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SHAPE DERIVATIVE OF THE CHEEGER CONSTANT

Enea Parini[1] and Nicolas Saintier[2,3]

Abstract. This paper deals with the existence of the shape derivative of the Cheeger constant $h_1(\Omega)$ of a bounded domain Ω. We prove that if Ω admits a unique Cheeger set, then the shape derivative of $h_1(\Omega)$ exists, and we provide an explicit formula. A counter-example shows that the shape derivative may not exist without the uniqueness assumption.

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1. Introduction

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. The *Cheeger constant* of Ω is defined as

$$h_1(\Omega) := \inf_{E \subset \Omega} \frac{P(E;\mathbb{R}^n)}{|E|}\cdot$$

Here $P(E;\mathbb{R}^n)$ is the distributional perimeter of E measured with respect to $\mathbb{R}^n$, while $|E|$ is the n-dimensional Lebesgue measure of E. A set $C \subset \Omega$ for which the infimum is attained is called a *Cheeger set*.

The problem of finding a Cheeger set for a given domain Ω has extensively received attention in the last decades, starting from the original work of Cheeger [5]. For an introductory survey on the Cheeger problem we refer to [18]; here we recall that for every bounded domain Ω with Lipschitz boundary there exists at least one Cheeger set. Uniqueness does not hold in general, but it is guaranteed if we assume Ω to be convex; in this case the Cheeger set turns out to be convex and of class $C^{1,1}$ (see [1]). The Cheeger constant can be obtained as the limit for $p \to 1$ of the first eigenvalue $\lambda_p(\Omega)$ of the p-Laplacian under Dirichlet boundary conditions (see [12]), and corresponds to the first eigenvalue of the 1-Laplacian (see [14]).

Shape analysis roughly consists in studying the regularity and the optimisation of a functional $J : \Omega \in \mathcal{A} \to J(\Omega) \in \mathbb{R}$ defined over some class $\mathcal{A}$ of subsets $\Omega \subset \mathbb{R}^n$. Due to its physical relevance, a particularly important class of functionals are the ones defined in terms of the eigenvalues of some operator. A lot of works have been dedicated for instance to the study of the dependence of the eigenvalues of the Laplacian as functions of the

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[1] LATP, Aix-Marseille Université, 39 rue Joliot Curie, 13453 Marseille cedex 13, France. enea.parini@univ-amu.fr

[2] Instituto de Ciencias, University Nac. Gral Sarmiento, J. M. Gutierrez 1150, C.P. 1613 Los Polvorines, Pcia de Bs. As, Argentina

[3] Dpto Matemática, FCEyN, University de Buenos Aires, Ciudad Universitaria, Pabellón I (1428) Buenos Aires, Argentina. nsaintie@dm.uba.ar; nsaintie@ungs.edu.ar

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domain under various boundary conditions. We refer for example to the monograph [11] for an introduction to the field of shape analysis.

In order to optimize J over $\mathcal{A}$ it is important to determine how sensitive is J under perturbation of a given set Ω. Given a smooth vector field $V \in C_c^\infty(\mathbb{R}^n;\mathbb{R}^n)$, define $F_t : \mathbb{R}^n \to \mathbb{R}^n$ as $F_t(x) = (Id + tV)(x)$. We then perturb Ω in the direction V by considering the sets $\Omega_t = F_t(\Omega)$. The shape derivative of J in the direction V at Ω is then defined as

$$J(\Omega, V)' := \lim_{t\to 0} \frac{J(\Omega_t) - J(\Omega)}{t}\cdot$$

For instance the shape derivative of the first eigenvalue $\lambda(\Omega)$ of the Laplacian with Dirichlet boundary condition is

$$\lambda(\Omega, V)' = -\int_{\partial\Omega} \left|\frac{\partial u}{\partial \nu}\right|^2 \langle V, \nu\rangle \, \mathrm{d}\mathcal{H}^{n-1},$$

where u is the unique positive normalized eigenfunction in Ω and ν is the unit exterior normal to $\partial\Omega$. This formula has been generalized in [8,16] to the first eigenvalue $\lambda_p(\Omega)$ of the p-Laplacian ($p > 1$):

$$\lambda_p(\Omega, V)' = -(p-1)\int_{\partial\Omega} \left|\frac{\partial u_p}{\partial \nu}\right|^p \langle V, \nu\rangle \, \mathrm{d}\mathcal{H}^{n-1}, \tag{1.1}$$

where u_p is the unique positive normalized eigenfunction in Ω.

