ESAIM: COCV 21 (2015) 60–72
DOI: 10.1051/cocv/2014017

**ESAIM: Control, Optimisation and Calculus of Variations**
www.esaim-cocv.org

# ON THE FABER–KRAHN INEQUALITY FOR THE DIRICHLET $p$-LAPLACIAN *

Anisa M.H. Chorwadwala[1], Rajesh Mahadevan[2] and Francisco Toledo[2]

**Abstract.** A famous conjecture made by Lord Rayleigh is the following: "The first eigenvalue of the Laplacian on an open domain of given measure with Dirichlet boundary conditions is minimum when the domain is a ball and only when it is a ball". This conjecture was proved simultaneously and independently by Faber [G. Faber, Beweiss dass unter allen homogenen Membranen von gleicher Fläche und gleicher Spannung die kreisförfegige den leifsten Grundton gibt. *Sitz. bayer Acad. Wiss.* (1923) 169–172] and Krahn [E. Krahn, Über eine von Rayleigh formulierte Minimaleigenschaftdes Kreises. *Math. Ann.* **94** (1924) 97–100.]. We shall deal with the $p$-Laplacian version of this theorem.

**Mathematics Subject Classification.** 35B06, 35B51, 35J92, 35P30, 49Q20.

Received July 20, 2013. Revised February 28, 2014.
Published online October 17, 2014.

## 1. Introduction

The $p$-Laplacian $\Delta_p$ is the non-linear operator defined as $\Delta_p f = \operatorname{div}(|\nabla f|^{p-2}\nabla f)$. We consider the following domain optimization problem:

Given the eigenvalue problem

$$\left.\begin{aligned} -\Delta_p u = \lambda |u|^{p-2}\, u \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega \end{aligned}\right\} \tag{1.1}$$

whose principal eigenvalue is

$$\lambda_1(\Omega) := \inf \left\{ \frac{\|\nabla\varphi\|^p_{L^p(\Omega)}}{\|\phi\|^p_{L^p(\Omega)}} \;\middle|\; \varphi \in W_0^{1,p}(\Omega) \setminus \{0\} \right\}, \tag{1.2}$$

we are interested in minimizing $\lambda_1(\Omega)$ among all bounded open sets $\Omega$ having a given volume (Lebesgue measure).

In the case $p = 2$, a famous conjecture made by Lord Rayleigh in 1894 and later proved by Faber [11] and Krahn [14] says that the ball minimizes this eigenvalue functional among all bounded domains with the same

*Keywords and phrases.* Symmetry, moving plane method, comparison principles, boundary point lemma.

* *This work was supported by Comisión Nacional de Investigación Científica y Tecnologíca [FONDECYT REGULAR Nº 1090305 and FONDECYT REGULAR Nº 1130595]; the National Board of Higher Mathematics, India; and Indian Institute of Science Education and Research, Pune.*

[1] Indian Institute of Science Education and Research, Pune, India. anisa@iiserpune.ac.in

[2] Departamento de Matemática, Univ. de Concepción, Concepción, Chile. rmahadevan@udec.cl;franciscotoledo@udec.cl

volume and is the only minimizer. Later, it has been shown that this holds also for any $p$ such that $1 < p < \infty$. It is quite standard to show using the Polya–Szego inequality for the Schwarz symmetrization that, given any bounded domain $\Omega$, the ball with the same volume has a smaller first Dirichlet eigenvalue than that of $\Omega$. However, proving that the ball is the only minimizer is non-trivial. Previously, this problem has been studied in a few papers using different approaches. One of these approaches is based on a result of Brothers and Ziemer [6] which establishes a criteria when equality holds in the Polya–Szego inequality and this can be found in Alvino, Ferone and Trombetti [2] (see Thm. 3.1). Another proof due to Bhattacharya [5], which is quite technical, is based on some estimates related to some inequalities of Talenti [21]. We follow a different, quite general approach where the symmetry question is treated by studying symmetry in a suitable overdetermined elliptic problem. Indeed, if a smooth domain with a connected boundary is a minimizing domain, then the normal derivative of the first eigenfunction has to be constant on the boundary, thus making the problem overdetermined.

Symmetry results for overdetermined boundary value problems for both linear and non-linear elliptic pdes has been studied by Serrin [20] using the moving plane method [4] whose idea goes back to Aleksandrov [1]. The strategy is to compare the solutions on either side of a moving plane, at some critical positions of the moving plane, by means of some comparison principles and then to obtain a Hopf type lemma which gives a contradiction to the overdetermined condition. We follow this strategy to give a proof of the uniqueness of the minimizer of the Dirichlet eigenvalue for the $p$-Laplacian in the class of bounded connected domains having a connected $C^{2,\gamma}$ boundary.

Overdetermined problems for non-linear elliptic operators have been treated in [9, 10, 12, 13] and some of the references included therein. Most of these results are again based on the principle of Serrin's paper. In Damascelli and Pacella [10], they obtain a symmetry result for an overdetermined problem for the $p$-Laplacian for $p \in (1, 2)$ for a general class of non-linear functions $f$ on the right-hand side. The symmetry results therein are based on weak comparison principles and strong comparison principles which require the non-degeneracy of the gradient of the solution, a condition which is not always satisfied for $p \in (2, \infty)$ for general $f$. Moreover, the proofs of the comparison principles are quite technical. However, we would like to point out that for the minimization of the first eigenvalue for the $p$-Laplacian this non-degeneracy condition is satisfied, in the light of a recent result of Lou [16], since, therein, it is shown that the critical set of the first eigenfunction is of measure zero. So, we may observe that the uniqueness of the minimizing domain for the eigenvalue problem also follows from the result of [10] for all $p \in (1, \infty)$.

In this article, we treat only the eigenvalue problem for which $f(s) = \lambda_1 s^{p-1}$ where $\lambda_1$ is the first eigenvalue for the domain for given $p \in (1, \infty)$. For such $f$, we can prove the weak comparison principle and the strong comparison principle needed for the Hopf lemma, in a much simpler way as compared to the approach of Damascelli and Pacella. In fact, we follow the approach of Cuesta and Takác [8] and improve upon a weak comparison principle, valid for any $p \in (1, \infty)$, proved therein for non-negative solutions of the $p$-Laplace equation with monotone non-decreasing nonlinearities in domains with at least $C^1$ boundary, for homogeneous Dirichlet boundary conditions. Our improvement (see Thm. 2.1) consists in relaxing the zero Dirichlet boundary condition as also the regularity assumption on the boundary. If we compare our paper with Serrin's paper, in Serrin's paper, the necessary weak comparison principles are assumed for the class of non-linear pdes being considered and the non-linear pde is assumed explicitly to be uniformly elliptic in the entire domain. The latter assumption does not hold for the $p$-Laplacian equation. We show that it is enough to use the ellipticity in a local way. The remaining ideas are as in Serrin's paper.

The results in [12, 13] are based on quite different principles.

In Section 2, we recall some useful facts. In particular, we recall some properties related to the first eigenvalue of the Dirichlet $p$-Laplacian and discuss the comparison principles. In Section 3, we prove Theorem 3.2, the main result, where we show the uniqueness in the Faber–Krahn inequality for the $p$-Laplacian. In Section 4, we provide the proof of the weak comparison principle being used. We also prove a proposition needed for verifying a hypotheses required for the application of the comparison principle.

