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OPTIMAL BLOWUP TIME FOR CONTROLLED ORDINARY DIFFERENTIAL EQUATIONS *

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Abstract. In this work, we study both minimal and maximal blowup time controls for some ordinary differential equations. The existence and Pontryagin's maximum principle for these problems are derived. As a key preliminary to prove our main results, due to certain monotonicity of the controlled systems, "the initial period optimality" for an optimal triplet is built up. This property reduces our blowup time optimal control problems (where the target set is outside of the state space) to the classical ones (where the target sets are in state spaces).

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1. Introduction

Due to its importance from both theoretical and applied points of view, the blowup phenomenon has been studied extensively. One of the most typical blowup models (*cf.* [1]) is given by $u_t - \Delta u = f(x, t, u, \nabla u)$, which describes the temperature distribution of a substance in a chemical reaction. In this model, the blowup phenomenon presents a dramatic increasing in the temperature which leads to the ignition of a chemical reaction. In the past 50 years, most studies on the blowup phenomenon of evolution equations focus on the existence of blowup solutions and the blowup rate (see [3, 5, 6, 9, 11], and references therein). It would be interesting to ask for the best/a good method controlling the blowup time. Two important and natural issues on this topic are to minimize and maximize respectively the blowup time with the aid of controls. These turn to the blowup time optimal control problems. As Barron and Liu mentioned in their paper (see [2]), although the researchers' initial interest is about the optimal control to the distributed systems, they met some difficulties. Hence, researchers discuss the relevant problems governed by ordinary differential equations.

To our best knowledge, the publications on the aforementioned subject are quite limited. The first study on this topic is due to Barron and Liu [2], where an optimal control problem to maximize the blowup time was

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discussed. The controlled system in [2] is an autonomous system. More precisely, it reads

$$\begin{cases} \dfrac{\mathrm{d}y(t)}{\mathrm{d}t} = f(y(t), u(t)),\ t > 0, \\ y(0) = x \in \mathbb{R}^n, \end{cases} \tag{1.1}$$

where $n \geq 1$, $u(\cdot)$ is a measurable function from $[0, +\infty)$ to some compact subset Z of $\mathbb{R}^q$. For some $p > 1$, f verifies that

$$\frac{x \cdot f(x,z)}{|x|^{p+1}} \to 1 \ \text{ uniformly in } \ z \in Z, \qquad \text{as} \quad |x| \to \infty. \tag{1.2}$$

Denote by $T_x(u)$ the right-ending point of the maximal interval of existence for the solution to equation (1.1) corresponding to a control u, the authors in [2] defined the following valued function:

$$V(x) = \sup_{u \in \mathcal{Z}} T_x(u), \qquad V : \mathbb{R}^n \longrightarrow [0, \infty], \tag{1.3}$$

where $\mathcal{Z} := \{\zeta : [0, +\infty) \to Z | \zeta \text{ is Lebesgue measurable}\}$. Through the dynamic programming principle, they proved that $V(\cdot)$ is the unique continuous viscosity solution to the Hamilton–Jacobi equation:

$$1 + \max_{z \in Z} D_x V(x) \cdot f(x,z) = 0. \tag{1.4}$$

From this, Pontryagin's maximum principle follows.

Another paper on this topic is due to Lin and Wang [7] where an optimal control problem to minimize the blowup time was studied. There the controlled system is the following special non-autonomous system:

$$\begin{cases} \dfrac{\mathrm{d}y(t)}{\mathrm{d}t} = |y(t)|^{p-1} y(t) + B(t)u(t),\ t > 0, \\ y(0) = y_0, \end{cases} \tag{1.5}$$

where $p > 1$, $y(t) \in \mathbb{R}^N$, $B(\cdot) \in L^\infty([0, +\infty); \mathbb{R}^{N \times M})$, $M, N \geq 1$, and $u(\cdot) \in \mathcal{U}_{ad}$,

$$\mathcal{U}_{ad} := \left\{ v : [0, +\infty) \to \mathbb{R}^M | u \text{ is Lebesgue measurable}, |u(t)| \leq \rho_0, \text{ for a.e. } t \in [0, +\infty) \right\}$$

for some fixed $\rho_0 > 0$. The authors proved in [7] the existence of optimal controls *via* the following strategy: they first verified the existence of optimal controls for a family of relevant problems (P_R), where the target sets are the sphere of the ball in $\mathbb{R}^n$, centered at the origin and of radius $R > 0$, and then by passing to the limit for $R \to \infty$ to get the existence of optimal controls for the origin problem. Also, they derived Pontryagin's maximum principle for an optimal control, through building up a new penalty functional.

In [8], Lou, Wen and Xu approached the problem in [7] by a different way. They proved that Pontryagin's maximum principle holds for at least one optimal relaxed control. Consequently, Pontryagin's maximum principle holds when the optimal relaxed control is unique and in this case it is the optimal control to the original problem.

However, it is difficult to derive Pontryagin's maximum principle for more general controlled systems, through utilizing the methods provided in either [7] or [8]. For instance, in [7], the key to derive Pontryagin's maximum principle is the application of Ekeland's variational principle to the functional $|y(t^* - \varepsilon; y_0, u)|^{1-p}$, where t^* is the optimal time and $y(\cdot; y_0, u)$ is the solution to the system (1.5). One can verify that $(1 - p)$ is the only one exponent making this approach valid.

On the other hand, we can see from Example 6.5 in Section 6 that Pontryagin's maximum principle for blowup time optimal control is not necessary to hold.

In this paper, we will study the minimal/maximal blowup time optimal control problem governed by the following system:

$$\begin{cases} \dfrac{\mathrm{d}y(t)}{\mathrm{d}t} = f(t, y(t), u(t)),\ t > 0, \\ y(0) = y_0. \end{cases} \tag{1.6}$$

Here, $f : [0, +\infty) \times \mathbb{R}^n \times U \to \mathbb{R}$, with (U, ρ) being a separable metric space, is given by

$$f(t, y, u) = G(t, |y|)\frac{y}{|y|} + A(t)y + b(t, u), \qquad \forall\, (t, y, u) \in [0, +\infty) \times \mathbb{R}^n \times U, \tag{1.7}$$

where $G(\cdot, \cdot)$ is a function on $[0, +\infty)^2$, $A(\cdot)$ is an $n \times n$-matrix-valued function on $[0, +\infty)$ and $b(\cdot, \cdot)$ is an n dimensional vector-valued function on $[0, +\infty) \times U$. We say that $y(\cdot)$ is a solution to (1.6) on $[0, T)$ (with $T > 0$) if $y(\cdot) \in C([0, T); \mathbb{R}^n)$ verifies

$$y(t) = y_0 + \int_0^t f(s, y(s), u(s))\, \mathrm{d}s, \qquad \forall\, t \in (0, T).$$

It is worth to mention that the system (1.6) covers the systems studied in [2, 7].

Denote

$$\begin{aligned}
\mathscr{U} &= \left\{ u(\cdot) : [0, +\infty) \to U \,\middle|\, u(\cdot) \quad \text{is measurable} \right\},\\
\mathscr{P} &= \left\{ (T, y(\cdot), u(\cdot)) \in (0, +\infty) \times C([0, T); \mathbb{R}^n) \times \mathscr{U} \,\middle|\, (1.6) \text{ holds on } [0, T) \right\},\\
\mathscr{P}_{ad} &= \left\{ (T, y(\cdot), u(\cdot)) \in \mathscr{P} \,\middle|\, \lim_{t \to T^-} |y(t)| = +\infty \right\},\\
\mathscr{U}_{ad} &= \left\{ u(\cdot) \in \mathscr{U} \,\middle|\, (T, y(\cdot), u(\cdot)) \in \mathscr{P}_{ad} \right\}.
\end{aligned} \tag{1.8}$$

Moreover, $\mathscr{P}$, $\mathscr{P}_{ad}$ and $\mathscr{U}_{ad}$ are named as the set of feasible triples, the set of admissible triples and the set of admissible controls, respectively.

If $\mathscr{U}_{ad} \neq \emptyset$, the corresponding minimal time optimal control problem is as£º

Problem (TI): Find $(\bar{t}, \bar{y}(\cdot), \bar{u}(\cdot)) \in \mathscr{P}_{ad}$ such that

$$\bar{t} = \inf_{(T, y(\cdot), u(\cdot)) \in \mathscr{P}_{ad}} T. \tag{1.9}$$

When $\mathscr{U}_{ad} = \mathscr{U}$, we can consider the maximal time optimal control problem:

Problem (TS): Find $(t^*, y^*(\cdot), u^*(\cdot)) \in \mathscr{P}_{ad}$ such that

$$t^* = \sup_{(T, y(\cdot), u(\cdot)) \in \mathscr{P}_{ad}} T. \tag{1.10}$$

The main purpose of this study is to derive Pontryagin's maximum principles for the above-mentioned two optimal control problems, *via* a new method. The key of our strategy is to bridge the gap between classical time optimal control problems (where the target sets are in state spaces) and the blowup ones (where the target set is outside of the state space). It is based on "the initial period optimality", which follows from a certain monotonicity of the controlled system. In plain language, says that if $\bar{u}(\cdot)$ is an optimal control for a time optimal control problem and $\bar{t}$ is the optimal time, then it is well-known that its terminal period is also optimal, *i.e.*, for any $T \in (0, \bar{t}\,)$, $\bar{u}(\cdot)|_{[T, \bar{t}\,)}$ is also an optimal control for the time optimal control problem restricted on $[T, \bar{t}\,)$. It is worth mentioning that in general, when the controlled system is time-varying, $\bar{u}(\cdot)|_{[0,T]}$ is no longer an optimal control for the time optimal control problem in the initial period. Fortunately, for our controlled system (1.6), we are able to derive the afore-mentioned initial period optimality. By this optimality, we can approach Pontryagin's maximum principles for Problem (TI) and (TS) by passing to the limit in Pontryagin's maximum principle for a class of classical time optimal control problems. When the controlled system is time-invariant, "the initial period optimality", as well as "the terminal period optimality", follows from the translation invariance of the trajectory at once. Thus, the condition for the time-invariant case could be weaker than that for the time-varying one.

