \{1, ..., s\})$ and the group G is discrete, we may conclude that $\alpha_{m_0+r}\phi_{m_0+r}\alpha_{m_0+r}^{-1} = \alpha_{m_0}\phi_{m_0}\alpha_{m_0}^{-1}$ for some m_0 and r, contradicting to our assumption that all $\phi_m(F)$ are non-conjugate subgroups in G.

We are left with the case when $d_m \to \infty$. Let $z_m \in \mathbf{H}^n$ be a point which realizes the minimum in (1) whose existence is proved in [Be]. Pick $\gamma_m \in M(n)$ such that $\gamma_m(z_m) = O \in \mathbf{H}^n$. Therefore, we have $d_m = \max_{1 \le i \le s} d_{\mathbf{H}^n}(O, \gamma_m \phi_m(f_i)\gamma_m^{-1}(O)) \to \infty$.

We may now apply Bestvina's construction for representations $(\gamma_m \phi_m \gamma_m^{-1})_m$, rescale the metric by d_m and get an $\mathbf{R}$-tree $(\mathcal{Z}, z_0)$ together with a small, non-trivial, isometric F-action.

This implies by Rips' theory [Be-F] that F would split over a virtually abelian subgroup which is impossible by the hypothesis. *QED.*

Remark. – The above compactification of the space $Def(G, M(n))$ of discrete faithful representations by $\mathbf{R}$-trees was first found in dimensions 2 and 3 by J. Morgan and P. Shalen [MS] and in arbitrary dimension n by J. Morgan [Mo2] (see also the discussion in the book [Ot]).

b) Let us suppose to the contrary that there is an infinite sequence of automorphisms ϕ_m of G representing distinct elements of the outer-automorphism group $Out(G)$. We will repeat the above argument for the sequence of discrete representations:

$\phi_m : G \to G \subset M(n)\,(m \in N)$. Let us put again $d_m = \min_{x \in \mathbf{H}^n} \max_{1 \le i \le l} d_{\mathbf{H}^n}(x, \phi_m(g_i)(x))$, where G is generated by $\{g_1, ..., g_l\}$.

Similarly to a), if $d_m \to \infty$ we get a small, isometric, non-trivial G-action on an $\mathbf{R}$-tree. Then again by Rips' theory this is impossible since our group G does not split over a virtually abelian subgroup. Hence, ϕ_m has a converging subsequence which we denote by the same letter. By applying once more Lemma 3.2, we have a point $x_0 \in \mathbf{H}^n$ and a sequence $(\alpha_m) \subset G$ such that

$$d_{\mathbf{H}^n}(x_0, \alpha_m \phi_m(g_i)\alpha_m^{-1}(x_0)) \le R < +\infty.$$

This means that after passing to a further subsequence the sequence $\alpha_{m_k}\phi_{m_k}\alpha_{m_k}^{-1}$ converges to a discrete faithful representation $\phi : G \to M(n)$.
All elements $\alpha_{m_k}\phi_{m_k}(g_i)\alpha_{m_k}^{-1}$ are in G, whose discreteness allows one to conclude that

$$\exists k_0\, \forall k > k_0\ \alpha_{m_k}\phi_{m_k}\alpha_{m_k}^{-1} = \alpha_{m_{k_0}}\phi_{m_{k_0}}\alpha_{m_{k_0}}^{-1}, \text{ for all } i \in \{1, ..., l\}.$$

Consequently, $\phi_{m_k}(g_i) = \phi_{m_{k_0}}(\eta_{m_k} \cdot g_i \cdot \eta_{m_k}^{-1})$, where $\eta_{m_k} = \alpha_{m_k}^{-1} \cdot \alpha_{m_{k_0}} \in G\,(k \in \mathbf{N})$, which implies that ϕ_{m_k} and $\phi_{m_{k_0}}$ represent the same element of $Out(G)\,(k > k_0)$ contradicting to our choice. The Lemma is proved. *QED.* □

LEMMA 3.4. – *Let $G \subset M(n)$ be a non-elementary Kleinian group, and $\varphi : G \to \Gamma$ an isomorphism onto a geometrically finite Kleinian group $\Gamma \subset M(n)$ which splits over none of its virtually abelian subgroup (including the trivial one). Then $\alpha G \alpha^{-1} \subset G$ implies $\alpha G \alpha^{-1} = G$ for any $\alpha \in M(n)$.*

Proof. – Suppose not. Then we have $\alpha G \alpha^{-1} < G$, where the symbol $<$ denotes proper embedding.