## 2. Preliminaries

In this section we recall some useful facts which will be used later on.

### Properties related to the first eigenvalue of the Dirichlet $p$-Laplacian

Given a bounded domain $\Omega$ in $\mathbb{R}^n$ with Lipschitz boundary, let $\lambda_1(\Omega)$ denote the first eigenvalue of the Dirichlet $p$-Laplacian defined through (1.2). Then, $\lambda_1(\Omega)$ is invariant under orthogonal transformations of the domain. For a bounded connected domain $\Omega$, it is known that the first eigenvalue is simple. It is also known that any eigenfunction for $\lambda_1$ has a strict sign in $\Omega$ and is the only eigenfunction with this property. We refer to Lindqvist [15] for these results. Hereafter, by first eigenfunction we shall refer to a positive eigenfunction whose $L^p$ norm is 1. In a domain $\Omega$ with $C^{2,\gamma}$ boundary, the normal derivative of the first eigenfunction is strictly negative on $\partial\Omega$ (*cf.* Sakaguchi [19], Lem. A.3). Furthermore, if $\Omega$ has a connected $C^{2,\gamma}$ boundary, then the first eigenfunction is known to belong to $C^{1,\alpha}(\overline{\Omega}) \cap C^{2,\gamma}(\overline{\Omega_\varepsilon})$ for some $\varepsilon > 0$, where $\Omega_\varepsilon = \{x \in \Omega : \mathrm{d}(x, \partial\Omega) < \varepsilon\}$ (see Barles [3], Thm. 1.3).

### Weak Comparison Principle for the $p$-Laplacian

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Let $\beta : \Omega \times \mathbb{R} \to \mathbb{R}$ be continuous and assume that there exists $\kappa \in [0,1]$ and $M > 0$ so that $\beta$ satisfies the growth assumption

$$\beta(x,s) \leq M\left(\kappa + |s|^{p-2}\right)|s| \quad \text{ for all } x \in \Omega, \text{ for all } s \text{ in } \mathbb{R}. \tag{2.1}$$

Further, we shall consider $\beta$ satisfying one of the two assumptions:
(b1) $s \longmapsto \beta(x,s)$ is locally Lipschitz on $\mathbb{R}\setminus\{0\}$ uniformly for $x \in \Omega$ and $\frac{\partial\beta}{\partial s} \leq 0$ for almost all $(x,s) \in \Omega\times\mathbb{R}$ or,
(b2) $\beta(x,s)$ is nondecreasing in $s$ for $(x,s) \in \Omega \times \mathbb{R}$ and $\beta(x,0) \geq 0$ for all $x \in \Omega$.

Let $f, g \in W^{-1,\frac{p}{p-1}}(\Omega)$, $f', g' \in W^{1-\frac{1}{p},p}(\partial\Omega)$ with $f \geq g$ in $\Omega$ (in the sense of distributions), $f' \geq g'$ on $\partial\Omega$. Let $u, v \in W^{1,p}(\Omega)$ solve (in the weak sense)

$$\begin{aligned} -\Delta_p u &= \beta(x,u) + f(x), & -\Delta_p v &= \beta(x,v) + g(x) & &\text{in } \Omega,\\ u &= f', & v &= g' & &\text{on } \partial\Omega. \end{aligned} \tag{2.2}$$

A question of interest is to know whether the following weak comparison result holds,

$$u \geq v \text{ almost everywhere in } \Omega. \tag{2.3}$$

For $\beta$ satisfying (b1), this has been shown by Tolksdorff [22].

For $\beta$ satisfying the hypothesis (b2), stronger conditions are needed for the comparison principle to hold. On a bounded domain in $\mathbb{R}^n$ with a $C^{1,\gamma}$ boundary, if it is assumed that

$(A-1)$ $\beta$ satisfies (b2), that is $\beta(x,s)$ is nondecreasing in $s$ for $(x,s) \in \Omega \times \mathbb{R}$, and $\beta(x,0) \geq 0$ for all $x \in \Omega$,
$(A-2)$ The problem

$$\begin{aligned} -\Delta_p u &= \beta(x,u) + f & &\text{in} & &\Omega,\\ u &= 0 & &\text{on} & &\partial\Omega. \end{aligned}$$

given $f \in L^\infty(\Omega)$, $f \geq 0$ in $\Omega$, admits a unique non-negative solution $u \in W_0^{1,p}(\Omega)$, and
$(A-3)$ $f, g \in L^\infty(\Omega)$, $0 \leq g \leq f$ on $\Omega$ and $0 = g' = f'$ on $\partial\Omega$.

Then it has been shown by Cuesta and Takác (see [8], Prop. 2.3) that *non-negative solutions of* (2.2) satisfy the weak comparison (2.3).

In our applications, we need to relax the zero Dirichlet boundary condition assumption as well as the $C^{1,\gamma}$ regularity assumption on the domain $\Omega$ in order to apply it in a piecewise smooth domain. So, we need the following variant of the result of Cuesta and Takác.

**Theorem 2.1.** *Let $u, v$ in $W^{1,p}(\Omega)$ be non-negative solutions weak solutions of (2.2) where $\Omega$ is just a Lipschitz domain and we assume that*

$(A-1)$ $\beta(x,s)$ *is nondecreasing in $s$ for $(x,s) \in \Omega \times \mathbb{R}$, and $\beta(x,0) \geq 0$ for all $x \in \Omega$,*
$(A-2')$ *The problem*

$$\begin{aligned} -\Delta_p w &= \beta(x,w) + f \qquad &\textit{in} \qquad &\Omega, \\ w &= f' \qquad &\textit{on} \qquad &\partial\Omega. \end{aligned}$$

*given $f \in W^{-1,\frac{p}{p-1}}(\Omega)$, $f \geq 0$ in $\Omega$ and $f' \in W^{1-\frac{1}{p},p}$ with $f' \geq 0$ on $\partial\Omega$, admits a unique non-negative solution $w \in W^{1,p}(\Omega)$.*
$(A-3')$ *$f, g \in W^{-1,\frac{p}{p-1}}(\Omega)$, $0 \leq g \leq f$ on $\Omega$ and $f', g' \in W^{1-\frac{1}{p},p}(\partial\Omega)$ with $0 \leq g' \leq f'$ on $\partial\Omega$.*

*Then, we conclude that $u \geq v$ almost everywhere on $\Omega$, that is the WCP holds.*

*Proof.* The proof of this theorem is inspired directly by the proof of Proposition 2.3 in [8]. It is given in Section 4. □

A positive definite matrix

Consider the strictly convex function $\Gamma : \mathbb{R}^n \to \mathbb{R}$ defined by $\Gamma(\xi) = \frac{|\xi|^p}{p}$. Then $\Gamma \in C^\infty(\mathbb{R}^n \setminus \{0\})$. Let $A = D\Gamma$, the gradient of $\Gamma$. Then $A = (A_1, A_2, \ldots, A_N) : \mathbb{R}^n \to \mathbb{R}^n$ is given by

$$A(\xi) = |\xi|^{p-2}\xi. \tag{2.4}$$

The Hessian matrix $\mathcal{A}(\xi) := \left[\frac{\partial A_i}{\partial \xi_j}(\xi)\right]_{i,j=1}^n = [D_{ij}\Gamma(\xi)]_{i,j=1}^n$ can be calculated and is found to be equal to $|\xi|^{p-2}\mathrm{Id} + (p-2)|\xi|^{p-4}\xi \otimes \xi$. It can be seen that $(p-1)|\xi|^{p-2}$ and $|\xi|^{p-2}$ are eigenvalues of $\mathcal{A}(\xi)$ with multiplicity one and $(n-1)$, respectively. Therefore, for any $\eta \in \mathbb{R}^n$, we have

$$\langle \mathcal{A}(\xi)\eta, \eta\rangle \geq \min\{1, p-1\}|\xi|^{p-2}|\eta|^2. \tag{2.5}$$

Thus $\mathcal{A}(\xi)$ is a positive definite matrix but which becomes degenerate or singular near $\xi = 0$ depending on whether $p > 2$ or $1 < p < 2$.

