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TIME-DEPENDENT MEAN-FIELD GAMES IN THE SUPERQUADRATIC CASE

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Abstract. We investigate time-dependent mean-field games with superquadratic Hamiltonians and a power dependence on the measure. Such problems pose substantial mathematical challenges as key techniques used in the subquadratic case, which was studied in a previous publication of the authors, do not extend to the superquadratic setting. The main objective of the present paper is to address these difficulties. Because of the superquadratic structure of the Hamiltonian, Lipschitz estimates for the solutions of the Hamilton−Jacobi equation are obtained here through a novel set of techniques. These explore the parabolic nature of the problem through the nonlinear adjoint method. Well-posedness is proven by combining Lipschitz regularity for the Hamilton−Jacobi equation with polynomial estimates for solutions of the Fokker−Planck equation. Existence of classical solutions is then established under conditions depending only on the growth of the Hamiltonian and the dimension. Our results also add to current understanding of superquadratic Hamilton−Jacobi equations.

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1. Introduction

The theory of mean-field games comprises a set of tools and methods, that aims at investigating differential games involving a (very) large number of rational, indistinguishable players. These were introduced in the independent works of Lasry and Lions [37–40] and Huang *et al.* [34, 35]. Since then, there has been intense research activity in this field, with several authors considering a variety of related problems. These include numerical methods [2, 3, 36], applications in economics [32, 41] and environmental policy [36], finite state problems [17, 27, 28], explicit models [33, 45], obstacle-type problems [19], congestion [18, 25], extended mean-field games [24, 29], probabilistic methods [13, 14], long-time behavior [8, 11] and weak solutions [9, 47, 48], to name only a few. For additional results, see also recent surveys in [1, 7, 42], or [23] and the references therein, and the College de France lectures by Lions [43, 44].

Keywords and phrases. Mean-field games, initial terminal value problem, superquadratic Hamiltonians, nonlinear adjoint method.

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A model time-dependent mean-field game (MFG) problem is given by

$$\begin{cases} -u_t + H(x, Du) = \Delta u + g(m) \\ m_t - \operatorname{div}(D_p H m) = \Delta m, \end{cases} \tag{1.1}$$

equipped with the initial-terminal conditions:

$$\begin{cases} u(x, T) = u_T(x) \\ m(x, 0) = m_0(x). \end{cases} \tag{1.2}$$

In the above, the terminal instant, $T > 0$, is fixed. To simplify the presentation, we consider the spatially periodic problem. For that, let $\mathbb{T}^d$ be the d-dimensional torus, identified as usual with the set $[0, 1]^d$. Then, we regard u and m as real valued functions defined over $\mathbb{T}^d \times [0, T]$. A typical Hamiltonian, H, and nonlinearity, g, satisfying the assumptions that will be detailed in Section 2 are:

$$H(x, p) \,=\, a(x) \left(1 + |p|^2\right)^{\frac{2+\mu}{2}} \,+\, V(x),$$

and

$$g(z) \,=\, z^\alpha,$$

where $0 \leq \mu < 1$ and a, $V \in \mathcal{C}^\infty(\mathbb{T}^d)$, a, $V > 0$, are given.

A fundamental question about MFG systems regards the existence of solutions. In the stationary setting, the first result in this direction was obtained in [37]. Smooth solutions were studied in [22] (see also [26] for a related problem), [29, 31]. In [38], the authors addressed for the first time the question of existence of weak-solutions to (1.1) and (1.2) for the first time. The planning problem was investigated in [47, 48], also in the framework of weak solutions. In the quadratic Hamiltonians case, existence of smooth solutions was established in [11]. We emphasize that the proof in [11] relies on a Hopf–Cole transformation and does not seem to extend to more general cases behaving like $|p|^2$ at infinity. As presented in [44], mean-field games with quadratic or subquadratic growth in the Hamiltonian, and the power nonlinearity $g(m) = m^\alpha$, have classical solutions under some bounds on α. In [30], the authors extended and improved substantially these results in the subquadratic setting. Also in the subquadratic setting, existence of smooth solutions was studied in [21] in the whole space, and in [20] for logarithmic nonlinearities.

To the best of our knowledge, superquadratic time-dependent mean-field games have not been studied in the literature before the present paper, nor can they be addressed by a minor extension of existing results. We stress that previous arguments regarding the existence of weak solutions do not extend to the superquadratic setting. Indeed, many of the key estimates for quadratic or subquadratic mean-field games are simply not valid for superquadratic Hamiltonians. For instance, the Gagliardo−Nirenberg estimates combined with the Crandall−Amann technique [4] are no longer valid due to the growth of the Hamiltonian. Consequently, in the superquadratic case, estimates for Hamilton−Jacobi (H-J) equations are substantially more delicate and require arguments quite distinct from the ones used in the quadratic or subquadratic cases. See, for instance, the recent developments concerning Hölder estimates in [5,6,10]. To show the existence of smooth solutions for the case of superquadratic Hamiltonians, we develop here a new class of Lipschitz estimates. These are proven by identifying additional regularizing effects, which combine the parabolic structure of the Hamilton−Jacobi equations with its stochastic optimal control origin. This is achieved by employing the nonlinear adjoint method [16] in a novel way. Recently, after this paper was written, a very important development regarding existence of weak solutions was obtained in [12]. In that paper, the existence of weak solutions for mean-field games which admit a variational formulation was obtained, and these results apply to some superquadratic problems. However, that our methods do not require a variational structure and can be easily modified to address a wide class of models. Moreover, for variational mean-field games, our methods yield stronger results as they prove the existence of smooth solutions to the corresponding variational problems which are convex but non-coercive.

Our main result is the following:

Theorem 1.1. *Assume that* A1–A10 *from Section* 2 *hold. Then there exists a* C^∞ *solution* (u, m) *to* (1.1) *under the initial-terminal conditions* (1.2)*, with* $m > 0$.

We observe that uniqueness of solutions to (1.1) and (1.2) follows from earlier results in [37, 38].

The key Assumptions A1 and A10 are discussed in Section 2.1. An outline of the proof of this theorem is described in Section 2.2. The various steps of the proof are detailed in the remaining sections. In particular, in Section 7, we establish Lipschitz regularity for H-J equations (see Thm. 2.3 stated in the next section).

2. Main assumptions and proof outline

We begin by discussing the main assumptions used in the present paper; these assumptions cover a range of relevant problems. This section ends with the statement of the key theorems and lemmas, as well as an outline of the proof of Theorem 1.1.

2.1. Assumptions

We assume our problem satisfies the following general hypotheses:

Assumption 1. The Hamiltonian $H : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}$ is C^∞ and

1. For fixed x, the map $p \mapsto H(x, p)$ is strictly convex;
2. Additionally, H satisfies the coercivity condition

$$\lim_{|p|\to\infty} \frac{H(x,p)}{|p|} = +\infty,$$

and, without loss of generality, we require further that $H(x, p) \geq 1$.

Assumption 2. The function $g : \mathbb{R}_0^+ \to \mathbb{R}$ is non-negative and increasing.
Finally, $u_0, m_0 \in C^\infty(\mathbb{T}^d)$ with $m_0 \geq 0$ and $\int_{\mathbb{T}^d} m_0 = 1$.

Since g is increasing and non-negative, it follows that there exists a convex increasing function, $G : \mathbb{R}_0^+ \to \mathbb{R}$, such that $g(z) = G'(z)$.

The Legendre transform of H is given by $L(x, v) = \sup_p (-p \cdot v - H(x, p))$. Then, if we set

$$\hat{L}(x,p) = D_pH(x,p)p - H(x,p), \tag{2.1}$$

by standard properties of the Legendre transform, $\hat{L}(x, p) = L(x, -D_pH(x, p))$.

Assumption 3. For some $c, C > 0$

$$\hat{L}(x,p) \geq cH(x,p) - C.$$

For convenience and definiteness, we choose g to be a power nonlinearity.

Assumption 4. $g(m) = m^\alpha$, for some $\alpha > 0$.