Put $G_n = \alpha^n G \alpha^{-n}\,(n \in \mathbf{Z})$. Then we obtain

$$G_n \supset G_{n+1}, \quad \alpha^n \notin G \quad (n \in \mathbf{Z}). \tag{2}$$

Since G_n does not split over a virtually abelian subgroup by Lemma 3.3 (a), it follows that all but finitely many G_n are conjugate by elements of $M(n)$. Therefore for some sequence $(m_k) \subset \mathbf{Z}$ one has

$G_{m_k} = \gamma_{m_k} G_{m_{k_0}} \gamma_{m_k}^{-1}$, $\gamma_{m_k} \in G$, or else $\alpha^{m_k} G \alpha^{-m_k} = \gamma_{m_k} \alpha^{m_{k_0}} G \alpha^{-m_{k_0}} \gamma_{m_k}^{-1}$, $(k \in \mathbf{N})$.

Therefore $G = \beta G \beta^{-1}$, where $\beta = \alpha^{-m_k} \gamma_{m_k} \alpha^{m_{k_0}} \in M(n)$ for each $k \in \mathbf{N}$. (3)

Since the assertion of Lemma 3.3.b is purely algebraic, it remains valid for groups isomorphic to geometrically finite ones, so we may apply it to the group G.

Thus,

$$\exists \xi \in G, \quad \exists p \in \mathbf{N} \quad \beta^p \gamma \beta^{-p} = \xi \gamma \xi^{-1} \quad \text{for all}\, \gamma \in G.$$

We have that for any $\gamma \in G : \beta_1 \gamma \beta_1^{-1} = \gamma$, where $\beta_1 = \xi^{-1} \beta^p$. Hence either $\beta_1 \equiv \mathrm{id}$ and $\xi = \beta^p$, or the group G leaves invariant the fixed point set of $\beta_1 \in M(n)$.

Since G is non-elementary, the latter case may happen only if β_1 is elliptic and G keeps some hyperbolic subspace $\mathbf{H}^r \subset \mathbf{H}^n$ $(r < n)$ invariant. Therefore, we can repeat the previous argument for the action of G on $\mathbf{H}^r$. This process will end for some $r > 2$ since, otherwise, for $r = 2$, we would get a Fuchsian group $G \subset PSL(2, \mathbf{R})$ splitting over $\mathbf{Z}$ which is impossible by the hypothesis.

Thus, without loss of generality we may assume that G is not leaving any geodesic subspace invariant and $\xi = \beta^p \in G$.

By using the above expression (3) for β, we get $\Pi_{i=1}^{p} (\alpha^{-m_k} \gamma_{m_k} \alpha^{m_{k_0}}) = \gamma \in G$.

Since $\gamma_{m_k} \in G$ and $\alpha^t \gamma_{m_k} \alpha^{-t} \in G_t$ $\forall t \in \mathbf{Z}$ we obtain after straightforward calculations that

$\alpha^{p \cdot (m_{k_0} - m_k)} \in G_l$, where $l = \min_{1 \le s \le p} \{(s-1) m_{k_0} - s m_k\}$, for any fixed $k \in \mathbf{N}$.

Hence, $\alpha^{p \cdot (m_{k_0} - m_k)} = \alpha^l \gamma_0 \alpha^{-l}$ for some $\gamma_0 \in G$ and, so $\alpha^{p (m_{k_0} - m_k)} = \gamma_0 \in G$ which is impossible by (2).

Thus the Lemma is proved. *QED.* □

Proof of Proposition 3.1. – Suppose that we a have a monomorphism $\psi : G \to G$ such that $G_1 = \psi(G) \subset G$. By iterating this map we get $G_m = \psi^m(G) \subset G_{m-1}$ $(m \in \mathbf{N})$. Then by applying Lemma 3.3.a to groups G_m (which do not split over virtually abelian subgroups either) we get an infinite sequence $r_k \in \mathbf{N}$ of indices such that $G_{m+r_k} = g_k G_m g_k^{-1}$ where $g_k \in G$ and $m \in \mathbf{N}$. Since $G_{m+r_k} \subset G_m$ we must have $G_{m+r_k} = G_m$ by Lemma 3.4.

It follows that $\psi^{r_k}(G) = G$ which implies that $\psi(G) = G$. Thus the Proposition is proved. *QED.* □

4. Examples

We provide below different types of examples to show that all conditions of Theorem 1.1 are essential.

PROPOSITION. – *For any natural numbers n and k such that $n \geq 3$ and $0 \leq k < n-1$, there exists a non-elementary, torsion-free, geometrically finite, freely indecomposable Kleinian group $\Gamma \subset Iso_+ \mathbf{H}^n \cong M(n)$ which splits as HNN-extension over an abelian parabolic subgroup of rank k and which is not co-Hopf.*

Remark. – Such examples are impossible for word-hyperbolic groups by the result of [Se].

Proof. – We will construct a group Γ acting on $\mathbf{H}^n$ which splits over an abelian subgroup H of rank $k = n - 2\,(n \geq 3)$ contained with infinite index in some bigger parabolic subgroup of Γ. To get all other values of $n - k$, it is enough to extend the action of Γ in the hyperbolic space $\mathbf{H}^m$ where $m > n$.

We start with a non-uniform lattice $F' \subset Iso_+\mathbf{H}^{n-1}$ having at least one maximal parabolic subgroup $H \subset F'$ of rank $k = n - 2$ (in particular F' can be taken as an arithmetic lattice).