The rest of the paper is organized as follows: Section 2 gives the existence of optimal controls (see Thms. 2.3 and 2.4). Section 3 proves Pontryagin's maximum principles for Problem (TI) (see Thms. 3.2 and 3.4). Section 4 verifies Pontryagin's maximum principles for Problem (TS) (see Thm. 4.1). Section 5 presents easier ways to derive Pontryagin's maximum principles for both (TI) and (TS) for the case where the controlled system is time-invariant. Section 6 provides some examples to illustrate our main results.

2. Existence of time optimal control problem

In this section, we will prove the existence of optimal controls to the problems (TI) and (TS). We impose the following assumptions:

(P1) (U,ρ) is a separable metric space;

(P2) The function $G(t,r)$ is measurable in $t\in[0,+\infty)$, continuously differentiable in $r\in[0,+\infty)$ and satisfies

$$G(t,0)=0,\qquad \forall\, t\in[0,+\infty). \tag{2.1}$$

Moreover, for any $M>0$,

$$\operatorname*{esssup}_{(t,r)\in[0,M]^2}\left|\frac{\partial G(t,r)}{\partial r}\right|<+\infty. \tag{2.2}$$

(P3) $A(\cdot)\in L^\infty_{loc}([0,+\infty);\mathbb{R}^{n\times n})$, *i.e.*, for any $T>0$,

$$\operatorname*{esssup}_{t\in[0,T]}\|A(t)\|<+\infty, \tag{2.3}$$

where $\|A\|$ represents the norm of an $n\times n$ matrix A: $\|A\|=\sup_{x\in S^{n-1}}|Ax|$.

(P4) The function $b(\cdot,\cdot)$ takes values in $\mathbb{R}^n$ and is a Carathéodory function, that is, it is measurable in the first variable and continuous in the second variable. Moreover,

$$\operatorname*{esssup}_{t\in[0,T]}\sup_{u\in U}|b(t,u)|<+\infty,\qquad \forall\, T>0 \tag{2.4}$$

and $U(t)\equiv\{b(t,u)|u\in U\}$ is a convex compact set.

(P5) There is an $R_0>0$ and a nonnegative function $\zeta(\cdot)$ (defined on $[R_0,+\infty)$) such that

$$G(t,r)-\|A(t)\|\,r-\sup_{u\in U}|b(t,u)|\geq\zeta(r),\quad \forall\,(t,r)\in[0,+\infty)\times[R_0,+\infty), \tag{2.5}$$

$$\int_{R_0}^{+\infty}\frac{1}{\zeta(r)}\,\mathrm{d}r<+\infty. \tag{2.6}$$

Because of (P2), for each u and each y_0, equation (1.6) has a unique solution over its maximal interval of existence. We denote this solution by $y(\cdot;u(\cdot))$ when y_0 is fixed. Two things are needed to be mentioned: First, Condition (2.1) is not necessary but only for the convenience; Second, it is only for the existence of optimal controls, but not for Pontryagin's maximum principle, to assume that $U(t)$ is convex and compact.

Before proving the existence of optimal controls, several lemmas are given in order:

Lemma 2.1. *Assume that* (P1)–(P4) *hold. Let* $(T,y(\cdot),u(\cdot))\in\mathscr{P}$, $\overline{\lim_{t\to T^-}}\,|y(t)|<+\infty$ *and* $u_k(\cdot)\in\mathscr{U}$ *be a sequence such that*

$$b(\cdot,u_k(\cdot))\to b(\cdot,u(\cdot)),\qquad \textit{weakly in }\ L^2([0,T+1];\mathbb{R}^n). \tag{2.7}$$

Then, there exist a $\delta\in(0,1)$ *and a* $K>0$, *such that* $y_k(\cdot)\equiv y(\cdot;u_k(\cdot))$, *which is the solution to equation* (1.6) *with the control* $u_k(\cdot)$, *exists on* $[0,T+\delta]$ *and satisfies*

$$|y_k(t)-y(t)|\leq 1,\qquad \forall\, t\in[0,T+\delta] \tag{2.8}$$

when $k\geq K$.

Proof. Let

$$v_k(t) = b(t, u_k(t)), \quad v(t) = b(t, u(t)), \qquad t \in [0, T+1].$$

Because of (P1)–(P4) and $\overline{\lim_{t\to T^-}} |y(t)| < +\infty$, it follows from the basic theory of ordinary differential equations that there exists a $\delta \in (0,1)$ such that $y(\cdot; u(\cdot))$ exists on $[0, T+\delta]$. Let

$$M_b = \operatorname*{esssup}_{t\in[0,T+1]} \sup_{u\in U} |b(t,u)|, \quad M_A = \operatorname*{esssup}_{t\in[0,T+1]} \|A(t)\|,$$

$$M = \max_{t\in[0,T+1]} |y(t)| + T + 2, \quad N = \operatorname*{esssup}_{(t,r)\in[0,M]^2} \left|\frac{\partial G(t,r)}{\partial r}\right|.$$

Write $\alpha = \dfrac{1}{3e^{(3N+M_A)(T+1)}}$. Let ℓ be an integer such that $\ell > \frac{2(T+1)M_b}{\alpha}$. By the weak convergence, it is not difficult to prove that for some $K > 0$,

$$\left|\int_0^{\frac{j(T+1)}{\ell}} (v_k(t) - v(t))\,\mathrm{d}t\right| \le \alpha, \quad \forall\, j = 1, 2, \ldots, \ell - 1; \quad k \ge K. \tag{2.9}$$

Thus,

$$\left|\int_0^t (v_k(s) - v(s))\,\mathrm{d}s\right| \le 2\alpha, \quad \forall\, t \in [0, T+1], \quad k \ge K. \tag{2.10}$$

We claim that when $k \ge K$,

$$|y_k(t)| < M, \qquad \forall\, t \in [0, T+\delta]. \tag{2.11}$$

Otherwise, for some $k \ge K$, there exists an $S \in (0, T+\delta]$, such that $|y_k(S)| = M$,

$$|y_k(t)| < M, \qquad \forall\, t \in [0, S). \tag{2.12}$$

We have

$$\begin{aligned}
|y_k(t) - y(t)| &= \left|\int_0^t \left(G(s, |y_k(s)|)\frac{y_k(s)}{|y_k(s)|} - G(s, |y(s)|)\frac{y(s)}{|y(s)|} + A(s)(y_k(s) - y(s)) + v_k(s) - v(s)\right)\mathrm{d}s\right| \\
&\le \left|\int_0^t (G(s, |y_k(s)|) - G(s, |y(s)|))\,\frac{y_k(s)}{|y_k(s)|}\,\mathrm{d}s\right| \\
&\quad + \left|\int_0^t \frac{G(s, |y(s)|)}{|y(s)|}\left(y_k(s) - y(s) + (|y(s)| - |y_k(s)|)\frac{y_k(s)}{|y_k(s)|}\right)\mathrm{d}s\right| \\
&\quad + M_A \int_0^t |y_k(s) - y(s)|\,\mathrm{d}s + 2\alpha \\
&\le (3N + M_A)\int_0^t |y_k(s) - y(s)|\,\mathrm{d}s + 2\alpha, \qquad \forall\, t \in [0, S].
\end{aligned}$$

Adopting Grownwall's inequality, we can get

$$|y_k(t) - y(t)| \le 3\alpha e^{(3N+M_A)t} \le 1, \qquad \forall\, t \in [0, S]. \tag{2.13}$$

In particular,

$$|y_k(S)| \le |y(S)| + 1 < M,$$

which leads to a contradiction since $|y_k(S)| = M$. Therefore, (2.11) holds. Further, we get (2.8) from (2.11) (see also (2.13)).

The proof is completed. ∎

Lemma 2.2. *Assume that* (P1)–(P4) *hold. Let* $(T, y_k(\cdot), u_k(\cdot)) \in \mathscr{P}$ *satisfy*

$$\lim_{k\to+\infty} |y_k(T)| = +\infty. \tag{2.14}$$

Then, $\bar{t} \le T$*, where* $\bar{t}$ *is the optimal time of Problem* (TI) (see (1.9)).