## 3. Uniqueness in Faber–Krahn inequality

The Faber–Krahn inequality $\lambda_1(\Omega) \geq \lambda_1(\Omega^*)$, where $\Omega$ is a bounded domain and $\Omega^*$ is a ball with the same volume, can be proved by an application of the Polya–Szegö inequality for the Schwarz symmetrization. In this section, we show that if $\Omega$ is any bounded connected open set with a *connected* $C^{2,\gamma}$ boundary then the above inequality is strict unless $\Omega$ is a copy of $\Omega^*$.

We begin with the following lemma whose proof is standard in shape optimization problems. It may be deduced from the first order necessary condition for an optimal shape using the formula for the shape derivative obtained in García Melián *et al.* [17]

**Lemma 3.1.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded connected domain with a* connected *$C^{2,\gamma}$ boundary which minimizes $\lambda_1$ among rectifiable domains of given volume. Then, there exists a negative constant $c$ such that the first eigenfunction $u$ (assumed to be positive), satisfies*

$$\frac{\partial u}{\partial n} = c \quad \textit{on} \quad \partial\Omega.$$

So, we are led to consider the following overdetermined problem. Let us denote $\lambda_1(\Omega)$ by $\lambda_1$. Consider a bounded connected domain $\Omega$ with a connected $C^{2,\gamma}$ boundary in $\mathbb{R}^n$. Suppose there exists a positive function $u$ satisfying

$$\begin{cases} -\Delta_p u = \lambda_1 u^{p-1} & \text{in } \Omega \\ \quad u = 0 & \text{in } \partial\Omega \\ \quad \frac{\partial u}{\partial n} = c & \text{in } \partial\Omega \end{cases} \tag{3.1}$$

in the weak sense. Must $\Omega$ be a ball? In the affirmative case, we will have shown that the ball is the unique minimizer of $\lambda_1$ among bounded connected domains with a connected $C^{2,\gamma}$ boundary having the same volume. Indeed, this result is proved in the next theorem.

**Theorem 3.2.** *Let $\Omega$ be a bounded connected domain with a connected $C^{2,\gamma}$ boundary in $\mathbb{R}^n$. Let us suppose that there exists a positive function $u$ satisfying (3.1) in the weak sense. Then $\Omega$ is a ball.*

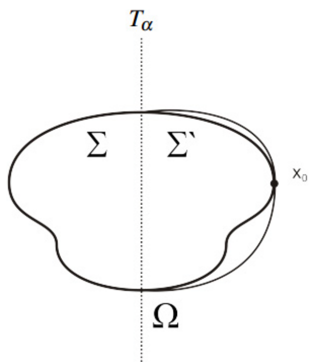
*Proof.* We recall, from the discussion in the first paragraph of Section 2, that $u$ is necessarily an eigenfunction for the eigenvalue $\lambda_1(\Omega)$ belongs to $C^{1,\alpha}(\overline{\Omega}) \cap C^{2,\gamma}(\overline{\Omega_\varepsilon})$ for some $\varepsilon > 0$, where $\Omega_\varepsilon = \{x \in \Omega : d(x, \partial\Omega) < \varepsilon\}$.

Let $T_0$ be a hyperplane in $\mathbb{R}^n$ not intersecting the domain $\Omega$. We suppose this plane to be moved continuously parallel to $T_0$ to new positions, until ultimately it begins to intersect $\Omega$. From that moment onward, at each stage the resulting plane $T$ cuts off from $\Omega$ a cap $\Sigma(T) = \Omega \cap T^-$, where $T^-$ is the half plane formed by $T$ containing $T_0$.

For any cap $\Sigma(T)$ thus formed, we let $\Sigma'(T)$ be its reflection in $T$. We note that $\Sigma'(T)$ is contained in $\Omega$ until:

(i) $\Sigma'(T)$ becomes internally tangent to the boundary of $\Omega$ at some point $P$ not on $T$, or

(ii) $T$ reaches a position where it is orthogonal to the boundary of $\Omega$.




Denote the plane $T$ when it reaches this position by $T_\alpha$ and $\Sigma(T_\alpha)$ by $\Sigma$ and its reflection across $T_\alpha$ by $\Sigma'$. We shall use $x'$ to denote the reflection of any point $x$ across $T_\alpha$ and $S'$ to denote the reflection of any set $S$ across $T_\alpha$. Our aim is to prove that $\Omega$ is symmetric with respect to $T_\alpha$, *i.e.*, $\Sigma' = \Omega \cap T_\alpha^+$, where $T_\alpha^+$ is the half plane formed by $T_\alpha$ not containing $T_0$. If we succeed in proving this, then we may conclude that for any given direction in $\mathbb{R}^n$, we can find a plane $T_\alpha$ about which $\Omega$ is symmetric. But the only domains which have this symmetry property are balls. Thus, our theorem would be proved.

For showing $\Sigma' = \Omega \cap T_\alpha^+$, since $(\Omega \cap T_\alpha^+)$ is *connected* and $\Sigma' \subset \Omega \cap T_\alpha^+$, it is enough to show that $\Sigma'$ is both open and closed in $\Omega \cap T_\alpha^+$. Now, given that $\Sigma'$, $\Omega \cap T_\alpha^+$ are open and $\Sigma' \subset \Omega \cap T_\alpha^+$, it is clear that $\Sigma'$ is open in $\Omega \cap T_\alpha^+$. Further, it can be deduced that $\partial_{\Omega \cap T_\alpha^+} \Sigma'$ (the boundary of $\Sigma'$ relative to $\Omega \cap T_\alpha^+$) is equal to $\partial\Sigma' \cap (\Omega \cap T_\alpha^+)$. So, $\Sigma'$ shall be closed in $\Omega \cap T_\alpha^+$ if we show that $\partial_{\Omega \cap T_\alpha^+} \Sigma' = \emptyset$, which would follow by proving $\partial\Sigma' \subseteq \partial(\Omega \cap T_\alpha^+)$. But, since $\partial\Sigma' = (\partial\Omega \cap \overline{T_\alpha^-})' \cup (\overline{\Omega} \cap T_\alpha)$ and $\partial(\Omega \cap T_\alpha^+) = (\partial\Omega \cap \overline{T_\alpha^+}) \cup (\overline{\Omega} \cap T_\alpha)$, it is sufficient to show that

$$\left(\partial\Omega \cap \overline{T_\alpha^-}\right)' \subseteq \partial\Omega \cap \overline{T_\alpha^+}. \tag{3.2}$$

We will assume that (3.2) does not hold and will reach a contradiction. This shall be obtained at the end of the following steps.