Consider the ball model of the hyperbolic space $B = \mathbf{B}^{n-1} = \{x \in \mathbf{R}^{n-1} : |x| < 1\}$ and the action of F' on B, and on its exterior $ext(B) = \overline{\mathbf{R}}^{n-1} \setminus \overline{B}$ as well using the reflection with respect to ∂B. Let $p = Fix(H)$ be the fixed point of the group H and $\Sigma \subset ext(B)$ a precisely invariant horosphere under H in F'. Denote by $\tau \in M(n)$ the reflection with respect to Σ and put $F =< F', F'' >$ where $F'' = \tau F' \tau^{-1}$. The group H acts cocompactly on Σ. We also have $\forall g \in F' \setminus H\; g\Sigma \cap \Sigma = \emptyset\,(i = 1, 2)$ and the same for F''. Therefore we can apply Maskit Combination I theorem [Ma] to conclude that the group $F \subset M(n)$ is Kleinian and is isomorphic to $F' *_H F''$.

Take now a small horoball $D_1 \subset B$ which is precisely invariant under H in F'. Then $\tau(D_1) = D_2$ is also precisely invariant under H in F''. Consider the parabolic element $\gamma = \tau \circ \tau_1$ where τ_1 is the reflection with respect to ∂D_1. The element γ belongs to the centre $\mathbf{Z}(H)$ of H in $M(n)$. To see this, we just send the point p to the infinity so that the sphere $\partial B = \mathbf{S}^{n-2}_\infty$ coincides with the hyperplane $\overline{\mathbf{R}}^{n-2} \subset \overline{\mathbf{R}}^{n-1}$. We have that $\gamma(x) = x + e_{n-1}$ where e_{n-1} is a vector orthogonal to $\mathbf{R}^{n-2}$. The group H is generated by elements $< h_1, ..., h_{n-2} >$ so that $h_i(x) = r_i(x) + b_i$ where $r_i \in O(n-2)$. Thus, we conclude that all r_i commute with γ, and so do $h_i\,(i \in \{1, ..., n-2\})$.

Let us now show that each closed ball $\overline{D}_i\,(i = 1, 2)$ is precisely invariant under H in F. We shall check this, say, for D_1.

Let $g \in F \setminus H$ so that $g = g_n \cdot g_{n-1} \cdot ... \cdot g_1$, where either $g_i \in F' \setminus H$, $g_{i+1} \in F'' \setminus H$ or $g_i \in F'' \setminus H$, $g_{i+1} \in F' \setminus H$.

If $g_1 \in F'' \setminus H$, then $g_1(D_1) \subset ext(\overline{B})$, and it thus follows easily that $g(D_1) \subset ext(\overline{B})$. This implies that if $g(\overline{D}_1) \cap \overline{D}_1 \neq \emptyset$ then $g(\overline{D}_1) \cap \overline{D}_1 = \{p\}$, so that $g(p) = p$ and $g \in H$.

If $g_1 \in F' \setminus H$ then $g_1(\overline{D}_1) \subset B \setminus \overline{D}_1$ and $g_2 \cdot g_1(\overline{D}_1) \subset ext(\overline{B})$. We repeat now the previous argument to show that $g(\overline{D}_1) \cap \overline{D}_1 = \emptyset$.

We can now apply Maskit Combination II theorem to affirm that the group $\Gamma =< F, \gamma \,|\, \gamma h \gamma^{-1} = h,\, h \in H >$ is Kleinian and is isomorphic to the HNN-extension $F*_H$ [Ma].

Indeed, γ maps the open ball D_1 to the exterior of the closed ball $\overline{D}_2$. Therefore, the pair (D_1, D_2) is precisely invariant under (H, H) in the group G, hence $\overline{D}_1$ and $\overline{D}_2$ are jointly γ-blocked closed topological discs in the terminology of [Ma], and our assertion follows.

Let us define a map $\phi : \Gamma \to \Gamma$ by putting $\phi|_F \equiv id$ and $\phi(\gamma) = \gamma^2$. It is now straightforward to see that ϕ is an isomorphism onto $\Gamma_1 =< F, \gamma^2 >$ due to the relation $\gamma^2 h \gamma^{-2} = h\,(h \in H)$.

Let us prove that $\gamma \notin \Gamma_1$, which follows from the consideration of normal forms in the HNN-extension. Indeed, if $\gamma \in \Gamma_1$, then it can be written under the following normal form: $\gamma^{2i_1} \cdot g_{k_1} \cdot \gamma^{2i_2} \cdot g_{k_2} \cdot ... \cdot \gamma^{2i_r} \cdot g_{k_r}$, where if $i_k < 0$ and $g_{k_i} \in H$ then $i_{k+1} \leq 0$; (*) if $i_k > 0$ and $g_{k_i} \in H$, then $i_{k+1} \geq 0$.