Proof. Write

$$v_k(t) = b(t, u_k(t)), \qquad t \in [0,T],\ k = 1, 2\dots$$

By (P4), $\{v_k(\cdot)\}$ is uniformly bounded in $L^\infty([0,T];\mathbb{R}^n)$. Hence, it is uniformly bounded in $L^2([0,T];\mathbb{R}^n)$. Thus, there is a subsequence, denoted in the same way, $v_k(\cdot) \rightharpoonup v(\cdot)$ weakly in $L^2([0,T];\mathbb{R}^n)$. Based on Mazur's Theorem (see [10], for example), there exists a sequence defined by a convex combination $\sum_{j=1}^{N_k} \alpha_{k,j} v_j(\cdot)$, which converges strongly to $v(\cdot)$ in $L^2([0,T];\mathbb{R}^n)$. Since $U(t)$ is a compact convex set, we get

$$v(t) \in U(t), \qquad t \in [0,T].$$

Then according to Filippov's Lemma (see [4], for example), there exists a $u(\cdot) \in \mathscr{U}$, such that

$$v(t) = b(t, u(t)), \qquad \text{a.e. } t \in [0,T].$$

We now suppose that $y(\cdot)$ is the solution to equation (1.6) corresponding to the control $u(\cdot)$. If $y(\cdot)$ blows up during $[0,T]$, the lemma is proved. Otherwise, $y(\cdot)$ exists on $[0,T]$. Then, Lemma 2.1 shows that there exist a $\delta > 0$ and a $K > 0$, such that $y_k(\cdot)$ exists and it is bounded uniformly on $[0, T+\delta]$ when $k \ge K$, which contradicts (2.14). Therefore, $y(\cdot)$ has to blow up in $[0,T]$. This proves our conclusion. □

Next, we have the following existence results for Problem (TI).

Theorem 2.3. *Assume that* (P1)–(P4) *hold and* $\mathscr{P}_{ad} \ne \emptyset$*. Then Problem* (TI) *admits at least one solution.*

Proof. Let $(T_k, y_k(\cdot), u_k(\cdot)) \in \mathscr{P}_{ad}$ be a minimizing sequence, that is,

$$\lim_{k\to+\infty} T_k = \bar{t}. \tag{2.15}$$

Then

$$T_k \ge \bar{t}, \qquad \forall\, k \ge 1.$$

Similar to the proof of Lemma 2.2, there is a $u(\cdot) \in \mathscr{U}$, such that along a subsequence,

$$b(\cdot, u_k(\cdot)) \to b(\cdot, u(\cdot)), \qquad \text{weakly in } L^2([0,\bar{t}];\mathbb{R}^n).$$

Let $y(\cdot) = y(\cdot; u(\cdot))$. Then we can easily see that $(\bar{t}, y(\cdot), u(\cdot)) \in \mathscr{P}$.

We claim that $y(\cdot)$ blows up at $\bar{t}$ [2], that is, $(\bar{t}, y(\cdot), u(\cdot)) \in \mathscr{P}_{ad}$. Otherwise, by Lemma 2.1, there exist a $\delta > 0$ and a $K > 0$, such that $y_k(\cdot)$ exists and it is bounded uniformly on $[0, \bar{t} + \delta]$ for $k \ge K$. This contradicts (2.15) since $y_k(\cdot)$ blows up at T_k.

Therefore, $(\bar{t}, y(\cdot), u(\cdot)) \in \mathscr{P}_{ad}$, and $(\bar{t}, y(\cdot), u(\cdot))$ is an optimal triple to Problem (TI). □

For Problem (TS), we have the following existence result:

Theorem 2.4. *Assume that* (P1)−(P5) *hold and* $\mathscr{U}_{ad} = \mathscr{U}$*. Suppose that* t^**, defined by* (1.10)*, is finite. Then, Problem* (TS) *admits at least one solution.*

[2]Based on the definition of $\bar{t}$, $y(\cdot; u(\cdot))$ cannot blow up before $\bar{t}$.

Proof. Let $(S_k, y_k(\cdot), u_k(\cdot)) \in \mathscr{P}_{ad}$ be a maximizing sequence, that is

$$\lim_{k\to+\infty} S_k = t^*. \tag{2.16}$$

Then

$$S_k \le t^*, \qquad \forall\, k \ge 1.$$

Similar to the proof of Lemma 2.2, there is a $u(\cdot) \in \mathscr{U}$, such that along a subsequence,

$$b(\cdot, u_k(\cdot)) \to b(\cdot, u(\cdot)), \qquad \text{weakly in } L^2([0,t^*];\mathbb{R}^n).$$

Let $y(\cdot) = y(\cdot\,; u(\cdot))$.

We claim that $y(\cdot)$ blows up at t^*. Otherwise $y(\cdot)$ blows up at some time $S < t^*$ since $\mathscr{U}_{ad} = \mathscr{U}$. By (2.6), there exists an $R > R_0$, such that

$$\int_R^\infty \frac{1}{\zeta(r)}\,\mathrm{d}r \le \frac{t^*-S}{2}\cdot \tag{2.17}$$

Since $y(\cdot)$ blows up at S, there is a $T < S$ such that

$$|y(T)| \ge R+1. \tag{2.18}$$

Since $\lim_{k\to+\infty} S_k = t^* > S$, we can apply Lemma 2.1 to find a constant $K > 0$ such that $S_k > S$ and

$$|y_k(T)| \ge R \tag{2.19}$$

for any $k \ge K$. Noting that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}|y_k(t)| &= G(t,|y_k(t)|) + \frac{1}{|y_k(t)|}\,\langle A(t)y_k(t) + b(t,u_k(t)), y_k(t)\rangle\\ &\ge G(t,|y_k(t)|) - \|A(t)\|\,|y_k(t)| - \max_{u\in U}|b(t,u)|\\ &\ge \zeta(|y_k(t)|), \qquad t\in[T,S_k),\ k\ge K,\end{aligned} \tag{2.20}$$

we have

$$\begin{aligned}\frac{t^*-S}{2} \ge \int_R^{+\infty}\frac{1}{\zeta(r)}\,\mathrm{d}r &\ge \int_{|y_k(T)|}^{+\infty}\frac{1}{\zeta(r)}\,\mathrm{d}r\\ &= \int_T^{S_k}\frac{1}{\zeta(|y_k(t)|)}\frac{\mathrm{d}}{\mathrm{d}t}|y_k(t)|\,\mathrm{d}t \ge S_k - T \ge S_k - S.\end{aligned} \tag{2.21}$$

Letting $k \to +\infty$ in the above leads to

$$\frac{t^*-S}{2} \ge t^* - S, \tag{2.22}$$

which contradicts the assumption $t^* > S$. Thus, the blowup time of $y(\cdot)$ is t^*. Therefore, $(t^*, y(\cdot), u(\cdot)) \in \mathscr{P}_{ad}$ and $(t^*, y(\cdot), u(\cdot))$ is an optimal triple of Problem (TS). □

3. Pontryagin's maximum principles for problem (TI)

In this section, we will discuss Pontryagin's maximum principle of Problem (TI). For simplicity, we relabel some previous assumptions.

(S1) (U,ρ) is a separable metric space;

(S2) The function $G(t,r)$ is measurable in $t \in [0,+\infty)$, continuously differentiable in $r \in [0,+\infty)$ and satisfies

$$G(t,0)=0, \qquad \forall\, t \in [0,+\infty). \tag{3.1}$$

Moreover, $\forall\, M > \alpha > 0$,

$$\operatorname*{esssup}_{(t,r)\in[0,M]^2} \left|\frac{\partial G(t,r)}{\partial r}\right| < +\infty, \tag{3.2}$$

$$\lim_{r\to+\infty} \operatorname*{essinf}_{t\in[\alpha,M]} \frac{G(t,r)}{r} = +\infty, \tag{3.3}$$

$$\varliminf_{r\to+\infty} \operatorname*{essinf}_{t\in[\alpha,M]} \frac{rG_r(t,r)}{G(t,r)} > 0. \tag{3.4}$$

(S3) There exists an $s_0 > 0$, a function $\varphi(\cdot) \in C^2(0,s_0)$ and a modulus of continuity $\omega(\cdot) \in C[0,+\infty)$ such that

$$\varphi(s) > 2, \quad \varphi'(s) < 0, \qquad \forall\, s \in (0,s_0), \tag{3.5}$$

$$\lim_{s\to 0^+} \varphi(s) = +\infty, \quad \lim_{s\to 0^+} \varphi'(s) = -\infty, \quad \lim_{s\to 0^+} \frac{\varphi(s)}{\varphi'(s)} = 0, \tag{3.6}$$

$$1 + \left|\frac{\varphi'(s)}{\varphi^2(s)}\right| + \left|\frac{\varphi(s)\varphi''(s)}{(\varphi'(s))^2}\right| \le \omega(s)\left(G_r(t,\varphi(s)) - \frac{G(t,\varphi(s))\varphi''(s)}{(\varphi'(s))^2}\right), \qquad \forall\,(t,s) \in [0,+\infty)\times(0,s_0). \tag{3.7}$$

(S4) For any $T > 0$,

$$\operatorname*{esssup}_{t\in[0,T]} \|A(t)\| < +\infty, \quad \operatorname*{esssup}_{t\in[0,T]} \sup_{u\in U} |b(t,u)| < +\infty. \tag{3.8}$$

To simplify the notations, we let, for each $\rho > 0$ and $s \in (0,s_0)$,

$$\begin{aligned}
\varOmega_T(\rho) &= \operatorname*{essinf}_{\substack{t\in[0,T]\\ r\ge\rho}} \frac{G(t,r)}{r},\ \omega_0(s) = \sup_{0<\tilde{s}<s} \frac{1}{\varphi(\tilde{s})},\ \omega_1(s) = \sup_{0<\tilde{s}<s} \frac{\varphi(\tilde{s})}{|\varphi'(\tilde{s})|}, \\
\tilde{\varOmega}(\rho) &= \operatorname*{essinf}_{r\ge\rho} \left(G_r(t,r) - \frac{G(t,r)\varphi''(\varPhi(r))}{(\varphi'(\varPhi(r)))^2}\right) \\
&= \operatorname*{essinf}_{\varphi(s)\ge\rho} \left(G_r(t,\varphi(s)) - \frac{G(t,\varphi(s))\varphi''(s)}{(\varphi'(s))^2}\right),
\end{aligned} \tag{3.9}$$

where $\varPhi(\cdot)$ is the inverse function of $\varphi(\cdot)$.