General results about the stability of the Cheeger constant $h_1(\Omega)$ as a function of Ω have been obtained in [10]. In particular the shape derivative was computed but only in the case $V(x) = \lambda x$, $\lambda \in \mathbb{R}$. The main purpose of this paper is to provide a formula for the shape derivative of $h_1(\Omega)$ in the case of an arbitrary deformation field V. Notice that setting $p = 1$ formally in (1.1) does not give any meaningful information. Indeed it is known that characteristic functions of Cheeger sets are, up to a multiplicative constant, normalized first eigenfunctions of the 1-Laplacian and they are obtained as limit of eigenfunctions of the p-Laplacian as p goes to 1 (see Sect. 2). Therefore, if C is a Cheeger set, the normal derivative should be thought as equal to $-\infty$ on $\partial\Omega \cap \partial C$, so that the integral in (1.1) would be infinite. This kind of problem has also been considered in [20] where the shape derivative of the best Sobolev constant for the embedding of $BV(\Omega)$ into $L^1(\partial\Omega)$ was computed. Let us mention finally that the other extreme case $p = +\infty$ corresponding to the first eigenvalue of the ∞-Laplacian has been recently studied in [7,17,19] for Dirichlet, Steklov and Neumann boundary condition respectively.

The main result of our paper is the following.

Theorem 1.1. *Let Ω be a bounded Lipschitz domain. Let $V \in C_c^\infty(\mathbb{R}^n;\mathbb{R}^n)$, and let $F_t : \mathbb{R}^n \to \mathbb{R}^n$ be the one-parameter family of diffeomorphisms defined by $F_t(x) = (Id + tV)(x)$. Set $\Omega_t = F_t(\Omega)$. Then*

$$\lim_{t\to 0} h_1(\Omega_t) = h_1(\Omega).$$

If moreover Ω admits a unique Cheeger set C then the shape derivative

$$h_1(\Omega, V)' = \lim_{t\to 0} \frac{h_1(\Omega_t) - h_1(\Omega)}{t}$$

exists and is given by

$$h_1(\Omega, V)' = \frac{1}{|C|}\int_{\partial^* C} (\mathrm{div}_{\partial C} V - h_1(\Omega)\langle V, \nu\rangle) \, \mathrm{d}\mathcal{H}^{n-1}, \tag{1.2}$$

where $\partial^* C$ is the reduced boundary of C, ν is the unit exterior normal vector on $\partial^* C$, and $\mathrm{div}_{\partial\Omega} V(x) = \mathrm{div}\, V(x) - (\nu(x), DV(x)\nu(x))$, $x \in \partial^*\Omega$, is the tangential divergence of V on $\partial\Omega$.

In the case where ∂C is of class $C^{1,1}$, this formula can be simplified:

Corollary 1.2. *If Ω admits a unique Cheeger set C and ∂C is of class $C^{1,1}$, then the shape derivative of $h_1(\Omega)$ is given by the formula*

$$h_1(\Omega, V)' = \frac{1}{|C|} \int_{\partial C \cap \partial \Omega} (\kappa - h_1(\Omega)) \langle V, \nu \rangle \, \mathrm{d}\mathcal{H}^{n-1}, \tag{1.3}$$

*where $\kappa(x) = div\, \nu$ is the sum of the principal curvatures of $\partial\Omega$ at the point x (*i.e. $(n-1)$ *times the mean curvature), and ν is the unit exterior normal to $\partial\Omega$.*

The assumption in the Corollary is in particular satisfied for every dimension n when Ω is convex (see [1]), or in dimension $n \leq 7$ when $\partial\Omega$ is of class $C^{1,1}$ and admits a unique Cheeger set C (see [4]). We point out that the uniqueness hypothesis is necessary. Indeed, at the end of this paper we provide a counter example of a domain admitting more than one Cheeger set, which is not shape differentiable for some choice of V. However, it is interesting to observe that the bounded domains Ω admitting a unique Cheeger set (and hence shape differentiable) are dense in the L^1 topology (see [4]).

2. Definitions and preliminary results

Let $\Omega \subset \mathbb{R}^n$ be an open set. The *total variation* in Ω of a function $u \in L^1(\Omega)$ is defined as

$$|Du|(\Omega) := \sup \left\{ \int_\Omega u \,\mathrm{div}\varphi \,\middle|\, \varphi \in C_c^1(\Omega; \mathbb{R}^n),\ \|\varphi\|_\infty \leq 1 \right\}.$$

A function u such that $|Du|(\Omega) < +\infty$ is said to be of *bounded variation*. The space of the functions of bounded variation will be denoted by $BV(\Omega)$. It can be easily proved that the total variation is lower semicontinuous with respect to the L^1-convergence (see [9]). Moreover, the following holds true. Suppose that Ω is a Lipschitz domain, and let $u \in BV(\Omega)$; if we denote by $\overline{u}$ the extension of u by zero outside Ω, then $\overline{u} \in BV(\mathbb{R}^n)$, and

$$|D\overline{u}|(\mathbb{R}^n) = |Du|(\Omega) + \int_{\partial\Omega} |u| \,\mathrm{d}\mathcal{H}^{n-1},$$

where $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure on $\partial\Omega$.

