ESAIM: COCV 19 (2013) 167–189
DOI: 10.1051/cocv/2012003

ESAIM: Control, Optimisation and Calculus of Variations
www.esaim-cocv.org

EPITAXIALLY STRAINED ELASTIC FILMS: THE CASE OF ANISOTROPIC SURFACE ENERGIES

Marco Bonacini[1]

Abstract. In the context of a variational model for the epitaxial growth of strained elastic films, we study the effects of the presence of anisotropic surface energies in the determination of equilibrium configurations. We show that the threshold effect that describes the stability of flat morphologies in the isotropic case remains valid for weak anisotropies, but is no longer present in the case of highly anisotropic surface energies, where we show that the flat configuration is always a local minimizer of the total energy. Following the approach of [N. Fusco and M. Morini, Equilibrium configurations of epitaxially strained elastic films: second order minimality conditions and qualitative properties of solutions. Preprint], we obtain these results by means of a minimality criterion based on the positivity of the second variation.

Mathematics Subject Classification. 74G55, 74K35, 74G65, 49Q20, 49K10.

Received June 22, 2011.
Published online 1st March 2012.

1. Introduction

The mechanism of epitaxial growth of an elastic film on a relatively thick substrate, in presence of a lattice mismatch at the interface between film and substrate, is understood to be governed by the competition of two opposing forms of energy, the bulk elastic energy and the surface energy. A proper variational formulation of the problem, which makes use of the tools of relaxation and geometric measure theory, is proposed in [2] and in [10]: here the film is modeled as a linear elastic solid grown on a flat substrate in a two-dimensional framework (corresponding to three-dimensional configurations with planar symmetry); equilibrium configurations correspond to minimizers of the total energy of the system, which is taken as the sum of the stored elastic energy and the energy of the free surface of the film. Due to the presence of these two competing forms of energy, flat morphologies become unstable after a critical value of the thickness of the film is reached: this threshold effect, known as Asaro-Grinfeld-Tiller (AGT) instability, is discussed in [14], while in [12] the authors determine analytically the critical threshold for the local minimality of the flat configuration using a minimality criterion based on the positivity of the second variation of the total energy. Concerning the regularity of local minimizers of the total energy, we refer to the recent paper [7], where it is proved that the profile of a volume constrained local minimizer may have at most a finite number of vertical cuts or cusp points, being of class C^1 away from

Keywords and phrases. Epitaxially strained crystalline films, anisotropic surface energy, second order minimality conditions, second variation.

[1] SISSA, Via Bonomea 265, 34136 Trieste, Italy. mbonacin@sissa.it

Article published by EDP Sciences

these singularities (see also [10], where the model is slightly different and a stronger notion of local minimality is considered).

In this paper we investigate the role played by the presence of anisotropy in the surface energy in determining the resulting equilibrium configurations: while in the analysis of [12] the surface term in the total energy was assumed to be isotropic, here we consider the free surface energy of the film to be of the form

$$\int_{\Gamma} \psi(\nu)\,\mathrm{d}\mathcal{H}^1, \tag{1.1}$$

where Γ is the free profile of the film, and ψ is a convex function of the normal ν to the surface of the film (see also [11], where a similar energy is considered in a slightly different context). The main information about the anisotropy is carried by the Wulff shape associated with ψ (see Sect. 2), which is the set that minimizes (1.1) under a volume constraint. We consider first the case of "weak" anisotropies, in which the surface density ψ satisfies a strong convexity condition (see (2.3)) and the corresponding Wulff shape is a regular set. Then we pass to the crystalline case, in which we assume that the boundary of the Wulff shape contains a horizontal facet intersecting the y-axis.

The main findings of our analysis are the following. In the case of regular anisotropies we observe the same qualitative behavior as in the isotropic case studied in [12]. Precisely, we show the existence of a volume threshold such that the flat configuration is a local minimizer for the total energy if and only if the volume is below the critical value (such a threshold is analytically determined). The situation is very different in presence of a crystalline anisotropy: in the main result of the paper (Thm. 2.9) we show that the flat configuration is *always* a strong local minimizer (see Def. 2.3) with respect to small L^∞-perturbations of the free profile, no matter how thick the film is. As in [12], these results are obtained by means of a sufficient condition for local minimality, expressed in terms of a suitable notion of second variation of the total energy.

The paper is organized as follows. In Section 2 we fix the notations, we describe the variational model and we state the main results. In Section 3 we compute the second variation of the total energy, and we start paving the way to the proof of the local minimality criterion, which will be completed in Section 4. Finally, Sections 5 and 6 are devoted to the proofs of the results concerning the stability of the flat configuration in the regular case and in the crystalline case, respectively.

2. Setting and main results

We start by describing the setting of the problem, as formulated in [2,12].

The reference configuration of the film is modeled as the subgraph of a lower semicontinuous function with finite pointwise total variation: given $b > 0$, we set

$$AP(0,b) := \{g : \mathbb{R} \to [0,+\infty) : g \text{ is lower semicontinuous and } b\text{-periodic}, \mathrm{Var}(g;0,b) < +\infty\},$$

where

$$\mathrm{Var}(g;0,b) := \sup\left\{ \sum_{i=1}^{k} |g(x_i) - g(x_{i-1})| : 0 < x_0 < x_1 < \ldots < x_k < b, k \in \mathbb{N} \right\}.$$

For an admissible profile $g \in AP(0,b)$, we introduce the sets

$$\begin{aligned}
\Omega_g &:= \{(x,y) : x \in (0,b),\ 0 < y < g(x)\},\\
\Gamma_g &:= \{(x,y) : x \in [0,b)\,,\ g^-(x) \le y \le g^+(x)\},\\
\Sigma_g &:= \{(x,y) : x \in [0,b)\,,\ g(x) < g^-(x),\ g(x) \le y \le g^-(x)\},
\end{aligned}$$

which will be referred to as the *reference configuration* of the film, the *free profile* of the film, and the set of *vertical cuts*, respectively (here $g^+(x) = g(x+) \vee g(x-)$ and $g^-(x) = g(x+) \wedge g(x-)$, where $g(x+)$ and $g(x-)$

denote the right and the left limits of g at x, respectively, which exist at every point). We consider also the b-periodic extension of the reference configuration:

$$\Omega_g^\# := \{(x,y) : x \in \mathbb{R},\, 0 < y < g(x)\}$$

(the sets $\Gamma_g^\#$, $\Sigma_g^\#$ are defined similarly). If g is Lipschitz, we denote by ν the exterior unit normal vector to Ω_g on Γ_g, and by $\tau = \nu^\perp$ the unit tangent vector to Γ_g (obtained rotating ν clockwise by $\frac{\pi}{2}$). Moreover, we denote by div_τ the tangential divergence on Γ_g, by ∂_τ, ∂_ν the tangential and normal derivative, and by D_τ, D_ν the tangential and normal gradient, respectively. Finally, for a sufficiently regular g the curvature of Γ_g will be denoted by

$$H = \mathrm{div}\nu = -\left(\frac{g'}{\sqrt{1+(g')^2}}\right)' \circ \pi_1 \qquad \text{on } \Gamma_g,$$

where $\pi_1 : \mathbb{R}^2 \to \mathbb{R}$ is the orthogonal projection on the x-axis.

In order to introduce the space of admissible elastic variations, we define for a given $g \in AP(0,b)$

$$LD_\#(\Omega_g;\mathbb{R}^2) := \{v \in L^2_{\mathrm{loc}}(\Omega_g^\#;\mathbb{R}^2) : v(x,y) = v(x+b,y) \text{ for } (x,y) \in \Omega_g^\#, E(v)|_{\Omega_g} \in L^2(\Omega_g;\mathbb{M}^{2\times 2})\},$$

where $E(v) := \frac{1}{2}(\nabla v + (\nabla v)^T)$ denotes the symmetrized gradient of v. We assign at the interface between the film and the substrate a boundary Dirichlet datum, which forces the film to be strained, of the form $u_0(x,0) = (e_0 x + q(x), 0)$, where $e_0 > 0$ and $q : \mathbb{R} \to \mathbb{R}$ is a b-periodic function of class C^∞ (the constant e_0 measures the lattice mismatch between film and substrate). Finally, let us introduce the following spaces of admissible pairs *film profile-deformation*:

$$\begin{aligned} Y(u_0;0,b) &:= \{(g,v) : g \in AP(0,b), v : \Omega_g^\# \to \mathbb{R}^2, v - u_0 \in LD_\#(\Omega_g;\mathbb{R}^2)\},\\ X(u_0;0,b) &:= \{(g,v) \in Y(u_0;0,b) : v(x,0) = u_0(x,0) \text{ for all } x \in \mathbb{R}\},\\ X_L(u_0;0,b) &:= \{(g,v) \in X(u_0;0,b) : g \text{ is Lipschitz continuous}\}. \end{aligned}$$

We consider the following notion of convergence in $Y(u_0;0,b)$: we say that a sequence (h_n,u_n) tends to (h,u) in Y iff

- $\sup_n \mathrm{Var}(h_n;0,b) < +\infty$;
- $d_H(\mathbb{R}^2_+ \setminus \Omega_{h_n}^\#, \mathbb{R}^2_+ \setminus \Omega_h^\#) \to 0$, where d_H is the Hausdorff distance defined as[2]

$$d_H(A,B) = \inf\{\varepsilon > 0 : A \subset \mathcal{N}_\varepsilon(B) \text{ and } B \subset \mathcal{N}_\varepsilon(A)\};$$

- $u_n \rightharpoonup u$ weakly in $H^1_{\mathrm{loc}}(\Omega_h^\#;\mathbb{R}^2)$

(note that this implies also that $h_n \to h$ in $L^1(0,b)$: see [10], Lem. 2.5). We have the following compactness theorem (see [2,10]):

Theorem 2.1. *Assume that* $(h_n,u_n) \in X(u_0;0,b)$ *satisfy*

$$\sup\left\{\int_{\Omega_{h_n}} |E(u_n)|^2\,\mathrm{d}z + \mathrm{Var}(h_n;0,b) + |\Omega_{h_n}|\right\} < +\infty.$$

Then there exists $(h,u) \in X(u_0;0,b)$ *such that, up to subsequences,* $(h_n,u_n) \to (h,u)$ *in* Y.

[2]Here $\mathcal{N}_\varepsilon(C)$ denotes the ε-neighborhood of a set C, and $\mathbb{R}^2_\pm = \{(x,y) : \pm y \geq 0\}$.

We are now ready to introduce the functional on X, which is the sum of the bulk elastic energy and of the energy of the free surface of the film. In our investigation, anisotropy is incorporated only in the surface term and neglected in the volume energy. This reflects the observation that surface anisotropy is more considerable than anisotropy in the elastic field. Hence we consider an elastic energy density of the form $W(u) := \frac{1}{2}\mathbb{C}E(u) : E(u)$, where

$$\mathbb{C}\xi := \begin{pmatrix} (2\mu+\lambda)\xi_{11} + \lambda\xi_{22} & 2\mu\xi_{12} \\ 2\mu\xi_{12} & (2\mu+\lambda)\xi_{22} + \lambda\xi_{11} \end{pmatrix} \qquad \text{for } \xi \in \mathbb{M}^{2\times 2}_{sym}.$$

Here μ and λ denote the Lamé coefficients, which are assumed to satisfy the ellipticity conditions $\mu > 0$, $\lambda + \mu > 0$ (note that $W(u) \geq \min\{\mu, \lambda+\mu\}|E(u)|^2$ and thus W is coercive).

We add to the elastic energy an anisotropic surface term: we consider a convex and positively homogeneous function of degree one $\psi : \mathbb{R}^2 \to [0, +\infty)$ satisfying the following condition:

$$m|z| \leq \psi(z) \leq M|z| \qquad \text{for every } z \in \mathbb{R}^2, \tag{2.1}$$

for some positive constants m and M.

Finally, we introduce the functional

$$\widetilde{F}(h,u) = \int_{\Omega_h} W(u)\,\mathrm{d}z + \int_{\Gamma_h} \psi(\nu)\,\mathrm{d}\mathcal{H}^1 \qquad \text{for } (h,u) \in X_L(u_0;0,b).$$

The functional $\widetilde{F}$, originally defined only for Lipschitz admissible profiles, can be extended to the whole space $X(u_0;0,b)$, by relaxation: we set for $(h,u) \in X(u_0;0,b)$

$$F(h,u) := \inf\Big\{ \liminf_{n\to\infty} \widetilde{F}(h_n,u_n) : (h_n,u_n) \in X_L(u_0;0,b), |\Omega_{h_n}| = |\Omega_h|, \, (h_n,u_n) \to (h,u) \text{ in } Y \Big\}.$$

The following theorem provides an explicit representation of the relaxed functional.