For $f = \begin{pmatrix} f^1 & f^2 & \dots & f^n \end{pmatrix}^\top$, denote

$$f_t = \begin{pmatrix} \frac{\partial f^1}{\partial t} \\ \frac{\partial f^2}{\partial t} \\ \vdots \\ \frac{\partial f^n}{\partial t} \end{pmatrix}, \quad f_y = \frac{\partial f}{\partial y} = \begin{pmatrix} \frac{\partial f^1}{\partial y_1} & \frac{\partial f^2}{\partial y_1} & \cdots & \frac{\partial f^n}{\partial y_1} \\ \frac{\partial f^1}{\partial y_2} & \frac{\partial f^2}{\partial y_2} & \cdots & \frac{\partial f^n}{\partial y_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f^1}{\partial y_n} & \frac{\partial f^2}{\partial y_n} & \cdots & \frac{\partial f^n}{\partial y_n} \end{pmatrix}.$$

We have the following lemma.

Lemma 3.1. *Assume that* (S2)–(S4) *hold. Let* $T > t_0 \geq 0$ *and* $g(\cdot) \in L^\infty([t_0, T]; \mathbb{R}^n)$*. Let* $\tilde{y}(\cdot)$ *and* $\hat{y}(\cdot)$ *be the solutions to the equation:*

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} = G(t, |y(t)|)\frac{y(t)}{|y(t)|} + A(t)y(t) + g(t), \qquad t > t_0 \tag{3.10}$$

over $[t_0, T]$ *and with the initial state* $y(t_0) = \tilde{y}_0$ *and* $y(t_0) = \hat{y}_0$*, respectively. Suppose that*

$$|\hat{y}_0| > \rho, \tag{3.11}$$

$$\Phi(|\hat{y}_0|) - \Phi(|\tilde{y}_0|) - \left|\frac{\hat{y}_0}{|\hat{y}_0|} - \frac{\tilde{y}_0}{|\tilde{y}_0|}\right| > 0, \tag{3.12}$$

where $\rho > 0$ *satisfies*

$$\Omega_T(\rho) \geq M + 1,\ \rho > 2M,\ M \equiv \operatorname*{esssup}_{t\in[t_0,T]} \max(|g(t)|, \|A(t)\|), \tag{3.13}$$

$$\omega_0(\Phi(\rho)) \leq 1, \quad \omega_1(\Phi(\rho)) \leq 1, \quad \omega(\Phi(\rho)) \leq \frac{1}{4M+1}$$

and

$$\tilde{\Omega}(\rho) \geq 18M.$$

Then, the function $\Phi(|\hat{y}(t)|) - \Phi(|\tilde{y}(t)|) - \left|\frac{\hat{y}(t)}{|\hat{y}(t)|} - \frac{\tilde{y}(t)}{|\tilde{y}(t)|}\right|$ *is monotonically increasing on* $[t_0, T]$*.*

Proof. Since $\Phi(\cdot)$ is monotonically decreasing, it follows from (3.12) that $|\tilde{y}_0| > |\hat{y}_0| > \rho$. As a solution to (3.10), $y(\cdot)$ satisfies

$$\begin{aligned}\frac{\mathrm{d}|y(t)|}{\mathrm{d}t} &= G(t, |y(t)|) + \left\langle A(t)y(t) + g(t), \frac{y(t)}{|y(t)|}\right\rangle \\ &\geq (\Omega_T(|y(t)|) - M)\,|y(t)| - M, \qquad \forall\, t \in [t_0, T].\end{aligned} \tag{3.14}$$

Hence, $|y(\cdot)|$ is monotonically increasing on $[t_0, T]$ when $|y(t_0)| > \rho$. Especially,

$$|\tilde{y}(t)| > \rho, \quad |\hat{y}(t)| > \rho, \qquad \forall\, t \in [t_0, T]. \tag{3.15}$$

Denote

$$X(t) = |\hat{x}(t)| - |\tilde{x}(t)|, \quad \Theta(t) = \hat{\theta}(t) - \tilde{\theta}(t),$$

where

$$\hat{\theta}(t) = \frac{\hat{y}(t)}{|\hat{y}(t)|}, \quad \tilde{\theta}(t) = \frac{\tilde{y}(t)}{|\tilde{y}(t)|}, \quad \hat{y}(t) = \varphi(|\hat{x}(t)|)\hat{\theta}(t), \quad \tilde{y}(t) = \varphi(|\tilde{x}(t)|)\tilde{\theta}(t),$$

or equivalently,

$$\hat{\theta}(t) = \frac{\hat{x}(t)}{|\hat{x}(t)|}, \quad \tilde{\theta}(t) = \frac{\tilde{x}(t)}{|\tilde{x}(t)|}, \quad \hat{x}(t) = \Phi(|\hat{y}(t)|)\hat{\theta}(t), \quad \tilde{x}(t) = \Phi(|\tilde{y}(t)|)\tilde{\theta}(t).$$

We have

$$\begin{aligned}\frac{\mathrm{d}X(t)}{\mathrm{d}t} &= \frac{G(t,\varphi(|\hat{x}(t)|))}{\varphi'(|\hat{x}(t)|)} - \frac{G(t,\varphi(|\tilde{x}(t)|))}{\varphi'(|\tilde{x}(t)|)} + \left\langle g(t), \frac{\hat{\theta}(t)}{\varphi'(|\hat{x}(t)|)} - \frac{\tilde{\theta}(t)}{\varphi'(|\tilde{x}(t)|)}\right\rangle \\ &\quad + \frac{\varphi(|\hat{x}(t)|)}{\varphi'(|\hat{x}(t)|)}\left\langle A(t)\hat{\theta}(t),\hat{\theta}(t)\right\rangle - \frac{\varphi(|\tilde{x}(t)|)}{\varphi'(|\tilde{x}(t)|)}\left\langle A(t)\tilde{\theta}(t),\tilde{\theta}(t)\right\rangle \\ &= \frac{G(t,\varphi(|\hat{x}(t)|))}{\varphi'(|\hat{x}(t)|)} - \frac{G(t,\varphi(|\tilde{x}(t)|))}{\varphi'(|\tilde{x}(t)|)} + \left(\frac{1}{\varphi'(|\hat{x}(t)|)} - \frac{1}{\varphi'(|\tilde{x}(t)|)}\right)\left\langle g(t),\hat{\theta}(t)\right\rangle \\ &\quad + \frac{1}{\varphi'(|\tilde{x}(t)|)}\langle g(t),\Theta(t)\rangle + \left(\frac{\varphi(|\hat{x}(t)|)}{\varphi'(|\hat{x}(t)|)} - \frac{\varphi(|\tilde{x}(t)|)}{\varphi'(|\tilde{x}(t)|)}\right)\left\langle A(t)\hat{\theta}(t),\hat{\theta}(t)\right\rangle \\ &\quad + \frac{\varphi(|\tilde{x}(t)|)}{\varphi'(|\tilde{x}(t)|)}\left(\left\langle A(t)\hat{\theta}(t),\Theta(t)\right\rangle + \left\langle A(t)\Theta(t),\tilde{\theta}(t)\right\rangle\right) \\ &\geq \frac{G(t,\varphi(|\hat{x}(t)|))}{\varphi'(|\hat{x}(t)|)} - \frac{G(t,\varphi(|\tilde{x}(t)|))}{\varphi'(|\tilde{x}(t)|)} \\ &\quad - M\left|\frac{1}{\varphi'(|\hat{x}(t)|)} - \frac{1}{\varphi'(|\tilde{x}(t)|)}\right| - M\left|\frac{\varphi(|\hat{x}(t)|)}{\varphi'(|\hat{x}(t)|)} - \frac{\varphi(|\tilde{x}(t)|)}{\varphi'(|\tilde{x}(t)|)}\right| \\ &\quad - 3M\omega_1(\Phi(\rho))\,|\Theta(t)| \end{aligned} \tag{3.16}$$

and

$$\begin{aligned}\frac{\mathrm{d}\Theta(t)}{\mathrm{d}t} &= A(t)\Theta(t) - \hat{\theta}(t)\hat{\theta}(t)^\top A(t)\hat{\theta}(t) + \tilde{\theta}(t)\tilde{\theta}(t)^\top A(t)\tilde{\theta}(t) \\ &\quad + \frac{1}{\varphi(|\hat{x}(t)|)}\left(g(t) - \hat{\theta}(t)\hat{\theta}(t)^\top g(t)\right) - \frac{1}{\varphi(|\tilde{x}(t)|)}\left(g(t) - \tilde{\theta}(t)\tilde{\theta}(t)^\top g(t)\right) \\ &= A(t)\Theta(t) - \Theta(t)\hat{\theta}(t)^\top A(t)\hat{\theta}(t) - \tilde{\theta}(t)\Theta(t)^\top A(t)\hat{\theta}(t) - \tilde{\theta}(t)\tilde{\theta}(t)^\top A(t)\Theta(t) \\ &\quad + \left(\frac{1}{\varphi(|\hat{x}(t)|)} - \frac{1}{\varphi(|\tilde{x}(t)|)}\right)\left(g(t) - \hat{\theta}(t)\hat{\theta}(t)^\top g(t)\right) \\ &\quad - \frac{1}{\varphi(|\tilde{x}(t)|)}\left(\Theta(t)\hat{\theta}(t)^\top g(t) + \tilde{\theta}(t)\Theta(t)^\top g(t)\right). \end{aligned} \tag{3.17}$$

Then,

$$\frac{\mathrm{d}|\Theta(t)|}{\mathrm{d}t} \leq M\left(4 + 2\omega_0(\Phi(\rho)\right)|\Theta(t)| + 2M\left|\frac{1}{\varphi(|\hat{x}(t)|)} - \frac{1}{\varphi(|\tilde{x}(t)|)}\right|. \tag{3.18}$$

We know from (3.12) that $X(t_0) - |\Theta(t_0)| > 0$. Denote

$$S = \sup\left\{\beta \in (t_0,T]\,\Big|\, X(t) - |\Theta(t)| > 0, \quad \forall\, t \in [t_0,\beta)\right\}.$$

Then,

$$X(t) - |\Theta(t)| > 0, \qquad \forall\, t \in [t_0, S).$$

Moreover, $X(S) - |\Theta(S)| = 0$ if $S < T$.