We observe that for this choice of g, our problems admit a variational structure that was explored in [12] in order to construct weak solutions. Our results can be generalized easily to the setting in which g depends simultaneously on m and x, or cases in which no variational structure exists, provided appropriate bounds on the growth and derivatives of H and g are assumed. This will not be pursued here to keep the presentation simple.

Assumption 5. H satisfies the following bounds

$$|D_x H|, |D^2_{xx} H| \leq CH + C,$$

and, for any symmetric matrix M, and any $\delta > 0$, there exists C_δ such that

$$\mathrm{Tr}(D^2_{px} HM) \leq \delta\,\mathrm{Tr}(D^2_{pp} HM^2) + C_\delta H.$$

Because $H \geq 1$, the inequality in the previous assumption is equivalent to $|D_x H|, |D^2_{xx} H| \leq \tilde{C} H$, for some constant $\tilde{C}$.

Assumption 6. $m_0 \geq \kappa_0$ for some $\kappa_0 \in \mathbb{R}^+$.

The preceding hypotheses are the same as the corresponding ones in [30]. The next group of assumptions is distinct and encodes the superquadratic nature of the Hamiltonian.

Assumption 7. For some $0 < \mu < 1$, the Hamiltonian satisfies

$$c_1 |p|^{2+\mu} + C_1 \leq H \leq c_2 |p|^{2+\mu} + C_2,$$

where c_i and C_i are non-negative constants.

Assumption 8. The following estimate holds:

$$|D_p H(x,p)|^2 \leq C|p|^\mu H(x,p) + C.$$

Assumption 9. H satisfies the following bounds:

$$\left|D^2_{xp} H\right|^2 \leq C H^{\frac{2+2\mu}{2+\mu}},$$

and, for any symmetric matrix M,

$$\left|D^2_{pp} HM\right|^2 \leq CH^{\frac{\mu}{2+\mu}}\,\mathrm{Tr}(D^2_{pp} HMM),$$

where μ and C are given constants.

Note that, in particular, the previous hypothesis implies that for any function u, $H(x, Du)$ satisfies the following estimates:

$$\left|\,\mathrm{div}(D_p H(x, Du))\right|^2 \leq CH^{\frac{\mu}{2+\mu}} \left(\mathrm{Tr}\, D^2_{pp} HD^2 u D^2 u\right) + CH^{\frac{2+2\mu}{2+\mu}}. \tag{2.2}$$

Assumption 10. The exponent α satisfies $\alpha < \dfrac{2}{\mathrm{d}(1+\mu) - 2}$.

2.2. Outline of the proof

The proof of Theorem 1.1 starts by considering a regularized version of (1.1). It consists of replacing $g(m)$ by

$$g_\epsilon(m) = \eta_\epsilon * g(\eta_\epsilon * m), \tag{2.3}$$

where η_ϵ is a standard, symmetric, mollifying kernel. This yields the regularized model:

$$\begin{cases} -u^\epsilon_t + H(x, Du^\epsilon) = \Delta u^\epsilon + g_\epsilon(m^\epsilon) \\ m^\epsilon_t - \mathrm{div}(D_p H m^\epsilon) = \Delta m^\epsilon. \end{cases} \tag{2.4}$$

For convenience, we set $g_0 = g$. The special structure of (2.3) makes it possible to prove estimates for (2.4) that are uniform in ϵ. Existence of C^∞ solutions for (2.4)–(1.2) follows from standard arguments using some of the ideas in [7], as detailed in [46].

The proof of Theorem 1.1 proceeds by considering polynomial estimates for $g_\epsilon(m^\epsilon)$ in terms of Du^ϵ as stated in the following theorem:

Theorem 2.1. *Let* (u^ϵ, m^ϵ) *be a solution of* (2.4). *Assume that* A1–A9 *hold. Let* $\theta > 1$, $0 \leq \upsilon \leq 1$. *For* $\beta_0 \in \left[1, \frac{\mathrm{d}(1+\mu)}{\mathrm{d}(1+\mu)-2}\right)$, *let* $\beta_{\upsilon,\theta} = \frac{\theta\beta_0}{\theta+\upsilon-\theta\upsilon}$ *and*

$$r_\theta = \frac{\mathrm{d}(\theta-1)+2}{2}\cdot \tag{2.5}$$

Suppose that

$$p_{\upsilon,\theta} = \frac{\beta_{\upsilon,\theta}}{\alpha} > 1. \tag{2.6}$$

Then, for $r = r_\theta$ *and* $p = p_{\upsilon,\theta}$, *we have*

$$\|g_\epsilon(m^\epsilon)\|_{L^\infty(0,T;L^p(\mathbb{T}^d))} \leq C + C\,\|Du^\epsilon\|_{L^\infty(0,T;L^\infty(\mathbb{T}^d))}^{\frac{(2+2\mu)r\upsilon\alpha}{\theta\beta_0}}, \tag{2.7}$$

where C *is independent of* ϵ.

Theorem 2.1 is proven in Section 4.3. Then, we establish L^∞ bounds for u^ϵ in terms of $g_\epsilon(m^\epsilon)$, as in the following Lemma:

Lemma 2.2. *Suppose* (u^ϵ, m^ϵ) *is a solution of* (2.4) *and* H *satisfies* A1. *Then, if* $p > \frac{d}{2}$,

$$\|u^\epsilon\|_{L^\infty(\mathbb{T}^d\times[0,T])} \leq C + C\|g_\epsilon(m^\epsilon)\|_{L^\infty(0,T;L^p(\mathbb{T}^d))}, \tag{2.8}$$

where C *is independent of* ϵ.

The proof of Lemma 2.2 is presented in Section 5. To estimate Du^ϵ in terms of $g_\epsilon(m^\epsilon)$, we apply the nonlinear adjoint method (see [16]), which yields the following estimate:

Theorem 2.3. *Suppose that* A1–A10 *hold. Let* (u^ϵ, m^ϵ) *be a solution of* (2.4) *and assume that* $p > d$. *Then,*

$$\begin{aligned}\|Du^\epsilon\|_{L^\infty(0,T;L^\infty(\mathbb{T}^d))} \leq{}& C + C\|g_\epsilon(m^\epsilon)\|_{L^\infty(0,T;L^p(\mathbb{T}^d))}^{\frac{1}{1-\mu}}\\ &+ C\|g_\epsilon(m^\epsilon)\|_{L^\infty(0,T;L^p(\mathbb{T}^d))}^{\frac{1}{1-\mu}}\|u\|_{L^\infty(0,T;L^\infty(\mathbb{T}^d))}^{\frac{1}{1-\mu}},\end{aligned} \tag{2.9}$$

where C *is independent of* ϵ.

This Theorem is established in Section 6. To prove Theorem 1.1, we combine the estimates in Theorem 2.1, Lemma 2.2 and Theorem 2.3, obtaining Lipschitz regularity for u^ϵ. This is done in Section 7. It follows from (2.7), combined with (2.9), that

$$\|Du^\epsilon\|_{L^\infty(\mathbb{T}^d\times[0,T])} \leq C + C\|Du^\epsilon\|_{L^\infty(\mathbb{T}^d\times[0,T])}^{\zeta},$$

where, if α is small enough, $\zeta < 1$. The precise bound for α is the one given in Assumption A10. Lastly, we obtain Lipschitz regularity in the following theorem:

Theorem 2.4. *Let* (u^ϵ, m^ϵ) *be a solution of* (2.4)–(1.2). *Suppose that* A1–10 *hold. Then,* $Du^\epsilon \in L^\infty(\mathbb{T}^d \times [0,T])$, *with bounds uniform in* ϵ.

We now present the proof of Theorem 1.1.