Theorem 2.2. *Let $\sigma = \psi(1,0) + \psi(-1,0)$. The following representation formula for F holds:*

$$F(h,u) = \int_{\Omega_h} W(u)\,\mathrm{d}z + \int_{\Gamma_h} \psi(\nu_h)\,\mathrm{d}\mathcal{H}^1 + \sigma\,\mathcal{H}^1(\Sigma_h) \tag{2.2}$$

where ν_h is the generalized outer normal to $\Omega_h^{\#} \cup \mathbb{R}^2_-$ at the points of its reduced boundary (which coincides, in the strip $[0,b) \times \mathbb{R}$, with Γ_h up to an $\mathcal{H}^1$-negligible set).

The proof can be obtained arguing as in [10], Theorem 2.8 and [2], Lemma 2.1, using Reshetnyak's lower semicontinuity and continuity theorems (see [1], Thms. 2.38 and 2.39) to treat the presence of anisotropy in the surface term (we refer also to the recent works [3,6] for related relaxation results in higher dimension).

We now define the notions of local minimizer, critical pair and flat configuration.

Definition 2.3. We say that $(h,u) \in X(u_0;0,b)$ is a *b-periodic local minimizer* for the functional F if there exists $\delta > 0$ such that

$$F(h,u) \leq F(g,v)$$

for all $(g,v) \in X(u_0;0,b)$ with $|\Omega_g| = |\Omega_h|$ and $\|g-h\|_\infty < \delta$; if the inequality is strict when $g \neq h$, we say that (h,u) is an *isolated b-periodic local minimizer*.

We will first study the situation in which the anisotropy is regular. Precisely, we make the following additional assumptions on the anisotropy ψ:

(R1) ψ is of class C^3 away from the origin;

(R2) the following strong convexity condition holds: for every $v \in \mathbb{S}^1$

$$\nabla^2\psi(v)[w,w] > c_0\,|w|^2 \qquad \text{for all } w \perp v, \tag{2.3}$$

for some constant $c_0 > 0$.

The results of Sections 3–5 are obtained under these hypotheses. We note for later use that by homogeneity

$$\nabla^2\psi(v)[v] = 0 \qquad \text{for every } v \in \mathbb{R}^2\setminus\{0\}. \tag{2.4}$$

Definition 2.4. We say that an element $(h,u) \in X(u_0;0,b)$ with $h \in C^2(\mathbb{R})$ is a *critical pair* for the functional F if u minimizes the elastic energy in Ω_h, that is, u satisfies the equation

$$\int_{\Omega_h} \mathbb{C}E(u) : E(w)\,\mathrm{d}z = 0 \qquad \text{for every } w \in A(\Omega_h), \tag{2.5}$$

where

$$A(\Omega_h) := \{w \in LD_\#(\Omega_h;\mathbb{R}^2) : w(\cdot,0) \equiv 0\},$$

and the following transmission condition holds:

$$W(u) + H^\psi = \text{const.} \qquad \text{on } \Gamma_h \cap \{y > 0\}. \tag{2.6}$$

Here H^ψ is the *anisotropic mean curvature* of Γ_h, defined as

$$H^\psi := \operatorname{div}(\nabla\psi \circ \nu) = \operatorname{div}_\tau(\nabla\psi \circ \nu)$$

(the second equality follows from $D\nu[\nu] = 0$).

Remark 2.5. The definition of critical pair is motivated by the Euler-Lagrange equation satisfied by a sufficiently regular (local) minimizer of F (see the formula for the first variation of F deduced in Step 1 of the proof of Theorem 3.1). Assuming more regularity in the anisotropy, we can apply standard results to deduce further regularity of a critical pair. In particular, if $h > 0$ and Γ_h is of class $C^{1,\alpha}$ for all $\alpha \in (0,1/2)$, then equation (2.5) (which is a linear elliptic system satisfying the Legendre-Hadamard condition) implies that $u \in C^{1,\alpha}(\overline{\Omega}_h)$ for all $\alpha \in (0,1/2)$ (see [12], Prop. 8.9). Moreover, if both ψ and u_0 are of class C^∞ (analytic, respectively), and equation (2.6) holds in the distributional sense, then (h,u) is of class C^∞ (analytic, respectively) by the results contained in [15], Section 4.2. Observe that condition (2.3) is exactly the assumption needed in the regularity result of [15].

Remark 2.6. We will make repeated use of the following explicit formula for the anisotropic mean curvature:

$$H^\psi(x,h(x)) = \big(\partial_1\psi(-h'(x),1)\big)'. \tag{2.7}$$

Indeed, from condition (2.4) it follows that $\nabla^2\psi(-h',1)[(-h',1)] = 0$, which in turn implies $\partial^2_{12}\psi(-h',1) = \partial^2_{11}\psi(-h',1)h'$; hence

$$\begin{aligned} H^\psi &= \operatorname{div}_\tau(\nabla\psi \circ \nu) = \partial_\tau(\nabla\psi \circ \nu)\cdot\tau \\ &= -\frac{h''}{1+h'^2}\Big[\partial^2_{11}\psi(-h',1) + h'\partial^2_{12}\psi(-h',1)\Big]\circ\pi_1 \\ &= -\Big(h''\,\partial^2_{11}\psi(-h',1)\Big)\circ\pi_1, \end{aligned}$$

which is (2.7).

Definition 2.7. The *flat configuration* corresponding to a given volume $d > 0$ and a boundary Dirichlet datum $u_0(x, 0) = (e_0 x, 0)$, $e_0 > 0$, is the pair $(\frac{d}{b}, v_{e_0})$ with

$$v_{e_0}(x, y) := e_0 \left(x, \frac{-\lambda}{2\mu + \lambda}\, y \right). \tag{2.8}$$

Notice that the flat configuration is a critical pair for F.

In Sections 3 and 4 we will prove a local minimality criterion for the functional F expressed in terms of the positivity of its second variation. The result will be established by implementing, in our anisotropic framework, the general strategy described in [12] to deal with the isotropic case. From this we will be able to deduce, in Section 5, a stability property for the flat configuration, showing that the qualitative results obtained in [12] hold also in the case of regular anisotropies: in particular we have a volume threshold of minimality, which can be determined analitically in terms of the *Grinfeld function* K, defined for $y \geq 0$ by

$$K(y) := \max_{n \in \mathbb{N}} \frac{1}{n} J(ny), \qquad J(y) := \frac{y + (3 - 4\nu_p) \sinh y \cosh y}{4(1 - \nu_p)^2 + y^2 + (3 - 4\nu_p) \sinh^2 y},$$

where $\nu_p = \frac{\lambda}{2(\lambda + \mu)}$ (the function K is strictly increasing and continuous, $K(y) \leq Cy$ for some positive constant C, and $\lim_{y \to +\infty} K(y) = 1$: see [12], Cor. 5.3). Precisely, we show:

Theorem 2.8. *For any $b > 0$ and $e_0 > 0$, let $d(b, e_0) \in (0, +\infty]$ be defined as $d(b, e_0) = +\infty$ if $0 < b \leq \frac{\pi}{4} \frac{(2\mu + \lambda)\, \partial^2_{11} \psi(0,1)}{e_0^2 \mu (\mu + \lambda)}$, and as the solution to*

$$K\left(\frac{2\pi d(b, e_0)}{b^2} \right) = \frac{\pi}{4} \frac{(2\mu + \lambda)\, \partial^2_{11} \psi(0, 1)}{e_0^2\, \mu\, (\mu + \lambda)} \frac{1}{b}$$

otherwise. Then the flat configuration $(\frac{d}{b}, v_{e_0})$ is an isolated b-periodic local minimizer for F if $0 < d < d(b, e_0)$, in the sense of Definition 2.3.

The threshold $d(b, e_0)$ is critical: indeed, for $d > d(b, e_0)$ there exists a sequence $(g_n, v_n) \in X(u_0; 0, b)$ such that $|\Omega_{g_n}| = d$, $\|g_n - \frac{d}{b}\|_\infty \leq \frac{1}{n}$ and $F(g_n, v_n) < F(\frac{d}{b}, v_{e_0})$.

We are now ready to state the main contribution of this note. As announced in the introduction, we will show that if we consider a less regular anisotropic surface density, whose Wulff shape has a flat facet parallel to the x-axis, we have a different qualitative behavior concerning the stability of the flat configuration. We recall (see [8,9,16]) that the *Wulff shape* associated to a function $\psi : \mathbb{S}^1 \to (0, +\infty)$ is the convex set

$$W_\psi = \{ z \in \mathbb{R}^2 : z \cdot v \leq \psi(v) \text{ for every } v \in \mathbb{S}^1 \}, \tag{2.9}$$

which coincides with the unique minimizer (up to translations) of the "anisotropic isoperimetric problem"

$$\min \left\{ \int_{\partial^* E} \psi(\nu_E)\, \mathrm{d}\mathcal{H}^1 : E \subset \mathbb{R}^2 \text{ has finite perimeter, } |E| = |W_\psi| \right\}.$$

Viceversa, every compact convex set K containing a neighborhood of the origin is the Wulff set associated with the convex function

$$\psi_K(v) = \sup\{ z \cdot v : z \in K \}. \tag{2.10}$$

Let us consider an anisotropy $\psi_c : \mathbb{R}^2 \to [0, +\infty)$ satisfying the following assumptions:

(C1) ψ_c is a positively 1-homogeneous and convex function;
(C2) the associated Wulff shape W_{ψ_c} contains a neighborhood of the origin;

(C3) the boundary of W_{ψ_c} contains a horizontal facet, precisely, a segment of the form $L = \{|x| \le a_1, y = a_2\}$ for some positive reals a_1, a_2.

Setting $\sigma_c = \psi_c(1,0) + \psi_c(-1,0)$, we consider the associated functional defined on $X(u_0; 0, b)$ as

$$F_c(h,u) = \int_{\Omega_h} W(u)\,\mathrm{d}z + \int_{\Gamma_h} \psi_c(\nu_h)\,\mathrm{d}\mathcal{H}^1 + \sigma_c\,\mathcal{H}^1(\Sigma_h).$$

The main result of the paper, which is proved in Section 6, is concerned with the stability of the flat configuration and shows that the presence of an horizontal facet in the Wulff shape eliminates the AGT instability. Precisely, we have:

Theorem 2.9. *Given any $b > 0$, $d > 0$, $e_0 > 0$, the flat configuration $(\frac{d}{b}, v_{e_0})$ corresponding to the volume d and the boundary Dirichlet datum $u_0(x,0) = (e_0 x, 0)$ is an isolated b-periodic local minimizer for F_c, in the sense of Definition 2.3.*

3. Second variation and $W^{2,\infty}$ local minimality

In this section, following [12], we introduce a suitable notion of second variation of the functional F along volume preserving deformations, in terms of which we will be able to state a local minimality criterion.

Let us assume that the anisotropy ψ satisfies conditions (R1) and (R2) of Section 2. Fix $(h,u) \in X(u_0; 0, b)$ with $h \in C^\infty(\mathbb{R})$, $h > 0$, and such that the displacement u minimizes the elastic energy in Ω_h. Given $\phi : \mathbb{R} \to \mathbb{R}$ of class C^∞, b-periodic and such that $\int_0^b \phi(x)\,\mathrm{d}x = 0$, define $h_t := h + t\phi$ for $t \in \mathbb{R}$ and let u_{h_t} be the elastic equilibrium in Ω_{h_t}. We define the *second variation of F at (h,u) along the direction ϕ* to be the value of

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left[F(h_t, u_{h_t})\right]|_{t=0}.$$

In the following theorem we compute explicitly the second variation defined as above. Denote by ν_t the outer unit normal vector to Ω_{h_t} on Γ_{h_t}, and by $H_t^\psi := \operatorname{div}(\nabla\psi \circ \nu_t)$ the anisotropic curvature of Γ_{h_t}. For any one-parameter family of functions $(g_t)_t$ we denote by $\dot g_t(x)$ the partial derivative with respect to t of the map $(t,x) \to g_t(x)$ (omitting the subscript when $t = 0$).