By (3.16)–(3.18), we get

$$
\begin{aligned}
\frac{\mathrm{d}(X(t)-|\Theta(t)|)}{\mathrm{d}t} &\geq \frac{G(t,\varphi(|\hat{x}(t)|))}{\varphi'(|\hat{x}(t)|)} - \frac{G(t,\varphi(|\tilde{x}(t)|))}{\varphi'(|\tilde{x}(t)|)} \\
&\quad -M\left|\frac{1}{\varphi'(|\hat{x}(t)|)} - \frac{1}{\varphi'(|\tilde{x}(t)|)}\right| - 2M\left|\frac{1}{\varphi(|\hat{x}(t)|)} - \frac{1}{\varphi(|\tilde{x}(t)|)}\right| \\
&\quad -M\left|\frac{\varphi(|\hat{x}(t)|)}{\varphi'(|\hat{x}(t)|)} - \frac{\varphi(|\tilde{x}(t)|)}{\varphi'(|\tilde{x}(t)|)}\right| \\
&\quad -M\left(4+2\omega_0(\Phi(\rho))+3\omega_1(\Phi(\rho))\right)|\Theta(t)| \\
&= \int_{|\tilde{x}(t)|}^{|\hat{x}(t)|}\left(G_r(t,\varphi(s)) - \frac{G(t,\varphi(s))\varphi''(s)}{(\varphi'(s))^2}\right)\mathrm{d}s \\
&\quad -M\left|\int_{|\tilde{x}(t)|}^{|\hat{x}(t)|}\left(1-\frac{\varphi(s)\varphi''(s)}{(\varphi'(s))^2}\right)\mathrm{d}s\right| \\
&\quad -M\left|\int_{|\tilde{x}(t)|}^{|\hat{x}(t)|}\frac{\varphi''(s)}{(\varphi'(s))^2}\,\mathrm{d}s\right| - 2M\left|\int_{|\tilde{x}(t)|}^{|\hat{x}(t)|}\frac{\varphi'(s)}{(\varphi(s))^2}\,\mathrm{d}s\right| \\
&\quad -M\left(4+2\omega_0(\Phi(\rho))+3\omega_1(\Phi(\rho))\right)|\Theta(t)| \\
&\geq \int_{|\tilde{x}(t)|}^{|\hat{x}(t)|}(1-2M\omega(\Phi(\rho))\left(G_r(t,\varphi(s)) - \frac{G(t,\varphi(s))\varphi''(s)}{(\varphi'(s))^2}\right)\mathrm{d}s - 7M|\Theta(t)| \\
&\geq \frac{1}{2}\tilde{\Omega}(\rho)X(t) - 9M|\Theta(t)| \\
&\geq 9M\left(X(t)-|\Theta(t)|\right) \\
&\geq 0, \qquad t\in[t_0,S].
\end{aligned} \tag{3.19}
$$

Therefore, $X(t)-|\Theta(t)|$ is monotonically increasing on $[t_0,S]$. Consequently $X(S)-|\Theta(S)|>0$. Thus, $S=T$. This completes the proof. □

With the aid of the above-mentioned lemma, we can now prove Pontryagin's maximum principle to Problem (TI).

Theorem 3.2. *Assume that* (S1)–(S4) *hold. Let* $(\bar{t},\bar{y}(\cdot),\bar{u}(\cdot))$ *be an optimal triple of* (TI). *Then, there exists a nontrivial solution* $\bar{\psi}(\cdot)\in C([0,\bar{t}\,);\mathbb{R}^n)$ *to the following equation:*

$$
\frac{\mathrm{d}\bar{\psi}(t)}{\mathrm{d}t} = -\left(\frac{G(t,|\bar{y}(t)|)}{|\bar{y}(t)|}I + \frac{|\bar{y}(t)|\,G_r(t,|\bar{y}(t)|)-G(t,|\bar{y}(t)|)}{|\bar{y}(t)|^3}\bar{y}(t)\bar{y}(t)^\top + A(t)^\top\right)\bar{\psi}(t), \quad t\in[0,\bar{t}\,) \tag{3.20}
$$

such that

$$
\left\langle\bar{\psi}(t),b(t,\bar{u}(t))\right\rangle = \max_{u\in U}\left\langle\bar{\psi}(t),b(t,u)\right\rangle, \quad \text{a.e. } t\in[0,\bar{t}\,). \tag{3.21}
$$

Moreover,

$$
\lim_{t\to\bar{t}^-}\bar{\psi}(t)=0. \tag{3.22}
$$

Proof. Write

$$
M = \operatorname*{esssup}_{t\in[0,\bar{t}\,]}\sup_{u\in U}\max(|b(t,u)|,\|A(t)\|).
$$

One can easily find a $\rho > 2M$ such that

$$\Omega_{\bar t}(\rho) \geq M+1, \quad \omega_0(\Phi(\rho)) \leq 1, \quad \omega_1(\Phi(\rho)) \leq 1, \quad \omega(\Phi(\rho)) \leq \frac{1}{4M+1},$$
$$\tilde{\Omega}(\rho) \geq 18M.$$

Meanwhile, there exists a $\delta > 0$ such that

$$|\bar y(t)| \geq \rho, \qquad t \in [\bar t - \delta, \bar t\,).$$

For each $z \in \mathbb{R}^n$ with $|z| > \rho$, we define

$$E_z \equiv \{\ell z | \ell \geq 1\}.$$

We now use Lemma 3.1 to show that for each $T \in [\bar t - \delta, \bar t\,)$, $(\bar y(\cdot), \bar u(\cdot))$ is an optimal pair of the following optimal control problem $(TI)_T$: To find a control $u(\cdot) \in \mathscr{U}$, such that the solution $y(\cdot)$ to

$$\begin{cases} \dfrac{\mathrm{d}y(t)}{\mathrm{d}t} = G(t, |y(t)|)\dfrac{y(t)}{|y(t)|} + A(t)y(t) + b(t, u(t)),\ t \in [0,T], \\ y(0) = y_0 \end{cases} \tag{3.23}$$

maximizes $|y(T)|^2$ with terminal constraint $y(T) \in E_{\bar y(T)}$.

Indeed, if the above statement did not hold, then there would exist a $\tilde u(\cdot) \in \mathscr{U}$ and an $\ell > 1$ such that

$$y(T; \tilde u(\cdot)) = \ell \bar y(T).$$

In this case, it holds that

$$\Phi(|\bar y(T)|) - \Phi(|y(T; \tilde u(\cdot)|) - \left|\frac{\bar y(T)}{|\bar y(T)|} - \frac{y(T; \tilde u(\cdot)}{|y(T; \tilde u(\cdot)|}\right| = \Phi(|\bar y(T)|) - \Phi(\ell |\bar y(T)|) > 0.$$

Define

$$\hat u(t) = \begin{cases} \tilde u(t),\ t \in [0,T], \\ \bar u(t),\ t \in [T, \bar t\,). \end{cases}$$

By Lemma 3.1, $\Phi(|\bar y(\cdot)|) - \Phi(|y(\cdot; \hat u(\cdot)|) - \left|\frac{\bar y(\cdot)}{|\bar y(\cdot)|} - \frac{y(\cdot; \hat u(\cdot)}{|y(\cdot; \hat u(\cdot)|}\right|$ increases in the existence interval of $y(\cdot; \hat u(\cdot))$ within $[T, \bar t)$. In particular, there is an $S < \bar t$ such that $\lim_{t \to S^-} \Phi(|y(t; \hat u(\cdot)|) = 0$, *i.e.*, $y(\cdot; \hat u(\cdot))$ blows up at S, which contradicts the optimality of $(\bar t, \bar y(\cdot), \bar u(\cdot))$ and proves the above statement.

