ESAIM: COCV 20 (2014) 1203–1213
DOI: 10.1051/cocv/2014012 **ESAIM: Control, Optimisation and Calculus of Variations**
www.esaim-cocv.org

IDENTIFICATION OF A WAVE EQUATION GENERATED BY A STRING*

Amin Boumenir[1]

Abstract. We show that we can reconstruct two coefficients of a wave equation by a single boundary measurement of the solution. The identification and reconstruction are based on Krein's inverse spectral theory for the first coefficient and on the Gelfand−Levitan theory for the second. To do so we use spectral estimation to extract the first spectrum and then interpolation to map the second one. The control of the solution is also studied.

Mathematics Subject Classification. 34A55, 34K29, 34L05.

Received June 27, 2013. Revised November 5, 2013.
Published online August 8, 2014.

1. Introduction

We are concerned with the identification and recovery of the mass of a Lebesgue Stieltjes measure $\mathrm{d}m(x)$ and a control function $p\,(t)$ appearing in the wave equation

$$\begin{cases} \dfrac{\partial^2}{\partial t^2}u(x,t)=\dfrac{\partial}{\partial m(x)}\dfrac{\partial^+}{\partial x^+}u\,(x,t)+p(t)u(x,t) & 0\le x\le b \\ u(x,0)=f(x),\quad u_t(x,0)=g(x), & \\ hu(0,t)-\dfrac{\partial^+}{\partial x^+}u(0,t)=0 & h\ge 0 \\ Hu(b,t)+\dfrac{\partial^+}{\partial x^+}u(b,t)=0 & H\ge 0 \end{cases} \tag{1.1}$$

from a single measurement

$$\{f(x),g(x)\}\to\{u(0,t),\ u(b,t) \qquad \text{for} \quad t>T\}\,. \tag{1.2}$$

We assume that the mass m of the string is right-continuous, nondecreasing so that $\mathrm{d}m(x)$ defines a Lebesgue−Stieltjes measure. If $\mathrm{d}m$ is absolutely continuous, *i.e.* m' is locally integrable, then the string operator, see (2.2) below, reduces to the classical Sturm−Liouville operator, $\frac{\partial}{\partial m(x)}\frac{\partial^+}{\partial x^+}=\frac{1}{m'(x)}\frac{\partial^2}{\partial x^2}$ except that $m'(x)$ is allowed to vanish on subintervals of $[0,b]$ where m is constant. *i.e.* $m'(x)\ge 0$. In terms of modeling,

Keywords and phrases. Inverse spectral methods; Krein string; Gelfand-Levitan theory.

* *To the memory of Professor R.W. Carroll.*

[1] Department of Mathematics, College of Science, Kuwait. University, P.O. Box 5969, 13060 Safat, Kuwait.
boumenir@sci.kuniv.edu.kw

$1/\sqrt{m'(x)}$ stands for the wave speed, and so its recovery unveils the nature of the medium. Thus depending on the nature of the mass, or medium, the string operator includes a wide variety of operators and spectra. If $h > 0$, then we say that we have elastic boundary conditions as $u(0,t)$ and $u_x(0,t)$ can vary. The case $h = \infty$ is known as a "killing" boundary condition [21], as it implies $u(0,t) = 0$, *i.e.* no motion at the boundary. Thus the parameters h and H control how much sound can be absorbed or reflected by the boundaries which are also described sometimes as either soft or hard.

We probe the medium by sending waves whose initial profile are given in (1.1) by $u(x, 0) = f(x)$, and speed $u_t(x, 0) = g(x)$. As the waves propagate inside, we measure their reflections on the wall, *i.e.* (1.2). Thus it is important that we let $h > 0$ or $H > 0$ so that sound waves can be heard across the boundary and this is why we shall avoid the case $h = H = 0$, *i.e.* Neumann boundary conditions as they act as sound insulators. The first question is how many measurements or readings of (1.2) are needed to recover both m and p? We show that we can do it for the setting (1.1) with a single measurement if we choose $f(x) = 1$ and $g(x) = x$. Recall that for the standard one dimensional wave equation with one coefficient [10],

$$u_{tt} = u_{xx} + q(x)u \quad \text{where} \quad 0 \le x \le b, \text{ and } t \ge 0 \tag{1.3}$$

and similar boundary conditions, it is shown that we can reconstruct the potential q from a finite, but unknown, number of measurements, N say. It was also shown that in case we have an *a priori* information on the upper bound of q then we can estimate the value of N and if in addition we know a lower bound on q, then at most 2 measurements are needed. In each case, the main issue is to find suitable initial conditions. The main tools used for (1.3) are asymptotics of eigenvalues and transformation operators that follow from the Gelfand−Levitan theory [22, 24]. Unfortunately for the string, as in (1.1), none of the techniques used for (1.3) is applicable. For example there are no known asymptotics for the eigenvalues of the string, except if we know the behavior of m as $x \to 0$, and it may even happen that the spectrum of a string is finite, see [11]. The lack of standard spectral features, such as transformation operators and asymptotics of eigenvalues for the string makes the inverse problem for (1.1) far more challenging than for the case in (1.3).

In [28, 29], Sini uses the Dirichlet−to−Neumann map to come up with Borg type uniqueness results for general Sturm−Liouville operators defined by $(py')' + qy = \lambda ry$, where $r > 0$. It is shown that if two operators differing by one coefficient have identical spectral sets then they must be equal. This settles the uniqueness issue. Thus the main issue in the inverse problem for (1.1), is whether we can extract a complete spectral set using a single measurement. Uniqueness would then follow from Sini's work. The aim of this paper is to show that when $h, H \ge 0$, and $h + H > 0$, we can identify m and p by listening only once through the walls of the boundary.

A possible application of (1.1) is to show how acoustic waves can be used to assess the condition of the wall thickness in an oil pipeline or clogged artery in a non invasive way. Also how to find a function p that helps focus and control high energy waves at certain points in the hope of clearing out any blockage.