**Step 1**

We introduce a new function $v$ defined in $\Sigma'$ by $v(x) = u(x')$, where $x'$ is the reflection of $x$, given any $x \in \Sigma'$, across $T_\alpha$. We shall first show that $u \geq v$ in $\Sigma'$.

We note that $u$ is such that:

$$\begin{aligned} -\Delta_p u &= \lambda_1 u^{p-1} \text{ in } \Sigma', \\ u &\geq 0 \text{ in } \partial\Sigma' = \left(\partial\Omega \cap \overline{T_\alpha^-}\right)' \cup \left(\overline{\Omega} \cap T_\alpha\right), \end{aligned}$$

while $v$ satisfies

$$\begin{aligned} -\Delta_p v &= \lambda_1 v^{p-1} \text{ in } \Sigma', \\ v &= 0 \text{ in } \left(\partial\Omega \cap \overline{T_\alpha^-}\right)', \\ v &= u \text{ in } \overline{\Omega} \cap T_\alpha, \end{aligned}$$

in the weak sense. We note that $u$ being the first eigenfunction in $\Omega$ it is non-negative and is in $C^{1,\alpha}(\overline{\Omega})$. So, $u$ and $v$ are non-negative, bounded on $\Sigma'$, satisfy a Dirichlet problem of the form (2.2) in the domain $\Sigma'$ with $\beta(s) = \lambda_1 s^{p-1}$ and with $f = g = 0$. The function $\beta$ satisfies the hypothesis $(A-1)$ as well as the growth hypothesis (2.1). The boundary values $f'$ and $g'$ on $\partial\Sigma'$ are just the restrictions of $u$ and $v$, respectively, to $\partial\Sigma'$. Since $u > 0$ in $\Omega$, from the above, we have clearly, $u \geq v \geq 0$ on the boundary $\partial\Sigma'$. By Proposition 4.1, which is proved in Section 4, the remaining hypothesis $(A-2')$ of Theorem 2.1 also holds. So, we are able to conclude that $u \geq v$ in $\Sigma'$.

**Step 2**

In this step, we will show that $w = u - v$ satisfies an elliptic variational inequality in $\Sigma'$.

From the fact that $u \geq v$ in $\Sigma'$, we conclude immediately that

$$-\mathrm{div}(A(\nabla u) - A(\nabla v)) = -\Delta_p u + \Delta_p v = \lambda_1 \left(u^{p-1} - v^{p-1}\right) \geq 0 \text{ in } \Sigma' \tag{3.3}$$

holds in the weak sense, where $A(\xi) = |\xi|^{p-2} \cdot \xi = D\Gamma(\xi)$ for $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$. Proceeding further, by the mean value theorem, we obtain

$$\begin{aligned} A(\nabla u) - A(\nabla v) &= A(t\nabla u + (1-t)\nabla v)|_0^1 \\ &= \int_0^1 \frac{d}{\mathrm{d}t} A(t\nabla u + (1-t)\nabla v)\ \mathrm{d}t \\ &= \int_0^1 \left(\langle (\nabla A_i)\,(t\nabla u(x) + (1-t)\nabla v(x))\mathrm{d}t, \nabla w\rangle\right)_{i=1}^n\ \mathrm{d}t \\ &= \left(\left\langle \int_0^1 (\nabla A_i)\,(t\nabla u(x) + (1-t)\nabla v(x))\mathrm{d}t, \nabla w \right\rangle\right)_{i=1}^n \end{aligned}$$

where $w = u - v$. Thus by (3.3)

$$-\sum_{i=1}^n \frac{\partial}{\partial x_i}\left(\sum_{j=1}^n \left(\int_0^1 D_{ij}\Gamma(t\nabla u(x) + (1-t)\nabla v(x))\mathrm{d}t\right)\frac{\partial w}{\partial x_j}\right) \geq 0$$

holds in the weak sense, where $[D_{ij}\Gamma]$ is the Hessian matrix of the convex function $\Gamma$ introduced in Section 2 and denoted by $\mathcal{A}$.

Let $a_{ij}(x) = \int_0^1 D_{ij}\Gamma(t\nabla u(x) + (1-t)\nabla v(x))\mathrm{d}t$. Since, $\mathcal{A}$ is a positive matrix (not necessarily strictly positive), so is $A(x) = ((a_{ij}(x)))$. We then introduce the linear operator $L = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial}{\partial x_j}\right)$. Then, the function $w = u - v$ satisfies the following variational inequality in the weak sense

$$L(-w) = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial w}{\partial x_j}\right) \geq 0 \text{ in } \Sigma'. \tag{3.4}$$

**Step 3**

We shall now show that for any point $x_0$ on $(\partial\Omega \cap \overline{T_\alpha^-})'$ we can choose a neighbourhood $U$ of $x_0$ in $\overline{\Omega}$ such that the matrix $A(x) = ((a_{ij}(x)))$ is uniformly positive definite there.

By (2.5), taking $K = \min\{1, p-1\}$, we have

$$\langle A(x)\eta, \eta\rangle \geq K\left(\int_0^1 |t\nabla u(x) + (1-t)\nabla v(x)|^{p-2}\ \mathrm{d}t\right)\ |\eta|^2. \tag{3.5}$$

At $x_0$, we can write

$$\nabla v\left(x_0\right) = \frac{\partial v}{\partial n}\left(x_0\right) n + \nabla_{\left(\partial\Omega\cap\overline{T_\alpha^-}\right)'} v\left(x_0\right)$$

where $\nabla_{(\partial\Omega\cap\overline{T_\alpha^-})'} v(x_0)$ is the tangential component of the gradient $\nabla v$ at $x_0$. Since $v = 0$ on $(\partial\Omega \cap \overline{T_\alpha^-})'$, we have $\nabla_{(\partial\Omega\cap\overline{T_\alpha^-})'} v(x_0) = 0$, thus

$$\nabla v\left(x_0\right) = \frac{\partial v}{\partial n}\left(x_0\right) n.$$

Let us define

$$g(t,x) = |t\nabla u(x) + (1-t)\nabla v(x)|^{p-2}. \tag{3.6}$$

As the normal derivative of $v$ at $x_0$ is such that $\frac{\partial v}{\partial n}(x_0) = c < 0$, we have $g(0, x_0) = |c|^{p-2} > 0$. Since $g(t,x)$ is continuous with respect to $t$ and $x$, we can find a neighbourhood $[0, t_0] \times U$ where $U$ is a neighbourhood of $x_0$ in $\overline{\Omega}$ and $0 < t_0 \leq 1$, such that

$$g(t,x) \geq \delta,\ \forall (t,x) \in [0, t_0] \times U$$

for some positive $\delta$. Therefore,

$$\int_0^1 g(t,x)\ \mathrm{d}t\ \ \geq \delta t_0. \tag{3.7}$$

From (3.5) and (3.7) we have our assertion.