Theorem 3.1. *Let (h,u), ϕ, and (h_t, u_{h_t}) be as above, and let φ be defined as $\varphi := \frac{\phi}{\sqrt{1+h'^2}} \circ \pi_1$. Then the function $\dot u$ belongs to $A(\Omega_h)$ and satisfies the equation*

$$\int_{\Omega_h} \mathbb{C}E(\dot u) : E(w)\,\mathrm{d}z = \int_{\Gamma_h} \operatorname{div}_\tau(\varphi\,\mathbb{C}E(u)) \cdot w\,\mathrm{d}\mathcal{H}^1 \qquad \text{for all } w \in A(\Omega_h). \tag{3.1}$$

Moreover, the second variation of F at (h,u) along the direction ϕ is given by

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}t^2} F(h_t, u_{h_t})|_{t=0} = &-\int_{\Omega_h} \mathbb{C}E(\dot u) : E(\dot u)\,\mathrm{d}z + \int_{\Gamma_h} (\nabla^2\psi \circ \nu)[D_\tau\varphi, D_\tau\varphi]\,\mathrm{d}\mathcal{H}^1 \\ &+ \int_{\Gamma_h} \left(\partial_\nu[W(u)] - HH^\psi\right)\varphi^2\,\mathrm{d}\mathcal{H}^1 - \int_{\Gamma_h} \left(W(u) + H^\psi\right)\partial_\tau\left((h' \circ \pi_1)\,\varphi^2\right)\mathrm{d}\mathcal{H}^1.\end{aligned} \tag{3.2}$$

Proof. The computation is carried out in [12], Theorem 3.2, in the case of an isotropic surface energy. The equation solved by $\dot u$ is deduced exactly in the same way, and also the same computation for the elastic energy yields

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left[\int_{\Omega_{h_t}} W(u_{h_t})\,\mathrm{d}z\right]\bigg|_{t=0} = &-\int_{\Omega_h} \mathbb{C}E(\dot u) : E(\dot u)\,\mathrm{d}z \\ &+ \int_{\Gamma_h} \partial_\nu[W(u)]\,\varphi^2\,\mathrm{d}\mathcal{H}^1 - \int_{\Gamma_h} W(u)\,\partial_\tau\left((h' \circ \pi_1)\,\varphi^2\right)\mathrm{d}\mathcal{H}^1.\end{aligned} \tag{3.3}$$

We are only left with the computation of the first and second derivatives of the surface energy.

Step 1. We compute the first variation of the surface term. Using the positive 1-homogeneity of ψ, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Gamma_{h_t}}\psi(\nu_t)\,\mathrm{d}\mathcal{H}^1 &= \frac{\mathrm{d}}{\mathrm{d}t}\int_0^b \psi(-h_t'(x),1)\,\mathrm{d}x\\ &= -\int_0^b \partial_1\psi(-h_t'(x),1)\phi'(x)\,\mathrm{d}x\\ &= \int_0^b \phi(x)H_t^{\psi}(x,h_t(x))\,\mathrm{d}x,\end{aligned}$$

where the last equality follows by integration by parts and by (2.7). We remark that the first variation of the complete functional F is

$$\frac{\mathrm{d}}{\mathrm{d}t}F(h_t,u_{h_t}) = \int_0^b \phi(x)\Big[W(u_{h_t}) + H_t^{\psi}\Big]|_{(x,h_t(x))}\,\mathrm{d}x.$$

Step 2. Before starting the computation of the second variation, we deduce some useful identities that will be used in the following. Observe first that, thanks to the fact that $D\nu[\nu]=0$, we have

$$D\nu = D_\tau\nu = H\tau\otimes\tau \qquad \text{on } \Gamma_h. \tag{3.4}$$

Moreover, for the same reason we have also $D\left(\nabla\psi\circ\nu\right)[\nu]=0$; differentiating we get

$$\partial_\nu\left(D\left(\nabla\psi\circ\nu\right)\right) = -D\left(\nabla\psi\circ\nu\right)D\nu,$$

thus

$$\begin{aligned}\partial_\nu H^{\psi} &= \partial_\nu\left[\operatorname{div}(\nabla\psi\circ\nu)\right] = \partial_\nu\left[\operatorname{trace}\left(D\left(\nabla\psi\circ\nu\right)\right)\right]\\ &= \operatorname{trace}\left[\partial_\nu\left(D\left(\nabla\psi\circ\nu\right)\right)\right] = -\operatorname{trace}\left[D\left(\nabla\psi\circ\nu\right)D\nu\right]\\ &= -HH^{\psi},\end{aligned}$$

where the last equality follows using (3.4).

Differentiating with respect to t the identity

$$\nu_t(x,y+t\phi(x)) = \frac{(-h_t'(x),1)}{\sqrt{1+(h_t'(x))^2}} \qquad \text{for } (x,y)\in\Gamma_h,$$

and evaluating the result at $t=0$, we get

$$\dot\nu + (\phi\circ\pi_1)\,\partial_2\nu = -\left(\frac{\phi'}{1+(h')^2}\circ\pi_1\right)\tau \qquad \text{on } \Gamma_h.$$

Now from this equality and from (3.4) we obtain

$$\dot\nu = -\left((\phi\circ\pi_1)\,H\tau_2 + \frac{\phi'}{1+(h')^2}\circ\pi_1\right)\tau = -D_\tau\varphi. \tag{3.5}$$

As a consequence of (2.4) we have $(\nabla^2\psi\circ\nu)[\nu,\dot\nu]=0$, and differentiating this identity in the direction ν we get

$$\nu\cdot\partial_\nu\big((\nabla^2\psi\circ\nu)[\dot\nu]\big) = -(\nabla^2\psi\circ\nu)[\dot\nu,\partial_\nu\nu] = 0$$

(recall that $\partial_\nu\nu=0$). Hence

$$\begin{aligned}\dot H^{\psi} = \frac{\partial}{\partial t}H_t^{\psi}|_{t=0} &= \frac{\partial}{\partial t}\big[\operatorname{div}(\nabla\psi\circ\nu_t)\big]|_{t=0} = \operatorname{div}\big((\nabla^2\psi\circ\nu)[\dot\nu]\big)\\ &= \operatorname{div}_\tau\big((\nabla^2\psi\circ\nu)[\dot\nu]\big) + \nu\cdot\partial_\nu\big((\nabla^2\psi\circ\nu)[\dot\nu]\big)\\ &= \operatorname{div}_\tau\big((\nabla^2\psi\circ\nu)[\dot\nu]\big) = -\operatorname{div}_\tau\big((\nabla^2\psi\circ\nu)[D_\tau\varphi]\big),\end{aligned}$$

where in the last equality we used (3.5).

Step 3. We finally pass to the second variation. Differentiating the formula for the first variation of the surface term with respect to t and evaluating at $t = 0$ we get

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left[\int_{\Gamma_{h_t}} \psi(\nu_t)\,\mathrm{d}\mathcal{H}^1\right]\bigg|_{t=0} &= \frac{\mathrm{d}}{\mathrm{d}t}\left[\int_0^b \phi(x) H_t^{\psi}(x, h_t(x))\,\mathrm{d}x\right]\bigg|_{t=0} \\ &= \int_0^b \phi(x)\dot{H}^{\psi}(x, h(x))\,\mathrm{d}x + \int_0^b \phi(x)\,\nabla H^{\psi}(x, h(x))\cdot(0, \phi(x))\,\mathrm{d}x \\ &= I_1 + I_2.\end{aligned}$$

Changing variables in I_1 and using the equality $\dot{H}^{\psi} = -\mathrm{div}_\tau\left((\nabla^2\psi\circ\nu)[D_\tau\varphi]\right)$ on Γ_h we obtain

$$I_1 = -\int_{\Gamma_h} \varphi\,\mathrm{div}_\tau\left((\nabla^2\psi\circ\nu)[D_\tau\varphi]\right)\,\mathrm{d}\mathcal{H}^1 = \int_{\Gamma_h} (\nabla^2\psi\circ\nu)[D_\tau\varphi, D_\tau\varphi]\,\mathrm{d}\mathcal{H}^1, \tag{3.6}$$

where the last equality follows by integration by parts, using the periodicity of φ.

For the second integral, we can decompose $\nabla H^{\psi} = (\partial_\nu H^{\psi})\nu + (\partial_\tau H^{\psi})\tau$, so that after a change of variables

$$\begin{aligned} I_2 &= \int_{\Gamma_h} (\partial_\nu H^{\psi})\varphi^2\,\mathrm{d}\mathcal{H}^1 + \int_{\Gamma_h} (\partial_\tau H^{\psi})(h'\circ\pi_1)\varphi^2\,\mathrm{d}\mathcal{H}^1 \\ &= -\int_{\Gamma_h} H H^{\psi}\varphi^2\,\mathrm{d}\mathcal{H}^1 - \int_{\Gamma_h} H^{\psi}\partial_\tau\left((h'\circ\pi_1)\varphi^2\right)\mathrm{d}\mathcal{H}^1, \end{aligned} \tag{3.7}$$

where we used the identity $\partial_\nu H^{\psi} = -HH^{\psi}$ satisfied on Γ_h and we integrated by parts in the last integral (using again the periodicity of the functions involved).

Collecting (3.3), (3.6) and (3.7), the formula in the statement follows. □

Let us introduce the following subspace of $H^1(\Gamma_h)$:

$$\widetilde{H}^1_{\#}(\Gamma_h) := \left\{\varphi \in H^1(\Gamma_h) :\ \varphi(0, h(0)) = \varphi(b, h(b)), \int_{\Gamma_h} \varphi\,\mathrm{d}\mathcal{H}^1 = 0\right\}$$

(note that the function φ defined in the statement of Thm. 3.1 belongs to this space). Having the formula for the second variation in hand, and observing that the last integral in (3.2) vanishes if (h, u) is a critical pair thanks to (2.6) and to the periodicity of the functions involved, we can define the quadratic form $\partial^2 F(h, u) : \widetilde{H}^1_{\#}(\Gamma_h) \to \mathbb{R}$ associated with the second variation at a critical pair (h, u) as

$$\partial^2 F(h,u)[\varphi] := -\int_{\Omega_h} \mathbb{C}E(v_\varphi) : E(v_\varphi)\,\mathrm{d}z + \int_{\Gamma_h} (\nabla^2\psi\circ\nu)[D_\tau\varphi, D_\tau\varphi]\,\mathrm{d}\mathcal{H}^1 + \int_{\Gamma_h} \left(\partial_\nu[W(u)] - HH^{\psi}\right)\varphi^2\,\mathrm{d}\mathcal{H}^1$$

for $\varphi \in \widetilde{H}^1_{\#}(\Gamma_h)$, where v_φ is the unique solution in $A(\Omega_h)$ to

$$\int_{\Omega_h} \mathbb{C}E(v_\varphi) : E(w)\,\mathrm{d}z = \int_{\Gamma_h} \mathrm{div}_\tau(\varphi\,\mathbb{C}E(u))\cdot w\,\mathrm{d}\mathcal{H}^1 \qquad \text{for every } w \in A(\Omega_h). \tag{3.8}$$

It is easily seen that the positive semi-definiteness of the quadratic form $\partial^2 F(h, u)$ is a necessary condition for local minimality (see [12], Cor. 3.4). On the other hand, we have the following minimality criterion (see [12], Thm. 4.6).

Theorem 3.2. *Let the anisotropy ψ satisfy (R1) and (R2), and let $(h,u)\in X(u_0;0,b)$, with $h\in C^\infty(\mathbb{R})$, $h>0$, be a critical pair for F such that*

$$\partial^2 F(h,u)[\varphi]>0 \qquad \text{for every } \varphi\in \widetilde{H}^1_{\#}(\Gamma_h)\backslash\{0\}. \tag{3.9}$$

Then there exists $\delta>0$ such that for any $(g,v)\in X(u_0;0,b)$, with $\|g-h\|_{W^{2,\infty}(0,b)}<\delta$, $|\Omega_g|=|\Omega_h|$ and $g\neq h$ we have

$$F(h,u)<F(g,v)$$

(we say that the critical pair (h,u) is an isolated $W^{2,\infty}$-local minimizer *for F).*

We remark that, if ψ is of class C^∞, the regularity assumption on h is not restrictive (see Rem. 2.5).

The strategy developed in [12] to prove the theorem (which, in turn, borrows some ideas from [4]) can be repeated here with some changes. We only recall what are the main steps, suggesting the modifications that are necessary to adapt the proof to our setting.

First of all, one can show that the positiveness condition (3.9) can be equivalently formulated in terms of the first eigenvalue of a suitable compact linear operator defined on $\widetilde{H}^1_{\#}(\Gamma_h)$. This is done by introducing the bilinear form on $\widetilde{H}^1_{\#}(\Gamma_h)$

$$(\varphi,\theta)_\sim := \int_{\Gamma_h}\left(\partial_\nu[W(u)]-HH^\psi\right)\varphi\,\theta\,\mathrm{d}\mathcal{H}^1+\int_{\Gamma_h}(\nabla^2\psi\circ\nu)[D_\tau\varphi,D_\tau\theta]\,\mathrm{d}\mathcal{H}^1 \tag{3.10}$$

which, if positive definite, defines an equivalent norm $\|\cdot\|_\sim$ on $\widetilde{H}^1_{\#}(\Gamma_h)$ (this can be shown using condition (2.3) and following the lines of the proof of [4], Prop. 4.2). Then, one has the following equivalent formulation of condition (3.9) (see [12], Prop. 3.6):

Proposition 3.3. *Condition (3.9) is satisfied if and only if the bilinear form $(\cdot,\cdot)_\sim$ is positive definite and the compact monotone self-adjoint operator $T:\widetilde{H}^1_{\#}(\Gamma_h)\to\widetilde{H}^1_{\#}(\Gamma_h)$, defined by duality as*

$$(T\varphi,\theta)_\sim := \int_{\Omega_h}\mathbb{C}E(v_\varphi):E(v_\theta)\,\mathrm{d}z=\int_{\Omega_h}\mathbb{C}E(v_\theta):E(v_\varphi)\,\mathrm{d}z$$

for every $\varphi,\theta\in\widetilde{H}^1_{\#}(\Gamma_h)$, satisfies $\lambda_1:=\max\{(T\varphi,\varphi)_\sim : \|\varphi\|_\sim=1\}<1$.