2. Preliminaries

To avoid jumps of the derivative at the end-points, we assume $m(0-) = m(0+)$ and $m(b-) = m(b+)$. Define the Hilbert space $L^2_{\mathrm{d}m}(0, b)$ whose inner product is given by $(f,\ h)_m = \int_0^b f(x)\overline{h(x)}\mathrm{d}m(x)$. The string operator is then defined by

$$\mathbb{S}y(x) := \frac{-\mathrm{d}y'(x+)}{\mathrm{d}m(x)} \qquad \text{for} \quad 0 \le x \le b \tag{2.1}$$

where $y'(x+)$ is the right derivative at x and $\mathbb{S}y(x) = f(x)$ for $f \in L^2_{\mathrm{d}m}(0, b)$ means [19]

$$y(x) = y(0) + y'(0)x - \int_0^x (x - t)\, f(t)\mathrm{d}m(t). \tag{2.2}$$

Note that m can also be a step function, with jumps m_k at x_k for $k = 1, \ldots, n-1$ in which case (2.1) reduces to a finite differences operator

$$\frac{\mathrm{d}}{\mathrm{d}m(x)} \frac{\mathrm{d}^+}{\mathrm{d}x^+} u(x_k) = \frac{1}{m_k} \left[\frac{u(x_{k+1}) - u(x_k)}{x_{k+1} - x_k} - \frac{u(x_k) - u(x_{k-1})}{x_k - x_{k-1}} \right].$$

Thus we can write the wave equation in (1.1) as an evolution equation in $L^2_{\mathrm{d}m}(0,b)$

$$\begin{cases} u''(t) = -\mathbb{S}u(t) + p(t)u(t). \\ u(0) = f \quad \text{and } u'(0) = g \end{cases} \tag{2.3}$$

where usually $f \in \mathrm{Dom}(\mathbb{S}) \subset L^2_{\mathrm{d}m}(0,b)$ and $g \in L^2_{\mathrm{d}m}(0,b)$. To express the solution u, we first denote by φ_n the normalized eigenfunctions of the string, *i.e.* $\|\varphi_n\|^2_m = 1$,

$$\begin{cases} \mathbb{S}\varphi_n(x) = \lambda_n \varphi_n(x) & 0 \le x \le b \\ h\varphi_n(0) - \varphi_n'(0) = 0 & h \ge 0 \\ H\varphi_n(b) + \varphi_n'(b) = 0 & H \ge 0. \end{cases} \tag{2.4}$$

Under the condition, $\varphi_n(0) > 0$ the set $\{\varphi_n\}_{n\ge1}$ is uniquely defined and forms an orthonormal eigenbasis in $L^2_{\mathrm{d}m}(0,b)$, since $\mathbb{S}$ is a self-adjoint operator there ([14], p. 153). Note that from (2.2) and (2.4), on the intervals where $\mathrm{d}m(x) = 0$ we have $\varphi_n''(x) = 0$ and so $\varphi_n(x) = a_n x + b_n$ is a linear function there. We shall see below that the eigenvalues λ_n are increasing $0 < \lambda_1 < \lambda_2 < \ldots$ and in case they are infinite and the limit-circle condition $\int_0^b x^2 \mathrm{d}m(x) < \infty$ holds, then $\lambda_n \to \infty$. Recall that a Stieltjes' string, whose mass has a finite number of jumps only, has a finite set of eigenvalues [11], and its reconstruction is based on continued fractions. The inverse spectral theory of the string, which is closely connected to Gaussian processes, and prediction theory, is based on function theory [14,19,21]. For the reconstruction and uniqueness of general strings, one needs the theory of DeBranges spaces ([14], Chapters 5 and 6).

Let $f, g \in L^2_{\mathrm{d}m}(0,b)$ and denote their Fourier coefficients

$$a_n = \int_0^b f(x)\varphi_n(x)\mathrm{d}m(x) \qquad \text{and} \qquad b_n = \int_0^b g(x)\varphi_n(x)\mathrm{d}m(x). \tag{2.5}$$

Then if $\sum |a_n|^2 \lambda_n < \infty$ then we can define a weak solution $u(.,t) \in L^2_{\mathrm{d}m}(0,b)$ of (2.3) by ([31], Sect. 29, 6, p. 403),

$$u(x,t) = \sum_{n\ge1} \left(a_n C(t,\lambda_n) + b_n S(t,\lambda_n)\right) \varphi_n(x) \qquad \text{for } \ 0 \le t \tag{2.6}$$

where the functions C and S are cosine and sine type functions as they satisfy $-y''(t) + p(t)y(t) = \lambda_n y(t)$ and the initial conditions

$$C(0,\lambda_n) = S'(0,\lambda_n) = 1 \quad \text{and} \ \ C'(0,\lambda_n) = S(0,\lambda_n) = 0. \tag{2.7}$$

At the lateral boundaries, $x = 0, \ \ x = b$ and $0 \le t$ we can read, as we shall see, the continuous functions

$$\begin{aligned} u(0,t) &= \sum_{n\ge1} \left(a_n C(t,\lambda_n) + b_n S(t,\lambda_n)\right) \varphi_n(0) \\ u(b,t) &= \sum_{n\ge1} \left(a_n C(t,\lambda_n) + b_n S(t,\lambda_n)\right) \varphi_n(b). \end{aligned} \tag{2.8}$$

In the sums (2.8) we must have $\varphi_n(0) \ne 0$, otherwise it would follow from (2.4), that $\varphi_n(0) = \varphi_n'(0) = 0$ and so $\varphi_n = 0$, and similarly $\varphi_n(b) \ne 0$. Thus if

$$a_n^2 + b_n^2 \ne 0 \qquad \text{for } n \ge 1 \tag{2.9}$$

then all eigenvalues λ_n are present in the sum (2.8). In other words if for a certain $i \in \mathbb{N}$, both $a_i = b_i = 0$ then λ_i would be missing in (2.8) and so cannot be extracted later from the reading. In what follows we show that (2.9) holds for a particular pair of initial conditions $f(x)$ and $g(x)$. Here we state the first result pertaining to a single measurement when

$$h,\ H \geq 0, \quad h + H \neq 0, \quad f(x) = 1 \quad \text{and} \quad g(x) = \begin{cases} x & \text{if} \quad h > 0 \\ \text{arbitrary} & \text{if} \quad h = 0. \end{cases} \tag{2.10}$$

Note that (2.10) excludes the case of Neumann boundary conditions. We first show that

Proposition 2.1. *Assume that* (2.10) *holds, then* $\lambda_n > 0$ *and* $a_n^2 + b_n^2 \neq 0$ *for all* $n \geq 1$.