Proof of Theorem 1.1. By Theorem 2.4, we have Lipschitz regularity for u^ϵ, uniformly in ϵ. Thus, the growth of the Hamiltonian plays no role in any further gain of regularity. Then, a number of additional estimates can be derived, see [30]. These ensure, in particular, that u^ϵ and m^ϵ are Hölder continuous, uniformly in ϵ. Thus, through some subsequence, we have that $u^\epsilon \to u$ in $\mathcal{C}^{0,\gamma}(\mathbb{T}^d \times [0,T])$ and $m^\epsilon \to m$ in $\mathcal{C}^{0,\gamma}(\mathbb{T}^d \times [0,T])$, as $\epsilon \to 0$. This shows that u is a (viscosity) solution of the first equation in (1.1). Furthermore, additional bounds on D^2u^ϵ provide enough compactness to conclude that m solves

$$m_t - \operatorname{div}(D_pH(x,Du)m) = \Delta m,$$

as a weak solution, *i.e.*,

$$\int_0^T \int_{\mathbb{T}^d} (-\phi_t + D_pHD\phi - \Delta\phi)\, m \mathrm{d}x\mathrm{d}t \;=\; 0,$$

for every $\phi \in \mathcal{C}_c^\infty(\mathbb{T}^d)$. By the results in [30], we have uniform bounds in every Sobolev space for (u^ϵ, m^ϵ). Finally, observing that (u,m) satisfies the same estimates as (u^ϵ, m^ϵ), we obtain existence of smooth solutions. □

The rest of this paper is organized as follows: the next Section presents some elementary estimates from [30]. In Section 4, we obtain higher integrability for m^ϵ, see Theorem 4.1. The proof of Theorem 2.1 is presented in Section 4.3. In Sections 5 and 6, we establish Lemma 2.2 and Theorem 2.3. Lipschitz regularity for the Hamilton−Jacobi equation is established in Section 7.

3. Elementary estimates

Next we recall several estimates for solutions of (2.4). These have appeared (either in the present form or in related versions) in [11, 15, 37, 38, 44]. For ease of presentation, we omit the proofs, which can be found in [30].

Proposition 3.1 (Stochastic Lax−Hopf estimate)**.** *Suppose that* A1 *holds. Let* (u^ϵ, m^ϵ) *be a solution to* (2.4). *Then, for any smooth vector field* $b : \mathbb{T}^d \times (t,T) \to \mathbb{R}^d$, *and any solution to*

$$\zeta_s + \operatorname{div}(b\zeta) = \Delta\zeta, \tag{3.1}$$

with $\zeta(x,t) = \zeta_0$, *we have the following upper bound:*

$$\begin{aligned}\int_{\mathbb{T}^d} u^\epsilon(x,t)\zeta_0(x)\mathrm{d}x \le \int_t^T \int_{\mathbb{T}^d} \big(L(y,b(y,s)) + g_\epsilon(m^\epsilon)(y,s)\big)\zeta(y,s)\mathrm{d}y\mathrm{d}s \\ + \int_{\mathbb{T}^d} u_T^\epsilon(y)\zeta(y,T).\end{aligned} \tag{3.2}$$

We notice that, for $b = -D_pH(x,Du)$, the inequality in (3.2) is attained.

Proposition 3.2 (First-order estimate)**.** *Assume that* A1−3 *hold. Let* (u^ϵ, m^ϵ) *be a solution of* (2.4). *Then,*

$$\int_0^T \int_{\mathbb{T}^d} cH(x, D_xu^\epsilon)m^\epsilon + G(\eta_\epsilon * m^\epsilon)\mathrm{d}x\mathrm{d}t \le CT + C\left\|u^\epsilon(\cdot,T)\right\|_{L^\infty(\mathbb{T}^d)}, \tag{3.3}$$

where $G' = g$.

Proposition 3.3 (Second-order estimate)**.** *Assume that* A1−6 *hold. Let* (u^ϵ, m^ϵ) *be a solution of* (2.4).

$$\int_0^T \int_{\mathbb{T}^d} g'(\eta_\epsilon * m^\epsilon)|D_x(\eta_\epsilon * m^\epsilon)|^2 + \operatorname{Tr}(D_{pp}^2 H(D_{xx}^2 u^\epsilon)^2)m^\epsilon \le C.$$

Corollary 3.4. *Assume that* A1−6 *hold. Let* (u^ϵ, m^ϵ) *be a solution of* (2.4). *Then,*

$$\int_0^T \|\eta_\epsilon * m^\epsilon\|^{\alpha+1}_{L^{\frac{2^*}{2}(\alpha+1)}(\mathbb{T}^d)}\mathrm{d}t \le C.$$

4. Regularity for the Fokker−Planck equation

Next, building upon the second-order estimate of Proposition 3.3, we obtain improved integrability for m^ϵ. In Section 4.2, the integrability of m^ϵ is controlled in terms of L^p norms of $D_pH(x, Du^\epsilon(x))$.

In the superquadratic case, further arguments yield uniform estimates for Du^ϵ in $L^\infty(\mathbb{T}^d \times [0,T])$ rather than in $L^r(0,T;L^p(\mathbb{T}^d))$, which was the space used in the subquadratic setting.

In this Section, the function H and its derivatives will be evaluated at $(x, Du^\epsilon(x))$; however, to ease the notation, we omit the argument.

4.1. Regularity by the second-order estimate

We begin by addressing the regularity of the Fokker−Planck equation by applying the second-order estimate from the previous section.

Theorem 4.1. *Assume that* A1−6 *and* A9 *hold. Let* (u^ϵ, m^ϵ) *be a solution of* (2.4)*. Then, for* $d \geq 2$, $\|m^\epsilon(\cdot,t)\|_{L^\infty(0,T;L^r(\mathbb{T}^d))}$ *is bounded for any* $1 \leq r < \frac{\mathrm{d}(1+\mu)}{\mathrm{d}(1+\mu)-2}$*, uniformly in* ϵ.

Proof. We omit the ϵ to simplify the notation. We start by defining an increasing sequence, β_n, such that $\|m(\cdot,t)\|_{L^{1+\beta_n}(\mathbb{T}^d)}$ is bounded. We set $\beta_0 = 0$ so that $\|m(\cdot,t)\|_{L^{1+\beta_0}(\mathbb{T}^d)} = 1 \leq C$.

At this point, it is critical to control $\int_{\mathbb{T}^d} \mathrm{div}(D_pH)m^{\beta+1}\mathrm{d}x$. This will be done using Assumption A9. In fact, using (2.2), we have:

$$\begin{aligned}
&\int_{\mathbb{T}^d} \mathrm{div}(D_pH)m^{\beta+1}\mathrm{d}x \\
&\leq \int_{\mathbb{T}^d} CH^{\frac{\mu}{2(2+\mu)}} m^{\frac{\mu}{2(2+\mu)}} \operatorname{Tr}\left(D^2_{pp}HD^2uD^2u\right)^{1/2} m^{1/2} m^{\beta+\frac{1}{(2+\mu)}} \\
&\quad + \int_{\mathbb{T}^d} CH^{\frac{1+\mu}{2+\mu}} m^{\frac{1+\mu}{2+\mu}} m^{\beta+\frac{1}{(2+\mu)}} \\
&\leq C_\delta \int_{\mathbb{T}^d} Hm + C_\delta \int_{\mathbb{T}^d} \operatorname{Tr}\left(D^2_{pp}HD^2uD^2u\right)m + \delta \int_{\mathbb{T}^d} m^{(2+\mu)\beta+1}.
\end{aligned} \tag{4.1}$$

We note that the time integral of the first two terms on the right-hand side of the previous inequalities is bounded by Propositions 3.2 and 3.3. Because of Sobolev's theorem, we proceed by examining the cases $d > 2$ and $d = 2$ separately.