The proof of this proposition relies, essentially, on the following representation formula of $\partial^2F(h,u)$ in terms of T:

$$\partial^2F(h,u)[\varphi]=(\varphi,\varphi)_\sim-(T\varphi,\varphi)_\sim. \tag{3.11}$$

Moreover, using (3.11) it is easily seen that condition (3.9) implies the existence of a constant $C>0$ such that

$$\partial^2F(h,u)[\varphi]\geq C\|\varphi\|^2_{H^1(\Gamma_h)} \qquad \text{for all } \varphi\in\widetilde{H}^1_{\#}(\Gamma_h). \tag{3.12}$$

Having this equivalent formulation in hand, the proof of Theorem 3.2 is obtained arguing similarly to [12], Proposition 4.5, with some natural modifications. Notice that the elliptic estimates provided by the technical lemmas [12], Lemmas 4.1, 4.4, are valid also in our setting, because they are concerned only with the volume term which we left unchanged. The main steps in the proof are the following.

Step 1. For g in a C^2-neighborhood of h, let v_g be the elastic equilibrium in Ω_g and consider a diffeomorphism $\Phi_g:\overline{\Omega}_h\to\overline{\Omega}_g$ of class C^2 such that Φ_g-Id is b-periodic in x, $\Phi_g(x,0)=(x,0)$, $\Phi_g(x,y)=(x,y+g_n(x)-h(x))$ in a neighborhood of $\overline{\Gamma}_h$, and $\|\Phi_g-Id\|_{C^2(\overline{\Omega}_h;\mathbb{R}^2)}\leq 2\|g-h\|_{C^2([0,b])}$. The same elliptic estimates proved in [12], Lemma 4.1, yield the following convergence (compare with [12], (4.21)):

$$\|\partial_{\nu_g}[W(v_g)]\circ\Phi_gJ_1\Phi_g-\partial_{\nu_h}[W(u)]\|_{H^{-\frac12}_{\#}(\Gamma_h)}\to 0 \quad \text{as } \|g-h\|_{C^2([0,b])}\to 0, \tag{3.13}$$

where $J_1\Phi_g$ denotes the 1-dimensional Jacobian of Φ_g on Γ_h.

Step 2. Let us introduce, for g in a C^2-neighborhood of h, a scalar product $(\cdot,\cdot)_{\sim,g}$ on $\widetilde{H}^1_\#(\Gamma_g)$ defined as in (3.10) with h replaced by g. We claim that the positivity condition (3.9) guarantees that it is possible to control the H^1-norm on Γ_g in terms of the norm associated with $(\cdot,\cdot)_{\sim,g}$, uniformly with respect to g in a C^2-neighborhood of h:

$$\|\varphi\|^2_{H^1(\Gamma_g)} \leq C\|\varphi\|^2_{\sim,g} \qquad \text{for every } \varphi \in \widetilde{H}^1_\#(\Gamma_g)$$

(here and in the following steps C denotes a generic positive constant, independent of g in a C^2-neighborhood of h, which may change from line to line). In fact, given $\varphi \in \widetilde{H}^1_\#(\Gamma_g)$, set $\tilde{\varphi} := (\varphi\circ\Phi_g)J_1\Phi_g$; then $\tilde{\varphi} \in \widetilde{H}^1_\#(\Gamma_h)$ and

$$\begin{aligned}
\|\varphi\|^2_{H^1(\Gamma_g)} &= \int_{\Gamma_h} \big(|\varphi\circ\Phi_g|^2 + |(\partial_{\tau_g}\varphi)\circ\Phi_g|^2\big) J_1\Phi_g \,\mathrm{d}\mathcal{H}^1 \\
&\leq (1+\delta_g)\int_{\Gamma_h}\big(\tilde{\varphi}^2 + (\partial_{\tau_h}\tilde{\varphi})^2\big)\,\mathrm{d}\mathcal{H}^1 \\
&\leq (1+\delta_g)\, C\, \|\tilde{\varphi}\|^2_{\sim},
\end{aligned}$$

where in the last inequality we used (3.11) and (3.12) to deduce that

$$\|\tilde{\varphi}\|^2_{\sim} \geq \partial^2 F(h,u)[\tilde{\varphi}] \geq C\|\tilde{\varphi}\|^2_{H^1(\Gamma_h)},$$

and δ_g is a constant depending only on $\|g-h\|_{C^2([0,b])}$, tending to 0 as $\|g-h\|_{C^2([0,b])}\to 0$.

Now, setting $a_h := \partial_{\nu_h}[W(u)] - HH^\psi$, $a_g := \partial_{\nu_g}[W(v_g)] - HH^\psi_g$ (we denote by H^ψ_g the anisotropic mean curvature of g), we obtain from Step 1 that

$$\|(a_g\circ\Phi_g)J_1\Phi_g - a_h(J_1\Phi_g)^2\|_{H^{-\frac12}(\Gamma_h)} \to 0 \qquad \text{as } \|g-h\|_{C^2([0,b])}\to 0.$$

Hence

$$\begin{aligned}
\|\tilde{\varphi}\|^2_{\sim} &= \int_{\Gamma_h}\big(a_h\tilde{\varphi}^2 + (\nabla^2\psi\circ\nu_h)[D_{\tau_h}\tilde{\varphi}, D_{\tau_h}\tilde{\varphi}]\big)\,\mathrm{d}\mathcal{H}^1 \\
&\leq \int_{\Gamma_h}(a_g\circ\Phi_g)(\varphi\circ\Phi_g)^2 J_1\Phi_g\,\mathrm{d}\mathcal{H}^1 + \int_{\Gamma_g}(\nabla^2\psi\circ\nu_g)[D_{\tau_g}\varphi, D_{\tau_g}\varphi]\,\mathrm{d}\mathcal{H}^1 + \delta_g\|\varphi\|^2_{H^1(\Gamma_g)} \\
&\quad + \|(a_g\circ\Phi_g)J_1\Phi_g - a_h(J_1\Phi_g)^2\|_{H^{-\frac12}(\Gamma_h)}\|(\varphi\circ\Phi_g)^2\|_{H^{\frac12}(\Gamma_h)} \\
&\leq \|\varphi\|^2_{\sim,g} + \delta_g\|\varphi\|^2_{H^1(\Gamma_g)} + C(1+\delta_g)\|(a_g\circ\Phi_g)J_1\Phi_g - a_h(J_1\Phi_g)^2\|_{H^{-\frac12}(\Gamma_h)}\|\varphi\|^2_{H^1(\Gamma_g)},
\end{aligned}$$

where, as before, δ_g tends to 0 as $\|g-h\|_{C^2}\to 0$, and in the last inequality we used the estimate

$$\|(\varphi\circ\Phi_g)^2\|_{H^{\frac12}(\Gamma_h)} \leq C\|(\varphi\circ\Phi_g)^2\|_{H^1(\Gamma_h)} \leq C\|(\varphi\circ\Phi_g)\|^2_{H^1(\Gamma_h)} \leq C(1+\delta_g)\|\varphi\|^2_{H^1(\Gamma_g)}.$$

Combining the previous estimates the claim follows.

Step 3. The previous step allows us to introduce a compact linear operator T_g also on $\widetilde{H}^1_\#(\Gamma_g)$, as we did for T on $\widetilde{H}^1_\#(\Gamma_h)$; denoting by $\lambda_{1,g}$ its first eigenvalue, one can prove, arguing exactly as in Step 3 of the proof of [12], Proposition 4.5, that

$$\limsup_{\|g-h\|_{C^2}\to 0} \lambda_{1,g} \leq \lambda_1 < 1,$$

where the last inequality follows by Proposition 3.3.

Step 4. We claim that the following estimate holds for g close to h in C^2:

$$F(h,u)+C\|\varphi_g\|^2_{H^1(\Gamma_g)}\le F(g,v_g),$$

where $\varphi_g:=\frac{g-h}{\sqrt{1+g'^2}}\circ\pi_1$. In order to prove this estimate, we define $h_t:=h+t(g-h)$ and u_t as the corresponding elastic equilibrium, and setting $f(t):=F(h_t,u_t)$, we can show that a careful estimate of the second variation combined with the previous steps yields

$$f''(t)>C(1-\lambda_1)\|\varphi_g\|^2_{H^1(\Gamma_g)} \tag{3.14}$$

for g sufficiently close to h in C^2. From this the claim will follow immediately, since (using $f'(0)=0$, being (h,u) a critical pair)

$$F(h,u)=f(0)=f(1)-\int_0^1(1-t)f''(t)\,\mathrm{d}t<F(g,v_g)-\frac{C(1-\lambda_1)}{2}\|\varphi_g\|^2_{H^1(\Gamma_g)}.$$

In order to prove (3.14), we have by Theorem 3.1

$$f''(t)=-(T_{h_t}\varphi_{g,t},\varphi_{g,t})_{\sim,h_t}+\|\varphi_{g,t}\|^2_{\sim,h_t}-\int_{\Gamma_{h_t}}(W(u_t)+H^\psi_t)\partial_{\tau_{h_t}}\big((h_t'\circ\pi_1)\varphi^2_{g,t}\big)\,\mathrm{d}\mathcal{H}^1 \tag{3.15}$$

where we set $\varphi_{g,t}:=\frac{g-h}{\sqrt{1+(h_t')^2}}\circ\pi_1$ and H^ψ_t denotes the anisotropic mean curvature of Γ_{h_t}. Using Steps 2, 3 and the fact that

$$\frac12\|\varphi_g\|^2_{H^1(\Gamma_g)}\le\|\varphi_{g,t}\|^2_{H^1(\Gamma_{h_t})}\le 2\|\varphi_g\|^2_{H^1(\Gamma_g)},$$

we deduce that

$$\begin{aligned}-(T_{h_t}\varphi_{g,t},\varphi_{g,t})_{\sim,h_t}+\|\varphi_{g,t}\|^2_{\sim,h_t}&\ge(1-\lambda_{1,h_t})\|\varphi_{g,t}\|^2_{\sim,h_t}\ge\frac{1-\lambda_1}{2}\|\varphi_{g,t}\|^2_{\sim,h_t}\\&\ge\frac{C(1-\lambda_1)}{2}\|\varphi_{g,t}\|^2_{H^1(\Gamma_{h_t})}\ge\frac{C(1-\lambda_1)}{4}\|\varphi_g\|^2_{H^1(\Gamma_g)}\end{aligned} \tag{3.16}$$

if $\|g-h\|_{C^2([0,b])}$ is sufficiently small. Moreover, since (h,u) is a critical pair, there exists a constant Λ such that $W(u)+H^\psi\equiv\Lambda$ on Γ_h, and it can be also shown that

$$\sup_{t\in(0,1]}\|W(u_t)+H^\psi_t-\Lambda\|_{L^\infty(\Gamma_{h_t})}\to0\quad\text{as } g\to h \text{ in } C^2. \tag{3.17}$$

We then have

$$\begin{aligned}-\int_{\Gamma_{h_t}}(W(u_t)+H^\psi_t)\partial_{\tau_{h_t}}\big((h_t'\circ\pi_1)\varphi^2_{g,t}\big)\,\mathrm{d}\mathcal{H}^1&=-\int_{\Gamma_{h_t}}(W(u_t)+H^\psi_t-\Lambda)\partial_{\tau_{h_t}}\big((h_t'\circ\pi_1)\varphi^2_{g,t}\big)\,\mathrm{d}\mathcal{H}^1\\&\ge -C\|W(u_t)+H^\psi_t-\Lambda\|_{L^\infty(\Gamma_{h_t})}\|\varphi_{g,t}\|^2_{H^1(\Gamma_{h_t})}\\&\ge -2C\|W(u_t)+H^\psi_t-\Lambda\|_{L^\infty(\Gamma_{h_t})}\|\varphi_g\|^2_{H^1(\Gamma_g)}.\end{aligned} \tag{3.18}$$

Hence (3.14) follows combining (3.15), (3.16) and (3.18), taking into account (3.17).

Step 5. Finally, using the estimate proved in Step 4, one obtains the $W^{2,\infty}$-local minimality by an approximation argument, as in [12], Theorem 4.6.

4. Improvement of the local minimality result

The improvement of the minimality Theorem 3.2 requires a careful review of the arguments developed in [12], Section 6, that lead to the following result.