The *perimeter* of a set $E \subset \Omega$ (measured with respect to $\mathbb{R}^n$) is defined as

$$P(E; \mathbb{R}^n) := |D\chi_E|(\mathbb{R}^n),$$

where χ_E is the characteristic function of E. The *Cheeger constant* of Ω is

$$h_1(\Omega) := \inf_{E \subset \Omega} \frac{P(E; \mathbb{R}^n)}{|E|},$$

where $|E|$ stands for the n-dimensional Lebesgue measure of E. A *Cheeger set* is a set $C \subset \Omega$ such that

$$\frac{P(C; \mathbb{R}^n)}{|C|} = h_1(\Omega).$$

The existence of a Cheeger set for every bounded Lipschitz domain Ω is proved *via* the direct method of the Calculus of Variations. Uniqueness does not hold in general; however, any convex body has a unique Cheeger set (see [1]). If C is a Cheeger set for Ω, then $\partial C \cap \Omega$ is analytic, up to a closed singular set of Hausdorff dimension $n-8$; at the regular points of $\partial C \cap \Omega$, the mean curvature is equal to $\frac{h_1(\Omega)}{n-1}$ (see *e.g.* [18], Prop. 4.2). Morever, if $\partial\Omega$ is of class $C^{1,1}$, then also ∂C enjoys the same regularity (see [4]); the same result holds if Ω is convex, as a consequence of the results in [21].

As an application of the coarea formula, $h_1(\Omega)$ can also be obtained as

$$h_1(\Omega) = \inf_{u \in BV(\Omega)\setminus\{0\}} \frac{|D\overline{u}|(\mathbb{R}^n)}{\|u\|_1}$$

or equivalently

$$h_1(\Omega) = \inf\left\{|D\overline{u}|(\mathbb{R}^n)\,|\, u \in BV(\Omega),\ \|u\|_1 = 1\right\}.$$

Therefore, $h_1(\Omega)$ can be seen as the first eigenvalue of the 1-Laplacian with Dirichlet boundary condition, which is defined formally as

$$\Delta_1 u = \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right),$$

and the characteristic functions of Cheeger sets are corresponding eigenfunctions. We refer to [14] for a thorough analysis of this problem. Here we observe that if Ω admits a unique Cheeger set C, then $u = \frac{1}{|C|}\chi_C$ is the unique nonnegative normalized eigenfunction of the 1-Laplacian, since every level set of a first eigenfunction is a Cheeger set (see [3], Thm. 2).

3. Proof of the main results

Recall that we are given a Lipschitz domain $\Omega \subset \mathbb{R}^n$ that we perturb in the direction of a smooth vector field $V \in C_c^\infty(\mathbb{R}^n;\mathbb{R}^n)$ in the sense that we consider the perturbed domains

$$\Omega_t = F_t(\Omega) \qquad \text{with} \qquad F_t(x) = (Id + tV)(x).$$

We let $h = h_1(\Omega)$ and $h_t = h_1(\Omega_t)$. We also assume that any function u defined in Ω (resp. Ω_t) is extended by 0 to $\mathbb{R}^n\setminus\overline{\Omega}$ (resp. $\mathbb{R}^n\setminus\overline{\Omega_t}$). With the notation of the previous section this means that $u = \bar{u}$.

We recall the change of variable formula for BV functions (see [9], Lem. 10.1). Let G_t be the inverse of F_t (which exists for small t). For an arbitrary function $u \in BV(\Omega)$, if we denote by v the function of $BV(\Omega_t)$ defined by $v(x) = u(G_t(x))$ we have the relations

$$\int_{\Omega_t} v(x)\,\mathrm{d}x = \int_\Omega u(y)|\det\, DF_t(y)|\,\mathrm{d}y$$

and

$$|Dv|(\mathbb{R}^n) = \int_{\mathbb{R}^n} |(DG_t)^T\sigma|\cdot|\det\, DF_t|\,\mathrm{d}|Du|,$$

where σ comes from the polar decomposition $Du = \sigma|Du|$.