Consider the case $d > 2$. Let $\beta_{n+1} = \frac{2}{\mathrm{d}(1+\mu)}(\beta_n + 1)$. Then, β_n is the nth partial sum of a geometric series with term $\dfrac{2^n}{d^n(1+\mu)^n}$. Therefore, $\lim\limits_{n\to\infty} \beta_n = \dfrac{2}{\mathrm{d}(1+\mu)-2}$. We define $q_n = \frac{2^*}{2}(\beta_{n+1}+1)$, where 2^* is the critical Sobolev exponent given by

$$2^* = \frac{2d}{d-2}\cdot$$

Hence, we have

$$\|m\|_{L^{(2+\mu)\beta_{n+1}+1}(\mathbb{T}^d)} \leq \|m\|^{1-\lambda_n}_{L^{1+\beta_n}(\mathbb{T}^d)} \|m\|^{\lambda_n}_{L^{q_n}(\mathbb{T}^d)},$$

where $\frac{\lambda_n}{q_n} + \frac{1-\lambda_n}{1+\beta_n} = \frac{1}{(2+\mu)\beta_{n+1}+1}$, and thus:

$$\lambda_n = \frac{q_n}{q_n - \beta_n - 1} \frac{(2+\mu)\beta_{n+1} - \beta_n}{1 + (2+\mu)\beta_{n+1}}\cdot \tag{4.2}$$

Since $\|m\|_{L^{1+\beta_n}(\mathbb{T}^d)} \leq C$, we get

$$\int_{\mathbb{T}^d} m^{(2+\mu)\beta_{n+1}+1}\mathrm{d}x = \|m\|^{(2+\mu)\beta_{n+1}+1}_{L^{(2+\mu)\beta_{n+1}+1}(\mathbb{T}^d)} \leq C\|m\|^{\lambda_n((2+\mu)\beta_{n+1}+1)}_{L^{q_n}(\mathbb{T}^d)}. \tag{4.3}$$

Setting $\beta = \beta_{n+1}$, from (4.1) and (4.3), we get for any $\tau \in [0, T]$

$$\begin{aligned}
&\int_{\mathbb{T}^d} m^{\beta_{n+1}+1}(x,\tau)\mathrm{d}x + \frac{4\beta_{n+1}}{\beta_{n+1}+1}\int_0^\tau\int_{\mathbb{T}^d} |D_x m^{\frac{\beta_{n+1}+1}{2}}(x,t)|^2 \mathrm{d}x\,\mathrm{d}t \\
&= \int_{\mathbb{T}^d} m^{\beta_{n+1}+1}(x,0)\mathrm{d}x + \beta\int_0^\tau\int_{\mathbb{T}^d} \mathrm{div}(D_pH)m^{\beta_{n+1}+1}\mathrm{d}x\mathrm{d}t \\
&\leq \int m^{\beta_{n+1}+1}(x,0)\mathrm{d}x + C_\delta\int_0^\tau\int_{\mathbb{T}^d} Hm \\
&+ C_\delta\int_0^\tau\int_{\mathbb{T}^d}\left(\mathrm{Tr}\, D^2_{pp}HD^2uD^2u\right)m + \delta\int_0^\tau \|m\|^{\lambda_n((2+\mu)\beta_{n+1}+1)}_{L^{q_n}(\mathbb{T}^d)}\mathrm{d}t. \qquad (4.4)
\end{aligned}$$

From the definition of β_{n+1}, it follows that $\lambda_n((2+\mu)\beta_{n+1}+1) = 1+\beta_{n+1}$. Hence,

$$\begin{aligned}
\|m\|^{\lambda_n((2+\mu)\beta_{n+1}+1)}_{L^{q_n}(\mathbb{T}^d)} &= \|m^{\frac{\beta_{n+1}+1}{2}}\|^2_{L^{2^*}(\mathbb{T}^d)} \\
&\leq C + C\left(\int_{\mathbb{T}^d}|D_x m^{\frac{\beta_{n+1}+1}{2}}|^2 + \int_{\mathbb{T}^d} m^{\beta_{n+1}+1}\right). \qquad (4.5)
\end{aligned}$$

Using elementary inequalities and $\int m\mathrm{d}x = 1$, we have for any $\zeta > 0$ that $\int_{\mathbb{T}^d} m^{\beta_{n+1}+1} \leq C_\zeta + \zeta\|m\|^{\beta_{n+1}+1}_{L^{q_n}(\mathbb{T}^d)}$. Thus, it follows that

$$\|m\|^{\beta_{n+1}+1}_{L^{q_n}(\mathbb{T}^d)} \leq C_\zeta + C\int_{\mathbb{T}^d}|D_x m^{\frac{\beta_{n+1}+1}{2}}|^2 + \zeta\|m\|^{\beta_{n+1}+1}_{L^{q_n}(\mathbb{T}^d)}.$$

From (4.4) and (4.5), with small enough δ and ζ, it follows that for some $\delta_1 > 0$,

$$\begin{aligned}
\int_{\mathbb{T}^d} m^{\beta_{n+1}+1}(x,\tau)\mathrm{d}x + \delta_1\int_0^\tau \|m\|^{\beta_{n+1}+1}_{L^{q_n}(\mathbb{T}^d)}\mathrm{d}t &\leq C + C\int_{\mathbb{T}^d} m^{\beta_{n+1}+1}(x,0)\mathrm{d}x \\
&+ C\int_0^\tau\int_{\mathbb{T}^d} Hm + C\int_0^\tau\int_{\mathbb{T}^d}\left(\mathrm{Tr}\, D^2_{pp}HD^2uD^2u\right)m.
\end{aligned}$$

Because the last two terms on the right-hand side are bounded by Propositions 3.2 and 3.3, we have the result.

Consider now the case $d = 2$. Let $1 < p < 1 + \dfrac{1}{\mu}$. As before, we inductively define β_n, starting with $\beta_0 = 0$. Letting $\beta_{n+1} := \dfrac{p-1}{p}(\beta_n + 1)$, we have that β_n is the nth partial sum of the geometric series with term $\dfrac{(p-1)^n}{p^n}$ and thus $\lim_{n\to\infty}\beta_n = p-1$. Let $q_n = \frac{p(\beta_{n+1}+1)}{1+\mu-\mu p}$. For λ_n as in (4.2), we have

$$\|m\|_{L^{(2+\mu)\beta_{n+1}+1}(\mathbb{T}^d)} \leq \|m\|^{1-\lambda_n}_{L^{1+\beta_n}(\mathbb{T}^d)}\|m\|^{\lambda_n}_{L^{q_n}(\mathbb{T}^d)}.$$

From the previous definitions, it follows that $\lambda_n((2+\mu)\beta_{n+1}+1) = 1+\beta_{n+1}$. Since $\|m\|_{1+\beta_n} \leq C$, we get

$$\int_{\mathbb{T}^d} m^{(2+\mu)\beta_{n+1}+1}\mathrm{d}x = \|m\|^{(2+\mu)\beta_{n+1}+1}_{L^{(2+\mu)\beta_{n+1}+1}(\mathbb{T}^d)} \leq C\|m\|^{\lambda_n((2+\mu)\beta_{n+1}+1)}_{L^{q_n}(\mathbb{T}^d)} = C\|m\|^{1+\beta_{n+1}}_{L^{q_n}(\mathbb{T}^d)}. \qquad (4.6)$$

As in (4.4), using (4.1), and (4.6) we get for any $\tau \in [0, T]$

$$\begin{aligned}
&\int_{\mathbb{T}^d} m^{\beta_{n+1}+1}(x,\tau)\mathrm{d}x + \frac{4\beta_{n+1}}{\beta_{n+1}+1}\int_0^\tau\int_{\mathbb{T}^d} |D_x m^{\frac{\beta_{n+1}+1}{2}}(x,t)|^2 \mathrm{d}x\,\mathrm{d}t \\
&\leq \int_{\mathbb{T}^d} m^{\beta_{n+1}+1}(x,0)\mathrm{d}x + C_\delta\int_0^\tau\int_{\mathbb{T}^d} Hm \\
&+ C_\delta\int_0^\tau\int_{\mathbb{T}^d}\left(\mathrm{Tr}\, D^2_{pp}HD^2uD^2u\right)m + \delta\int_0^\tau \|m\|^{1+\beta_{n+1}}_{L^{q_n}(\mathbb{T}^d)}\mathrm{d}t. \qquad (4.7)
\end{aligned}$$