Theorem 4.1. *Let the anisotropy ψ satisfy (R1) and (R2) and let $(h,u)\in X(u_0;0,b)$, with $h\in C^\infty(\mathbb{R})$, $h>0$, be a critical pair for F such that condition (3.9) is satisfied. Then (h,u) is an isolated b-periodic local minimizer for F, in the sense of Definition 2.3.*

As in [12], the proof is achieved by considering a sequence of penalized minimum problems: let (g_n,v_n) be a solution to

$$\min\left\{F(k,w)+\Lambda\big||\Omega_k|-|\Omega_h|\big| : (k,w)\in X(u_0;0,b),\, k\geq h-\frac{1}{n}\right\}.$$

Assuming by contradiction that we can find a sequence of pairs $(\tilde g_n,\tilde v_n)\in X(u_0;0,b)$ such that $|\Omega_{\tilde g_n}|=|\Omega_h|$, $F(\tilde g_n,\tilde v_n)<F(h,u)$ and $\|\tilde g_n-h\|\leq\frac{1}{n}$, we then have, since $(\tilde g_n,\tilde v_n)$ is an admissible competitor for the penalized problem,

$$F(g_n,v_n)\leq F(g_n,v_n)+\Lambda\big||\Omega_{g_n}|-|\Omega_h|\big|\leq F(\tilde g_n,\tilde v_n)<F(h,u).$$

The conclusion will follow by showing, *via* regularity estimates, that the functions g_n are converging to h in $W^{2,\infty}$, a contradiction with the $W^{2,\infty}$-local minimality of (h,u) given by Theorem 3.2.

We start to carry out the previous strategy with an approximation lemma which can be easily deduced from the second part of the proof of [2], Lemma 2.1, by means of Reshetnyak's Continuity Theorem.

Lemma 4.2. *Given any $h\in AP(0,b)$ with $h=h^-$, there exists a sequence of b-periodic and Lipschitz functions $h_n\uparrow h$ pointwise such that*

$$\lim_{n\to+\infty}\int_{\Gamma_{h_n}}\psi(\nu_{h_n})\,\mathrm{d}\mathcal{H}^1=\int_{\Gamma_h}\psi(\nu_h)\,\mathrm{d}\mathcal{H}^1.$$

Another preliminary result that we will need in the following is an easy consequence of condition (2.3).

Lemma 4.3. *For any $\xi\in\mathbb{R}$ we have*

$$\partial^2_{11}\psi(\xi,1)\geq\frac{c_0}{(1+\xi^2)^{\frac{3}{2}}},$$

where c_0 is the constant appearing in (2.3).

Proof. We split the vector $(1,0)$ into its components parallel and orthogonal to the direction $(\xi,1)$:

$$(1,0)=\frac{\xi}{\sqrt{1+\xi^2}}\left(\frac{\xi}{\sqrt{1+\xi^2}},\frac{1}{\sqrt{1+\xi^2}}\right)+\left(1-\frac{\xi^2}{1+\xi^2},-\frac{\xi}{1+\xi^2}\right).$$

From this decomposition, using (2.4), we get

$$\begin{aligned}\partial^2_{11}\psi(\xi,1)&=\nabla^2\psi(\xi,1)\big[(1,0),(1,0)\big]\\&=\frac{1}{\sqrt{1+\xi^2}}\nabla^2\psi\left(\frac{\xi}{\sqrt{1+\xi^2}},\frac{1}{\sqrt{1+\xi^2}}\right)\left[\left(1-\frac{\xi^2}{1+\xi^2},-\frac{\xi}{1+\xi^2}\right),\left(1-\frac{\xi^2}{1+\xi^2},-\frac{\xi}{1+\xi^2}\right)\right]\\&\geq\frac{c_0}{\sqrt{1+\xi^2}}\left|\left(1-\frac{\xi^2}{1+\xi^2},-\frac{\xi}{1+\xi^2}\right)\right|^2=\frac{c_0}{(1+\xi^2)^{\frac{3}{2}}},\end{aligned}$$

which is the inequality in the statement. □

Remark 4.4. Using the previous lemma and formula (2.7), a straightforward computation shows that the anisotropic mean curvature of a circumference of radius ρ is bounded from below by the constant $\frac{c_0}{\rho}$.

We now prove an "anisotropic version" of [12], Lemma 6.5.

Lemma 4.5. *Let $h \in C^\infty(\mathbb{R})$ be a b-periodic function, and let $\Lambda_0 = \|H^\psi\|_{L^\infty(\Gamma_h)}$, where H^ψ denotes the anisotropic mean curvature of Γ_h. Then for any admissible profile $k \in AP(0,b)$*

$$\int_{\Gamma_k} \psi(\nu_k)\,\mathrm{d}\mathcal{H}^1 + \Lambda_0 \int_0^b |k-h|\,\mathrm{d}x \geq \int_{\Gamma_h} \psi(\nu_h)\,\mathrm{d}\mathcal{H}^1.$$

Proof. If k is Lipschitz, then using the 1-homogeneity and convexity of ψ we get

$$\begin{aligned}\int_{\Gamma_k} \psi(\nu_k)\,\mathrm{d}\mathcal{H}^1 - \int_{\Gamma_h} \psi(\nu_h)\,\mathrm{d}\mathcal{H}^1 &= \int_0^b \big[\psi(-k',1) - \psi(-h',1)\big]\,\mathrm{d}x \\ &\geq \int_0^b (h'-k')\,\partial_1\psi(-h',1)\,\mathrm{d}x = \int_0^b |k-h|\,\mathrm{sign}(k-h)\,H^\psi(x,h(x))\,\mathrm{d}x \\ &\geq -\Lambda_0 \int_0^b |k-h|\,\mathrm{d}x,\end{aligned}$$

where we integrated by parts using the periodicity of h, k and formula (2.7). If $k \in AP(0,b)$ and $\Sigma_k = \emptyset$, then the conclusion follows by approximation using Lemma 4.2. Finally, if $\Sigma_k \neq \emptyset$, one can simply replace k with k^- (for which $\Sigma_{k^-} = \emptyset$ and $\Gamma_{k^-} = \Gamma_k$), and apply again Lemma 4.2. □

One essential point in the regularization procedure which leads to the $W^{2,\infty}$ convergence is that the solutions to the penalized problems that we will consider satisfy an inner ball condition (see also [5]):

Lemma 4.6 (uniform inner ball condition)**.** *Let $h \in AP(0,b) \cap C^2(\mathbb{R})$, $\Lambda > 0$, $d > 0$; let $(g,v) \in X(u_0;0,b)$ be a solution to*

$$\min\Big\{F(k,w) + \Lambda\big||\Omega_k| - d\big| : (k,w) \in X(u_0;0,b),\ k \geq h\Big\}.$$

Then there exists $\rho_0 = \rho_0(\Lambda, h)$ such that for every $\rho < \rho_0$ and for every $z \in \Gamma_g \cup \Sigma_g$ there exists a ball $B_\rho(z_0) \subset \Omega_g^\# \cup \mathbb{R}^2_-$ such that $\partial B_\rho(z_0) \cap (\Gamma_g \cup \Sigma_g) = \{z\}$.

Proof. As in [12], Lemma 6.7, the proof is based on a suitable isoperimetric inequality which in our anisotropic framework reads as follows (see [12], Lem. 6.6):

> *let $k \in AP(0,b)$, $B_\rho(z_0) \subset \Omega_k^\# \cup \mathbb{R}^2_-$, and let $z_1 = (x_1,y_1)$, $z_2 = (x_2,y_2)$ be points in $\partial B_\rho(z_0) \cap (\Gamma_k^\# \cup \Sigma_k^\#)$ (with $x_1 < x_2$). Let $S = (x_1,x_2) \times \mathbb{R}$, let γ be the shortest arc on $\partial B_\rho(z_0)$ connecting z_1 and z_2 (if z_1 and z_2 are antipodal, the arc which stays above), let γ' be the arc on $\Gamma_k^\# \cup \Sigma_k^\#$ connecting z_1 and z_2, and let D be the region enclosed by $\gamma \cup \gamma'$. Then*
>
> $$\int_{\Gamma_k^\# \cap S} \psi(\nu_k)\,\mathrm{d}\mathcal{H}^1 + \psi(-1,0)\big(k(x_1+) - y_1\big) + \psi(1,0)\big(k(x_2-) - y_2\big) - \int_\gamma \psi(\nu)\,\mathrm{d}\mathcal{H}^1 \geq \frac{c_0}{\rho}\,|D|, \qquad (4.1)$$
>
> *where c_0 is the constant appearing in (2.3).*

Let us prove (4.1). Assume first that k is Lipschitz in $[x_1, x_2]$: let h be the function in (x_1, x_2) whose graph coincides with γ, then arguing as in the proof of Lemma 4.5 we obtain (observe that $k(x_1) = h(x_1)$, $k(x_2) = h(x_2)$, and $k \geq h$)

$$\begin{aligned}\int_{\Gamma_k^{\#}\cap S} \psi(\nu_k)\,\mathrm{d}\mathcal{H}^1 - \int_{\Gamma_h\cap S} \psi(\nu_h)\,\mathrm{d}\mathcal{H}^1 &= \int_{x_1}^{x_2} \big[\psi(-k',1) - \psi(-h',1)\big]\,\mathrm{d}x \\ &\geq \int_{x_1}^{x_2} (h'-k')\,\partial_1\psi(-h',1)\,\mathrm{d}x = \int_{x_1}^{x_2} (k-h)\,\big(\partial_1\psi(-h',1)\big)'\,\mathrm{d}x \\ &\geq \frac{c_0}{\rho}\int_{x_1}^{x_2} (k-h)\,\mathrm{d}x,\end{aligned}$$

which is (4.1) (in the last inequality we used Rem. 4.4). For a general k, we can proceed by approximation using the following property: given $g : [x_1, x_2] \to \mathbb{R}$ lower semicontinuous with finite total variation, there exists a sequence of Lipschitz functions $g_n : [x_1, x_2] \to \mathbb{R}$ such that $g_n(x_1) = g(x_1)$, $g_n(x_2) = g(x_2)$, $g_n \to g$ in $L^1((x_1, x_2))$, and

$$\int_{\Gamma_{g_n}\cap S} \psi(\nu_{g_n})\,\mathrm{d}\mathcal{H}^1 \to \int_{\Gamma_g\cap S} \psi(\nu_g)\,\mathrm{d}\mathcal{H}^1 + \psi(-1,0)\big(g(x_1+) - g(x_1)\big) + \psi(1,0)\big(g(x_2-) - g(x_2)\big).$$

This can be obtained from [12], Lemma 6.2, using Reshetnyak's continuity theorem. Thus (4.1) follows.

Now the proof of the lemma can be obtained arguing exactly as in [12], Lemma 6.7, taking $\rho_0 < \min\{c_0/\Lambda, 1/\|h''\|_\infty\}$. In particular, one can use (4.1) to show that, if $B_{\rho_0}(z) \subset \Omega_g^{\#} \cup \mathbb{R}^2_-$, then $\partial B_{\rho_0}(z) \cap (\Gamma_g^{\#} \cup \Sigma_g^{\#})$ is empty or consists of a single point. Then, the conclusion follows by showing that

$$\bigcup\Big\{B_{\rho_0}(z) : B_{\rho_0}(z) \subset \Omega_g^{\#} \cup \mathbb{R}^2_-\Big\} = \Omega_g^{\#} \cup \mathbb{R}^2_-$$

as in [5], Lemma 2, or [10], Proposition 3.3, Step 2. □

The following proposition contains the main regularization result which allows us to get $W^{2,\infty}$-convergence of the sequence of penalized minima.

Proposition 4.7. *Let $(h, u) \in X(u_0; 0, b)$, $h > 0$, be a critical pair for F. Let $\Lambda > \Lambda_0 := \|H^\psi\|_{L^\infty(\Gamma_h)}$, where H^ψ is the anisotropic mean curvature of Γ_h. Let $(g_n, v_n) \in X(u_0; 0, b)$ be a solution to the penalization problem*

$$\min\Big\{F(g, v) + \Lambda\big||\Omega_g| - |\Omega_h|\big| : (g, v) \in X(u_0; 0, b),\ g \geq h - a_n\Big\} \tag{4.2}$$

where $(a_n)_n$ is a sequence of positive numbers converging to zero. Assume also that $g_n \to h$ in $L^1(0, b)$, $\nabla v_n \rightharpoonup \nabla u$ in $L^2_{\mathrm{loc}}(\Omega_h; \mathbb{R}^2 \times \mathbb{R}^2)$,

$$\lim_{n\to+\infty} \int_{\Gamma_{g_n}} \psi(\nu_{g_n})\,\mathrm{d}\mathcal{H}^1 = \int_{\Gamma_h} \psi(\nu_h)\,\mathrm{d}\mathcal{H}^1, \qquad \lim_{n\to+\infty} \mathcal{H}^1(\Sigma_{g_n}) = 0, \tag{4.3}$$

$$\text{and} \quad \lim_{n\to+\infty} \int_{\Omega_{g_n}} W(v_n)\,\mathrm{d}z = \int_{\Omega_h} W(u). \tag{4.4}$$

Then $g_n \in W^{2,\infty}(0, b)$ for n large enough, and $g_n \to h$ in $W^{2,\infty}(0, b)$.