Proof of Theorem 1.1. Let $u \in BV(\Omega)$ be a nonnegative eigenfunction for h such that $\|u\|_1 = 1$ in the sense that u is an extremal in (2) (which is known to exist). Consider the function $w_t \in BV(\Omega_t)$ defined as $w_t = u \circ G_t$. Then

$$|Dw_t|(\mathbb{R}^n) = \int_{\mathbb{R}^n} |(DG_t)^T\sigma|\cdot|\det\, DF_t|\,\mathrm{d}|Du|,$$

where σ comes from the polar decomposition $Du = \sigma|Du|$. Since $|\sigma| = 1$ $|\nabla u|-$ a.e., and $DF_t \to Id$ uniformly as $t \to 0$, so that $|\det\, DF_t| \to 1$ uniformly, we have using (2) and the above change of variable formula that

$$h_t \le \frac{|Dw_t|(\mathbb{R}^n)}{\displaystyle\int_{\Omega_t} w_t} = \frac{\displaystyle\int_{\mathbb{R}^n} |(DG_t)^T|\cdot|\det\, DF_t|\,\mathrm{d}|Du|}{\displaystyle\int_\Omega u(y)|\det\, DF_t(y)|\,\mathrm{d}y} = (1 + o(1))\frac{\displaystyle\int_{\mathbb{R}^n}\mathrm{d}|Du|}{\displaystyle\int_\Omega u(y)\,\mathrm{d}y}.$$

It follows that

$$\limsup_{t\to 0} h_t \le h$$

Let $u_t \in BV(\Omega_t)$ be a nonnegative extremal for h_t such that $\|u_t\|_1 = 1$. Consider the function $v_t \in BV(\Omega)$ defined as $v_t = u_t \circ F_t$. Then

$$\begin{aligned} |Dv_t|(\mathbb{R}^n) &= \int_{\mathbb{R}^n} |(DF_t)^T \sigma_t| \cdot |\det\, DG_t| \, \mathrm{d}|Du_t| \le (1+o(1)) \int_{\mathbb{R}^n} \mathrm{d}|Du_t| \\ &= (1+o(1))h_t \\ &\le h + o(1), \end{aligned} \tag{3.1}$$

and

$$\int_\Omega v_t \, \mathrm{d}x = \int_{\Omega_t} u_t |\det\, DF_t^{-1}| \, \mathrm{d}x = 1 + o(1). \tag{3.2}$$

Therefore (v_t) is bounded in $BV(\mathbb{R}^n)$. Since the embedding of $BV(\mathbb{R}^n)$ into $L^1_{loc}(\mathbb{R}^n)$ is compact, it follows that there exists a function $v \in BV(\mathbb{R}^n)$ such that (up to a subsequence), $v_t \to v$ a.e.. We deduce first that $v = 0$ in $\mathbb{R}^n \backslash \Omega$, then, using (3.2), that

$$\int_\Omega v \, \mathrm{d}x = \lim_{t\to 0} \int_\Omega v_t \, \mathrm{d}x = 1,$$

and eventually according to (3.1), that

$$|Dv|(\mathbb{R}^n) \le \liminf_{t\to 0} |Dv_t|(\mathbb{R}^n) \le h.$$

Letting $v = v_{|\Omega}$, it follows that $\int_\Omega v \, \mathrm{d}x = 1$, and

$$h \le |Dv|(\mathbb{R}^n) \le \liminf_{t\to 0} |Dv_t|(\mathbb{R}^n) = h.$$

It follows that

$$\lim_{t\to 0} h_t = h,$$

and that v is an extremal for h.

We assume from now on that Ω admits a unique Cheeger set $C \subset \Omega$. As a consequence, the only nonnegative normalized extremal for h is $|C|^{-1}\chi_C$; this follows from the fact that every level set of an extremal is a Cheeger set (see [3], Thm. 2). In particular $u = v = |C|^{-1}\chi_C$. Therefore $v_t \to u$ in $L^1(\Omega)$ and

$$\lim_{t\to 0} |Dv_t|(\mathbb{R}^n) = |Du|(\mathbb{R}^n).$$

By [2], Proposition 3.13, this implies that

$$\lim_{t\to 0} \int_{\mathbb{R}^n} \phi \, \mathrm{d}|Dv_t| = \int_{\mathbb{R}^n} \phi \, \mathrm{d}|Du|$$

for any $\phi \in C_c(\mathbb{R}^n)$.

Let us prove the differentiability. Using $w_t = u \circ G_t$ as a test-function for h_t, we obtain

$$h_t - h \le \frac{\displaystyle\int_{\mathbb{R}^n} |(DG_t)^T \sigma| \cdot |\det\, DF_t| \, \mathrm{d}|Du|}{\displaystyle\int_\Omega u(y) |\det\, DF_t(y)| \, \mathrm{d}y} - h.$$