Next, by the classical Pontryagin's maximum principle for the Problem $(TI)_T$, there exists a nontrivial pair $(\varphi_{0,T}, \varphi_T(\cdot)) \in \mathbb{R} \times C([0,T]; \mathbb{R}^n)$ such that

$$\begin{aligned} &\varphi_{0,T} \leq 0, \\ &\frac{\mathrm{d}\varphi_T(t)}{\mathrm{d}t} = -\left(\frac{G(t, |\bar y(t)|)}{|\bar y(t)|} I + \frac{|\bar y(t)|\, G_r(t, |\bar y(t)|) - G(t, |\bar y(t)|)}{|\bar y(t)|^3} \bar y(t) \bar y(t)^\top + A(t)^\top\right) \varphi_T(t), \quad t \in [0,T], \end{aligned} \tag{3.24}$$

$$\langle \varphi_T(t), b(t, \bar u(t)) \rangle = \max_{u \in U} \langle \varphi_T(t), b(t,u) \rangle, \qquad \text{a.e. } t \in [0,T] \tag{3.25}$$

and

$$\langle \varphi_T(T) + \varphi_{0,T} \bar y(T), q - \bar y(T) \rangle \geq 0, \qquad \forall\, q \in E_{\bar y(T)}. \tag{3.26}$$

Obviously, (3.26) implies

$$\langle \varphi_T(T) + \varphi_{0,T} \bar y(T), \bar y(T) \rangle \geq 0. \tag{3.27}$$

If $\varphi_{0,T} = 0$, then $\varphi_T(\cdot) \neq 0$ because of the non-triviality. If $\varphi_{0,T} \neq 0$, then it follows from (3.27) that

$$\langle \varphi_T(t), \bar{y}(T) \rangle \geq -\varphi_{0,T} |\bar{y}(T)|^2 > 0,$$

which indicates $\varphi_T(\cdot) \neq 0$. Therefore, $\varphi_T(\cdot) \neq 0$ always holds. Thus, by replacing $\varphi_T(\cdot)$ by $\dfrac{\varphi_T(\cdot)}{|\varphi_T(0)|}$ if necessary, we can suppose that $|\varphi_T(0)| = 1$. While (3.24)–(3.27) remain true.

Given $\varepsilon > 0$, $\{\varphi_T(\cdot)\}_{\bar{t} > T > 0}$ is equicontinuous on $[0, \bar{t} - \varepsilon]$. Therefore there is a subsequence, denoted in the same way, such that it converges uniformly to $\bar{\psi}(\cdot)$ on $[0, \bar{t} - \varepsilon]$ when $T \to \bar{t}^-$. Hence, (3.20)–(3.21) are verified.

Furthermore, by (3.4), there exists a $T_0 \in (0, \bar{t}\,)$ and a $c \in (0, 1)$ such that

$$|\bar{y}(t)|\, G_r(t, |\bar{y}(t)|) \geq c\, G(t, |\bar{y}(t)|), \qquad \forall\, t \in [T_0, \bar{t}\,).$$

By (3.20), we get

$$\begin{aligned} \frac{1}{2} \frac{\mathrm{d} |\bar{\psi}(t)|^2}{\mathrm{d}t} &= -\frac{G(t, |\bar{y}(t)|)}{|\bar{y}(t)|} |\bar{\psi}(t)|^2 - \frac{|\bar{y}(t)|\, G_r(t, |\bar{y}(t)|) - G(t, |\bar{y}(t)|)}{|\bar{y}(t)|^3} \left\langle \bar{y}(t), \bar{\psi}(t) \right\rangle^2 \\ &\quad - \left\langle A(t) \bar{\psi}(t), \bar{\psi}(t) \right\rangle \\ &\leq -\left(\frac{c\, G(t, |\bar{y}(t)|)}{|\bar{y}(t)|} - M \right) |\bar{\psi}(t)|^2, \quad t \in [T_0, \bar{t}\,). \end{aligned} \tag{3.28}$$

On the other hand, since

$$\begin{aligned} \frac{1}{2} \frac{\mathrm{d} |\bar{y}(t)|^2}{\mathrm{d}t} &= \frac{G(t, |\bar{y}(t)|)}{|\bar{y}(t)|} |\bar{y}(t)|^2 + \langle A(t) \bar{y}(t), \bar{y}(t) \rangle + b(t, \bar{u}(t)), \bar{y}(t) \\ &\leq \left(\frac{G(t, |\bar{y}(t)|)}{|\bar{y}(t)|} + M \right) |\bar{y}(t)|^2 + M |\bar{y}(t)|, \qquad \forall\, t \in [0, \bar{t}\,) \end{aligned}$$

and $\lim\limits_{t \to \bar{t}^-} |\bar{y}(t)| = +\infty$, we conclude

$$\lim_{t \to \bar{t}^-} \int_0^t \left(\frac{G(s, |\bar{y}(s)|)}{|\bar{y}(s)|} + M \right) \mathrm{d}s = +\infty.$$

This, along with (3.27), yields

$$\lim_{t \to \bar{t}^-} \int_{T_0}^t \frac{G(s, |\bar{y}(s)|)}{|\bar{y}(s)|} \,\mathrm{d}s = +\infty.$$

From the above inequality and (3.28), the desired equality (3.22) follows immediately. This completes the proof. □

We will show later that any optimal triple of Problem (TS) also satisfies the above theorem. For this purpose, we need some further observation on the optimal triple of problem (TI). First of all, we assume

(S5) For almost all $t \in [0, +\infty)$, $b(t, U)$ is a convex set which contains 0 as its interior point. Meanwhile, for any $x \in \partial\,(b(t, U))$, there exists a unique $\lambda \in S^{n-1}$, such that

$$\langle \lambda, y - x \rangle \leq 0, \qquad \forall\, y \in b(t, U). \tag{3.29}$$

Remark 3.3. Suppose that U is a closed ball (in $\mathbb{R}^m$) which contains its interior point 0. Further assume that

$$b(t, u) = B(t) u, \qquad \forall\, t \in [0, +\infty), u \in U \tag{3.30}$$

and for each $t \in [0, +\infty)$, $B(t) \in \mathbb{R}^{n \times m}$ has the full row rank. Then (S5) holds.

Theorem 3.4. *Assume that* (S1)–(S5) *hold and* $(\bar{t}, \bar{y}(\cdot), \bar{u}(\cdot))$ *is an optimal triple of Problem* (TI)*. Then, there exists a nontrivial solution* $\bar{\psi}(\cdot) \in C([0,\bar{t}); \mathbb{R}^n)$ *to equation* (3.20) *such that* (3.21)–(3.22) *hold. Moreover, the following transversality condition*

$$\left\langle \bar{\psi}(t), \bar{y}(t) \right\rangle > 0, \qquad \forall\, t \in (\bar{t}-\delta, \bar{t}\,) \tag{3.31}$$

holds for some $\delta \in (0, \bar{t}\,)$.

Proof. We will use the same symbols as those used in the proof of Theorem 3.2. We need only to prove (3.31). If $\bar{t}-\delta \le T_1 < T_2 < \bar{t}$, then it follows from (3.25) that

$$\langle \varphi_{T_i}(t), b(t,u) - b(t,\bar{u}(t)) \rangle \le 0, \qquad \text{a.e. } t \in [0, T_1], \quad i = 1, 2. \tag{3.32}$$

Since $\varphi_{T_i}(t) \ne 0$ $(\forall\, t \in [0, T_i],\ i = 1, 2)$, $b(t, \bar{u}(t))$ is a boundary point of $b(t, U)$ for almost all $t \in [0, T_i]$. Thus, (S5) and (3.32) imply that

$$\varphi_{T_1}(t) = c\varphi_{T_2}(t), \qquad \text{a.e. } t \in [0, T_1] \tag{3.33}$$

for some constant $c > 0$. By the continuity of $\varphi_{T_i}(\cdot)$ on $[0, T_1]$ and $|\varphi_{T_1}(0)| = |\varphi_{T_2}(0)| = 1$, we get

$$\varphi_{T_1}(t) = \varphi_{T_2}(t), \qquad t \in [0, T_1]. \tag{3.34}$$

Consequently,

$$\bar{\psi}(t) = \varphi_T(t), \qquad \forall\, t \in [0, T],\ T \in [\bar{t}-\delta, \bar{t}\,) \tag{3.35}$$

since $\varphi_T(\cdot)$ converges uniformly to $\bar{\psi}(\cdot)$ on $[0, \bar{t}-\varepsilon]$ for any $\varepsilon > 0$. Especially,

$$\left\langle \bar{\psi}(T), \bar{y}(T) \right\rangle = \langle \varphi_T(T), \bar{y}(T) \rangle \ge -\varphi_{0,T} |\bar{y}(T)|^2 \ge 0, \qquad \forall\, T \in [\bar{t}-\delta, \bar{t}\,)\,. \tag{3.36}$$

On the other hand, since 0 is an interior point in $b(t, U)$ for almost all $t \in [0, \bar{t}\,)$,

$$\left\langle \bar{\psi}(t), b(t, \bar{u}(t)) \right\rangle = \max_{u \in U} \left\langle \bar{\psi}(t), b(t,u) \right\rangle > 0, \qquad \text{a.e. } t \in (0, \bar{t}\,). \tag{3.37}$$

Then

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \left\langle \bar{\psi}(t), \bar{y}(t) \right\rangle &= -\frac{|\bar{y}(t)|\, G_r(t, |\bar{y}(t)|) - G(t, |\bar{y}(t)|)}{|\bar{y}(t)|} \left\langle \bar{\psi}(t), \bar{y}(t) \right\rangle + \left\langle \bar{\psi}(t), b(t, \bar{u}(t)) \right\rangle \\ &> -\frac{|\bar{y}(t)|\, G_r(t, |\bar{y}(t)|) - G(t, |\bar{y}(t)|)}{|\bar{y}(t)|} \left\langle \bar{\psi}(t), \bar{y}(t) \right\rangle, \qquad \text{a.e. } t \in (0, \bar{t}\,). \end{aligned} \tag{3.38}$$

By (S2),

$$g(t) \equiv \int_0^t \frac{|\bar{y}(s)|\, G_r(s, |\bar{y}(s)|) - G(s, |\bar{y}(s)|)}{|\bar{y}(s)|} \,\mathrm{d}s \tag{3.39}$$

is well-defined in $(0, \bar{t}\,)$. It follows from (3.38) that

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(e^{g(t)} \left\langle \bar{\psi}(t), \bar{y}(t) \right\rangle \right) > 0, \qquad \forall\, t \in (0, \bar{t}\,). \tag{3.40}$$

Finally, (3.31) follows easily from (3.36) and (3.40). We complete the proof. □

4. Maximum principles to problem (TS)

In the statement of the following theorem, we need
(S5′) For almost all $t \in [0, +\infty)$, the origin of $\mathbb{R}^n$ is an interior point of $b(t, U)$.