It is well-known (see e.g. [Ma]) that such a normal form cannot be the identity unless it is trivial. By multiplying the above normal form $(*)$ by γ^{-1} we obtain now a normal form in Γ, not equal to the identity either, meaning that $\gamma \notin \Gamma_1$.

The group Γ_1 is a proper subgroup of Γ isomorphic to itself. The Proposition is proved. *QED.* □

Remark. – In dimension 3 one can construct such examples by using the hyperbolization theorem.

PROPOSITION 4.2 (Example 2). – *There exists a geometrically finite, torsion-free, non-elementary, freely indecomposable Kleinian group $G \subset M(n)$ which contains infinitely many conjugacy classes of subgroups isomorphic to a fixed non-elementary, freely indecomposable subgroup $F \subset G$. In particular, if $n = 3$ then we can take F to be isomorphic to $\pi_1(S_g)$, where S_g is closed surface of genus $g > 2$.*

Proof. – The idea of this example in dimension $n = 3$ is essentially due to W.Thurston (see 8.8.6 in [Th1]) we will clarify and generalize it to arbitrary dimension.

We start from the following lemma.

LEMMA 4.3. – *Suppose $B = A*_H = \{A, b : bHb^{-1} = H\}$ is a HNN extension of the group A over the subgroup H. If $F \subset A$ is a subgroup which splits over the same subgroup H as amalgamated free product $F = F' *_H F''$ such that $F' \neq H$ and $F'' \neq H$ then the groups $F_n = F' *_H b^n F'' b^{-n}$ $(n = 1, 2, 3, \ldots)$ are mutually non-conjugate in B.*

Proof. [1] – Let B act on the Bass-Serre tree $\mathcal{G}$ corresponding to the splitting $B = A*_H$. We can write each element $f \in F_n$ as $f = a_0 b^n a_1 b^{-n} a_2 b^n a_3 b^{-n} \ldots\ a_{2p}$. Since both splittings of B and F_n are over the same subgroup H we can affirm that $a_{2s} \in F' \setminus H$ and $a_{2s+1} \in F'' \setminus H$. We note that each term like $b^n a_{2s+1} b^{-n}$ gives a contribution of $2n$ to the translation length of the element f. Therefore the set of all translation lengths of the elements of F_n is $2n \cdot \mathbf{N}$. On the other hand this set is invariant under conjugation in B so we conclude that all subgroups F_n are mutually non-conjugate. Thus the Lemma is proved. *QED.*

To finish the proof of the Proposition we shall apply lemma 4.3 to the group $\Gamma = F*_H$ obtained in Proposition 4.1 (by putting $F = A$). Since $F = F' *_H F''$, we obtain again that all the groups $F_n = F' *_H (\gamma^n F'' \gamma^{-n})$ are not conjugate in the whole group Γ (in the notations of Proposition 4.1).

The Proposition is proved. *QED.* □

5. Proper conjugation of topologically tame Kleinian groups

In this section, we shall prove the latter half of Theorem 1.3.

Let $G \subset PSL_2\mathbf{C}$ be a finitely generated, torsion-free, non-elementary, topologically tame Kleinian group. Seeking a contradiction, we suppose that there is an $\alpha \in PSL_2\mathbf{C}$ such that $\alpha G \alpha^{-1}$ is contained in G as a proper subgroup.

[1] This elegant "arboreal" proof was suggested to us by the referee.

Since $\mathbf{H}^3/(\alpha G\alpha^{-1})$ is isometric to $\mathbf{H}^3/G$, the group $\alpha G\alpha^{-1}$ is also topologically tame. Let $\iota : \alpha G\alpha^{-1} \to G$ be the inclusion. Associated with ι, there is a covering $p : \mathbf{H}^3/(\alpha G\alpha^{-1}) \to \mathbf{H}^3/G$. Since ι takes parabolics in $\alpha G\alpha^{-1}$ to those of G, the covering map p induces one between the non-cuspidal parts $p : (\mathbf{H}^3/\alpha G\alpha^{-1})_0 \to (\mathbf{H}^3/G)_0$. (Note that we abuse the notation slightly and use the same symbol p as the original covering.) By proving that p is a homeomorphism, we shall prove that $\alpha G\alpha^{-1}$ is equal to G, and get a contradiction. Let $(C, \partial_0 C)$ be a relative core of $(\mathbf{H}^3/G)_0$, and $(C_1, \partial C_1)$ that of $(\mathbf{H}^3/\alpha G\alpha^{-1})_0$.

Let e be a geometrically infinite end of $(\mathbf{H}^3/\alpha G\alpha^{-1})_0$. Let S_0 be a frontier component of C_1 facing e. Since $\alpha G\alpha^{-1}$ is topologically tame, the end e has a neighbourhood homeomorphic to $S_0 \times \mathbf{R}$. By Canary's covering theorem ([Ca]), if we take a sufficiently small neighbourhood E of e, which we can assume to be homeomorphic to $S_0 \times \mathbf{R}$, the restriction $p|E$ is a finite-sheeted covering to its image, and the image is a neighbourhood of a geometrically infinite end of $(\mathbf{H}^3/G)_0$.