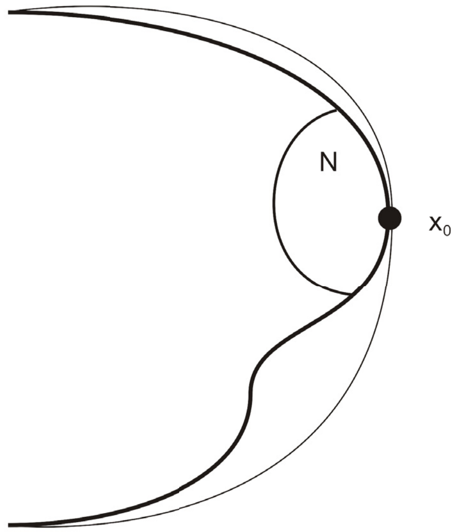
**Step 4**

Let $x_0$ be a point common to $(\partial\Omega \cap \overline{T_\alpha^-})'$ and $\partial\Omega \cap \overline{T_\alpha^+}$ where $(\partial\Omega \cap \overline{T_\alpha^-})'$ is internally tangent to $\partial\Omega \cap \overline{T_\alpha^+}$. We show that there is a neighbourhood of $x_0$ in $\overline{\Sigma}$ where the variational inequality (3.4) holds in a strong sense.

At such an $x_0$, since we have $\frac{\partial u}{\partial n} = c < 0$ on $\partial\Omega$, we have both $\frac{\partial u}{\partial n}(x_0) = c < 0$ and $\frac{\partial v}{\partial n}(x_0) = c < 0$. So, for any $t \in [0,1]$, the convex combination $t\frac{\partial u}{\partial n}(x_0) + (1-t)\frac{\partial v}{\partial n}(x_0) = c < 0$. Resolving the vectors $\nabla u(x_0)$ and $\nabla v(x_0)$ in the common normal direction $n(x_0)$ (taking it as the $n$th component) and in the tangential component, we see that they belong to the convex open set $\{\xi \in \mathbb{R}^n : \xi_n < c/2\}$. Moreover, by the regularity results in Barles [3], $u$ is $C^{2,\gamma}$ in a neighbourhood of $\overline{\Sigma}$. Thus, $u$ and $v$ are $C^{2,\gamma}$ in a neighbourhood of $x_0$ in $\overline{\Sigma}$. So, it follows, by the continuity of $\nabla u$ and $\nabla v$, that there is a neighbourhood $V$ of $x_0$ in $\overline{\Sigma}$ such that $\nabla u(x)$ and $\nabla v(x)$ belong to $\{\xi \in \mathbb{R}^n : \xi_n < c/2\}$ for all $x$ in $V$. By the convexity of $\{\xi \in \mathbb{R}^n : \xi_n < c/2\}$, it follows that $t\nabla u(x) + (1-t)\nabla v(x) \neq 0$ for all $t \in [0,1]$ and for all $x$ in $V$. This permits us to conclude that $\Gamma$ is continuously differentiable at all points $\xi = t\nabla u(x) + (1-t)\nabla v(x)$ for all $t \in [0,1]$ and for all $x$ in $V$ and for this reason, the matrix function $A$ is differentiable. Since, $w = u - v$ is $C^{2,\gamma}$ in the same neighbourhood, it follows that the variational inequality (3.4) holds in a strong sense in $V$.

**Step 5**

Since we assume (3.2) to be false, in either of the two cases (i) or (ii), there exists a point $(\partial\Omega \cap \overline{T_\alpha^-})'$ which belongs to $\partial\Omega \cap \overline{T_\alpha^+}$ where $(\partial\Omega \cap \overline{T_\alpha^-})'$ is internally tangent to $\partial\Omega \cap \overline{T_\alpha^+}$ and such that every neighbourhood of $x_0$ in $\overline{\Omega}$ contains points of $(\partial\Omega \cap \overline{T_\alpha^-})'$ which are inside $\Omega$.




Using the result of Step 3, choose a neighbourhood $U$ of $x_0$ in $\overline{\Omega}$ so that $L$ is uniformly elliptic in $U$ and a neighbourhood $V$ of $x_0$ in $\overline{\Sigma}$ as in Step 4. Take $N$ to be the open subset $U \cap V \cap \Sigma'$. By Step 1, we have $-w \leq 0$ in $N$ and we observe from Step 4 that $-w$ satisfies the elliptic variational inequality $L(-w) \geq 0$ in $N$ in the strong sense. So, by Theorem 5 (Sect. 3, Chap. 2) in [18], the maximum value 0 (which is attained at $x_0$) of the non-positive function $-w$ cannot be attained at an interior point of $N$ unless $w$ is constant and equal to 0. However, $w$ cannot be identically equal to 0 in $N$ for the following reason. By the assertion at the beginning of this step and the choice of $N$, there exists points of $\partial N$ near to $x_0$ which are inside $\Omega$. At such points $u > 0$ by the strict positivity of the first eigenfunction and $v = 0$ by the boundary condition on $\partial\Sigma' \cap \partial\Omega$, which gives $w > 0$ there. By the continuity of $w$, we shall also have $w > 0$ at some interior points of $N$. So, $w$ is non-constant in $N$ and, by the above mentioned theorem from [18], we conclude that

$$-w < 0 \qquad \text{in} \qquad N.$$

**Step 6**

We now complete the proof by showing that if (3.2) does not hold this leads to a contradiction. Depending on whether the case (i) occurs or case (ii) occurs, the arguments will be different. The case (i) can be handled with the help of a standard Hopf lemma but the second case, case (ii), is slightly more involved.

We first consider the case (i). In this case, at such an $x_0$, the boundary $\partial\Sigma'$ is regular and so there is a sphere at $x_0$ which is interiorly tangent to $\partial\Sigma'$. Since $w$ is $C^2$ in $N$ due to the result of Barles [3], satisfies the elliptic variational inequality (3.4) in $N$, $w(x_0) = 0$ and, from the last step, we have $w > 0$ in $N$, we conclude by the Hopf lemma, Theorem 7 on page 65 in [18] that

$$\frac{\partial w}{\partial n}(x_0) < 0.$$

This implies that

$$\frac{\partial u}{\partial n}(x_0) < \frac{\partial v}{\partial n}(x_0)\,.$$

It is a contradiction, since $\frac{\partial u}{\partial n}(x_0) = \frac{\partial v}{\partial n}(x_0) = c$.

Now, we consider the case (ii) and $x_0$ belonging to the plane $T_\alpha$. In this situation, although at $x_0$ the interior sphere condition is satisfied with respect to $\Omega$ this is not satisfied with respect to $\Sigma'$, since $x_0$ is a corner to this domain. A different argument has to be given so as to produce the contradiction. The strategy of Serrin for nonlinear uniformly elliptic equations is to show first that there is vanishing of second order of $w$ at $x_0$ and then show that for any direction $\xi$ which leaves $\Omega$ at $x_0$, non-tangentially, one has

$$\frac{\partial w}{\partial \xi}(x_0) < 0 \qquad \text{or} \qquad \frac{\partial^2 w}{\partial^2 \xi}(x_0) < 0. \tag{3.8}$$

This will be in contradiction to the vanishing of second order of the function $w$ at $x_0$.