Proof. Multiply (2.4) by φ_n to obtain

$$\begin{aligned}\lambda_n \int_0^b \varphi_n^2(x)\mathrm{d}m(x) &= -\int_0^b \varphi_n''(x)\varphi_n(x)\mathrm{d}x = -\varphi_n'(b)\varphi_n(b) + \varphi_n'(0)\varphi_n(0) + \int_0^b \varphi_n'^2(x)\mathrm{d}x \\ &= H\varphi_n^2(b) + h\varphi_n^2(0) + \int_0^b \varphi_n'^2(x)\mathrm{d}x > 0\end{aligned}$$

and by (2.10) we deduce that $\lambda_n > 0$.

Next we compute the coefficients $a_n,\ b_n$ in (2.5) first for the case $h > 0$, *i.e.* $f(x) = 1$, and $g(x) = x$. Since $\lambda_n \neq 0$, use the fact that $\mathrm{d}\varphi_n' = -\lambda_n \varphi_n(x)\mathrm{d}m(x)$ to obtain

$$\begin{aligned}-\lambda_n a_n &= -\int_0^b \lambda_n \varphi_n(x)\mathrm{d}m(x) = \int_0^b \mathrm{d}\varphi_n'(x) = \varphi_n'(b) - \varphi_n'(0) \\ -\lambda_n b_n &= -\int_0^b x\lambda_n\varphi_n(x)\mathrm{d}m(x) = \int_0^b x\mathrm{d}\varphi_n'(x) = \varphi_n(0) - \varphi_n(b) + b\varphi_n'(b).\end{aligned} \tag{2.11}$$

Thus if $a_n = b_n = 0$, together with the boundary condition (2.4) and (2.11) yield a homogeneous system in $(\varphi_n(0),\ \varphi_n'(0),\ \varphi_n(b),\ \varphi_n'(b))$

$$\begin{cases} \varphi_n'(0) - \varphi_n'(b) = 0 \\ \varphi_n(0) - \varphi_n(b) + b\varphi_n'(b) = 0 \\ h\varphi_n(0) - \varphi_n'(0) = 0 \\ H\varphi_n(b) + \varphi_n'(b) = 0 \end{cases}$$

which has a nontrivial solution and yet its determinant, by (2.10), satisfies $H + h + Hbh \geq h + H > 0$, which is impossible.

For the next case, if $h = 0$, then by (2.4) we have $\varphi_n'(0) = 0$. Next if $a_n = 0$, then (2.11) yields $\varphi_n'(b) = 0$ which forces $\varphi_n(b) = 0$ since $H \neq 0$. Thus $\varphi_n = 0$ which is impossible and therefore $a_n \neq 0$ for all $n \geq 1$. □

We now show that (2.6) defines a weak solution and its traces, defined by (2.8), exist and are continuous.

3. Weak solution

Without loss of generality, by Proposition 2.1, we can take $h = 0$, and $f(x) = 1$ and $g(x) = 0$ as initial conditions so that no eigenvalue is missing from (2.6). We next show that these conditions generate a weak solution defined by

$$u(x,t) = \sum_{n\geq 1} a_n C(t, \lambda_n)\varphi_n(x) \tag{3.1}$$

and its traces on the boundary $u(0,t)$ and $u(b,t)$ are continuous and so readable. Recall that (2.11) $a_n\lambda_n = -\varphi_n'(b) \neq 0$. For the convergence of the series in (3.1), we use the fact that since

$$\begin{cases} -C''(t,\lambda_n) + p(t)C(t,\lambda_n) = \lambda_n C(t,\lambda_n) \\ C(t,\lambda_n) = 1 \quad \text{and} \quad C'(t,\lambda_n) = 0 \end{cases} \tag{3.2}$$

then for large λ_n, we have [22]

$$C(t,\lambda_n) = \cos\left(t\sqrt{\lambda_n}\right) + o(1) = O(1) \quad \text{and } C'(t,\lambda_n) = O\left(\sqrt{\lambda_n}\right). \tag{3.3}$$

Also recall that, from the inverse spectral theory of the string [14,19], a necessary and sufficient for the existence of a string with mass m is simply that

$$\int_0^\infty \frac{1}{1+\lambda}\mathrm{d}\Gamma(\lambda) < \infty,$$

where Γ is the spectral function associated with the rescaled eigenfunctions $\varphi_n(x)/\varphi_n(0)$. The fact that there are no asymptotics of eigenvalues or eigenfunctions required is a key difference between the Gelfand−Levitan and Krein theories. To proceed further we can express Γ by just rewriting the eigenfunction expansion for $\psi \in L^2_{\mathrm{d}m}(0,b)$ as

$$\psi(x) = \sum_{n\geq 1}\left(\int_0^b \psi(\eta)\varphi_n(\eta)\mathrm{d}m(\eta)\right)\varphi_n(x) = \sum_{n\geq 1}\left(\int_0^b \psi(\eta)\frac{\varphi_n(\eta)}{\varphi_n(0)}\mathrm{d}m(\eta)\right)\frac{\varphi_n(x)}{\varphi_n(0)}\varphi_n^2(0).$$