By Sobolev's theorem, we get

$$\|m\|^{1+\beta_{n+1}}_{L^{q_n}(\mathbb{T}^d)} = \|m^{\frac{\beta_{n+1}+1}{2}}\|^2_{L^{\frac{2q_n}{\beta_{n+1}+1}}(\mathbb{T}^d)} \leq C\int_{\mathbb{T}^d} |D_x m^{\frac{\beta_{n+1}+1}{2}}|^2 \mathrm{d}x + C_\zeta + \zeta\|m\|^{\beta_{n+1}+1}_{L^{q_n}(\mathbb{T}^d)}. \tag{4.8}$$

From (4.7) and (4.8), with small enough δ and ζ, we have for some $\delta_1 > 0$ that

$$\int_{\mathbb{T}^d} m^{\beta_{n+1}+1}(x,\tau)\mathrm{d}x + \delta_1\int_0^\tau \|m\|^{\beta_{n+1}+1}_{L^{q_n}(\mathbb{T}^d)}\mathrm{d}t \leq C + C\int_{\mathbb{T}^d} m^{\beta_{n+1}+1}(x,0)\mathrm{d}x + C\int_0^\tau\int_{\mathbb{T}^d} Hm$$
$$+ C\int_0^\tau\int_{\mathbb{T}^d}\left(\mathrm{Tr}\, D^2_{pp}HD^2uD^2u\right)m.$$

Notice that the last two terms on the right-hand side are bounded because of Propositions 3.2 and 3.3. We have then established the result. □

4.2. Regularity by L^p estimates

Now we bound m^ϵ in $L^\infty([0,T],L^p(\mathbb{T}^d))$ with estimates depending polynomially on the L^∞-norm of D_pH. Because explicit expressions will be needed, we prove them in detail. For ease of presentation, we omit the ϵ in the proofs of this section.

We start by setting $1 \leq \beta_0 < \frac{\mathrm{d}(1+\mu)}{\mathrm{d}(1+\mu)-2}$, and we consider $\beta_1 \doteq \theta\beta_0$ for some fixed $\theta > 1$.

Lemma 4.2. *Assume that (u^ϵ, m^ϵ) is a solution of (2.4) and let $\beta \geq \beta_0$ for $\beta_0 > 1$ fixed. Then*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^d}(m^\epsilon)^\beta(t,x)\mathrm{d}x \leq C\left\||D_pH|^2\right\|_{L^\infty(\mathbb{T}^d)}\int_{\mathbb{T}^d}(m^\epsilon)^\beta(t,x)\mathrm{d}x - c\int_{\mathbb{T}^d}\left|D_x\left((m^\epsilon)^{\frac{\beta}{2}}\right)\right|^2\mathrm{d}x. \tag{4.9}$$

Lemma 4.3. *We have that*

$$\int_{\mathbb{T}^d}(m^\epsilon)^{\beta_1}(\tau,x)\mathrm{d}x \leq \left(\int_{\mathbb{T}^d}(m^\epsilon)^{\beta_0}(\tau,x)\,\mathrm{d}x\right)^{\theta\kappa}\left(\int_{\mathbb{T}^d}(m^\epsilon)^{\frac{2^*\beta_1}{2}}(\tau,x)\,\mathrm{d}x\right)^{\frac{2(1-\kappa)}{2^*}},$$

where κ is given by

$$\kappa = \frac{2}{d(\theta-1)+2}\cdot \tag{4.10}$$

Proof. Hölder's inequality gives

$$\left(\int_{\mathbb{T}^d} m^{\beta_1}\right)^{\frac{1}{\beta_1}} \leq \left(\int_{\mathbb{T}^d} m^{\beta_0}\right)^{\frac{\kappa}{\beta_0}}\left(\int_{\mathbb{T}^d} m^{\frac{2^*}{2}\beta_1}\right)^{\frac{(1-\kappa)}{\frac{2^*}{2}\beta_1}},$$

where $\frac{1}{\theta\beta_0} = \frac{\kappa}{\beta_0} + \frac{2(1-\kappa)}{2^*\theta\beta_0}$. By rearrangement of the exponents, the inequality in the statement follows. The expression for κ follows from the previous identity. □

Lemma 4.4. *Let κ be defined by (4.10). Then,*

$$\left(\int_{\mathbb{T}^d}(m^\epsilon)^{\beta_1}\right)^{(1-\kappa)} \leq C + \delta\left\|(m^\epsilon)^{\frac{\beta_1}{2}}\right\|^{2(1-\kappa)}_{L^{2^*}(\mathbb{T}^d)}.$$

Proof. Let $\lambda = \frac{2}{d(\beta_1-1)+2}$. Then,

$$\int_{\mathbb{T}^d}\left|m^{\beta_1}\right|\mathrm{d}x \leq \left(\int_{\mathbb{T}^d} m\mathrm{d}x\right)^{\beta_1\lambda}\left(\int_{\mathbb{T}^d} m^{\frac{2^*\beta_1}{2}}\mathrm{d}x\right)^{\frac{2(1-\lambda)}{2^*}}.$$

Because m is a probability measure for every $t \in [0,T]$, we obtain

$$\left(\int_{\mathbb{T}^d} m^{\beta_1} \mathrm{d}x\right)^{(1-\kappa)} \le \left\|m^{\frac{\beta_1}{2}}\right\|_{L^{2^*}(\mathbb{T}^d)}^{2(1-\kappa)(1-\lambda)}.$$

Finally, because $(1-\lambda) < 1$, a further application of Young's inequality weighted by δ establishes the result. □

Proposition 4.5. *We have that*

$$\int_{\mathbb{T}^d} (m^\epsilon)^{\beta_1} \mathrm{d}x \le \left[\int_{\mathbb{T}^d} (m^\epsilon)^{\beta_0} \mathrm{d}x\right]^{\theta\kappa} \left[C + C\left(\int_{\mathbb{T}^d} \left|D_x\left((m^\epsilon)^{\frac{\beta_1}{2}}\right)\right|^2 \mathrm{d}x\right)^{(1-\kappa)}\right],$$

where κ is given by (4.10).

Proof. Sobolev's theorem implies that

$$\left\|(m)^{\frac{\beta_1}{2}}\right\|_{L^{2^*}(\mathbb{T}^d)}^{2(1-\kappa)} \le C\left(\int_{\mathbb{T}^d} \left|D_x\left((m)^{\frac{\beta_1}{2}}\right)\right|^2\right)^{(1-\kappa)} + C\left(\int_{\mathbb{T}^d} \left|(m)^{\beta_1}\right|\right)^{(1-\kappa)}. \tag{4.11}$$

Using Lemma 4.4, we obtain

$$\begin{aligned}\left(\int_{\mathbb{T}^d} m^{\frac{2^*}{2}\beta_1}(\tau,x)\,\mathrm{d}x\right)^{\frac{2}{2^*}(1-\kappa)} &= \left\|m^{\frac{\beta_1}{2}}\right\|_{L^{2^*}(\mathbb{T}^d)}^{2(1-\kappa)} \\ &\le C\left(\int_{\mathbb{T}^d} \left|D_x\left(m^{\frac{\beta_1}{2}}\right)\right|^2\right)^{(1-\kappa)} + C. \end{aligned} \tag{4.12}$$

By combining inequality (4.12) with Lemma 4.3, the result follows. □

Next, we control the derivative with respect to the time of $\|m^\epsilon\|_{L^{\beta_1}(\mathbb{T}^d)}^{\beta_1}$.

Proposition 4.6. *Let (u^ϵ, m^ϵ) be a solution of (2.4). If κ is given as in (4.10), then*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^d} (m^\epsilon)^{\beta_1} \mathrm{d}x \le C + C\left\||D_pH|^2\right\|_{L^\infty(\mathbb{T}^d)}^r \left(\int_{\mathbb{T}^d} (m^\epsilon)^{\beta_0} \mathrm{d}x\right)^\theta, \tag{4.13}$$

where $r = \frac{1}{\kappa}$.