Observe that

$$|\det\, DF_t(y)| = 1 + t.\mathrm{div}\, V(y) + o(t),$$

and

$$|(DG_t(y))^T\sigma(y)| = |\sigma(y)| - t\langle\sigma(y), DV(y).\sigma(y)\rangle + o(t),$$

where $o(t)$ is uniform in y. Therefore

$$\begin{aligned}h_t - h &\leq \frac{h + t\displaystyle\int_{\mathbb{R}^n}(\text{div } V - \langle\sigma, DV\sigma\rangle)\,\mathrm{d}|Du| + o(t)}{1 + t\displaystyle\int_{\Omega} u \,\text{div } V + o(t)} - h \\ &= \frac{t\left(\displaystyle\int_{\mathbb{R}^n}(\text{div } V - \langle\sigma, DV\sigma\rangle)\,\mathrm{d}|Du| - h\int_{\Omega} u \,\text{div } V\right)}{1 + t\displaystyle\int_{\Omega} u \,\text{div } V + o(t)}.\end{aligned}$$

We used the fact that $|\sigma| = 1$ $|Du|-$ a.e. and u is a normalized extremal for h. It follows that

$$\limsup_{t\to 0^+}\frac{h_t - h}{t} \leq \int_{\mathbb{R}^n}(\text{div } V - \langle\sigma, DV\sigma\rangle)\,\mathrm{d}|Du| - h\int_{\Omega} u \,\text{div } V,$$

and

$$\liminf_{t\to 0^-}\frac{h_t - h}{t} \geq \int_{\mathbb{R}^n}(\text{div } V - \langle\sigma, DV\sigma\rangle)\,\mathrm{d}|Du| - h\int_{\Omega} u \,\text{div } V.$$

Let us now prove the opposite inequality. We use v_t as a test-function for h, and we obtain

$$h_t - h = \int_{\mathbb{R}^n}\mathrm{d}|Du_t| - h \geq \int_{\mathbb{R}^n}|(DG_t)^T\sigma_t|\cdot|\det\, DF_t|\,\mathrm{d}|Dv_t| - \frac{\int_{\mathbb{R}^n}\mathrm{d}|Dv_t|}{\int_{\Omega} v_t},$$

where σ_t is such that $Du_t = \sigma_t|Du_t|$. We can also write

$$h_t - h \geq \int_{\mathbb{R}^n}\mathrm{d}|Dv_t| + t\int_{\mathbb{R}^n}(\text{div } V - \langle\sigma_t, DV\sigma_t\rangle)\,\mathrm{d}|Dv_t| - \frac{\int_{\mathbb{R}^n}\mathrm{d}|Dv_t|}{\int_{\Omega} v_t} + o(t).$$

Since div $V \in C_c(\mathbb{R}^n)$, we have

$$\lim_{t\to 0}\int_{\mathbb{R}^n}\text{div } V\,\mathrm{d}|Dv_t| = \int_{\mathbb{R}^n}\text{div } V\,\mathrm{d}|Du|.$$

Observe also that

$$\int_{\Omega} v_t = 1 - t\int_{\mathbb{R}^n} u_t \text{ div } V + o(t) = 1 - t\int_{\mathbb{R}^n} u \text{ div } V + o(t).$$

so that

$$\begin{aligned}\frac{\int_{\mathbb{R}^n}\mathrm{d}|Dv_t|}{\int_{\Omega} v_t} &= \int_{\mathbb{R}^n}\mathrm{d}|Dv_t| + t\left(\int_{\mathbb{R}^n}\mathrm{d}|Dv_t|\right)\left(\int_{\Omega} u \text{ div } V\right) + o(t) \\ &= \int_{\mathbb{R}^n}\mathrm{d}|Dv_t| + th\int_{\Omega} u \text{ div } V + o(t),\end{aligned}$$

where we used the fact that $|Dv_t|(\mathbb{R}^n) = h + o(1)$. Hence,

$$h_t - h \geq t\left(\int_{\mathbb{R}^n}\text{div } V\,\mathrm{d}|Du| - h\int_{\Omega} u \text{ div } V - \int_{\mathbb{R}^n}\langle\sigma_t, DV\sigma_t\rangle\,\mathrm{d}|Dv_t|\right) + o(t)$$

Since $Dv_t \rightharpoonup^* Du$ and $|Dv_t|(\mathbb{R}^n) \to |Du|(\mathbb{R}^n)$, we have, according to Reshetnyak's Theorem (see [2], Thm. 2.39), that

$$\lim_{t\to 0}\int_{\mathbb{R}^n} f(x,\sigma_t(x))\,\mathrm{d}|Dv_t| = \int_{\mathbb{R}^n} f(x,\sigma(x))\,\mathrm{d}|Du| \qquad \text{for any } f\in C_b(\mathbb{R}^n\times S^{n-1}).$$

It follows in particular that

$$\lim_{t\to 0}\int_{\mathbb{R}^n}\langle\sigma_t, DV\sigma_t\rangle\,\mathrm{d}|Dv_t| = \int_{\mathbb{R}^n}\langle\sigma, DV\sigma\rangle\,\mathrm{d}|Du|.$$