Thus, when the eigenfunctions are $\varphi_n(x)/\varphi_n(0)$, the spectral function is simply given by

$$\Gamma(\lambda) = \sum_{\lambda_n<\lambda}\varphi_n^2(0) \tag{3.4}$$

and

$$\int_0^\infty \frac{1}{1+\lambda}\mathrm{d}\Gamma(\lambda) = \sum_{n\geq 1}\frac{1}{1+\lambda_n}\varphi_n^2(0) < \infty \quad \textit{i.e.} \quad \left(\frac{\varphi_n(0)}{\sqrt{\lambda_n}}\right) \in \ell^2. \tag{3.5}$$

Proposition 3.1. *Assume that* $h = 0$, $\sup_{n\geq 1}\max_{0\leq x\leq b}|\varphi_n(x)/\varphi_n(0)| < \infty$, $f(x) = 1$ *and* $g(x) = 0$, *then* (2.6) *defines a weak solution* u *which is continuous over* $[0,b]\times[0,T]$ *and* $\nabla u(.,t) \in L^2_{\mathrm{d}m}(0,b)$.

Proof. It is enough to study the convergence of the sequence of classical solutions

$$u^{(k)}(x,t) = \sum_{n=1}^{n=k} a_n C(t,\lambda_n)\varphi_n(x)$$

and show that it converges to u in the weak sense in $L^2_{\mathrm{d}m}(0,b)\times L^2(0,T)$. First we can rewrite it as

$$u^{(k)}(x,t) = \sum_{n=1}^{n=k}\frac{-\varphi_n'(b)}{\lambda_n}C(t,\lambda_n)\varphi_n(x) = H\sum_{n=1}^{n=k}\frac{\varphi_n(b)}{\varphi_n(0)}\frac{\varphi_n^2(0)}{\lambda_n}C(t,\lambda_n)\frac{\varphi_n(x)}{\varphi_n(0)}. \tag{3.6}$$

From (3.5), Weiertrass M-test and

$$\left|\frac{\varphi_n(b)}{\varphi_n(0)}\frac{\varphi_n^2(0)}{\lambda_n}C(t,\lambda_n)\frac{\varphi_n(x)}{\varphi_n(0)}\right| \leq \frac{\varphi_n^2(0)}{\lambda_n}$$

we deduce that $u^{(k)}$ converges uniformly to u over $[0,b]\times[0,T]$, and so u, given by (3.1) is also continuous. Next we look at its derivatives in $L^2_{\mathrm{d}m}(0,b)$ .We have

$$u_t^{(k)}(x,t)=H\sum_{n=1}^{n=k}\frac{\varphi_n(b)}{\varphi_n(0)}\frac{\varphi_n(0)}{\sqrt{\lambda_n}}\frac{C'(t,\lambda_n)}{\sqrt{\lambda_n}}\varphi_n(x)$$

and since

$$\left|\frac{\varphi_n(b)}{\varphi_n(0)}\frac{\varphi_n(0)}{\sqrt{\lambda_n}}\frac{C'(t,\lambda_n)}{\sqrt{\lambda_n}}\right|\le\frac{|\varphi_n(0)|}{\sqrt{\lambda_n}}$$

again by (3.5) we deduce that $u_t(.,t)\in L^2_{\mathrm{d}m}(0,b)$ for each t and by (3.3) it follows that $u_t\in L^2_{\mathrm{d}m}(0,b)\times L^2(0,T)$.

Finally we show that $u_x(.,t)\in L^2_{\mathrm{d}x}(0,b)$ for each $t>0$. Use (3.6) to see that

$$\begin{aligned}\int_0^b\left(u_x^{(k)}(\eta,t)\right)^2\mathrm{d}\eta&=-H\left(u_x^{(k)}(b,t)\right)^2+\int_0^b\mathbb{S}u^{(k)}(\eta,t)\ \ u^{(k)}(\eta,t)\,\mathrm{d}m(\eta)\\&\le\int_0^b\mathbb{S}u^{(k)}(\eta,t)\ \ u^{(k)}(\eta,t)\,\mathrm{d}m(\eta)=\sum_{n=1}^{n=k}\frac{\varphi_n'^2(b)}{\lambda_n}C^2(t,\lambda_n)\\&\le H^2\sum_{n=1}^{n=k}\frac{\varphi_n^2(b)}{\varphi_n^2(0)}\frac{\varphi_n^2(0)}{\lambda_n}C^2(t,\lambda_n)\\&\le M(t)\sum_{n=1}^{n=k}\frac{\varphi_n^2(0)}{\lambda_n}<\infty.\end{aligned}$$

Having $\nabla u\in L^2(0,b)\times[0,T]$ allows us to say that u is a generalized solution, see [31]. The trace $u(0,t)$ is continuous since the sequence

$$u^{(k)}(0,t)=\sum_{n=1}^{n=k}a_nC(t,\lambda_n)\,\varphi_n(0)=H\sum_{n=1}^{n=k}\frac{\varphi_n(b)}{\varphi_n(0)}\frac{\varphi_n^2(0)}{\lambda_n}C(t,\lambda_n)$$

converges uniformly by (3.3) and the M-test , as we clearly have

$$\sum_{n\ge1}\left|\frac{\varphi_n(b)}{\varphi_n(0)}\frac{\varphi_n^2(0)}{\lambda_n}C(t,\lambda_n)\right|\le M_1(t)\sum_{n\ge1}\frac{\varphi_n^2(0)}{\lambda_n}<\infty$$

where $M_1(t)=\sup_{n\ge1}\left|\frac{\varphi_n(b)}{\varphi_n(0)}C(t,\lambda_n)\right|$. Similarly for the other trace $u(b,t)$, we also have

$$\begin{aligned}u^{(k)}(b,t)&=\sum_{n=1}^{n=k}a_nC(t,\lambda_n)\,\varphi_n(b)=H\sum_{n=1}^{n=k}\frac{\varphi_n^2(b)}{\lambda_n}C(t,\lambda_n)\\&\le\sum_{n=1}\left|\frac{\varphi_n^2(b)}{\varphi_n^2(0)}\frac{\varphi_n^2(0)}{\lambda_n}C(t,\lambda_n)\right|\le M_2(t)\sum_{n=1}\frac{\varphi_n^2(0)}{\lambda_n}.\end{aligned}$$