Theorem 4.1. *Assume that* (S1)–(S4) *hold and* $(t^*, y^*(\cdot), u^*(\cdot))$ *is an optimal triple of Problem* (TS). *Then there exists a nontrivial solution* $\psi^*(\cdot) \in C([0, t^*); \mathbb{R}^n)$ *to the following equation*

$$\frac{\mathrm{d}\psi^*(t)}{\mathrm{d}t} = -\left(\frac{G(t, |y^*(t)|)}{|y^*(t)|} I + \frac{|y^*(t)|\, G_r(t, |y^*(t)|) - G(t, |y^*(t)|)}{|y^*(t)|^3} y^*(t) y^*(t)^\top + A(t)^\top\right) \psi^*(t), \quad t \in [0, t^*) \tag{4.1}$$

such that

$$\langle \psi^*(t), b(t, u^*(t)) \rangle = \max_{u \in U} \langle \psi^*(t), b(t, u) \rangle, \quad \text{a.e. } t \in [0, t^*) \tag{4.2}$$

and

$$\lim_{t \to t^*-} \psi^*(t) = 0. \tag{4.3}$$

Furthermore, if (S5′) *holds, then*

$$\langle \psi^*(t), y^*(t) \rangle < 0, \qquad \forall\, t \in [0, t^*). \tag{4.4}$$

Proof. The proof is similar to that of Theorems 3.2 and 3.4. Write

$$M = \operatorname*{esssup}_{t \in [0, t^*]} \sup_{u \in U} \max(|b(t, u)|, \|A(t)\|).$$

One can easily find a $\rho > 2M$ such that

$$\begin{aligned} &\mathit{\Omega}_{t^*}(\rho) \geq M + 1, \quad \omega_0(\mathit{\Phi}(\rho)) \leq 1, \quad \omega_1(\mathit{\Phi}(\rho)) \leq 1, \quad \omega(\mathit{\Phi}(\rho)) \leq \frac{1}{4M + 1}, \\ &\tilde{\mathit{\Omega}}(\rho) \geq 18M. \end{aligned}$$

Then, there is a $\delta > 0$ such that

$$|y^*(t)| \geq 2\rho, \qquad \forall\, t \in [t^* - \delta, t^*).$$

For $|z| > \rho$, denote[3]

$$E_z \equiv \left\{ \ell z \middle| \frac{1}{2} \leq \ell \leq 1 \right\}. \tag{4.5}$$

We now use Lemma 3.1 to show that for any $T \in [t^* - \delta, t^*)$, $(y^*(\cdot), u^*(\cdot))$ is an optimal pair of the following problem $(TS)_T$: To find a control $u(\cdot) \in \mathscr{U}$, such that the solution $y(\cdot)$ to the equation

$$\begin{cases} \dfrac{\mathrm{d}y(t)}{\mathrm{d}t} = G(t, |y(t)|) \dfrac{y(t)}{|y^*(t)|} + A(t) y(t) + b(t, u(t)), \; t \in [0, T], \\ \\ y(0) = y_0 \end{cases} \tag{4.6}$$

minimizes $|y(T)|^2$ with the terminal constraint $y(T) \in E_{y^*(T)}$.

Seeking for a contradiction, we suppose that there did exist a $\tilde{u}(\cdot) \in \mathscr{U}$ and an $\ell \in [\frac{1}{2}, 1)$, such that

$$y(T; \tilde{u}(\cdot) = \ell y^*(T).$$

[3]In the proof of Theorem 3.2, it will also work if E_z was defined by (4.5) there.

In this case

$$\Phi(|y(T;\tilde{u}(\cdot)|)-\Phi(|y^*(T)|)-\left|\frac{y(T;\tilde{u}(\cdot)}{|y(T;\tilde{u}(\cdot)|}-\frac{y^*(T)}{|y^*(T)|}\right|=\Phi(\ell|y^*(T)|)-\Phi(|y^*(T)|)>0.$$

We set

$$\hat{u}(\cdot)=\begin{cases}\tilde{u}(t), & t\in[0,T],\\ u^*(t), & t\in[T,t^*).\end{cases}$$

Then, by Lemma 3.1, $\Phi(|y(\cdot;\hat{u}(\cdot)|)-\Phi(|y^*(\cdot)|)-\left|\frac{y(\cdot;\hat{u}(\cdot)}{|y(\cdot;\hat{u}(\cdot)|}-\frac{y^*(\cdot)}{|y^*(\cdot)|}\right|$ is monotonically increasing in the existence interval of $y(\cdot;\hat{u}(\cdot))$ within $[T,t^*)$. In particular, we know that $y(\cdot;\hat{u}(\cdot))$ exists on $[0,t^*]$ and $\lim_{t\to t^{*-}}\Phi(|y(t^*;\hat{u}(\cdot)|)>0$, which contradicts the optimality of $(t^*,y^*(\cdot),u^*(\cdot))$. Hence, the above statement holds.

According to classical Pontryagin's maximum principle for Problem $(TS)_T$, there is a nontrivial pair $(\varphi_{0,T},\varphi_T(\cdot))\in\mathbb{R}\times C([0,T];\mathbb{R}^n)$ such that

$$\varphi_{0,T}\le 0, \tag{4.7}$$

$$\frac{\mathrm{d}\varphi_T(t)}{\mathrm{d}t}=-\left(\frac{G(t,|y^*(t)|)}{|y^*(t)|}I+\frac{|y^*(t)|\,G_r(t,|y^*(t)|)-G(t,|y^*(t)|)}{|y^*(t)|^3}y^*(t)y^*(t)^\top+A(t)^\top\right)\varphi_T(t),\quad t\in[0,T], \tag{4.8}$$

$$\langle\varphi_T(t),b(t,u^*(t))\rangle=\max_{u\in U}\langle\varphi_T(t),b(t,u)\rangle,\quad \text{a.e. } t\in[0,T] \tag{4.9}$$

and

$$\langle\varphi_T(T)-\varphi_{0,T}y^*(T),q-y^*(T)\rangle\ge 0,\qquad \forall\, q\in E_{y^*(T)}. \tag{4.10}$$

Obviously, (4.10) ensures

$$\langle\varphi_T(T),y^*(T)\rangle=\varphi_{0,T}|y^*(T)|^2. \tag{4.11}$$

When $\varphi_{0,T}=0$, we get $\varphi_T(\cdot)\ne 0$ from the non-triviality. When $\varphi_{0,T}\ne 0$, we get from (4.11) that

$$\langle\varphi_T(t),y^*(T)\rangle<0,$$

which leads to $\varphi_T(\cdot)\ne 0$ for this case. In summary, we conclude that $\varphi_T(\cdot)\ne 0$.

Thus, we can reset $\varphi_T(\cdot)$ such that $|\varphi_T(0)|=1$. Moreover, we have (4.8)–(4.9) and the following inequality

$$\langle\varphi_T(T),y^*(T)\rangle\le 0. \tag{4.12}$$

Next, similar to the proof of Theorem 3.2, we get that, at least by taking a subsequence, $\{\varphi_T(\cdot)\}_{t^*>T>0}$ convergence uniformly to $\psi^*(\cdot)$ on $[0,t^*-\varepsilon]$ for any $\varepsilon>0$ when $T\to\bar{t}^*$. Then we get the conjugate function $\psi^*(\cdot)$ and (4.1)–(4.3).

When (S5′) holds, similar to (3.40), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{h(t)}\langle\varphi_T(t),\bar{y}(t)\rangle\right)>0,\qquad \text{a.e. } t\in(0,T),\ T\in[t^*-\delta,t^*), \tag{4.13}$$

where

$$h(t)\equiv\int_0^t\frac{|y^*(s)|\,G_r(s,|y^*(s)|)-G(s,|y^*(s)|)}{|y^*(s)|}\,\mathrm{d}s,\qquad t\in[0,t^*). \tag{4.14}$$

Combining(4.12) with (4.13), we get

$$\langle\varphi_T(t),y^*(t)\rangle<0,\qquad \text{a.e. } t\in[0,T). \tag{4.15}$$

Therefore,

$$\langle \psi^*(t), y^*(t) \rangle \le 0, \qquad \forall\, t \in [0, t^*). \tag{4.16}$$

Then, since it also holds that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{h(t)} \langle \psi^*(t), \bar{y}(t) \rangle \right) > 0, \qquad \text{a.e. } t \in [0, t^*), \tag{4.17}$$

we get (4.4). □

Remark 4.2. In Theorem 4.1, if (S5′) is replaced by the following condition: $(S5'')$ For almost all $t \in [0, +\infty)$,

$$0 \in \overline{b(t, U)}, \tag{4.18}$$

then instead of (4.4), we have the following transversality result:

$$\langle \psi^*(t), y^*(t) \rangle \le 0, \qquad \forall\, t \in [0, t^*). \tag{4.19}$$

5. Results for autonomous systems

When the controlled system (1.6) is time-invariant, both the statement and the proof of Pontryagin's maximum principle can be simplified. In fact, due to the invariance of the system, for each $T \in (0, \bar{t}\,)$, the restriction of an optimal triple $(\bar{t}, \bar{y}(\cdot), \bar{u}(\cdot))$ (of Problem(TI)/(TS)) over $[0, T]$ is an optimal triplet to the new time optimal control problem that steers y_0 to the target set $\{\bar{y}(T)\}$ in the minimal/maximal time. This new problem is a classical time optimal control problem whose maximum principle can be derived by the standard way. By passing to the limit for $T \to \bar{t}^-$ in Pontryagin's maximum principle for the new problem, we can derive the desired one for Problem (TI)/(TS).