We thus obtain

$$\limsup_{t\to 0^+}\frac{h_t-h}{t} \geq \int_{\mathbb{R}^n}(\mathrm{div}\ V - \langle\sigma, DV\sigma\rangle)\,\mathrm{d}|Du| - h\int_{\Omega} u\ \mathrm{div}\ V$$

and

$$\liminf_{t\to 0^-}\frac{h_t-h}{t} \leq \int_{\mathbb{R}^n}(\mathrm{div}\ V - \langle\sigma, DV\sigma\rangle)\,\mathrm{d}|Du| - h\int_{\Omega} u\ \mathrm{div}\ V.$$

Therefore

$$h_1(\Omega,V)' = \lim_{t\to 0^+}\frac{h_t-h}{t}$$

exists, and

$$h_1(\Omega,V)' = \int_{\mathbb{R}^n}(\mathrm{div}\ V - \langle\sigma, DV\sigma\rangle)\,\mathrm{d}|Du| - h\int_{\Omega} u\,\mathrm{div}\ V.$$

Since $u=|C|^{-1}\chi_C$, we have that $|Du| = |C|^{-1}\mathcal{H}^{n-1}_{|\partial^* C}$ as a measure. We can thus rewrite the previous formula as

$$\begin{aligned} h_1(\Omega,V)' &= \frac{1}{|C|}\left(\int_{\partial^* C}(\mathrm{div}\ V - \langle\sigma, DV\sigma\rangle)\,\mathrm{d}\mathcal{H}^{n-1} - h\int_C \mathrm{div}\ V\right)\\ &= \frac{1}{|C|}\int_{\partial^* C}(\mathrm{div}\ V - \langle\sigma, DV\sigma\rangle - h\langle V,\nu\rangle)\,\mathrm{d}\mathcal{H}^{n-1}, \end{aligned}$$

where ν is the unit exterior normal to $\partial^* C$, and σ is given by $Du=\sigma|Du|$. We observe that $\sigma=-\nu$ $\mathcal{H}^{n-1}$–a.e. on $\partial^* C$. Recall that

$$\mathrm{div}\,V(x) - (\nu(x), DV(x)\nu(x)) = \mathrm{div}_{\partial C}\,V(x), \qquad x\in\partial^* C,$$

is the tangential divergence of V on $\partial^* C$ (see *e.g.* [11], Def. 5.4.6). We thus obtain that

$$h_1(\Omega,V)' = \frac{1}{|C|}\int_{\partial^* C}(\mathrm{div}_{\partial C}\,V - h\langle V,\nu\rangle)\,\mathrm{d}\mathcal{H}^{n-1} \tag{3.3}$$

which ends the proof of Theorem 1.1. □

Proof of Corollary 1.2. Suppose that Ω admits a unique Cheeger set C which is $C^{1,1}$. The unit exterior normal vector ν to ∂C is thus defined at every point and is Lipschitz continuous. Its components are thus differentiable at $\mathcal{H}^{n-1}$ almost every point of ∂C; moreover, the quantity $\kappa := \mathrm{div}_{\partial C}\nu$ belongs to $L^\infty(\partial C)$ and it can be seen as the distributional curvature of ∂C. Indeed one can easily adapt [11], Section 5.4.3 to the case of $C^{1,1}$ domains to obtain

$$\mathrm{div}_{\partial C}\,V = \mathrm{div}_{\partial C}\,V_{\partial C} + \kappa(V,\nu) \qquad \mathcal{H}^{n-1}-a.e.,$$

where $V_{\partial C} = V-(V,\nu)\nu$ is the tangential part of V, and

$$\int_{\partial C}\mathrm{div}_{\partial C}\,V_{\partial C}\,\mathrm{d}\mathcal{H}^{n-1} = 0.$$

Therefore it holds

$$\int_{\partial C} \operatorname{div}_{\partial C} V = \int_{\partial C} \kappa \langle V, \nu \rangle$$

and we can rewrite (3.3) as

$$\begin{aligned} h_1(\Omega, V)' &= \frac{1}{|C|} \int_{\partial C} (\operatorname{div}_{\partial C} V - h_1(\Omega)\langle V, \nu \rangle) \, \mathrm{d}\mathcal{H}^{n-1} \\ &= \frac{1}{|C|} \int_{\partial C} (\kappa - h_1(\Omega))\langle V, \nu \rangle \, \mathrm{d}\mathcal{H}^{n-1} \\ &= \frac{1}{|C|} \int_{\partial C \cap \partial \Omega} (\kappa - h_1(\Omega))\langle V, \nu \rangle \, \mathrm{d}\mathcal{H}^{n-1} \end{aligned}$$

since $\kappa = h_1(\Omega)$ in $\partial C \cap \Omega$. We then deduce (1.3). □

We complete this section providing some explicit examples of computation of shape derivatives.