Thus both traces $u(0,t)$ and $u(b,t)$ are continuous functions. □

Remark 3.2. In case m is smooth then we have $\varphi_n''(x)+\lambda_n m'(x)\varphi_n(x)=0$, and the assumption on the boundedness of φ_n follows easily from the WKB method,

$$\varphi_n(x)\sim c_1\exp\left(-i\sqrt{\lambda_n m'(x)}\right)+c_2\exp\left(i\sqrt{\lambda_n m'(x)}\right)\qquad\text{as}\ \ \lambda_n\to\infty.$$

4. Recovering $m(x)$

We now want to extract the spectral data of the string by readings on the boundary of the solution given by (3.1)

$$u(0,t) = \sum_{n\geq 1} a_n C(t,\lambda_n)\,\varphi_n(0) \qquad \text{and } u(b,t) = \sum_{n\geq 1} a_n C(t,\lambda_n)\,\varphi_n(b) \qquad \text{for } t > T$$

where $C(t,\lambda_n)$ are defined by (3.2). Here the reading $u(0,t)$ carries information for both $m(x)$ and $p(t)$, and it is difficult to filter one out from the other. Basically we face two problems. The first is that not knowing p, we would not know the profile for $C(t,\lambda_n)$. Next even if we knew p and the functions $C(t,\lambda_n)$ are still unknown because the λ_n are. For this reason if we want to recover p over the interval $[0,\ T]$, we shall assume that $p(t) = \kappa \leq 0$ for $t > T$, where κ is a given constant. This is usually known in control theory, as bang-bang controller. Thus when p is quiet, we can "listen" to the solution for $t > T$, which is given by

$$u(x,t) = \sum_{n\geq 1} a_n \left[C(T,\lambda_n)\cos\left((t-T)\sqrt{\lambda_n-\kappa}\right) + C'(T,\lambda_n)\frac{\sin\left((t-T)\sqrt{\lambda_n-\kappa}\right)}{\sqrt{\lambda_n-\kappa}}\right]\varphi_n(x).$$

The readings for $t > T$, are then

$$\begin{aligned} u(0,t) &= \sum_{n\geq 1} \alpha_n \cos\left((t-T)\sqrt{\lambda_n-\kappa}\right) + \beta_n \sin\left((t-T)\sqrt{\lambda_n-\kappa}\right) \\ u(b,t) &= \sum_{n\geq 1} \widetilde{\alpha}_n \cos\left((t-T)\sqrt{\lambda_n-\kappa}\right) + \widetilde{\beta}_n \sin\left((t-T)\sqrt{\lambda_n-\kappa}\right) \end{aligned} \tag{4.1}$$

where

$$\begin{aligned} \alpha_n &= a_n C(T,\lambda_n)\,\varphi_n(0) & \beta_n &= a_n C'(T,\lambda_n)\,\varphi_n(0)/\sqrt{\lambda_n-\kappa} \\ \widetilde{\alpha}_n &= a_n C(T,\lambda_n)\,\varphi_n(b) & \widetilde{\beta}_n &= a_n C'(T,\lambda_n)\,\varphi_n(b)/\sqrt{\lambda_n-\kappa}. \end{aligned} \tag{4.2}$$

Using the shifted Laplace transform

$$\mathcal{L}\left(u(0,t)\right)(s) = \int_T^\infty \mathrm{e}^{-s(t-T)} u(0,t)dt = \sum_{n\geq 1} \frac{s\alpha_n + \beta_n\sqrt{\lambda_n-\kappa}}{s^2+\lambda_n-\kappa}. \tag{4.3}$$

We can read the poles $\pm i\sqrt{\lambda_n-\kappa}$, as the zeros of $1/\mathcal{L}\left(u(0,t)\right)(s)$, and from their residues compute both sequences $\pm i\alpha_n + \beta_n$ since $\lambda_n - \kappa > 0$. Solving the system would yield both α_n and β_n and similarly we could repeat the same procedure to evaluate $\widetilde{\alpha}_n, \widetilde{\beta}_n$ from $1/\mathcal{L}\left(u(b,t)\right)(s)$. Thus we can read from (4.3) the following data

$$\left\{\lambda_n,\ \alpha_n,\ \beta_n,\ \widetilde{\alpha}_n, \widetilde{\beta}_n\right\}_{n\geq 1}. \tag{4.4}$$

Taking the ratios $\widetilde{\alpha}_n/\alpha_n$ or $\widetilde{\beta}_n/\beta_n$ we would get the complete sequence $\{\varphi_n(b)/\varphi_n(0)\}_{n\geq 1}$. This is possible as $C(T,\lambda_n)$, $C'(T,\lambda_n)$ cannot be both zero.