Since this is a local argument at $x_0$, it is enough to work in a neighbourhood of $x_0$. By the $C^{2,\gamma}$ regularity of the first eigenfunction $u$ in a neighbourhood of $\partial\Omega$ the eigenvalue problem may be written in the form

$$-a(u, |\nabla u|)\Delta u - h(u, |\nabla u|)u_i u_j u_{ij} = f(u, |\nabla u|) \tag{3.9}$$

with

$$a(s,\xi) = |\xi|^{p-2} \quad h(s,\xi) = (p-2)|\xi|^{p-4} \quad \text{ and } f(s,\xi) = \beta(s) = \lambda_1 s^{p-1}. \tag{3.10}$$

Further, the equation is uniformly elliptic in a neighbourhood of $x_0$ since $|\nabla u(x_0)| > 0$ and $\nabla u$ is continuous. Now, by the comments of Serrin [20] in the beginning of Section 3 (p. 310), the equation is invariant under the reflection across the plane $T_\alpha$ and the second order partial derivatives $u_{nn}$ in any rectangular coordinate frame can be determined in terms of the remaining second order partial derivatives. This is used in showing that $u$ and $v$ coincide upto to the second order in their Taylor expansion at a point like $x_0$ in case (ii). Also, the elliptic variational inequality (3.4) for $w = u - v$ in $\Sigma'$ may seen to match that obtained following the steps used in Section 4 (p. 315) of Serrin [20] starting from the eigenvalue equation written above as a uniformly elliptic equation in a neighbourhood of $x_0$. So, the hypothesis of Lemma 2, Serrin [20] are verified by the matrix $A$. So, by this lemma, one of the alternatives (3.8) holds. This contradicts the fact that $w$ must vanish to the second order at such a point. So, the proof is complete. □

**Remark 3.3.** As a corollary we may deduce in a straightforward way that the ball is the unique minimizer for $\lambda_1$ among domains with the same surface area. Historically, this result was proved by Courant [7] for the Laplacian before the proof of the Faber–Krahn inequality.

## 4. Auxiliary results

*Proof of Theorem 2.1.* In view of the weakened hypotheses on the regularity of the domain and the data, the solutions are not necessarily bounded as in the proof of Proposition 2.3 [8]. So, we use a function space setting which is more natural. Indeed, let us denote $L^p_+(\Omega) = \{\varphi \in L^p(\Omega) \,|\, \varphi \geq 0 \text{ a.e. in } \Omega\}$. Given $f \in W^{-1,\frac{p}{p-1}}(\Omega)$ and $f' \in W^{1-\frac{1}{p},p}(\partial\Omega)$ with $f' \geq 0$ on $\partial\Omega$ and $\varphi \in L^p_+(\Omega)$, by the growth condition (2.1), $\beta(\varphi) \in L^{\frac{p}{p-1}}(\Omega)$. We may then define the nonlinear operator $T_{f,f'}$ by letting $T_{f,f'}(\varphi) = \zeta$, where $\zeta$ is the unique weak solution of

$$\begin{aligned} -\Delta_p \zeta &= \beta(\varphi) + f \quad && \text{in } \Omega, \\ \zeta &= f' && \text{on } \partial\Omega. \end{aligned} \tag{4.1}$$

By the continuity of $\beta$ in the $s$ variable, the nonlinear operator $\varphi \mapsto \beta(\varphi) + f$ is continuous from $L^p(\Omega) \to L^{\frac{p}{p-1}}(\Omega)$. The non-linear, solution operator $S : W^{-1,\frac{p}{p-1}}(\Omega) \to W^{1,p}(\Omega)$ which maps $\xi \in W^{-1,\frac{p}{p-1}}(\Omega)$ to the unique solution $\eta$

$$\begin{aligned} -\Delta_p \eta &= \xi \quad && \text{in } \Omega, \\ \eta &= f' && \text{on } \partial\Omega, \end{aligned} \tag{4.2}$$

given $f' \in W^{1-\frac{1}{p},p}(\partial\Omega)$, is continuous. Also the inclusion of $W^{1,p}(\Omega)$ into $L^p(\Omega)$ is continuous. So, the operator $T_{f,f'}$ which is the composition of these three continuous operators is continuous. Since, by the assumption

$(A-1)$, $\beta(\varphi(x)) \geq 0$ for almost all $x \in \Omega$, by appealing to the WCP in Tolksdorff [22] we conclude that indeed $T_{f,f'}(\varphi) = \zeta \geq 0$. So, we conclude that $T_{f,f'}$ maps $L^p_+(\Omega)$ into itself.

*Claim.* Let $f_1, f_2 \in W^{-1,\frac{p}{p-1}}(\Omega)$ and $\varphi_1, \varphi_2 \in L^p_+(\Omega)$. If $f_1 \leq f_2$, $\varphi_1 \leq \varphi_2$ and $f'_1 \leq f'_2$ in their respective spaces, then $T_{f_1,f'_1}(\varphi_1) \leq T_{f_2,f'_2}(\varphi_2)$ holds almost everywhere in $\Omega$.
Indeed, using the condition (b2), we conclude that $f^*_1 \leq f^*_2$ where $f^*_i := \beta(\varphi_i) + f_i$, $i = 1, 2$. If $\zeta_i = T_{f_1,f'_1}(\varphi_i)$, $i = 1, 2$, then

$$\begin{aligned} -\Delta_p \zeta_1 &= f^*_1, & -\Delta_p \zeta_2 &= f^*_2 & &\text{in } \Omega,\\ \zeta_1 &= f'_1, & \zeta_2 &= f'_2 & &\text{on } \partial\Omega. \end{aligned}$$

So, again by the weak comparison result proved in [22] we obtain $\zeta_1 \leq \zeta_2$ almost everywhere in $\Omega$. This proves the claim.
Now, let $u, v$ be non-negative solutions of the non-linear pdes in (2.2). To begin with, by the uniquness assumption $(A-2')$ we have, $T_{f,f'}(u) = u$ and $T_{g,g'}(v) = v$. Now, using the claim we obtain that the inequalities,

$$0 \leq T_{f,f'}(0) \leq T_{f,f'}(u) = u, \qquad 0 \leq T_{g,g'}(0) \leq T_{g,g'}(v) = v$$

hold almost everywhere in $\Omega$. We can then show, by an inductive application of the claim, that there exists a set of measure zero $Z$ in $\Omega$ outside of which the following chains of inequalities hold

$$0 \leq T_{f,f'}(0) \leq T^2_{f,f'}(0) \leq \cdots \leq T^n_{f,f'}(0) \leq \cdots \leq u = T_{f,f'}(u) \tag{4.3}$$

$$0 \leq T_{g,g'}(0) \leq T^2_{g,g'}(0) \leq \cdots \leq T^n_{g,g'}(0) \leq \cdots \leq v = T_{g,g'}(v) \tag{4.4}$$

It follows from the above inequalities that, for all $n$, $T^n_{f,f'}(0)$ and $T^n_{g,g'}(0)$ are non-negative and bounded pointwise, respectively, by the functions $u$ and $v$ in $\Omega \setminus Z$. Thus, the pointwise limits $u^*(x) = \lim_{n\longrightarrow\infty} \left[T^n_{f,f'}(0)\right](x)$ and $v^*(x) = \lim_{n\longrightarrow\infty} \left[T^n_{g,g'}(0)\right](x)$ exist almost everywhere in $\Omega$. Since, $u$ and $v$ bound, respectively, the sequence of functions $T^n_{f,f'}(0)$ and $T^n_{g,g'}(0)$, using the dominated convergence theorem, we conclude that the convergence of $T^n_{f,f'}(0)$ to $u^*$ and the convergence of $T^n_{g,g'}(0)$ to $v^*$ are also in $L^p$. So, by the continuity of the operators $T_{f,f'}$ and $T_{g,g'}$ on $L^p$ we conclude that the relations $T_{f,f'}(u^*) = u^*$ and $T_{g,g'}(v^*) = v^*$ hold. So, by the uniqueness assumption in $(A-2')$, it follows that $u^* = u$ and $v^* = v$.
Again, by applying the claim above, inductively, for any $n \geq 1$, we obtain $T^n_{g,g'}(0) \leq T^n_{f,f'}(0)$ almost everywhere in $\Omega$. Therefore, upon taking the limit as $n$ goes to infinity we obtain $v \leq u$ almost everywhere in $\Omega$. This proves the theorem. □

Now, we prove a proposition which establishes the hypotheses $(A-2')$ needed for the application of the weak comparison principle in Step 1 of Theorem 3.2. Let $\lambda_1$ be the first eigenvalue of the Dirichlet $p$-Laplacian as in (1.2) on a bounded domain $\Omega$. Let $\mathcal{O}$ be an open proper subset of $\Omega$.