Proof. We review the proof of [12], Theorem 6.9, underlining the main changes needed to treat the present situation.

Step 1. We show that $\sup_{[0,b]} |g_n - h| \to 0$ as $n \to +\infty$. We may assume that $\overline{\Gamma_{g_n} \cup \Sigma_{g_n}}$ converge in the Hausdorff metric (up to subsequences) to some compact connected set K containing Γ_h. We claim that $\mathcal{H}^1(K \setminus \Gamma_h) = 0$.

In fact, the approximate normal vector ν_K is defined at $\mathcal{H}^1$-a.e. point of K, coinciding with ν_h on Γ_h, and applying [13], Theorem 3.1, we get

$$\int_{\Gamma_h}\psi(\nu_h)\,\mathrm{d}\mathcal{H}^1 \le \int_K \psi(\nu_K)\,\mathrm{d}\mathcal{H}^1 \le \liminf_{n\to+\infty}\int_{\Gamma_{g_n}}\psi(\nu_{g_n})\,\mathrm{d}\mathcal{H}^1 + M\mathcal{H}^1(\Sigma_{g_n}) = \int_{\Gamma_h}\psi(\nu_h)\,\mathrm{d}\mathcal{H}^1,$$

from which the claim immediately follows. Now, since K is the Hausdorff limit of graphs, for every $x\in[0,b)$ the section $K\cap(\{x\}\times\mathbb{R})$ is connected; hence $\mathcal{H}^1(K\setminus\Gamma_h)=0$ implies that $K=\overline{\Gamma}_h$. The uniform convergence of g_n to h follows using this equality, the definition of Hausdorff convergence and the continuity of h.

Step 2. We have $g_n\in C^0([0,b])$ and $\Sigma_{g_n,c}=\emptyset$ for n large enough, where

$$\Sigma_{g_n,c} := \{(x,g_n(x)) : x\in[0,b)\,,\; g_n(x)=g_n^-(x),\;(g_n)'_+(x) = -(g_n)'_-(x) = +\infty\}$$

is the set of *cusps*. The argument relies only on the inner ball condition, proved above (Lem. 4.6), and can be obtained repeating word for word the second step in the proof of [12], Theorem 6.9.

Step 3. We claim that $g_n\in C^1([0,b])$ for n large enough. In fact, using again the inner ball condition we first obtain that g_n is Lipschitz and admits left and right derivatives at every point, which are left and right continuous respectively: this is proved in [5], Lemma 3, (notice that the second situation described in the quoted result can be excluded thanks to the fact that $\Sigma_{g_n}\cup\Sigma_{g_n,c}=\emptyset$, as proved in the previous step).

From this we can also obtain the following decay estimate for v_n: for all $z_0\in\Gamma_{g_n}$ there exists $c_n>0$, a radius $r_n>0$ and an exponent $\alpha_n\in(1/2,1)$ such that

$$\int_{B_r(z_0)\cap\Omega_{g_n}}|\nabla v_n|^2\,\mathrm{d}z \le c_n r^{2\alpha_n} \tag{4.5}$$

for all $r<r_n$ (see [10], Thm. 3.12).

Finally, the argument which leads to the C^1 regularity of g_n goes as follows. It consists in showing that the left and right tangent lines at any point z_0 coincide, comparing the energy of (g_n,v_n) with the energy of a suitable competitor obtained by replacing the graph of g_n in a neighborhood of z_0 with an affine function. Assume by contradiction that the left and right tangent lines at a point $z_0=(x_0,g_n(x_0))\in\Gamma_{g_n}$ are distinct, and form an angle $\theta\in(0,\pi)$. Extend v_n out of Ω_{g_n} to a function $\tilde v_n$ which still satisfies the estimate

$$\int_{B_r(z_0)}|\nabla\tilde v_n|^2\,\mathrm{d}z \le c_n r^{2\alpha_n}. \tag{4.6}$$

For $r<r_n$, consider the points $z_r'=(x_r',g_n(x_r'))$, $z_r''=(x_r'',g_n(x_r''))$ on $\Gamma_{g_n}\cap\partial B_r(z_0)$ such that the arcs γ_r', γ_r'' on Γ_{g_n} connecting z_r' to z_0, and z_r'' to z_0 respectively, are contained in $\Gamma_{g_n}\cap B_r(z_0)$. Let s be the affine function whose graph connects z_r' and z_r'', denote by ν_r, ν_r' and ν_r'' the upper-pointing normals to the segments $[z_r',z_r'']$, $[z_r',z_0]$ and $[z_r'',z_0]$ respectively and define

$$\tilde g_n(x) = \begin{cases} g_n(x) & \text{if } x\in[0,b)\setminus(x_r',x_r''),\\ \max\{s(x),h(x)-a_n\} & \text{if } x\in(x_r',x_r''). \end{cases}$$

Then $(\tilde g_n,\tilde v_n)$ is an admissible competitor in problem (4.2), and by the minimality of (g_n,v_n) we get

$$\begin{aligned}
0 &\ge F(g_n,v_n) + \Lambda\big||\Omega_{g_n}|-|\Omega_h|\big| - F(\tilde g_n,\tilde v_n) - \Lambda\big||\Omega_{\tilde g_n}|-|\Omega_h|\big| \\
&\ge \int_{\gamma_r'\cup\gamma_r''}\psi(\nu_{g_n})\,\mathrm{d}\mathcal{H}^1 - \int_{\Gamma_{\tilde g_n}\cap((x_r',x_r'')\times\mathbb{R})}\psi(\nu_{\tilde g_n})\,\mathrm{d}\mathcal{H}^1 - \int_{B_r(z_0)}W(\tilde v_n)\,\mathrm{d}z - \Lambda|\Omega_{g_n}\triangle\Omega_{\tilde g_n}| \\
&\ge |z_r'-z_0|\,\psi(\nu_r') + |z_r''-z_0|\,\psi(\nu_r'') - |z_r'-z_r''|\,\psi(\nu_r) \\
&\quad - \int_{(x_r',x_r'')\cap\{h>s+a_n\}}\big(\psi(-h'(x),1)-\psi(-s',1)\big)\,\mathrm{d}x - c_n r^{2\alpha_n} - \Lambda\pi r^2,
\end{aligned} \tag{4.7}$$

where we used (4.6) and the inequality

$$\int_{\gamma_r'\cup\gamma_r''}\psi(\nu_{g_n})\,\mathrm{d}\mathcal{H}^1 \geq |z_r'-z_0|\,\psi(\nu_r')+|z_r''-z_0|\,\psi(\nu_r''),$$

which can be deduced arguing as in the proof of Lemma 4.5.

Now, observe that $|z_r'-z_0|\nu_r'+|z_r''-z_0|\nu_r''=|z_r'-z_r''|\nu_r$; therefore, applying [11], Proposition 8.1, (notice that the assumption (2.3) guarantees that the sublevel set $\{\psi\leq 1\}$ is strictly convex) we get

$$|z_r'-z_0|\,\psi(\nu_r')+|z_r''-z_0|\,\psi(\nu_r'')-|z_r'-z_r''|\,\psi(\nu_r)\geq r\,\omega(1-\nu_r'\cdot\nu_r''),$$

where $\omega:[0,2]\to[0,+\infty)$ is a modulus of continuity. From (4.7) we deduce

$$r\,\omega(1-\nu_r'\cdot\nu_r'')\leq\int_{(x_r',x_r'')\cap\{h>s+a_n\}}\big(\psi(-h'(x),1)-\psi(-s',1)\big)\,\mathrm{d}x+c_n'r^{2\alpha_n},$$

and, in turn,

$$\omega(1-\nu_r'\cdot\nu_r'')\leq 2\,\mathrm{Lip}(\psi)\,\mathrm{osc}_{(x_r',x_r'')}h'+c_n'r^{2\alpha_n-1},$$

and since $\alpha_n>\frac{1}{2}$ and h' is continuous, letting $r\to 0$ we obtain $\omega(1-\nu_r'\cdot\nu_r'')\to 0$, which is a contradiction since $\nu_r'\cdot\nu_r''\to\cos\theta<1$. This completes the proof of the C^1 regularity of g_n.

Step 4. We have $g_n\to h$ in $C^1([0,b])$. The purely geometric argument that leads to this claim relies only on the inner ball condition, and is contained in the fourth step of the proof of [12], Theorem 6.9.

Step 5. We now prove that for all $\alpha\in(0,1/2)$, $g_n\to h$ in $C^{1,\alpha}([0,b])$, $v_n\in C^{1,\alpha}(\overline{\Omega}_{g_n})$ for n large enough, and $\sup_n\|v_n\|_{C^{1,\alpha}(\overline{\Omega}_{g_n})}<+\infty$.

The first claim follows by a comparison argument. Fix any point $z_0=(x_0,g_n(x_0))\in\Gamma_{g_n}$, $r>0$, denote by γ_r the open arc contained in Γ_{g_n} of endpoints z_0 and $(x_0+r,g_n(x_0+r))$, and define $\tilde g_n$ as

$$\tilde g_n(x)=\begin{cases} g_n(x) & \text{if } x\in[0,b)\setminus(x_0,x_0+r),\\ \max\{s(x),h(x)-a_n\} & \text{if } x\in(x_0,x_0+r),\end{cases}$$

where s is the affine function whose graph connects z_0 and $(x_0+r,g_n(x_0+r))$. Then, comparing the energies of g_n and $\tilde g_n$ (as we did in Step 3), one can see that inequality (6.8) in [12] becomes in our case

$$\int_{x_0}^{x_0+r}\psi(-g_n',1)\,\mathrm{d}x-\int_{x_0}^{x_0+r}\psi(-s',1)\,\mathrm{d}x\leq c'r^{2\sigma}. \tag{4.8}$$

Now, observe that for every a,b there exists a point ξ in the interval $[a\wedge b,a\vee b]$ such that

$$\begin{aligned}\psi(b,1)-\psi(a,1)&=\partial_1\psi(a,1)\,(b-a)+\frac{1}{2}\,\partial_{11}^2\psi(\xi,1)\,(b-a)^2\\ &\geq\partial_1\psi(a,1)\,(b-a)+\frac{c_0(b-a)^2}{2(1+\xi^2)^{3/2}}\\ &\geq\partial_1\psi(a,1)\,(b-a)+\frac{c_0(b-a)^2}{2(1+\max\{a^2,b^2\})^{3/2}}\end{aligned} \tag{4.9}$$

(in the first inequality we used Lem. 4.3). Applying (4.9) with $a=-\frac{1}{r}\int_{x_0}^{x_0+r}g_n'\,\mathrm{d}x$ and $b=-g_n'(x)$, integrating in (x_0,x_0+r) and using (4.8), we get

$$\frac{c_0}{2(1+M_1^2)^{3/2}}\frac{1}{r}\int_{x_0}^{x_0+r}\left(g_n'(x)-\frac{1}{r}\int_{x_0}^{x_0+r}g_n'\,\mathrm{d}s\right)^2\mathrm{d}x\leq\frac{1}{r}\int_{x_0}^{x_0+r}\psi(-g_n',1)\,\mathrm{d}x-\frac{1}{r}\int_{x_0}^{x_0+r}\psi(-s',1)\,\mathrm{d}x\leq c'r^{2\sigma-1}.$$

From this inequality, arguing as in Step 5 of the proof of [12], Theorem 6.9, it follows that the sequence $(g_n)_n$ is equibounded in $C^{1,\sigma-\frac{1}{2}}([0,b])$ for all $\sigma\in(1/2,1)$, thus proving the first claim. The other claims are obtained using standard elliptic estimates (see [12], Prop. 8.9).

Step 6. The conclusion ($g_n\to h$ in $W^{2,\infty}(0,b)$) follows using the Euler-Lagrange equation (2.6) satisfied by a critical pair.