In order to recover the mass m, we need to use the complete spectral data

$$\Xi := \left\{\lambda_n,\ \frac{\varphi_n(b)}{\varphi_n(0)}\right\}_{n\geq 1}$$

to construct the norming constants. To this end define the function

$$\delta(\lambda) = H\prod_{n\geq 1}\left(1-\frac{\lambda}{\lambda_n}\right)$$

then the norming constants are in fact given by the well-known formula, see [2]

$$\frac{1}{\varphi_n^2(0)} = \int_0^b (\varphi_n(x)/\varphi_n(0))^2 \,\mathrm{d}m(x) = -\delta'(\lambda_n)\,\varphi_n(b)/\varphi_n(0).$$

Thus we can reconstruct the spectral function by (3.4) and by Krein inverse spectral theory, we can uniquely reconstruct the mass m, see the backward problem in [14,19]. Recall that there is no uniqueness in the Neumann case. Thus we have proved the following

Proposition 4.1. *Assume that $p(t) = \kappa \le 0$ for $t \ge T$, $h = 0$, $f(x) = 1$ and $g(x) = 0$ then we can recover the mass m uniquely from one reading of $\{u(0,t),\ u(b,t)\}$ for $t\ > T$.*

Remark 4.2. The reading over (T,∞) and the use of the Laplace transform are for simplicity only. If we do not know when p is constant, then we can guess and pick a value for T "large enough". If the formula for the Laplace transform does not fit (4.3), then it means that our guess was wrong and we need to read at a later time. If p is a control function then we usually have some information about its behaviour. If the value of κ is unknown, then we would need the asymptotics of the large λ_n to find it. This requires an *a priori* information on the behavior of m as $x \to 0$, [7]. If we know that $p(t) = \kappa$ on $(T_1,\ T_2)$, then we could use spectral estimation methods to read the sequence (λ_n), see the window problem [27] or the method of pencil operators as in [4]. Again this methods require for example that the set of exponentials is a minimal set in $L^2(0,T)$.

Observe that the above proposition is still valid in the particular case $p = 0$, where we can use more initial conditions as required by (2.10).

Corollary 4.3. *Assume that $p = 0$ in (1.1) and (2.10) holds then we can recover m from a single measurement $u(0,t)$ and $u(b,t)$ for $t\ > 0$.*

Proof. We only need notice that from (2.6) with $p = 0$, we have the simple expression

$$u(x,t) = \sum_{n\ge 1} \left(a_n \cos\left(t\sqrt{\lambda_n}\right) + b_n \sin\left(t\sqrt{\lambda_n}\right)\right)\varphi_n(x)$$

and the same procedure using the Laplace transform, as in (4.3), applies and so we can read off $\lambda_n,\ a_n$, and b_n and form the spectral function (3.4). If it is known that $\exp\left(i\sqrt{\lambda_n}\right)$ form a minimal set in $L^2(0,\tau)$ then we need only read $u(0,t)$ and $u(b,t)$ for $0 \le t \le \tau$. □

Once m has been recovered, we then know φ_n, $\varphi_n(0)$, and $\varphi_n(b)$ explicitly for the next step, which is to recover $p(t)$ over $(0,\ T)$.

5. The recovery of $p(t)$

We now show how recover $p(t)$ for $t \in (0,\ T)$ from the data in (4.4). After having recovered m, we can reconstruct the sequences of eigenfunctions $\varphi_n,\ a_n,\ \varphi_n(0)$ and so we can extract from (4.2) the sequence of coefficients

$$\Theta := \{C(T,\lambda_n),\ C'(T,\lambda_n)\}_{n\ge 1}. \tag{5.1}$$

Note that, from the Gelfand−Levitan theory, since p is assumed to be locally integrable, there exists a continuous kernel K such that

$$C(T,\lambda_n) = \cos\left(T\sqrt{\lambda_n}\right) + \int_0^T K(T,\eta)\cos\left(\eta\sqrt{\lambda_n}\right)\mathrm{d}\eta$$

$$C'(T,\lambda_n) = -\sqrt{\lambda_n}\sin\left(T\sqrt{\lambda_n}\right) + K(T,T)\cos\left(T\sqrt{\lambda_n}\right) + \int_0^T K_x(T,\eta)\cos\left(\eta\sqrt{\lambda_n}\right)\mathrm{d}\eta. \tag{5.2}$$

It follows that the coefficients $C(T,\lambda_n)$ and $C'(T,\lambda_n)+\sqrt{\lambda_n}\sin\left(T\sqrt{\lambda_n}\right)$ are bounded for $n\to\infty$

$$C(T,\lambda_n)=O(1)\qquad\text{and}\quad C'(T,\lambda_n)+\sqrt{\lambda_n}\sin\left(T\sqrt{\lambda_n}\right)=O(1)\,. \tag{5.3}$$

In order to reconstruct the function p, by the Gelfand−Levitan inverse spectral theory we shall interpolate both functions $C(T,\lambda)$ and $C'(T,\lambda)$ from their extracted values at λ_n. To this end we need to use the eigensolutions of the initial value problem, Kramer's theorem [11,32],

$$\begin{cases}\mathbb{S}\varphi(x,\lambda)=\lambda\varphi(x,\lambda)\\ \varphi(0,\lambda)=1\quad\text{and}\quad\varphi'(0,\lambda)=h\end{cases}$$

then $\varphi(x,\lambda)$ is an entire function of λ and is related to φ_n in (2.4) when $h=0$, by

$$\varphi_n(x)=\varphi_n(0)\,\varphi(x,\lambda_n)\,. \tag{5.4}$$

Since φ_n are normalized by $\|\varphi_n\|=1$, (5.4) helps verify again (3.4) since

$$\|\varphi(x,\lambda_n)\|\,|\varphi_n(0)|=1.$$

Proposition 5.1. *We can reconstruct the function $C(T,.)$ and $C'(T,.)$ from the spectral set Θ in (5.1).*

Proof. Since $C(T,\lambda_n)$ are bounded, by (3.4), the integral is also

$$\int_0^\infty\frac{C^2(T,\lambda)}{(1+\lambda)^2}\mathrm{d}\Gamma(\lambda)=\sum_{n\geq1}\frac{C^2(T,\lambda_n)}{(1+\lambda_n)^2}\varphi_n^2(0)\leq O(1)\sum_{n\geq1}\frac{1}{(1+\lambda_n)}\varphi_n^2(0)<\infty.$$