**Proposition 4.1.** *Given $f' \in W^{1-\frac{1}{p},p}(\partial\mathcal{O})$ and $f' \geq 0$ on $\partial\mathcal{O}$, the problem*

$$\left.\begin{aligned} -\Delta_p w &= \lambda_1 |w|^{p-2} w & &\text{in } \mathcal{O},\\ w &= f' & &\text{on } \partial\mathcal{O}. \end{aligned}\right\} \tag{4.5}$$

*admits a unique non-negative solution.*

*Proof.* Let us first prove that if a solution exists then it is non-negative. Let $u$ be a solution of the above problem. As $u \geq 0$ on $\partial\mathcal{O}$, we obtain that $u^- \in W^{1,p}_0(\mathcal{O})$. Therefore, taking $u^-$ as a test function, we have

$$\int_{\mathcal{O}} |\nabla u|^{p-2} \langle \nabla u, \nabla u^- \rangle \,\mathrm{d}x = \lambda_1 \int_{\mathcal{O}} |u|^{p-2} u u^- \,\mathrm{d}x.$$

From this we obtain

$$\int_{\mathcal{O}} |\nabla u^-|^p \,\mathrm{d}x = \lambda_1 \int_{\mathcal{O}} |u^-|^p \,\mathrm{d}x.$$

We cannot have $u^- \neq 0$, for otherwise, from the variational characterization of the first eigenvalue we would obtain that $\lambda_1(\mathcal{O}) \leq \lambda_1 = \lambda_1(\Omega)$. However, since $\mathcal{O}$ is a proper open subset of $\Omega$ we always have $\lambda_1(\Omega) < \lambda_1(\mathcal{O})$. So, we conclude that $u^- = 0$ showing that $u \geq 0$ on $\mathcal{O}$.

**Existence.** We denote by $f'$ again a $W^{1,p}(\mathcal{O})$ function whose trace on $\partial\mathcal{O}$ is $f'$. We can then obtain a weak solution of (4.5) by minimizing the functional $J(w) = \int_{\mathcal{O}} |\nabla w|^p \,\mathrm{d}x - \lambda_1(\Omega) \int_{\mathcal{O}} |w|^p \,\mathrm{d}x$ on the affine space $A := W_0^{1,p}(\mathcal{O}) + f'$. Indeed, if $w$ in $A$ is a minimizer of $J$ then we shall have

$$0 = \frac{d}{\mathrm{d}t}\bigg|_{t=0} J(w + t\,\varphi) = \int_{\mathcal{O}} |\nabla w|^{p-2} \langle \nabla w, \nabla\varphi\rangle \,\mathrm{d}x - \lambda_1 \int_{\mathcal{O}} |w|^{p-2} w\,\varphi \,\mathrm{d}x \quad \forall\, \varphi \in \mathcal{C}_0^1(\mathcal{O}). \tag{4.6}$$

which is just the weak formulation of (4.5). As $A$ is a closed convex subset of the reflexive Banach space $W^{1,p}(\mathcal{O})$, for showing the existence of a minimizer of $J$ on $A$, it is enough to prove that $J$ is coercive and weakly sequentially lower semi-continuous on $A$.

$J$ is weakly sequentially lower semi-continuous on $A$: this is true since $\int_\Omega |\nabla w|^p \mathrm{d}x$ is lower semicontinuous for the weak topology on $W^{1,p}(\mathcal{O})$ and $\int_\Omega |w|^p \mathrm{d}x$ is continuous for the weak topology on $W^{1,p}(\mathcal{O})$ due to the compact inclusion of $W^{1,p}(\mathcal{O})$ in $L^p(\mathcal{O})$.
$J$ is coercive on $A$: Let $w_n := f' + \varphi_n \in A$ be a sequence such that $\|w_n\|_{W^{1,p}(\mathcal{O})} \longrightarrow \infty$ as $n \to \infty$. If $\int_{\mathcal{O}} |w_n|^p \,\mathrm{d}x$ is a bounded sequence, then the coercivity is immediate.

So, let us assume that $\int_{\mathcal{O}} |w_n|^p \,\mathrm{d}x \to \infty$ as $n \to \infty$. We may write $w_n := f' + \varphi_n$ with $\varphi_n \in W_0^{1,p}(\mathcal{O})$. Let $B_n := \frac{\int_{\mathcal{O}} |w_n|^p \,\mathrm{d}x}{\int_{\mathcal{O}} |\varphi_n|^p \,\mathrm{d}x}$. It can be argued, using the triangle inequality, that $\int_{\mathcal{O}} |\varphi_n|^p \,\mathrm{d}x \to \infty$ and $B_n \to 1$ as $n \to \infty$. From the Poincaré inequality on $\mathcal{O}$, we conclude that $\int_{\mathcal{O}} |\nabla\varphi_n|^p \,\mathrm{d}x \to \infty$ as $n \to \infty$. Setting $A_n := \frac{\int_{\mathcal{O}} |\nabla w_n|^p \,\mathrm{d}x}{\int_{\mathcal{O}} |\nabla\varphi_n|^p \,\mathrm{d}x}$, we obtain using the triangle inequality, that $\int_{\mathcal{O}} |\nabla w_n|^p \,\mathrm{d}x \to \infty$ and $A_n \to 1$ as $n \to \infty$. Now,

$$\begin{aligned} J\,(w_n) &= A_n \left( \int_{\mathcal{O}} |\nabla\varphi_n|^p \,\mathrm{d}x - \lambda_1(\Omega) \frac{B_n}{A_n} \int_{\mathcal{O}} |\varphi_n|^p \,\mathrm{d}x \right) \\ &\geq A_n \left( 1 - \frac{B_n}{A_n} \frac{\lambda_1(\Omega)}{\lambda_1(\mathcal{O})} \right) \int_{\mathcal{O}} |\nabla\varphi_n|^p \,\mathrm{d}x \end{aligned} \tag{4.7}$$

where the last inequality has been obtained by applying Poincaré inequality in the domain $\mathcal{O}$. Since we have $0 < \lambda_1(\Omega) < \lambda_1(\mathcal{O})$, since $A_n$ and $B_n$ converge to 1 as $n \to \infty$, it follows that $A_n \left(1 - \frac{B_n}{A_n} \frac{\lambda_1(\Omega)}{\lambda_1(\mathcal{O})}\right)$ is bounded below by a positive constant $C > 0$. Once again, we have the coercivity of $J$.