Before considering the general case, let us consider a special case when all the ends of $(\mathbf{H}^3/G)_0$ (hence also all the ends of $(\mathbf{H}^3/\alpha G\alpha^{-1})_0$) are geometrically infinite. In this case, since each end has a neighbourhood in which p is a finite-sheeted, the covering p itself is also finite-sheeted. Since the number of ends of $(\mathbf{H}^3/\alpha G\alpha^{-1})_0$ is equal to that of $(\mathbf{H}^3/G)_0$, this implies that p is a homeomorphism unless there is a compact surface Σ such that the pair $(C, \partial_0 C)$ is homeomorphic as a pair to $(\Sigma \times I, \partial\Sigma \times I)$, or $(\Sigma \tilde{\times} I, \partial\Sigma \times I)$ if Σ is non-orientable. In these exceptional cases, by considering the genus of Σ, we can see that p must be a homeomorphism. Thus we have completed the proof of the theorem in this special case.

Now allow $(\mathbf{H}^3/G)_0$ to have also a geometrically finite end. As we assumed $\alpha G\alpha^{-1}$ is a proper subgroup of G, we have a proper descending sequence $G > \alpha G\alpha^{-1} > \alpha^2 G\alpha^{-2} > \cdots$. Let G_n denote $\alpha^n G\alpha^{-n}$, and $p_n : \mathbf{H}^3/G_{n+1} \to \mathbf{H}^3/G_n$ be the covering projection associated with the inclusion. (Again we use the same symbol p_n also to denote the induced covering from $(\mathbf{H}^3/G_{n+1})_0$ to $(\mathbf{H}^3/G_n)_0$.) Associated with the isomorphism from G_n to G_{n+1} which the conjugation by α induces, there is an isometry $k_n : \mathbf{H}^3/G_n \to \mathbf{H}^3/G_{n+1}$. Let P_{n-1} denote $p_0 \circ \ldots \circ p_{n-1}$ and K_{n-1} denote $k_{n-1} \circ \ldots \circ k_0$. Let $(C_n, \partial_0 C_n)$ be a relative core of $(\mathbf{H}^3/G_n)_0$, which we take to be $K_{n-1}(C, \partial_0 C)$.

Let e be a geometrically finite end of $(\mathbf{H}^3/G)_0$. Let S be the frontier component of the convex core of $\mathbf{H}^3/G$ whose restriction to $(\mathbf{H}^3/G)_0$ is S_0 facing e. Let $f_n : S \to \mathbf{H}^3/G$ be a pleated surface defined by $f_n = P_{n-1} \circ K_{n-1}|S \; (= p_0 \circ p_1 \circ \ldots \circ p_{n-1} \circ k_{n-1} \circ \ldots \circ k_1 \circ k_0|S) : S \to \mathbf{H}^3/G$. We shall prove $\{f_n\}$ converges uniformly on every compact set of S after taking a subsequence.

Since f_n maps cusps of S to those of $\mathbf{H}^3/G$, we have only to prove that the images of the non-cuspidal part of S remain to be contained in a fixed compact subset of $\mathbf{H}^3/G$. (Refer to Canary-Epstein-Green [CEG].) As the isometry type of $f_n(S)$, with respect to the path metrics induced from the hyperbolic metric on $\mathbf{H}^3/G$, is independent of n, we can assume that $f_n(S)$ intersects the cuspidal part of $\mathbf{H}^3/G$ only at its own cuspidal part. Furthermore by the same reason, it is sufficient to prove that $f_n(S_0)$ does not tend to an end (after taking a subsequence).

Suppose, on the contrary, that $\{f_n(S_0)\}$ tends to an end e' of $(\mathbf{H}^3/G)_0$. This implies, in particular, that the end e' is geometrically infinite. Note that the number of the geometrically infinite ends of $(\mathbf{H}^3/G_n)_0$ is the same as that of $(\mathbf{H}^3/G)_0$ because these two manifolds are isometric. Since every geometrically infinite end of $(\mathbf{H}^3/G_n)_0$ has a neighbourhood which finitely covers that of $(\mathbf{H}^3/G)_0$ by P_{n-1}, we have an end e'_n of $(\mathbf{H}^3/G_n)_0$ with a neighbourhood E'_n parametrized by $\Sigma_n \times \mathbf{R}$ for a compact surface Σ_n, which finitely covers a neighbourhood of e'.