In fact, setting $K_n=\{x\in[0,b]:g_n(x)=h(x)-a_n\}$ and assuming without loss of generality that $A_n=(0,b)\setminus K_n$ is not empty, it is easily seen that $g_n'(x)=h'(x)$ for every $x\in K_n$, while for $x\in A_n$ the following Euler-Lagrange equations are satisfied by g_n and h respectively:

$$\begin{aligned}\big(\partial_1\psi(-g_n'(x),1)\big)'&=-W(v_n)(x,g_n(x))+\lambda_n,\\ \big(\partial_1\psi(-h'(x),1)\big)'&=-W(u)(x,h(x))+\lambda,\end{aligned}$$

for some Lagrange multipliers λ_n, λ (the first equation follows by the minimality of (g_n,v_n), the second one by the fact that (h,u) is a critical pair: see (2.6)). Observe that, thanks to the results contained in [1], Section 7.7, g_n' is a Lipschitz function for all n. Now using the fact that the anisotropic mean curvature is expressed as a derivative (see (2.7)), we first deduce from the previous equations that, splitting A_n into the union of its connected components $(\alpha_{i,n},\beta_{i,n})$,

$$\begin{aligned}\lambda_n|A_n|-\int_{A_n}W(v_n)(x,g_n(x))\,\mathrm{d}x&=\sum_i\int_{\alpha_{i,n}}^{\beta_{i,n}}\big(\partial_1\psi(-g_n'(x),1)\big)'\,\mathrm{d}x\\ &=\sum_i\Big(\partial_1\psi(-g_n'(\beta_{i,n}),1)-\partial_1\psi(-g_n'(\alpha_{i,n}),1)\Big)\\ &=\sum_i\Big(\partial_1\psi(-h'(\beta_{i,n}),1)-\partial_1\psi(-h'(\alpha_{i,n}),1)\Big)\\ &=\int_{A_n}\big(\partial_1\psi(-h'(x),1)\big)'\,\mathrm{d}x=\lambda|A_n|-\int_{A_n}W(u)(x,h(x))\,\mathrm{d}x,\end{aligned}$$

which, in turn, gives

$$\lambda_n-\lambda=\frac{1}{|A_n|}\int_{A_n}\Big[W(v_n)(x,g_n(x))-W(u)(x,h(x))\Big]\,\mathrm{d}x.$$

From assumption (4.4) and Step 5 one can deduce that $W(v_n)(\cdot,g_n(\cdot))\to W(u)(\cdot,h(\cdot))$ uniformly in $[0,b]$, hence we conclude that $\lambda_n\to\lambda$. Now the Euler-Lagrange equations, the convergence $\lambda_n\to\lambda$ and the uniform convergence of $W(v_n)(\cdot,g_n(\cdot))$ to $W(u)(\cdot,h(\cdot))$ imply

$$\big(\partial_1\psi(-g_n'(x),1)\big)'\to\big(\partial_1\psi(-h'(x),1)\big)'\quad\text{uniformly in }[0,b].$$

Finally, from this we deduce that $g_n''\to h''$ in $L^\infty(0,b)$, since

$$\|g_n''-h''\|_{L^\infty(0,b)}=\left\|\frac{\big(\partial_1\psi(-g_n',1)\big)'}{\partial^2_{11}\psi(-g_n',1)}-\frac{\big(\partial_1\psi(-h',1)\big)'}{\partial^2_{11}\psi(-h',1)}\right\|_{L^\infty(0,b)}\to0$$

(using the fact that the denominators are uniformly bounded away from 0 by Lem. 4.3). This concludes the proof of the proposition. □

Proof of Theorem 4.1. By contradiction, let $(\tilde g_n,\tilde v_n)\in X(u_0;0,b)$, with $|\Omega_{\tilde g_n}|=|\Omega_h|$, be such that $0<\|\tilde g_n-h\|_{L^\infty(0,b)}\le\frac{1}{n}$ and

$$F(\tilde g_n,\tilde v_n)\le F(h,u).\tag{4.10}$$

Fix $\Lambda > \max\{\Lambda_0, W_0\}$, where Λ_0 is defined in Proposition 4.7 and

$$W_0 = \frac{1}{b}\int_0^b W(U_0(x,y))\,\mathrm{d}x, \qquad U_0(x,y) = u_0(x,0) + e_0\left(0, \frac{-\lambda}{2\mu+\lambda}\,y\right)$$

(notice that W_0 is finite since u_0 is Lipschitz), and let (g_n, v_n) be a solution to the minimum problem

$$\min\left\{F(g,v) + \Lambda\big||\Omega_g| - |\Omega_h|\big| : (g,v)\in X(u_0;0,b),\, g \geq h - \frac{1}{n}\right\}; \tag{4.11}$$

then

$$F(g_n,v_n) \leq F(g_n,v_n) + \Lambda\big||\Omega_{g_n}| - |\Omega_h|\big| \leq F(\tilde{g}_n,\tilde{v}_n) \leq F(h,u). \tag{4.12}$$

We claim that $(g_n,v_n)\to(h,u)$ in Y, up to subsequences. In fact, by (4.12) we have a uniform bound

$$\int_{\Omega_{g_n}} |E(v_n)|^2\,\mathrm{d}z + \mathrm{Var}(g_n;0,b) + |\Omega_{g_n}| \leq C$$

(the bound on the variation of g_n follows using condition (2.1), which gives a uniform bound on $\mathcal{H}^1(\Gamma_{g_n})$), so that by Theorem 2.1 we have $(g_n,v_n)\xrightarrow{Y}(k,v)\in X(u_0;0,b)$ up to subsequences. Taken any $(g,w)\in X(u_0;0,b)$ with $g\geq h$ (it is an admissible competitor for all the penalized problems) we have, by the l.s.c. of F with respect to the convergence in Y and the minimality of (g_n,v_n),

$$F(k,v) + \Lambda\big||\Omega_k| - |\Omega_h|\big| \leq \liminf_{n\to+\infty}\big(F(g_n,v_n) + \Lambda\big||\Omega_{g_n}| - |\Omega_h|\big|\big) \leq F(g,w) + \Lambda\big||\Omega_g| - |\Omega_h|\big|. \tag{4.13}$$

From the previous inequality with $(g,w)=(h,v)$ we get, since $k\geq h$,

$$\int_{\Gamma_k}\psi(\nu_k)\,\mathrm{d}\mathcal{H}^1 + \Lambda\int_0^b |k-h| \leq \int_{\Gamma_h}\psi(\nu_h)\,\mathrm{d}\mathcal{H}^1,$$

from which it follows $k=h$ by Lemma 4.5 (using $\Lambda>\Lambda_0$), and in turn $v=u$. Thus the claim is proved.

Moreover, using again (4.13) with $(g,w)=(h,u)$, combined with the l.s.c. of the volume energy and of the map $g\to\int_{\Gamma_g}\psi(\nu_g)\,\mathrm{d}\mathcal{H}^1$ with respect to the convergence in Y (the second one follows from Reshetnyak's lower semicontinuity theorem), we deduce that conditions (4.3) and (4.4) hold. By Proposition 4.7 we can conclude that $g_n\to h$ in $W^{2,\infty}(0,b)$.

We now deal with the volume constraint. Suppose first by contradiction that $|\Omega_{g_n}|<|\Omega_h|$. In this case, consider the competitor $(\bar{g}_n,\bar{v}_n)$, where $\bar{g}_n = g_n + (|\Omega_h| - |\Omega_{g_n}|)/b$ and

$$\bar{v}_n(x,y) = \begin{cases} U_0(x,y) & \text{if } 0\leq y < (|\Omega_h| - |\Omega_{g_n}|)/b,\\ v_n\Big(x, y - \frac{|\Omega_h|-|\Omega_{g_n}|}{b}\Big) + e_0\Big(0, \frac{-\lambda(|\Omega_h|-|\Omega_{g_n}|)}{b(2\mu+\lambda)}\Big) & \text{if } y \geq (|\Omega_h| - |\Omega_{g_n}|)/b, \end{cases}$$

for $(x,y)\in\Omega_{\bar{g}_n}$: then

$$F(\bar{g}_n,\bar{v}_n) + \Lambda\big||\Omega_{\bar{g}_n}| - |\Omega_h|\big| - F(g_n,v_n) - \Lambda\big||\Omega_{g_n}| - |\Omega_h|\big| = \big(|\Omega_h| - |\Omega_{g_n}|\big)(W_0 - \Lambda) < 0$$

(since $\Lambda > W_0$), which contradicts the minimality of (g_n,v_n).

Thus, $|\Omega_{g_n}|\geq|\Omega_h|$ for every n. We define $\hat{g}_n := g_n - (|\Omega_{g_n}| - |\Omega_h|)/b$, so that

$$|\Omega_{\hat{g}_n}| = |\Omega_h|, \quad \hat{g}_n\to g \text{ in } W^{2,\infty}(0,b), \quad \text{and} \quad F(\hat{g}_n,v_n)\leq F(h,u).$$

From the isolated $W^{2,\infty}$-minimality of (h,u) (given by Thm. 3.2) we get $(\hat{g}_n,v_n)=(h,u)$ for n large. By (4.12) this implies that $F(g_n,v_n)=F(\tilde{g}_n,\tilde{v}_n)=F(h,u)$ for n large, thus the pair $(\tilde{g}_n,\tilde{v}_n)$ is a solution to the minimum problem (4.11). Hence, the previous compactness argument applied now to the sequence $(\tilde{g}_n,\tilde{v}_n)$ instead of (g_n,v_n) leads to $\tilde{g}_n\to h$ in $W^{2,\infty}$, which contradicts (4.10) since (h,u) is an isolated $W^{2,\infty}$-local minimizer. □

5. Stability of the flat configuration

Now we come to the study of the stability of the flat configuration $(\frac{d}{b}, v_{e_0})$. We start by noticing that we can consider without loss of generality variations in the subspace

$$\widetilde{H}^1_0(\Gamma_{d/b}) := \{\varphi \in \widetilde{H}^1_\#(\Gamma_{d/b}) : \varphi(0, d/b) = \varphi(b, d/b) = 0\},$$

(see [12], Rem. 4.8); in turn, this space can be identified with

$$\widetilde{H}^1_0(0,b) := \left\{ \varphi \in H^1(0,b) : \varphi(0) = \varphi(b) = 0,\ \int_0^b \varphi = 0 \right\}.$$

Observe moreover that the quadratic form associated with the second variation of the functional F at the flat configuration is given by

$$\partial^2 F\,(d/b, v_{e_0})\,[\varphi] = -\int_{(0,b)\times(0,\frac{d}{b})} \mathbb{C}E(v_\varphi) : E(v_\varphi)\,\mathrm{d}z + \partial^2_{11}\psi(0,1)\int_0^b {\varphi'}^2(x)\,\mathrm{d}x$$

for all $\varphi \in \widetilde{H}^1_0(0,b)$, where $v_\varphi \in A(\Omega_{d/b})$ is the solution to

$$\int_{(0,b)\times(0,\frac{d}{b})} \mathbb{C}E(v_\varphi) : E(w)\,\mathrm{d}z = \tau \int_0^b \varphi'(x)\, w_1(x, d/b)\,\mathrm{d}x \quad \text{for every } w = (w_1, w_2) \in A(\Omega_{d/b})$$

with $\tau = \frac{4\mu(\mu+\lambda)e_0}{2\mu+\lambda}$. Observe that, by Lemma 4.3, the coefficient $\partial^2_{11}\psi(0,1)$ is strictly positive.

Proof of Theorem 2.8. Arguing as in the proof of [12], Theorem 5.1, we get an explicit expression of the second variation in terms of the Fourier coefficients of φ, namely

$$\partial^2 F(d/b, v_{e_0})[\varphi] = \sum_{n\in\mathbb{Z}} n^2 \varphi_n \varphi_{-n} \left[\partial^2_{11}\psi(0,1) - \frac{\tau^2(1-\nu_p) b J(2\pi n d/b^2)}{2\pi\mu n} \right], \tag{5.1}$$

where the φ_n's are the Fourier coefficients of φ in $(0,b)$. Now by definition of K

$$\sup_{n\in\mathbb{Z}} \frac{\tau^2(1-\nu_p) b J(2\pi n d/b^2)}{2\pi\mu n} \gtrless \partial^2_{11}\psi(0,1) \iff K\left(\frac{2\pi d}{b^2}\right) \gtrless \frac{\pi}{4}\,\frac{(2\mu+\lambda)\partial^2_{11}\psi(0,1)}{e_0^2\mu(\mu+\lambda)}\,\frac{1}{b}\,,$$

which implies by (5.1)

$$\partial^2 F(d/b, v_{e_0})[\varphi] > 0 \quad \forall \varphi \in \widetilde{H}^1_0(0,b) \iff K\left(\frac{2\pi d}{b^2}\right) < \frac{\pi}{4}\,\frac{(2\mu+\lambda)\partial^2_{11}\psi(0,1)}{e_0^2\mu(\mu+\lambda)}\,\frac{1}{b}\,,$$

$$K\left(\frac{2\pi d}{b^2}\right) > \frac{\pi}{4}\,\frac{(2\mu+\lambda)\partial^2_{11}\psi(0,1)}{e_0^2\mu(\mu+\lambda)}\,\frac{1}{b} \implies \partial^2 F(d/b, v_{e_0})[\varphi] < 0 \quad \text{for some } \varphi \in \widetilde{H}^1_0(0,b)\,.$$

Then the conclusion follows by Theorem 4.1. ☐

Remark 5.1. It can be interesting to study what can be said, in this anisotropic contest, about the issue of the *global minimality* of the flat configuration, that is, whether $(\frac{d}{b}, v_{e_0})$ minimizes F among *all* b-periodic competitors satisfying the same volume constraint. One can check that the corresponding result proved in the first part of [12], Theorem 2.11, can be extended to anisotropic functionals (under the assumptions (R1)–(R2) on ψ), with no particular changes in the proof: precisely, we have that for every $b > 0$ and $e_0 > 0$ there exists $d_{\mathrm{glob}}(b, e_0) \in (0, d(b, e_0)]$ such that the flat configuration $(\frac{d}{b}, v_{e_0})$ is a b-periodic global minimizer if and only if $d \le d_{\mathrm{glob}}(b, e_0)$, and it is the *unique* global minimizer if $d < d_{\mathrm{glob}}$.