Thus the values $C(T,\lambda_n)/(1+\lambda_n)$ can be seen as the Fourier coefficients of a function $h\in L^2_{\mathrm{d}m}(0,b)$, *i.e.*

$$\frac{C(T,\lambda_n)}{(1+\lambda_n)}=\int_0^b h(x)\varphi(x,\lambda_n)\,\mathrm{d}m(x),\quad\text{where}\qquad h(x)=\sum_{n\geq1}\frac{C(T,\lambda_n)}{(1+\lambda_n)}\varphi(x,\lambda_n)\,\varphi_n^2(0)$$

Thus, by Kramer's sampling theorem [11,32], we get a unique analytic extention defined by

$$\gamma_1(\lambda):=\frac{C(T,\lambda)}{(1+\lambda)}=\int_0^b h(x)\varphi(x,\lambda)\,\mathrm{d}m(x)\quad\text{for all }\lambda\in\mathbb{C}. \tag{5.5}$$

Similarly we would interpolate the function

$$\gamma_2(\lambda):=\frac{C'(T,\lambda)+\sqrt{\lambda}\sin\left(T\sqrt{\lambda}\right)}{1+\lambda} \tag{5.6}$$

and γ_1 and γ_2 are φ-Fourier transform. From γ_1 and γ_2 in (5.5) and (5.6) we can get $C(T,\lambda)$ and $C'(T,\lambda)$ □

The eigenvalues μ_n of a $\kappa-$family of Sturm−Liouville problems can be generated as the zeros of the function

$$\Delta_\kappa(\mu)=C'(T,\mu)+\kappa C(T,\mu)=(1+\mu)\left[\gamma_2(\mu)+\kappa\gamma_1(\mu)\right]-\sqrt{\mu}\sin\left(T\sqrt{\mu}\right)=0 \tag{5.7}$$

and $\kappa\in\mathbb{R}$. Having evaluated the μ_n, we can then use the traces $C(T,\mu_n)$, and $C'(T,\mu_n)$ to compute the norming constants which would then deliver the spectral function of the Sturm−Liouville problem

$$\begin{cases}-C''(t,\mu_n)+p(t)C(t,\mu_n)=\mu_nC(t,\mu_n)\qquad 0\leq t\leq T\\ C'(0,\mu_n)=0\quad\text{and}\quad C'(T,\mu_n)+\kappa C(T,\mu_n)=0\end{cases}$$

and then finally p by the Gelfand−Levitan theory. Thus by combining Propositions 3.1 and 5.1, and the above construction we have proved the main result.

Proposition 5.2. *Assume that* $p(t) = \kappa \leq 0$ *for* $t > T$, $h = 0$, $f(x) = 1$ *and* $g(x) = 0$ *then we can uniquely recover the mass* m *and* $p(t)$ *for* $0 < t < T$ *from a single measurement of* $\{u(0,t),\ u(b,t)\}$ *for* $t > T$.

Remark 5.3. With few modifications we can also treat the case $h > 0$. By (2.10) we need to use initial condition $f(x) = 1$ and $g(x) = x$ and the solution given in (2.6). The interpolation should remain the same.

5.1. Numerical methods

We now briefly explain how the above procedure can be implemented numerically. An important class of strings, whose spectra are finite, are known as Stieltjes strings. Their construction is simple, explicit and based on the well-known Stieltjes continued fractions. Because the mass $m(x)$ is a step function over a finite interval, its recovery is very fast as it requires arithmetic operations only. The key approximation result ([14], Sect. 5.8) is that general strings, whose spectra are infinite, are seen as the limit of an increasing sequence of longer Stieltjes strings. Thus good approximations are always possible, and can be obtained by simply using a set of algebraic rules. There are also examples where full explicit constructions are possible, and can serve as benchmark for numerical methods. For example we refer to [30] where a smooth mass $m(x)$ is approximated by step functions, *i.e.* Stieltjes strings. A different and direct approach, based on identification, approximates analytic masses by Taylor polynomials, [7], and contains several numerical examples.

For the Gelfand−Levitan theory, the algorithms are different as they are based on the idea of transformation operators, which lead to integral equations. Because of their important applications in geophysics, vibrations, solitons, the inverse spectral transform, they got more attention and more numerical methods are available for the standard Sturm−Liouville operator, and sometimes by just reversing direct methods. The numerical side of truncation errors, stability, convergence rates has also been studied by various authors, and can be found in [1, 5, 6, 13, 13, 15, 23, 26], just to name a few. There are fast root finding methods based on sampling theory for the computation of spectra, with guaranteed error bounds, through characteristic functions such as (5.7), see [8]. For a reference that treat the numerical aspects of the wave equation, but with nonclassical boundary conditions, such as $\delta(x)$, we mention [18].

We now look at the control question which follows by simply reversing the identification process of p.

6. Controlling the solution

Assume that we have identified the string operator $\mathbb{S}$, and we now want to find a controller $p \in L^1(0,T)$ so we can steer the solution to a certain given reachable target $r(x)$ at a given time T and with a certain speed $s(x)$

$$\begin{cases} u(x,T) = r(x) = \sum_{n\geq 1} r_n \varphi_n(x) \in L^2_{\mathrm{d}m}(0,b) \\ u_t(x,T) = s(x) = \sum_{n\geq 1} s_n \varphi_n(x) \in L^2_{\mathrm{d}m}(0,b). \end{cases} \tag{6.1}$$

The condition on the Fourier coefficients translates into $\sum_{n\geq 1} r_n^2 \varphi_n^2(0) < \infty$ and $\sum_{n\geq 1} s_n^2 \varphi_n^2(0) < \infty$. From (3.1), by identifying the Fourier coefficients, we have the system

$$a_n C(T,\lambda_n) = r_n \quad \text{and} \quad a_n C'(T,\lambda_n) = s_n \quad \text{for } n \geq 1 \tag{6.2}$$

Assume that $a_n \neq 0$, which is needed for the identification of m, we end end up with an interpolation problem where we are given the sequences $C(T,\lambda_n) = r_n/a_n$ and $C'(T,\lambda_n) = s_n/a_n$ *i.e.* the set

$$\Theta := \{C(T,\lambda_n),\ C'(T,\lambda_n)\}_{n\geq 1} \tag{6.3}$$

which is treated in the previous section. We state the result which follows by reversing the identification of p.