Taking sufficiently large n and $t \in \mathbf{R}$, we can see that there are finite-sheeted coverings $q : \tilde{S}_0 \to S_0$ and $\rho : \tilde{\Sigma}_n \to \Sigma_n$ such that $f_n \circ q(\tilde{S}_0)$ is homotopic to $P_{n-1}(\rho(\tilde{\Sigma}_n) \times \{t\})$, because of the following. Since e' is topologically tame, there is a compact surface S' such that e' has a neighbourhood homeomorphic to $S' \times \mathbf{R}$, which we denote itself by $S' \times \mathbf{R}$ by slightly abusing the notation. As $P_{n-1}|E_n$ is finite-sheeted covering to its image, for sufficiently large $t \in \mathbf{R}$, the restriction $P_{n-1}(\Sigma_n \times \{t\})$ is homotopic to a finite-sheeted covering of $S' \times \{\text{pt.}\}$. On the other hand, if we take a sufficiently large n, the surface $f_n(S_0)$ is contained in $S' \times \mathbf{R}$, and since f_n maps cusps to cusps, the surface $f_n(S_0)$ is homotopic to a finite-sheeted covering of $S' \times \{\text{pt.}\}$ as we can see by using the fact that both S_0 and S' are compact and the boundary of S_0 is mapped into that of $S' \times \mathbf{R}$ by f_n. Thus we have proved what we claimed in the first sentence of this paragraph.

By lifting $f_n \circ q(\tilde{S}_0), P_{n-1}(\rho(\tilde{\Sigma}_n) \times \{t\})$, and a homotopy between them to $(\mathbf{H}^3/G_n)_0$, we can see that $K_{n-1} \circ q(\tilde{S}_0)$ and $\rho(\tilde{\Sigma}_n) \times \{t\}$ are homotopic. Recall that since K_{n-1} is an isometry, $K_{n-1}(S_0)$ is a frontier component of the convex core of $(\mathbf{H}^3/G_n)_0$ facing a geometrically finite end, in particular it is homotopic to a frontier component of a relative core C_n of $(\mathbf{H}^3/G_n)_0$. Thus in this situation, finite-sheeted coverings of two frontier components of a relative C_n, one facing a geometrically finite end, i.e., $K_{n-1}(S_0)$, and the other facing the end e'_n, which we shall denote by Ξ_n, are homotopic. (Note that for a topologically tame end whose end is parametrized by $X \times \mathbf{R}$ and which faces a frontier component X' of a core, X and X' are homotopic. This can be proved, for instance, by using the uniqueness of core proved by McCullough-Miller-Swarup [MMS].) Obviously this is possible only when C_n has exactly two frontier components. Furthermore even when C_n has two frontier components, by considering a relative version of characteristic compression body defined by Bonahon [Bo] as we shall see in the following, we can see that this is possible only when the frontier components of C_n are incompressible, which implies that both $\rho(\tilde{S}_0)$ and $q(\tilde{\Sigma}_n) \times \{t\}$ are incompressible.

Suppose, on the contrary, that C_n has a compressible frontier component, say that Ξ_n is compressible. Consider a "relative" characteristic compression body W_n of C_n containing Ξ_n as the exterior boundary, that is, the union of Ξ_n and disjoint regular neighbourhoods of non-parallel compression discs for Ξ_n. Then, since the surfaces $K_{n-1}(S_0)$ and Ξ_n are separated by the interior boundary of W_n, which is incompressible, and the groups $i_*\pi_1(K_{n-1}(S_0))$ and $i_*\pi_1(\Xi_n)$ are commensurable, it follows that the group $i_*\pi_1(\Xi_n)$ is commensurable with the fundamental group of a component of the interior boundary of W_n (where i_* is the induced map on the fundamental groups by the inclusion). This is a contradiction, and we have proved that the frontier components of C_n are incompressible.

As $\rho(\tilde{\Sigma}_n) \times \{t\}$ is homotopic to a finite-sheeted covering of Ξ_n, let $\rho' : \tilde{\Xi}_n \to \Xi_n$ denote a covering homotopic to $\rho(\tilde{\Sigma}_n) \times \{t\}$. Take a covering $\tilde{C}_n$ of C_n associated with $q_\#(\pi_1(\tilde{S}_0)) = \rho'_\#(\pi_1(\tilde{\Xi}_n))$. Then both q and ρ' are lifted to $\tilde{C}_n$ as homeomorphisms into the boundary of $\tilde{C}_n$, whose image we shall denote by $\overline{S}_0$ and $\overline{\Xi}_n$. In $\tilde{C}_n$, the surfaces $\overline{S}_0$ and $\overline{\Xi}_n$ are homotopic, hence bound a trivial I-bundle between them. Since $\overline{S}_0$ and $\overline{\Xi}_n$ lie on the boundary, it follows that $\tilde{C}_n$ itself is homeomorphic to $\overline{\Xi}_n \times I$. This implies that

C_n is a trivial I-bundle over a frontier component since it is covered by a trivial I-bundle and has two frontier components. (Refer, for instance, Hempel [He].)