**Uniqueness.** Suppose $u, v$ are two different solutions of (4.6) in $A$. Let $w_1 := \nabla \log u$ and $w_2 := \nabla \log v$. As $f(x) = |x|^p$ is a strictly convex function we have

$$|w_1|^p \geq |w_2|^p + p\;|w_2|^{p-2}\,\langle w_2, w_2 - w_1\rangle \tag{4.8}$$

and equality holds if and only if $w_1 = w_2$. If we prove that $w_1 = w_2$ then we are done because in that case we will have $0 = \nabla \log u - \nabla \log v = \nabla \log\left(\frac{u}{v}\right)$. That is, $\log\left(\frac{u}{v}\right) = k$ for some constant $k$. As a result we get $u = e^k\, v$. But as $u \equiv v = f' \not\equiv 0$ on $\partial\mathcal{O}$ we get $u \equiv v$ in $\mathcal{O}$. Therefore, it suffices to prove that

$$|w_1|^p = |w_2|^p + p\;|w_2|^{p-2}\,\langle w_2, w_2 - w_1\rangle. \tag{4.9}$$

The proof of (4.9) is the same as the proof of Lemma 3.1 in Lindqvist [15]. We include the proof here for completeness. The function $u$ solves (4.5). If we could use $u - v^p\, u^{1-p}$ as a test function in the equation for $u$ and, use $v - u^p\, v^{1-p}$ as a test function in (4.5) with $v$ as a solution, after integrating by parts and summing the two identities, the resulting new identity can be reduced to

$$\begin{aligned} 0 = \int_{\mathcal{O}} \Big[ & u^p \left\{ |w_1|^p - |w_2|^p - p\,|w_2|^{p-2} \langle w_2, w_2 - w_1 \rangle \right\} \\ & + v^p \left\{ |w_2|^p - |w_1|^p - p\,|w_1|^{p-2} \langle w_1, w_1 - w_2 \rangle \right\} \Big]\, \mathrm{d}x. \end{aligned} \tag{4.10}$$

by using the following:

$$\nabla(u - v^p\, u^{1-p}) = \left\{ 1 + (p-1) \left( \frac{v}{u} \right)^p \right\} \nabla u - p\, \left( \frac{v}{u} \right)^{p-1} \nabla v,$$

and,

$$\nabla \left( v - u^p\, v^{1-p} \right) = \left\{ 1 + (p-1) \left( \frac{u}{v} \right)^p \right\} \nabla v - p\, \left( \frac{u}{v} \right)^{p-1} \nabla u.$$

But by (4.8) the integrand in (4.10) is non-negative (being the sum of two non-negative terms) and so, it follows from (4.10) that this integrand is equal to zero almost everywhere in $\mathcal{O}$. Therefore, each of the terms in the integrand must be zero. This would prove (4.9).

In general, in the above argument, instead of the test functions $u - v^p\, u^{1-p}$ and $(v - u^p\, v^{1-p})$ one has to use the test functions $u_\varepsilon - v_\varepsilon^p\, u_\varepsilon^{1-p}$ and $(v_\varepsilon - u_\varepsilon^p\, v_\varepsilon^{1-p})$, respectively, where $u_\varepsilon = u + \varepsilon$ and $v_\varepsilon = v + \varepsilon$. Then, by a similar calculation and after passing to the limit as $\varepsilon \to 0$ one obtains (4.9). □

*Acknowledgements.* The authors thank S. Prashanth (TIFR-CAM, Bangalore, India) for the notes on comparison principles, and thank Jorge García–Melian (ULL, Spain) for helpful references. The authors also would like to thank the referees for a critical revision of the manuscript which has led to a much improved version of the article.

## References

[1] A.D. Aleksandrov, Uniqueness theorems for surfaces in the large. (Russian) *Vestnik Leningrad Univ.* **13** (1958) 5–8.

[2] A. Alvino, V. Ferone, and G. Trombetti, On the properties of some nonlinear eigenvalues. *SIAM J. Math. Anal.* **29** (1998) 437–451.

[3] G. Barles, Remark on uniqueness results of the first eigenvalue of the $p$-Laplacian. In vol. 384 of *Annales de la faculté des sciences de Toulouse* (1988) 65–75.

[4] H. Berestycki and L. Nirenberg, On the moving plane method and the sliding method. *Boll. Soc. Brasiliera Mat. Nova Ser.* **22** (1991) 1–37.

[5] T. Bhattacharya, A proof of the Faber Krahn inequality for the first eigenvalue of the $p$-Laplacian. *Ann. Mat. Pura Appl. Ser.* **177** (1999) 225–231.

[6] J. Brothers, and W. Ziemer, Minimal rearrangements of Sobolev functions. *Journal fur die reine und angewandle Mathematik* **384** (1988) 153–179.

[7] R. Courant, Beweis des Satzes, dass von allen homogenen Membranen gegebenen Umfantes und gegebener Spannung die kreisfrmige den tiefsten Grundton besizt. *Math. Z.* **3** (1918) 321–28.

[8] M. Cuesta and P. Takác, A strong comparison principle for positive solutions of degenerate elliptic equations. *Differ. Integral Eq.* **13** (2000) 721–746.

[9] L. Damascelli, Comparison theorems for some quasilinear degenerate elliptic operators and applications to symmetry and monotonicity results. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **15** (1998) 493–516.

[10] L. Damascelli and F. Pacella, Monotonicity and symmetry results for $p$-Laplace equations and applications. *Adv. Differ. Equ.* **5** (2000) 1179–1200.

[11] G. Faber, Beweiss dass unter allen homogenen Membranen von gleicher Fläche und gleicher Spannung die kreisförfegige den leifsten Grundton gibt. *Sitz. bayer Acad. Wiss.* (1923) 169–172.

[12] I. Fragalà, F. Gazzola and B. Kawohl, Overdetermined problems with possibly degenerate ellipticity, a geometric approach. *Math. Zeit.* **254** (2006) 117–132.

[13] A. Farina and B. Kawohl, Remarks on an overdetermined problem. *Calc. Var. Partial Differ. Eq.* **31** (2008) 351–357.

[14] E. Krahn, Über eine von Rayleigh formulierte Minimaleigenschaftdes Kreises. *Math. Ann.* **94** (1924) 97–100.

[15] P. Lindqvist, On a nonlinear eigenvalue problem, Department of Mathematics. Norwegian University of Sciencie and technology N-7491, Trondheim, Norway.

[16] H. Lou, On singular sets of solutions to $p$-Laplace equations. *Chinese Ann. Math.* **29** 521–530 (2008).

[17] J. García Melián and S. de Lis, On the perturbation of eigenvalues for the $p$-Laplacian. *C.R. Acad. Sci. Paris* **332** (2001) 893–898.

[18] M.H. Protter and H.F. Weinberger, Maximum Principles in Differential Equations. Prentice-Hall (1967).

[19] S. Sakaguchi, Concavity properties of solutions to some degenerate quasilinear elliptic equations. *Ann. Scuo. Normale Sup. di Pisa* **14** (1987) 403–421.

[20] J. Serrin, A symmetry problem in potential theory. *Arch. Rational Mech. Anal.* **43** (1971) 304–318.

[21] G. Talenti, Nonlinear elliptic equations, rearrangements of functions and Orlicz spaces. *Ann. Mat. Pura. Appl.* **120** (1979) 159–184.

[22] P. Tolksdorff, On the Dirichlet Problem for quasilinear equations. *Commun. Partial Differ. Eq.* **8** (1983) 773–817.