Example 3.1 (the ball)**.** Let $\Omega = B_R$ be the ball of radius R, and V is a vector field such that $V(x) = \nu(x)$ on ∂B_R, we have that $\frac{\mathrm{d}h_t}{\mathrm{d}t}(0) = \left[\frac{d}{dr} h_1(B_r)\right](R)$. Since $h_1(B_r) = \frac{n}{r}$, we obtain using (1.3) that

$$h_1(\Omega, V)' = \frac{n\omega_n R^{n-1}}{\omega_n R^n} \cdot \left(\frac{n-1}{R} - \frac{n}{R} \right) = -\frac{n}{R^2}$$

as expected. Now let V be a volume-preserving perturbation; formula (1.3) becomes

$$h_1(\Omega, V)' = -\frac{1}{|\Omega|} \int_{\partial \Omega} \langle V, \nu \rangle \, \mathrm{d}\mathcal{H}^{n-1} = -\frac{1}{|\Omega|} \int_{\Omega} \operatorname{div} V = 0$$

in accordance with the well-known fact that the ball minimizes $h_1(\Omega)$ among all bounded domains with fixed volume.

Example 3.2 (The annulus)**.** As another simple example take $\Omega = A_{r,R} = B_R \backslash \bar{B}_r$, the annulus $\{r < |x| < R\}$, $r < R$. According to [6, 13], $A_{r,R}$ coincides with its Cheeger set so that

$$h_1(A_{r,R}) = \frac{|\partial A_{r,R}|}{|A_{r,R}|} = n \frac{R^{n-1} + r^{n-1}}{R^n - r^n} \cdot$$

Taking $V(x) = \nu(x)$, we have by direct computation that

$$\frac{\mathrm{d}}{\mathrm{d}t} h_1(A_{r-t,R+t})_{|t=0} = n \frac{-R^{2n-2} - r^{2n-2} - (n-1)r^{n-2}R^n - (n-1)R^{n-2}r^n - 2n(rR)^{n-1}}{(R^n - r^n)^2},$$

which coincides with formula (1.3):

$$h_1(\Omega, V)' = \left(\frac{n-1}{R} - h_1(A_{r,R}) \right) \frac{|\partial B_R|}{|A_{r,R}|} - \left(\frac{n-1}{r} + h_1(A_{r,R}) \right) \frac{|\partial B_r|}{|A_{r,R}|} \cdot$$

In dimension 2 this example can be generalized to curved annulus:

Example 3.3 (Curved annulus in the plane)**.** Let Γ be a smooth planar closed curve with no self-intersection, and $\Omega = \Sigma_{\Gamma,a} = \{x \in \mathbb{R}^2, \, dist(x, \Gamma) < a\}$ its tubular neighborhood of width a. We take a so small that Ω has no self-intersection. According to [15], $h_1(\Omega) = \frac{1}{a}$ and Ω itself is the unique Cheeger set. We take $V = \nu$. Then $\Omega_t = \Sigma_{\Gamma,a+t}$ and $h(\Omega, V)' = -\frac{1}{a^2} = -h_1(\Omega)^2$ which coincides with formula (1.3):

$$h_1(\Omega, V)' = \frac{1}{|\Omega|} \int_{\partial \Omega} (\kappa - h_1(\Omega)) \, \mathrm{d}\mathcal{H}^{n-1}$$

since $\int_{\partial \Omega} \kappa = 2\pi \chi(\Omega) = 0$ according to the Gauss−Bonnet formula.

Example 3.4 (the square)**.** We eventually provide an example where the Cheeger set is a proper subset of Ω. According to [13] a rectangle $R_{a,b} \subset \mathbb{R}^2$ of edges $2a$ and $2b$ has a unique Cheeger set C with

$$h_1(R_{a,b}) = \frac{4-\pi}{2(a+b) - 2\sqrt{(a-b)^2 + \pi ab}} \tag{3.4}$$

(see *e.g.* one of the two squares in Fig. 1). We take $\Omega = [0,1] \times [0,1] = R_{1/2,1/2}$ and $V(x,y) = (\eta(x), 0)$ with $\eta : \mathbb{R} \to [0,1]$ smooth with compact support in $(1-\delta, 1+\delta)$, δ small, and $\eta(x) = 1$ for $x \in (1-\delta/2, 1+\delta/2)$. Then $\Omega_t = (0, 1+t) \times (0,1)$ for sufficiently small t. It follows by direct computations from (3.4) that