Thus in this case, $(\mathbf{H}^3/G_n)_0$ is homeomorphic to $\Sigma_n \times \mathbf{R}$, as Γ_n is topologically tame. As a neighbourhood of an end of $(\mathbf{H}^3/G_n)_0$ carries all the elements of the fundamental group then, the fact that there is a geometrically infinite end whose neighbourhood finitely covers its image implies that P_{n-1} is itself a finite-sheeted covering. Since both $(\mathbf{H}^3/G_n)_0$ and $(\mathbf{H}^3/G)_0$ are homeomorphic to $\Sigma_n \times \mathbf{R}$, the covering is a homeomorphism and $G_n = G$. This contradicts our assumption. Thus we have proved that $\{f_n\}$ must converge after taking a subsequence.

Let $\{f_{n_j}\}$ be a convergent subsequence of $\{f_n\}$. Then there exists an integer j_0 such that the map f_{n_j} and $f_{n_{j'}}$ are homotopic if $j > j' \geq j_0$. Let T be the frontier component of C_{j_0} corresponding to S_0, i.e., which faces the same end as S_0 does. Let $T_n = K_{n-1}(T)$, which is a frontier component of the relative core C_n. Since f_{n_j} and $f_{n_{j'}}$ are homotopic, we can see that so are $p_{n_{j'}} \circ \cdots \circ p_{n_j-1}(T_{n_j})$ and $T_{n_{j'}}$ by lifting a homotopy to $\mathbf{H}^3/G_{n'_j}$. Then it follows that if $j > j' \geq j_0$, then $p_{n_{j'}} \circ \cdots \circ p_{n_j-1}$ maps a subgroup of $\pi_1(C_{n_j}) \cong G_{n_j}$ corresponding to $\pi_1(T_{n_j})$ to that of $\pi_1(C_{n_{j'}})$ corresponding to $\pi_1(T_{n_{j'}})$.

Take another geometrically finite end of $\mathbf{H}^3/G$ facing a frontier component T' of C and repeat the same argument. Let T'_n denote $K_{n-1}(T')$. Then we can see that if we take a subsequence $\{n_l\}$ of $\{n_j\}$, there exists a l_0 such that if $l > l' \geq l_0 \geq j_0$, then $p_{n_{l'}} \circ \ldots \circ p_{n_l-1}$ maps subgroups of $\pi_1(C_{n_l})$ corresponding to $\pi_1(T_{n_l})$ and $\pi_1(T'_{n_l})$ to those of $\pi_1(C_{n_{l'}})$ corresponding to $\pi_1(T_{n_{l'}})$ and $\pi_1(T'_{n_{l'}})$ respectively.

We repeat the same for all the geometrically finite ends of $\mathbf{H}^3/G$, taking geometrically finite ends one by one, and choosing a subsequence each time. Then we obtain a subsequence $\{n_i\}$ of $\{n\}$ and an i_0 such that if $i > i' \geq i_0$, then for every frontier component F facing a geometrically finite end and $F_n = k_n(F)$, the covering map $p_{n_{i'}} \circ \ldots \circ p_{n_i-1}$ maps a subgroup of $\pi_1(C_{n_i}) \cong G_{n_i}$ corresponding to $\pi_1(F_{n_i})$ to that of $\pi_1(C_{n_{i'}})$ corresponding to $\pi_1(F_{n_{i'}})$. Combining this with the fact that the restriction of $p_{n_{i'}} \circ \ldots \circ p_{n_i-1}$ to a neighbourhood of each geometrically infinite end is a finitely-sheeted covering to its image, and recalling that the cusps are preserved by this map, we can see that $p_{n_{i'}} \circ \ldots \circ p_{n_i-1}$ can be homotoped to a map $q_{i,i'}$ whose restriction to C_{n_i} is a covering map to $C_{n_{i'}}$ by Theorem 13.6 in Hempel [He].

Since both C_{n_i} and $C_{n_{i'}}$ are compact and their frontier components have negative Euler numbers, counting the number of the frontier components or considering the Euler number as before, we can see that $q_{i,i'}|C_{n_i}$ must be a homeomorphism. It follows that $p_{n_{i'}} \circ \ldots \circ p_{n_i-1}$ induces an isomorphism between the fundamental groups, which means that $G_{n_i} = G_{n_{i'}}$. This contradicts our assumption. Thus we have completed the proof of (b) in Theorem 1.3.

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REFERENCES

[B] R. BAER, *Groups without proper isomorphic quotient groups,* (*Bull. Amer. Math. Soc.*, Vol. 50, 1944, pp. 267-278).

[Bo] F. BONAHON, *Cobordism of automorphisms of surfaces,* (*Ann. Sci. Ecole Norm. Sup.*, Vol. 16 (4), 1983, pp. 237-270).

[Be] M. BESTVINA, *Degenerations of the hyperbolic space,* (*Duke Math. J.*, Vol. 56, 1988, pp. 143-161).

[Be-F] M. BESTVINA and M. FEIGHN, *Stable actions of groups on real trees,* (*Inv. Math.*, Vol. 121, 1995, pp. 287-321).

[Ca] R. CANARY, *A covering theorem for hyperbolic 3-manifold and its applications,* (*Topology*, Vol. 35, 1996, pp. 751-778).