Proof. Using Lemma 4.2 with $\beta \equiv \beta_1$, and applying Proposition 4.5, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^d} m^{\beta_1} \le{}& C\left\||D_pH|^2\right\|_{L^\infty(\mathbb{T}^d)} \left(\int_{\mathbb{T}^d} m^{\beta_0}\right)^{\theta\kappa}\left[C\left(\int_{\mathbb{T}^d}\left|D_x\left(m^{\frac{\beta_1}{2}}\right)\right|^2 \mathrm{d}x\right)^{(1-\kappa)} + C\right] \\ &- c\int_{\mathbb{T}^d}\left|D_x\left(m^{\frac{\beta_1}{2}}\right)\right|^2 \\ \le{}& C\left\||D_pH|^2\right\|_{L^\infty(\mathbb{T}^d)} \left(\int_{\mathbb{T}^d} m^{\beta_0}\right)^{\theta\kappa}\left(\int_{\mathbb{T}^d}\left|D_x\left(m^{\frac{\beta_1}{2}}\right)\right|^2\right)^{(1-\kappa)} \\ &+ C\left\||D_pH|^2\right\|_{L^\infty(\mathbb{T}^d)} \left(\int_{\mathbb{T}^d} m^{\beta_0}\right)^{\theta\kappa} - c\int_{\mathbb{T}^d}\left|D_x\left(m^{\frac{\beta_1}{2}}\right)\right|^2 \\ \le{}& C\left\||D_pH|^2\right\|_{L^\infty(\mathbb{T}^d)} \left(\int_{\mathbb{T}^d} m^{\beta_0}\right)^{\theta\kappa} + C\left\||D_pH|^2\right\|_{L^\infty(\mathbb{T}^d)}^r \left(\int_{\mathbb{T}^d} m^{\beta_0}\right)^{\theta} \\ \le{}& C + C\left\||D_pH|^2\right\|_{L^\infty(\mathbb{T}^d)}^r \left(\int_{\mathbb{T}^d} m^{\beta_0}\right)^{\theta},\end{aligned}$$

where the last two inequalities follow by applying Young's inequality with ε for the conjugate exponents r and s given by $s = \frac{1}{1-\kappa}$ and $r = \frac{1}{\kappa}$. □

Corollary 4.7. *Suppose that* (u^ϵ, m^ϵ) *is a solution of* (2.4). *Let* r *be given as in Proposition* 4.6. *Then,*

$$\int_{\mathbb{T}^d} m^{\beta_1}(\tau,x)\mathrm{d}x \le C + C\left\| |D_pH|^2 \right\|^r_{L^\infty(\mathbb{T}^d)}.$$

Proof. Integrate (4.13) in time over (τ, T). This yields

$$\int_{\mathbb{T}^d} m^{\beta_1}(\tau,x)\,\mathrm{d}x \le C\left\| |D_pH|^2 \right\|^r_{L^\infty(\mathbb{T}^d)} \int_0^\tau \left(\int_{\mathbb{T}^d} m^{\beta_0}\mathrm{d}x\right)^\theta \mathrm{d}t + C. \tag{4.14}$$

From Proposition 4.1, we have $\int_{\mathbb{T}^d} m^{\beta_0}(\tau,x)\,\mathrm{d}x \le C$. The result is then established. □

4.3. Interpolated bounds

We now obtain estimates for m^ϵ in terms of the L^∞-norm of Du^ϵ by interpolating previous results.

Lemma 4.8. *Let* (u^ϵ, m^ϵ) *be a solution of* (2.4). *Assume that* A1−6 *and* A9 *hold. Assume further that* θ, p, $r > 1$, $0 \le \upsilon \le 1$ *are such that* (2.5)−(2.6). *Let* $\beta_\upsilon = \frac{\theta\beta_0}{\theta+\upsilon-\theta\upsilon}$, *where* $\beta_0 \in \left[1, \frac{\mathrm{d}(1+\mu)}{\mathrm{d}(1+\mu)-2}\right)$. *Then,*

$$\|g(m^\epsilon)\|_{L^\infty(0,T;L^p(\mathbb{T}^d))} \le C + C\left\| |D_pH|^2 \right\|^{\frac{r\upsilon\alpha}{\theta\beta_0}}_{L^\infty(0,T;L^\infty(\mathbb{T}^d))}.$$

Proof. As before, we omit the ϵ in the proof. Hölder's inequality gives

$$\left(\int_{\mathbb{T}^d} m^{\beta_\upsilon}\right)^{\frac{1}{\beta_\upsilon}} \le \left(\int_{\mathbb{T}^d} m^{\beta_0}\right)^{\frac{1-\upsilon}{\beta_0}} \left(\int_{\mathbb{T}^d} m^{\theta\beta_0}\right)^{\frac{\upsilon}{\theta\beta_0}},$$

since $\frac{1}{\beta_\upsilon} = \frac{1-\upsilon}{\beta_0} + \frac{\upsilon}{\theta\beta_0}$.

Theorem 4.1 ensures that

$$\int_{\mathbb{T}^d} m^{\beta_\upsilon} \le C\left(\int_{\mathbb{T}^d} m^{\theta\beta_0}\right)^{\frac{\upsilon}{\theta+\upsilon-\theta\upsilon}}.$$

On the other hand, Corollary 4.7 gives

$$\int_{\mathbb{T}^d} m^{\theta\beta_0} \le C + C\left\| |D_pH|^2 \right\|^r_{L^\infty(0,T;L^\infty(\mathbb{T}^d))},$$

which in turn leads to

$$\int_{\mathbb{T}^d} m^{\beta_\upsilon} \le C + C\left\| |D_pH|^2 \right\|^{\frac{r\upsilon}{(\theta+\upsilon-\theta\upsilon)}}_{L^\infty(0,T;L^\infty(\mathbb{T}^d))}.$$

Note that $\|g(m)\|_{L^p(\mathbb{T}^d)} = \left(\int_{\mathbb{T}^d} m^{\alpha p}\right)^{\frac{1}{p}}$. Because of (2.6), it follows that

$$\|g(m)\|_{L^p(\mathbb{T}^d)} \le C + C\left\| |D_pH|^2 \right\|^{\frac{r\upsilon}{p(\theta+\upsilon-\theta\upsilon)}}_{L^\infty(0,T;L^\infty(\mathbb{T}^d))}.$$

By noticing that $p(\theta+\upsilon-\theta\upsilon) = \frac{\theta\beta_0}{\alpha}$, because of (2.6), the Lemma is established. □

Proof of Theorem 2.1. Theorem 2.1 follows from Lemmas 4.8 and A8. □

5. Bounds for the Hamilton–Jacobi equation

Next, we control $\|u^\epsilon\|_{L^\infty(\mathbb{T}^d\times[0,T])}$ in terms of $\|g_\epsilon(m^\epsilon)\|_{L^\infty([0,T],L^p(\mathbb{T}^d))}$. Because we already have lower bounds for u^ϵ, since $g \geq 0$, see [30], it suffices in what follows to obtain the upper bounds.

We start by presenting the proof of Lemma 2.2.

Proof of Lemma 2.2. For ease of notation, we omit the ϵ in m^ϵ. By using Proposition 3.1 with $b = 0$ and $\zeta_0 = \theta(\cdot,\tau) = \delta_x$, $0 \leq \tau < T$, we obtain the estimate

$$\begin{aligned} u(x,\tau) \leq & (T-\tau)\max_{z\in\mathbb{T}^d} L(z,0) \\ & + \int_\tau^T \int_{\mathbb{T}^d} g_\epsilon(m)(y,t)\theta(y,t-\tau)\mathrm{d}y\mathrm{d}t + \int_{\mathbb{T}^d} u(y,T)\theta(y,T-\tau)\mathrm{d}y. \end{aligned}$$

The main issue is to control

$$\int_\tau^T \int_{\mathbb{T}^d} g_\epsilon(m)(y,t)\theta(y,t-\tau)\mathrm{d}y\mathrm{d}t. \tag{5.1}$$

For $\dfrac{1}{p}+\dfrac{1}{q}=1$, the heat kernel satisfies

$$\|\theta(\cdot,t)\|_q \leq \frac{C}{t^{\frac{d}{2p}}}\cdot$$

Hence,

$$\int_{\mathbb{T}^d} g_\epsilon(m)(y,t)\theta(y,t-\tau)\mathrm{d}y \leq \frac{C}{(t-\tau)^{\frac{d}{2p}}}\|g_\epsilon(m(\cdot,t))\|_{L^p(\mathbb{T}^d)}.$$