$$h_1(\Omega, V)' = -\frac{1}{2} h_1(\Omega).$$

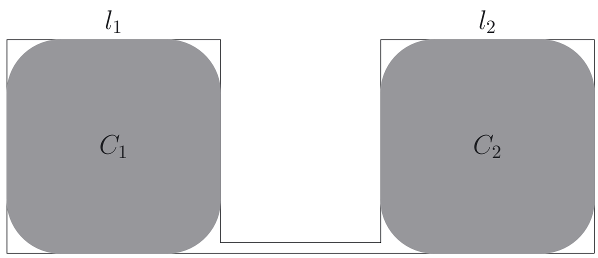

FIGURE 1. If $l_1 = l_2$, the Cheeger sets are given by C_1, C_2 and $C_1 \cup C_2$.

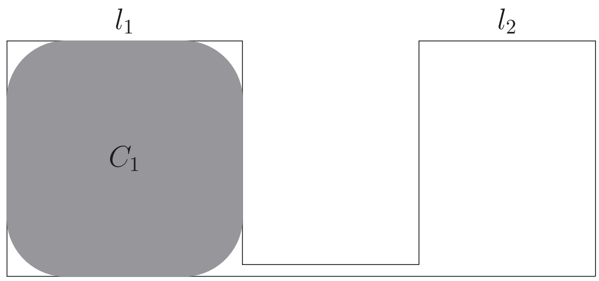

FIGURE 2. If $l_1 > l_2$, the only Cheeger set is given by C_1.

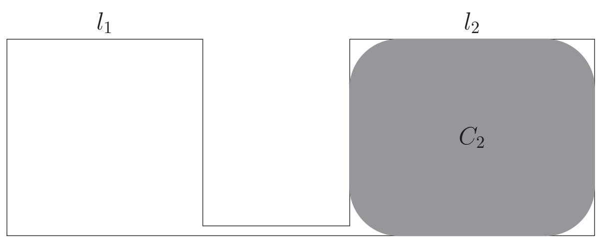

FIGURE 3. If $l_2 > l_1$, the only Cheeger set is given by C_2.

Since $\partial C \cap \Omega$ is made of arc of circle of radius $1/h_1(\Omega)$, it is easily seen that

$$|C| = 1 - \frac{4-\pi}{h_1(\Omega)^2} = \frac{4\sqrt{\pi} - 2\pi}{4-\pi},$$

$$\mathcal{H}^1(\partial C \cap S) = 1 - \frac{2}{h_1(\Omega)} = \frac{2\sqrt{\pi} - \pi}{4-\pi},$$

where $S := \{1\} \times [0,1]$. It follows that

$$h_1(\Omega, V)' = -h_1(\Omega)\frac{\mathcal{H}^1(\partial C \cap S)}{|C|},$$

which is formula (1.3) since $\kappa = 0$ on $\partial C \cap \partial\Omega$, $\langle V, \nu\rangle = 1$ on S and $\langle V, \nu\rangle = 0$ on $\partial\Omega \setminus S$.

4. A counter-example to the differentiability of $h_1(\Omega)$

If Ω does not admit a unique Cheeger set, then $h_1(\Omega)$ is in general not differentiable. As a counter example, we consider the "barbell domain", made of two equal rectangles R_1 and R_2 linked by a thin strip (see Fig. 1), defined as

$$\Omega = ([0,1] \times [0,1]) \cup ([1,2] \times [0,\varepsilon]) \cup ([2,3] \times [0,1]),$$

where $\varepsilon > 0$ is sufficiently small. Let V be a smooth vector field such that:

- V is supported in $[3-\delta, 3+\delta] \times [-\delta, 1+\delta]$ for some small δ;
- $V(x,y) = f(x,y)\overrightarrow{e_1}$ for some smooth nonnegative function f satisfying $f(3,y) = 1$ for $y \in [0,1]$.

In other words, V shifts the far right edge of Ω to the right. For small positive values of t, $h_1(\Omega_t)$ behaves like the Cheeger constant of a rectangle obtained by enlarging R_2. Recalling formula (3.4) which gives the Cheeger constant of a rectangle $R_{a,b}$ of edges $2a$ and $2b$, we see that $\frac{\partial}{\partial b}h_1(R_{a,b}) < 0$. Therefore

$$\lim_{t\to 0^+} \frac{h_1(\Omega_t) - h_1(\Omega)}{t} < 0.$$

For small negative values of t, $h_1(\Omega_t) = h_1(R_1) = h_1(\Omega)$ so that

$$\lim_{t\to 0^-} \frac{h_1(\Omega_t) - h_1(\Omega)}{t} = 0.$$

It follows that $h_1(\Omega)$ is not differentiable at $t = 0$.

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