[CEG] R. D. CANARY, D. B. A. EPSTEIN, P. Green, Notes on notes of Thurston, *Analytical and geometric aspects of hyperbolic spaces* 3-92 (*London Math. Soc. Lecture Note Ser.* Vol. 111 Cambridge Univ. Press 1987).

[De] T. DELZANT, *L'image d'un groupe dans un groupe hyperbolique,* (*Comment. Math. Helvetici* Vol. 70, 1995, pp. 267-284).

[GW] F. GONZALES-ACUNA and W. WHITTEN, *Embeddings of 3-manifold groups,* (*Mem. Amer. Math. Soc.*, No 474).

[GLW] F. GONZALES-ACUNA, R. LITHERLAND and W. WHITTEN, *Cohopficity of Seifert bundle groups,* (*Trans. Amer. Math. Society*, Vol. 341, 1994, No 1, pp. 143-155).

[Gro] M. GROMOV, *Hyperbolic groups, in Essays in Group Theory,* Ed. : S.M. Gersten, M.S.R.I. Pub. 8, Springer Verlag 1987.

[H] M. HEINS, *On a problem of H. Hopf,* (*Jour. de. Math.*, Vol. 37, 1958, pp. 153-160).

[He] J. HEMPEL, *3-manifolds,* (*Ann. Math. Studies*, 86, Princeton Univ).

[Ka] M. KAPOVICH, *Notes on Thurston hyperbolization theorem,* Preprint, University of Utah, 1997 (book in preparation).

[JMP] T. JORGENSEN, A. MARDEN, C. POMMERENKE, *Two examples of covering surfaces, In: Riemann Surfaces and Related Topics,* I. KRA and B. MASKIT ed., pp 305-319, (*Ann. Math. Studies*, 97, Princeton, 1978).

[Mar] A. MARDEN, *The geometry of finitely generated Kleinian groups,* (*Ann. Math.*, Vol. 99, 1974, pp. 383-462).

[Ma] B. MASKIT, *Kleinian groups,* Springer Verlag, 1987.

[Mc] D. MCCULLOUGH, *Compact submanifolds of 3-manifolds with boundary,* (*Quart. J. Math. Oxford* Vol. 37, 1986, pp. 299-307).

[MMS] D. MCCULLOUGH, A. MILLER and G. A. SWARUP, *Uniqueness of cores of non-compact 3-manifolds,* (*J. London Math. Soc.*, Vol. 32, 1985, pp. 548-556).

[Mo1] J. W. MORGAN, *On Thurston's uniformization theorem for three-dimensional manifolds, in "The Smith Conjecture",* edited by H. BASS and J. W. MORGAN, (Academic Press, 1984, pp. 37-125).

[Mo2] J.W. MORGAN, *Group actions on trees and compactifications of the space of classes of* $SO(n,1)$*-representations,* (*Topology*, Vol. 25, 1986, pp. 1-33).

[MS] J.W. MORGAN and P. SHALEN, Valuations, trees and degeneration of hyperbolic structures, (*Ann. of Math.*, Vol. 122, 1985, pp. 398-476).

[Ot] J.P. OTAL, *Le théorème d'hyperbolisation pour les varétés fibrées de dimension* 3, (*Astérisque*, No 235, 1996, 206 p).

[Pa] F. PAULIN, *Outer automorphisms of hyperbolic groups and small actions on* **R***-trees, In: "Arboreal group theory",* R. ALPERIN ed., pp. 331-343, (*Pub M.S.R.I.*, 19, Springer-Verlag 1991).

[PW] L. POTYAGAILO and S. WANG, *On the cohopficity of 3-manifold and Kleinian groups,* preprint (*IHES*, 1992).

[RS] E. RIPS and Z. SELA, *Structure and rigidity in hyperbolic groups I,* (*Geom. and Func. Anal.*, Vol. 4, 1994, pp. 337-371).

[Sc] G. P. SCOTT, *Compact submanifolds of 3-manifolds,* (*J. London Math. Soc.*, Vol. 7, 1973, pp. 246-250).

[Se] Z. SELA, *Structure and Rigidity in (Gromov) hyperbolic groups and discrete groups in rank 1 Lie group, II.* (*Geom. and Func. Anal.*, Vol. 7, 1997, pp. 561-593).

[Th1] W. THURSTON, *Geometry and topology on* 3*-manifolds,* Preprint Princeton, 1978.

[Th2] W. THURSTON, *Hyperbolic structures on* 3*-manifolds I: Deformation of acylindrical manifolds,* (*Ann. of Math.*, Vol. 124, 1986, pp. 203-246).

[WW] S. Wang and Y. Wu, *A covering invariant of graph manifolds,* (*Proc. London Math. Soc.*, Vol. 68 (3), 1994, pp. 203-224).

[WZ] S. Wang S. and Q. Zhou, *On the proper conjugation of Kleinian groups,* (*Geom. Dedicata*, Vol. 56, No 2, 1995, pp. 145-154).

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