6. The crystalline case

This section is devoted to the proof of the main result of the paper, Theorem 2.9, which deals with the stability of the flat configuration in the crystalline case. Let ψ_c be a crystalline anisotropy satisfying conditions (C1)–(C3) of Section 2. The strategy will be the following. First of all, we show that we do not lose in generality if we prove the theorem for crystalline anisotropies of a particular form (namely, whose Wulff shape is a rectangle with sides parallel to the coordinate axes). Then, we conclude using an approximation argument combined with the results obtained in the previous sections for the regular case.

Proof of Theorem 2.9. We divide the proof into three steps.

Step 1. From the assumptions on ψ_c it follows that we can find $0 < b_1 \le a_1$, $b_2 > 0$ such that the rectangle $R = \{(x,y) : |x| \le b_1, -b_2 \le y \le a_2\}$ is contained in the Wulff shape W_{ψ_c}. Denote by ψ_R the function whose Wulff shape is R, given by

$$\psi_R(\nu_1,\nu_2) = \begin{cases} b_1|\nu_1| + a_2|\nu_2| & \text{if } \nu_2 \ge 0, \\ b_1|\nu_1| + b_2|\nu_2| & \text{if } \nu_2 < 0, \end{cases}$$

(see Eq. (2.10)), and by F_R the functional corresponding to this anisotropic surface density. Note that, since $R \subset W_{\psi_c}$, by (2.10) it follows immediately that $\psi_R \le \psi_c$; moreover

$$\psi_R(0,1) = a_2 = \psi_c(0,1) \tag{6.1}$$

(concerning the second equality see, for instance, [8], Prop. 3.5 (iv)).

Step 2. We introduce a family of "approximating" functionals, defined as follows. We consider, for $\varepsilon > 0$, the family of anisotropic surface densities $\psi_\varepsilon(x,y) = b_1\sqrt{\varepsilon^2y^2 + x^2} + (a_2 - b_1\varepsilon)|y|$, and the associated functionals

$$F_\varepsilon(h,u) = \int_{\Omega_h} W(u)\,\mathrm{d}z + \int_{\Gamma_h} \psi_\varepsilon(\nu_h)\,\mathrm{d}\mathcal{H}^1 + 2b_1\,\mathcal{H}^1(\Sigma_h).$$

The functions ψ_ε converge monotonically as $\varepsilon \to 0^+$ to ψ_R in $\mathbb{R} \times [0,+\infty)$: indeed, it is sufficient to observe that for $(x,y) \in \mathbb{R} \times [0,+\infty)$

$$\begin{aligned}\psi_\varepsilon(x,y) &= b_1\sqrt{\varepsilon^2y^2 + x^2} + (a_2 - b_1\varepsilon)y \\ &= \frac{b_1^2x^2}{b_1\sqrt{\varepsilon^2y^2 + x^2} + b_1\varepsilon y} + a_2y \nearrow b_1|x| + a_2y = \psi_R(x,y).\end{aligned} \tag{6.2}$$

From a geometrical point of view, this means that the Wulff shapes associated with the functions ψ_ε are converging monotonically from the interior to the corresponding one associated with ψ_R in the upper half-plane (see Fig. 1).

Consider now the functionals $\widehat{F}_\varepsilon$ corresponding to the regular surface densities $\hat\psi_\varepsilon(x,y) = b_1\sqrt{\varepsilon^2y^2 + x^2}$; the functions $\hat\psi_\varepsilon$ satisfy all the assumptions considered in the regular case: in particular, condition (2.3) follows after some computations from the formula

$$\nabla^2\hat\psi_\varepsilon(v)[w,w] = \frac{b_1}{\sqrt{v_1^2 + \varepsilon^2v_2^2}}\left[(w_1^2 + \varepsilon^2w_2^2) - \frac{(v_1w_1 + \varepsilon^2v_2w_2)^2}{v_1^2 + \varepsilon^2v_2^2}\right],$$

where $v = (v_1,v_2)$ and $w = (w_1,w_2)$. The general analysis developed in the first part of the paper applies to the functional $\widehat{F}_\varepsilon$: in particular, since $\partial^2_{11}\hat\psi_\varepsilon(0,1) = \frac{b_1}{\varepsilon}$, from Theorem 2.8 it follows that, given any $b > 0$ and $e_0 > 0$, there exists $\varepsilon_0 = \varepsilon_0(b,e_0) > 0$ such that if $0 < \varepsilon \le \varepsilon_0$ the flat configuration $(\frac{d}{b}, v_{e_0})$ is an isolated L^∞-local minimizer for $\widehat{F}_\varepsilon$ for every volume $d > 0$. The same is true also for F_ε, since the energies F_ε and $\widehat{F}_\varepsilon$ differ only by a constant value: $F_\varepsilon = \widehat{F}_\varepsilon + (a_2 - b_1\varepsilon)b$.

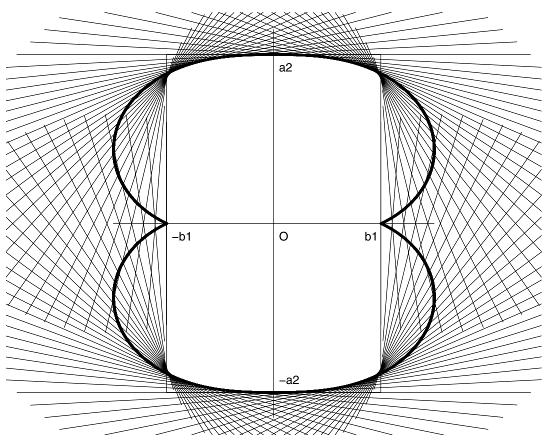

FIGURE 1. The Wulff shape corresponding to the anisotropy ψ_ε is an approximation from the interior of the symmetric rectangle $R_0 = \{|x| \le b_1, |y| \le a_2\}$. To construct the Wulff shape associated with a function ψ, consider at every point $\psi(\nu)\nu$, $\nu \in \mathbb{S}^1$, of the polar plot of ψ (the bold curve in the figure), the line orthogonal to the radius vector and passing through that point: the Wulff shape is the intersection of all the halfplanes containing the origin and whose boundary is one of these lines (see (2.9)).

Step 3. Given $b > 0$, $d > 0$, $e_0 > 0$, let $\varepsilon_0 = \varepsilon_0(b, e_0)$ be as above, and let $\delta > 0$ be such that the flat configuration minimizes the energy F_{ε_0} among all competitors satisfying the volume constraint whose L^∞ distance from the flat configuration is less than δ.

Then, for all $(g, v) \in X(u_0; 0, b)$ such that $|\Omega_g| = d$ and $0 < \|g - \frac{d}{b}\|_\infty < \delta$ we have, using condition (6.1),

$$\begin{aligned} F_c\left(\frac{d}{b}, v_{e_0}\right) &= \int_{\Omega_{d/b}} W(v_{e_0})\,\mathrm{d}z + b\,\psi_c(0,1) = \int_{\Omega_{d/b}} W(v_{e_0})\,\mathrm{d}z + b\,\psi_R(0,1) \\ &= F_R\left(\frac{d}{b}, v_{e_0}\right) = F_{\varepsilon_0}\left(\frac{d}{b}, v_{e_0}\right) < F_{\varepsilon_0}(g,v) \le F_R(g,v) \le F_c(g,v) \end{aligned}$$

where the first inequality follows from the local minimality of the flat configuration for F_{ε_0}, the second one is a straight consequence of (6.2) and the last one follows using $\psi_R \le \psi_c$. From the previous chain of inequalities the conclusion follows. □

Remark 6.1. Concerning the global minimality of the flat configuration in the crystalline case, an argument similar to the one used in the previous proof combined with the result stated in Remark 5.1 shows that, for every $b > 0$ and $e_0 > 0$, the flat configuration $(\frac{d}{b}, v_{e_0})$ is a global minimizer if the volume d is sufficiently small.

Remark 6.2. A natural question arising from the previous analysis is whether in the crystalline case the flat configuration is always a *global* minimizer. This is in fact not true, at least if the interval of periodicity is sufficiently large. Indeed, we first recall that in [12], Proposition 2.12, was proved that, for b sufficiently large, the threshold of global minimality is strictly smaller than the threshold of local minimality. The same comparison argument used to prove that result shows that, if ψ_R is an anisotropy whose associated Wulff shape is a rectangle (as in Step 1 of the proof of Theorem 2.9), then for every $s > 0$ there exists $b > 0$ such that one can construct a b-periodic competitor (g, v) whose energy is strictly below the energy of the flat configuration (s, v_{e_0}): indeed, it is sufficient to observe that the surface energy corresponding to ψ_R coincides, up to constant factors, with the isotropic surface energy when evaluated on the flat configuration and on the competitor constructed in the proof of [12], Proposition 2.12. Finally, the same is true for a general anisotropy ψ_c satisfying assumptions (C1)–(C3): in fact, one can always find a rectangle R containing the associated Wulff shape whose upper side contains the horizontal facet, in such a way that

$$\psi_R(0,1) = \psi_c(0,1), \qquad \psi_c \leq \psi_R,$$

hence $F_c(g,v) \leq F_R(g,v) < F_R(s,v_{e_0}) = F_c(s,v_{e_0})$.

Acknowledgements. I am very grateful to Massimiliano Morini for proposing the subject of this work and for multiple helpful discussions.

References

[1] L. Ambrosio, N. Fusco and D. Pallara, *Functions of bounded variation and free discontinuity problems.* Oxford University Press, New York (2000).

[2] E. Bonnetier and A. Chambolle, Computing the equilibrium configuration of epitaxially strained crystalline films. *SIAM J. Appl. Math.* **62** (2002) 1093–1121.

[3] A. Braides, A. Chambolle and M. Solci, A relaxation result for energies defined on pairs set-function and applications. *ESAIM: COCV* **13** (2007) 717–734.

[4] F. Cagnetti, M.G. Mora and M. Morini, A second order minimality condition for the Mumford-Shah functional. *Calc. Var. Partial Differential Equations* **33** (2008) 37–74.

[5] A. Chambolle and C.J. Larsen, C^∞ regularity of the free boundary for a two-dimensional optimal compliance problem. *Calc. Var. Partial Differential Equations* **18** (2003) 77–94.

[6] A. Chambolle and M. Solci, Interaction of a bulk and a surface energy with a geometrical constraint. *SIAM J. Math. Anal.* **39** (2007) 77–102.

[7] B. De Maria and N. Fusco, Regularity properties of equilibrium configurations of epitaxially strained elastic films. Submitted paper (2011)

[8] I. Fonseca, The Wulff theorem revisited. *Proc. Roy. Soc. London Ser. A* **432** (1991) 125–145.

[9] I. Fonseca and S. Müller, A uniqueness proof for the Wulff theorem. *Proc. Roy. Soc. Edinburgh* **119A** (1991) 125–136.

[10] I. Fonseca, N. Fusco, G. Leoni and M. Morini, Equilibrium configurations of epitaxially strained crystalline films: existence and regularity results. *Arch. Rational Mech. Anal.* **186** (2007) 477–537.

[11] I. Fonseca, N. Fusco, G. Leoni and V. Millot, Material voids in elastic solids with anisotropic surface energies. *J. Math. Pures Appl.* **96** (2011) 591–639.

[12] N. Fusco and M. Morini, Equilibrium configurations of epitaxially strained elastic films: second order minimality conditions and qualitative properties of solutions. *Arch. Rational Mech. Anal.* **203** (2012) 247–327.

[13] A. Giacomini, A generalization of Gołąb's theorem and applications to fracture mechanics. *Math. Models Methods Appl. Sci.* **12** (2002) 1245–1267.

[14] M.A. Grinfeld, The stress driven instability in elastic crystals: mathematical models and physical manifestations. *J. Nonlinear Sci.* **3** (1993) 35–83.

[15] H. Koch, G. Leoni and M. Morini, On optimal regularity of free boundary problems and a conjecture of De Giorgi. *Comm. Pure Appl. Math.* **58** (2005) 1051–1076.

[16] J. Taylor, Crystalline variational problems. *Bull. Amer. Math. Soc.* **84** (1978) 568–588.