Thus, if $d < 2p$, we have

$$\int_\tau^T \int_{\mathbb{T}^d} g_\epsilon(m)(y,t)\theta(y,t-\tau)\mathrm{d}y\mathrm{d}t \leq C\|g_\epsilon(m)\|_{L^\infty(0,T;L^p(\mathbb{T}^d))}. \qquad \square$$

6. Regularity by the adjoint method

The aim of this section is to obtain estimates for $\|Du^\epsilon\|_{L^\infty(0,T;L^\infty(\mathbb{T}^d))}$. The key tools are the adjoint method [16] and the methods developed in [22] (see also [29]). In what follows, we obtain Lipschitz estimates for the solutions of the Hamilton–Jacobi equation in terms of $L^\infty(0,T;L^p(\mathbb{T}^d))$ norms of the nonlinearity g. This result is important not only for its role in the realm of mean-field game theory, but it also adds to the current understanding of the regularity of superquadratic Hamilton–Jacobi equations. For some related results, see [5,6,10], where the authors investigate Hölder's regularity.

Our main *a priori* estimate is the following:

Theorem 6.1. *Suppose that* A1–A9 *hold. Let* (u^ϵ,m^ϵ) *be a solution of* (2.4) *and assume that* $p > d$. *Then,*

$$\begin{aligned} \|Du^\epsilon\|_{L^\infty(0,T;L^\infty(\mathbb{T}^d))} \leq & C + C\|u^\epsilon\|_{L^\infty(0,T;L^\infty(\mathbb{T}^d))} + C\|g_\epsilon(m^\epsilon)\|_{L^\infty(0,T;L^p(\mathbb{T}^d))} \\ & \times \left(\|Du^\epsilon\|^\mu_{L^\infty(0,T;L^\infty(\mathbb{T}^d))}(1+\|u^\epsilon\|_{L^\infty(0,T;L^\infty(\mathbb{T}^d))})\right), \end{aligned}$$

where μ *is the exponent given by Assumption* A8.

Proof. For convenience, the proof of the theorem proceeds in the four steps below. $\square$

We omit the superscript ϵ for the solution (u^ϵ, m^ϵ) in the following proofs.

Step 1. The adjoint equation is the following partial differential equation

$$\rho_t - \Delta\rho - \operatorname{div}(D_p H \rho) = 0 \tag{6.1}$$

for which we choose the initial data $\rho(\cdot,\tau) = \delta_{x_0}$. Using this and the first equation in (2.4), we have the following representation formula for u:

$$u(x_0,\tau) = \int_\tau^T \int_{\mathbb{T}^d} (D_p H D_x u - H + g_\epsilon(m))\rho + \int_{\mathbb{T}^d} u(x,T)\rho(x,T). \tag{6.2}$$

Corollary 6.2. *Suppose that* A1–A9 *hold. Let* (u^ϵ, m^ϵ) *be a solution of* (2.4)*. Let* ρ *solve* (6.1) *with initial data* $\rho(\cdot,\tau) = \delta_{x_0}$*. Then,*

$$\int_\tau^T \int_{\mathbb{T}^d} H\rho + \int_\tau^T \int_{\mathbb{T}^d} g_\epsilon(m)\rho \leq C + C\left[u(x_0,\tau) - \int_{\mathbb{T}^d} u(x,T)\rho(x,T)\right]. \tag{6.3}$$

Proof. It suffices to use Assumption A3 in (6.2). □

Corollary 6.3. *Suppose that* A1–A9 *hold. Let* (u^ϵ, m^ϵ) *be a solution of* (2.4)*. Let* ρ *solve* (6.1) *with initial data* $\rho(\cdot,\tau) = \delta_{x_0}$*. Then,*

$$\int_\tau^T \int_{\mathbb{T}^d} H\rho \leq C + C\|u^\epsilon\|_{L^\infty(0,T;L^\infty(\mathbb{T}^d))}. \tag{6.4}$$

Proof. The result follows from Corollary 6.2 and the positivity of g. □

Step 2. We have, using the ideas from [22]:

Proposition 6.4. *Suppose that* A1–A9 *hold. Let* (u^ϵ, m^ϵ) *be a solution of* (2.4)*. Let* ρ *solve* (6.1) *with initial data* $\rho(\cdot,\tau) = \delta_{x_0}$*. Then, for* $0 < \nu < 1$*,*

$$\int_\tau^T \int_{\mathbb{T}^d} |D\rho^{\nu/2}|^2 \mathrm{d}x\,\mathrm{d}t \leq C + C\|Du^\epsilon\|^\mu_{L^\infty(0,T;L^\infty(\mathbb{T}^d))}\Big(1 + \|u^\epsilon\|_{L^\infty(0,T;L^\infty(\mathbb{T}^d))}\Big),$$

where μ *is the exponent given in Assumption* A8*.*

Proof. Multiply (6.1) by $\nu\rho^{\nu-1}$. Then,

$$\frac{\partial\rho^\nu}{\partial t} - \nu\rho^{\nu-1}\operatorname{div}(D_p H(x,Du)\rho) = \nu\rho^{\nu-1}\Delta\rho. \tag{6.5}$$

We now integrate the previous identity on $[\tau,T]\times\mathbb{T}^d$. Since $\rho(\cdot,t)$ is a probability measure and we have $0<\nu<1$, it follows that: $\int_{\mathbb{T}^d} \rho^\nu(x,t)\mathrm{d}x \leq 1$. Consequently, the integral of the first term of the left-hand side of (6.5) is bounded. We also have:

$$\begin{aligned}\left|\int_\tau^T \int_{\mathbb{T}^d} \nu\rho^{\nu-1}\operatorname{div}(D_p H(x,Du)\rho)\mathrm{d}x\mathrm{d}t\right| &= c_\nu \left|\int_\tau^T \int_{\mathbb{T}^d} \rho^{\nu/2}\rho^{\nu/2-1} D\rho D_p H \mathrm{d}x\mathrm{d}t\right| \\ &\leq \zeta \int_\tau^T \int_{\mathbb{T}^d} |D(\rho^{\nu/2})|^2 \mathrm{d}x\mathrm{d}t + C_{\zeta,\nu}\int_\tau^T \int_{\mathbb{T}^d} |D_p H|^2 \rho^\nu \mathrm{d}x\mathrm{d}t,\end{aligned}$$

for any $\zeta > 0$, with $C_{\zeta,\nu}$ depending only on ζ and ν. Because $0 < \nu < 1$, we have $\rho^\nu \leq C_\delta + \delta\rho$, for any $\delta > 0$ and suitable C_δ. Using Assumption A8, it follows from Proposition 3.2 and Corollary 6.3 that

$$C\int_\tau^T\int_{\mathbb{T}^d}|D_pH|^2\rho^\nu \mathrm{d}x\,\mathrm{d}t \leq C + C_\delta\int_\tau^T\int_{\mathbb{T}^d}|Du|^\mu H\mathrm{d}x\mathrm{d}t + \delta\int_\tau^T\int_{\mathbb{T}^d}|Du|^\mu H\rho\mathrm{d}x\mathrm{d}t$$
$$\leq C + C\|Du\|^\mu_{L^\infty(0,T;L^\infty(\mathbb{T}^d))}\Big(1+\|u\|_{L^\infty(0,T;L^\infty(\mathbb{T}^d))}\Big).$$

The integral of the right-hand side of (6.5) is

$$\nu(1-\nu)\int_\tau^T\int_{\mathbb{T}^d}|D\rho|^2\rho^{\nu-2}\mathrm{d}x\mathrm{d}t = \frac{4(1-\nu)}{\nu}\int_\tau^T\int_{\mathbb{T}^d}|D(\rho^{\nu/2})|^2\mathrm{d}x\mathrm{d}t.$$