Proposition 6.1. *Assume that condition* (5.3) *holds for the set* (6.3) *which is defined by the sequences* $\{r_n,\ s_n,\ a_n\}$, *where* $a_n \neq 0$ *in* (6.2)*, then we can reconstruct a controller* $p \in L^1(0,T)$ *such that* (6.1) *holds.*

Acknowledgements. The author sincerely thanks the referees for their valuable comments.

References

[1] L.E. Andersson, Algorithms for solving inverse eigenvalue problems for Sturm−Liouville equations, *Inverse Methods in Action* edited by P.C. Sabatier. Springer-Verlag (1990) 138–145.

[2] F. Al-musallam and A. Boumenir, Identification and control of a heat equation. *Int. J. Evolution Equations* **6** (2013) 85–100.

[3] S.A. Avdonin and A. Bulanova, Boundary control approach to the spectral estimation problem: the case of multiple poles. *Math. Control Signals Systems* **22** (2011) 245–265.

[4] S.A. Avdonin, F. Gesztesy and A. Makarov, Spectral estimation and inverse initial boundary value problems. Vol. 4 of *Inverse Probl. Imaging* (2010) 1–9.

[5] V. Barcilon, Iterative solution of the inverse Sturm−Liouville problem. *J. Math. Phys.* **15** (1974) 429–436.

[6] A. Boumenir, The recovery of analytic potentials. *Inverse Probl.* **15** (1999) 1405–1423.

[7] A. Boumenir, The reconstruction of an analytic string from two spectra. *Inverse Probl.* **20** (2004) 833–846.

[8] A. Boumenir, Computing Eigenvalues of periodic Sturm−Liouville problems by the Shannon-Whittaker sampling theorem. *Math. Comput.* **68** (1999) 1057–1066.

[9] A. Boumenir and Vu Kim Tuan, Recovery of the heat coefficient by two measurements. *Inverse Probl. Imaging* **5** (2011) 775–791

[10] A. Boumenir and Vu Kim Tuan, An inverse problem for the wave equation. *J. Inverse Ill-Posed Probl.* **19** (2011) 573–592.

[11] A. Boumenir and A.I. Zayed, Sampling with a String. *J. Fourier Anal. Appl.* **8** (2002) 211–232.

[12] A. Boumenir and R. Carroll, Toward a general theory of transmutation. *Nonlin. Anal.* **26** (1996) 1923–1936.

[13] R. Carroll, F. Santosa, On the complete recovery of geophysical data. *Math. Methods Appl. Sci.* **4** (1982) 33–73.

[14] H. Dym and H.P. McKean, Gaussian processes, function theory, and the inverse spectral problem. Vol. 31 of *Probab. Math. Stat.* Academic Press, New York, London (1976).

[15] G.M.L. Gladwell, Inverse Problems in Vibration, series: Solid Mechanics and Its Applications, 2nd edition. Springer (2004).

[16] O.H. Hald, The inverse Sturm−Liouville problem with symmetric potentials. *Acta Math.* **141** (1978) 263–291.

[17] S. Hansen and E. Zuazua, Exact controllability and stabilization of a vibrating string with an interior point mass. *SIAM J. Control Optim.* **33** (1995) 1357–1391.

[18] S.I. Kabanikhin, A. Satybaev and M. Shishlenin, Direct Methods of Solving Multidimensional Inverse Hyperbolic Problems, Series: *Inverse and Ill-Posed Problems Series.* De Gruyter **48** (2005).

[19] I.S. Kac and M.G. Krein, On the spectral functions of the String. *Amer. Math. Soc. Transl.* **103** (1974) 19–102.

[20] V. Komornik and P. Loreti, Fourier Series in Control Theory. *Springer Monogr. Math.* Springer (2005)

[21] U. Küchler, K. Neumann An extension of Krein's inverse spectral theorem to strings with non-reflecting left boundaries. *Lect. Notes Math.* **1485** (1991) 354–373.

[22] B.M. Levitan and M.G. Gasymov, Determination of a differential equation by two spectra. *Russian Math. Surv.* **19** (1964) 3–63.

[23] J.R. McLaughlin, Stability theorems for two inverse spectral problems. *Inverse Probl.* **4** (1988) 529–540.

[24] V. Marchenko, Sturm−Liouville Operators and Applications. *Oper. Theory Adv. Appl.*, vol. 22. Birkhäuser, Basel (1986).

[25] Y. Privat, E. Trélat and E. Zuazua, Optimal Observation of the One-dimensional Wave Equation. *J. Fourier Anal. Appl.* **19** (2013) 514–544.

[26] W. Rundell and P.E. Sacks, Reconstruction techniques for classical inverse Sturm−Liouville problem. *Math. Comput.* **58** (1992) 161–183.

[27] T.I. Seidman, S.A. Avdonin and S.A. Ivanov, The 'window problem' for series of complex exponentials. *J. Fourier Anal. Appl.* **6** (2000) 233–254.

[28] M. Sini, On the one-dimensional Gelfand and Borg−Levinson spectral problems for discontinuous coefficients. *Inverse Probl.* **20** (2004) 1371–1386.

[29] M. Sini, Some uniqueness results of discontinuous coefficients for the one-dimensional inverse spectral problem. *Inverse Probl.* **19** (2003) 871–894.

[30] G. Turchetti and G. Sagretti, Stieltjes Functions and Approximation Solutions of an Inverse Problem. *Springer Lect. Notes Phys.* **85** (1978) 123-33.

[31] V.S. Valdimirov, Equations of Mathematical Physics. Marcel Dekker, New York (1971).

[32] A. Zayed, Advances in Shannon's Sampling Theory. CRC Press, Boca Raton, FL (1993).