More precisely, we consider

$$\begin{cases} \dfrac{\mathrm{d}y(t)}{\mathrm{d}t} = f(y(t), u(t)),\ u(t) t > 0, \\ y(0) = y_0. \end{cases} \tag{5.1}$$

The following conditions are imposed:

(A1) (U, ρ) is a separable metric space;

(A2) The function $f(y, u)$ is continuous in (y, u) and continuously differentiable in $y \in \mathbb{R}^n$. Meanwhile,

$$|y(0, u)| \le L, \qquad \forall\, u \in U \tag{5.2}$$

for some $L > 0$. Futther, for any $R > 0$, there exists an $L_R > 0$ such that

$$|f_y(y, u)| \le L_R, \qquad \forall\, |y| \le R; u \in U. \tag{5.3}$$

We have the following result.

Theorem 5.1. *Assume that* (A1)–(A2) *hold and* $(\bar{t}, \bar{y}(\cdot), \bar{u}(\cdot))$ *is an optimal triple of Problem* (TI)/(TS). *Then, there exists a nontrivial solution* $\bar{\psi}(\cdot) \in C([0, \bar{t}\,]; \mathbb{R}^n)$ *to the following equation*

$$\frac{\mathrm{d}\bar{\psi}(t)}{\mathrm{d}t} = -f_y(\bar{y}(t), \bar{u}(t))\bar{\psi}(t), \quad t \in [0, \bar{t}\,), \tag{5.4}$$

such that

$$\left\langle \bar{\psi}(t), f(\bar{y}(t), \bar{u}(t)) \right\rangle = \max_{u \in U} \left\langle \bar{\psi}(t), f(\bar{y}(t), u) \right\rangle, \quad \text{a.e. } t \in [0, \bar{t}\,). \tag{5.5}$$

Proof. The translation invariance of the autonomous systems ensures that for any $T \in (0, \bar{t}\,)$, $(T, \bar{y}(\cdot), \bar{u}(\cdot))$ is an optimal triple of the following optimal control problem: to find $(t^*, y^*(\cdot), u^*(\cdot)) \in \mathscr{P}_{ad}^T$ such that

$$t^* = \inf_{(t,y(\cdot),u(\cdot))\in\mathscr{P}_{ad}^T} t \quad \Big/ \quad t^* = \sup_{(t,y(\cdot),u(\cdot))\in\mathscr{P}_{ad}^T} t, \tag{5.6}$$

where

$$\mathscr{P}_{ad}^T = \Big\{ (t, y(\cdot), u(\cdot)) \in (0, t] \times C([0, +\infty); \mathbb{R}^n) \times \mathscr{U} \,\Big|\, (5.1) \text{ holds on } [0, t),\, y(t) = \bar{y}(T) \Big\}. \tag{5.7}$$

Thus, by classical Pontryagin's maximum principle, there exists a nontrivial solution $\bar{\psi}_T(\cdot) \in C([0, T]; \mathbb{R}^n)$ to the following equation

$$\frac{\mathrm{d}\bar{\psi}_T(t)}{\mathrm{d}t} = -f_y(\bar{y}(t), \bar{u}(t))\bar{\psi}_T(t), \qquad t \in [0, T] \tag{5.8}$$

such that

$$\left\langle \bar{\psi}_T(t), f(\bar{y}(t), \bar{u}(t)) \right\rangle = \max_{u\in U} \left\langle \bar{\psi}_T(t), f(\bar{y}(t), u) \right\rangle, \quad \text{a.e. } t \in [0, T]. \tag{5.9}$$

Because of the non-triviality of $\bar{\psi}_T(\cdot)$, we can set $|\bar{\psi}_T(0)| = 1$. Thus, for any $\varepsilon > 0$, $\bar{\psi}_T(\cdot)$ is equicontinuous on $[0, \bar{t} - \varepsilon]$. Therefore, $\{\varphi_T(\cdot)\}$ has a subsequence that converges uniformly to $\bar{\psi}(\cdot)$ on $[0, \bar{t} - \varepsilon]$ for any $\varepsilon > 0$ when $T \to \bar{t}^-$. Then, by passing to the limit for $T \to \bar{t}^-$ in (5.8) and (5.8), we obtain (5.4)–(5.5). This completes the proof. □

Remark 5.2. In Theorem 5.1, we do not impose any growth condition on the nonlinear term. In fact, growth conditions are contained in the existence of optimal controls.

Remark 5.3. In the time-invariance case, some weaker conditions than those imposed in Theorem 4.1 can imply the transversality condition. But the proof is quite technique. We will not discuss it here.

6. Some examples

The conditions in Theorems 3.4 and 4.1 concern mainly about the functions G and φ. Though quite complicated, they are satisfied by many interesting and important systems. We will present some of them in what follows.

We always assume that $p > 1$, $\beta > \dfrac{1}{p-1}$ and $g(\cdot)$ is measurable in $[0, +\infty)$, satisfying

$$0 < \operatorname*{essinf}_{t\in[\alpha,T]} g(t) \le \operatorname*{esssup}_{t\in[\alpha,T]} g(t) < +\infty, \qquad \forall\, T > \alpha > 0. \tag{6.1}$$

Example 6.1. Let

$$G(t, r) = g(t) r^p, \qquad t \in [0, +\infty), r \ge 0.$$

By taking

$$\varphi(s) = s^{-\beta}, \qquad s > 0, \tag{6.2}$$

one can directly verify that (S2)–(S3) hold.

Example 6.2. Let

$$G(t, r) = g(t) r \ln^p(1 + r), \qquad t \in [0, +\infty), r \ge 0.$$

We take

$$\varphi(s) = \exp\left(s^{-\beta}\right), \qquad s > 0. \tag{6.3}$$

Then (S2)–(S3) hold.

Example 6.3. Let

$$G(t,r) = g(t)\left(e^{(p-1)r} - 1\right), \qquad t \in [0,+\infty), r \geq 0.$$

Choose

$$\varphi(s) = \ln(1+s^{-\beta}), \qquad s > 0. \tag{6.4}$$

Then (S2)–(S3) hold.

Remark 6.4. In Examples 1 and 3, $\varphi(\cdot)$ can be defined by (6.3), one needs only the positivity of β to guarantee (S2)–(S3).

Similarly, in Example 3, if $\beta > 0$ and $\varphi(\cdot)$ is defined by (6.2), then (S2)–(S3) hold.

We end the paper with the next example, which shows that Pontryagin's maximum principle does not hold in that case.

Example 6.5. Consider

$$\begin{cases} \dfrac{\mathrm{d}y(t)}{\mathrm{d}t} = \sec^2 t + u(t)\sec^2\dfrac{t}{2}, \; t > 0, \\ y(0) = 0, \end{cases} \tag{6.5}$$

where $u(t) \in [0,1]$. Let $u^*(\cdot) \equiv 1$, $u_*(\cdot) \equiv 0$. Then the corresponding states are

$$y^*(t) = \tan t + 2\tan\frac{t}{2}, \quad y_*(t) = \tan t, \qquad t \in \left[0, \frac{\pi}{2}\right)\cdot$$

Both $y^*(\cdot)$ and $y_*(\cdot)$ blow up at $t = \frac{\pi}{2}$. Moreover, for any measurable $u(\cdot)$ satisfying $0 \leq u(\cdot) \leq 1$, it holds that

$$y_*(t) \leq y(t; u(\cdot)) \leq y^*(t), \qquad t \in \left[0, \frac{\pi}{2}\right)\cdot$$

Therefore, any control is an optimal control for the related blowup time optimal control problem. However, when $\bar{u}(\cdot) \equiv \frac{1}{2}$ and $\bar{y}(\cdot)$ denotes the corresponding state, there is no non-trivial solution of

$$\frac{\mathrm{d}\bar{\psi}(t)}{\mathrm{d}t} = -f_y(t, \bar{y}(t), \bar{u}(t))\bar{\psi}(t), \qquad t \in \left[0, \frac{\pi}{2}\right), \tag{6.6}$$

such that

$$\left\langle \bar{\psi}(t), f(t, \bar{y}(t), \bar{u}(t)) \right\rangle = \max_{u \in U} \left\langle \bar{\psi}(t), f(t, \bar{y}(t), u) \right\rangle, \quad \text{a.e. } t \in \left[0, \frac{\pi}{2}\right), \tag{6.7}$$

where

$$f(t,y,u) = \sec^2 t + u\sec^2\frac{t}{2}\cdot$$

In fact, (6.7) implies $\bar{\psi}(\cdot) = 0$, a.e. $t \in [0, \frac{\pi}{2})$.

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