Gathering the previous estimates, we get

$$\frac{4(1-\nu)}{\nu}\int_\tau^T\int_{\mathbb{T}^d}|D(\rho^{\nu/2})|^2\mathrm{d}x\mathrm{d}t \leq C + \zeta\int_\tau^T\int_{\mathbb{T}^d}|D(\rho^{\nu/2})|^2\mathrm{d}x\mathrm{d}t$$
$$+ C\|Du\|^\mu_{L^\infty(0,T;L^\infty(\mathbb{T}^d))}\left(1+\|u\|_{L^\infty(0,T;L^\infty(\mathbb{T}^d))}\right).$$

By choosing small enough ζ, we obtain the result. □

Step 3. To finish the proof of Theorem 6.1, we now fix a unit vector, $\xi \in \mathbb{R}^d$. We differentiate the first equation of (2.4) in the ξ direction and multiply it by ρ. Integrating by parts and using (6.1), we obtain:

$$u_\xi(x_0,\tau) = \int_\tau^T\int_{\mathbb{T}^d} -D_\xi H\rho + (g_\epsilon(m))_\xi\rho + \int_{\mathbb{T}^d}u_\xi(x,T)\rho(x,T).$$

Note that

$$\left|\int_{\mathbb{T}^d}u_\xi(x,T)\rho(x,T)\right| \leq \|u_\xi(\cdot,T)\|_{L^\infty(\mathbb{T}^d)}.$$

Using Corollary 6.3 and Assumption A5, we have

$$\int_\tau^T\int_{\mathbb{T}^d}|D_\xi H|\rho \leq C + C\int_\tau^T\int_{\mathbb{T}^d}H\rho \leq C + C\|u\|_{L^\infty(0,T;L^\infty(\mathbb{T}^d))}.$$

Thus it remains to bound

$$\int_\tau^T\int_{\mathbb{T}^d}(g_\epsilon(m))_\xi\rho. \tag{6.6}$$

This will be done in the next step.

Step 4. To bound (6.6), we integrate by parts, from which it follows that:

$$\left|\int_\tau^T\int_{\mathbb{T}^d}(g_\epsilon(m))_\xi\rho\right| \leq \int_\tau^T\int_{\mathbb{T}^d}g_\epsilon(m)\rho^{1-\beta}|\rho^{\beta-1}D\rho|$$
$$\leq C\int_\tau^T\|g_\epsilon(m)\|_a\|\rho^{1-\beta}\|_b\|D\rho^\beta\|_2,$$

for any $2 \leq a, b \leq \infty$ satisfying $\frac{1}{a}+\frac{1}{b}+\frac{1}{2} = 1$. From this, we get, for $\beta = \frac{\nu}{2}$, with $0 < \nu < 1$,

$$\left|\int_\tau^T\int_{\mathbb{T}^d}g_\epsilon(m)_\xi\rho\right| \leq C\|g(m)\|_{L^\infty(\tau,T;L^a(\mathbb{T}^d))}\|\rho^{1-\frac{\nu}{2}}\|_{L^2(\tau,T;L^b(\mathbb{T}^d))}\|D\rho^{\frac{\nu}{2}}\|_{L^2(\tau,T;L^2(\mathbb{T}^d))}.$$

From Proposition 6.4, we have a bound for $\|D\rho^{\frac{\nu}{2}}\|_{L^2(\tau,T;L^2(\mathbb{T}^d))}$. Therefore, it suffices to estimate $\|\rho^{1-\frac{\nu}{2}}\|_{L^2(\tau,T;L^b(\mathbb{T}^d))}$. We have now to estimate

$$\int_\tau^T \left(\int_{\mathbb{T}^d} \rho^{b(1-\frac{\nu}{2})}\right)^{\frac{2}{b}}.$$

Given $0<\kappa<1$, we define b by

$$\frac{1}{b(1-\frac{\nu}{2})} = 1-\kappa+\frac{\kappa}{\frac{2^*\nu}{2}}\cdot \tag{6.7}$$

We will choose κ appropriately so that $b>2$ holds. Additionally, it follows trivially from (6.7) that $1<b(1-\frac{\nu}{2})<\frac{2^*}{2}\nu$, and so by Hölder's inequality, we have:

$$\left(\int_{\mathbb{T}^d} \rho^{b(1-\frac{\nu}{2})}\right)^{\frac{1}{b(1-\frac{\nu}{2})}} \leq \left(\int_{\mathbb{T}^d}\rho\right)^{1-\kappa}\left(\int_{\mathbb{T}^d}\rho^{\frac{2^*\nu}{2}}\right)^{\frac{2\kappa}{2^*\nu}}.$$

Recall that by Sobolev's inequality, $\left(\int_{\mathbb{T}^d}\rho^{\frac{2^*\nu}{2}}\right)^{\frac{2}{2^*}} \leq C+C\int_{\mathbb{T}^d}|D\rho^{\frac{\nu}{2}}|^2$. Choose now $\kappa=\frac{\nu}{2-\nu}$. Note that if $0<\nu<1$, we have $0<\kappa<1$. Then,

$$\begin{aligned}\left\|\rho^{1-\frac{\nu}{2}}\right\|_{L^2(0,T;L^b(\mathbb{T}^d))} &\leq \left[C+C\int_0^T\int_{\mathbb{T}^d}|D\rho^{\frac{\nu}{2}}|^2\right]^{\frac{1}{2}}\\ &\leq \left[C+C\|Du\|^\mu_{L^\infty(0,T;L^\infty(\mathbb{T}^d))}\Big(1+\|u\|_{L^\infty(0,T;L^\infty(\mathbb{T}^d))}\Big)\right]^{\frac{1}{2}}.\end{aligned}$$

Also, using Proposition 6.4, we have

$$\|D\rho^{\frac{\nu}{2}}\|_{L^2(0,T;L^2(\mathbb{T}^d))} \leq \left[C+C\|Du\|^\mu_{L^\infty(0,T;L^\infty(\mathbb{T}^d))}\left(1+\|u\|_{L^\infty(0,T;L^\infty(\mathbb{T}^d))}\right)\right]^{\frac{1}{2}}.$$

It remains to check that it is possible to choose ν such that $b>2$. Indeed, for $\frac{d-1}{d}<\nu<1$, we have $\frac{d-1}{d+1}<\kappa<1$ and $b=\frac{2d}{3d-2d\nu-2}>2$. Note that a is given by $a=\frac{d}{d(\nu-1)+1}$. Thus, if $p>d$, we have, for ν close enough to 1, that $p>a$ and, therefore, this ends the proof of Theorem 6.1.

The result in Theorem 6.1 can be further simplified, as stated in Theorem 2.3. We now present its proof.

Proof of Theorem 2.3. By referring to Lemma 2.2, Theorem 6.1 becomes

$$\begin{aligned}\|Du\|_{L^\infty(\mathbb{T}^d\times[0,T)} \leq\, & C+C\|g_\epsilon\|_{L^\infty(0,T;L^p(\mathbb{T}^d))}\\ &+C\|g_\epsilon\|_{L^\infty(0,T;L^p(\mathbb{T}^d))}\|Du\|^\mu_{L^\infty(\mathbb{T}^d\times[0,T])}\\ &+C\|g_\epsilon\|_{L^\infty(0,T;L^p(\mathbb{T}^d))}\|Du\|^\mu_{L^\infty(\mathbb{T}^d\times[0,T])}\|u^\epsilon\|_{L^\infty(\mathbb{T}^d\times[0,T])}.\end{aligned}$$

Young's inequality then yields

$$\begin{aligned}\|Du\|_{L^\infty(\mathbb{T}^d\times[0,T])} \leq\, & C+C\|g_\epsilon\|_{L^\infty(0,T;L^p(\mathbb{T}^d))}+C\|g_\epsilon\|^{\frac{1}{1-\mu}}_{L^\infty(0,T;L^p(\mathbb{T}^d))}\\ &+C\|g_\epsilon\|^{\frac{1}{1-\mu}}_{L^\infty(0,T;L^p(\mathbb{T}^d))}\|u^\epsilon\|^{\frac{1}{1-\mu}}_{L^\infty(\mathbb{T}^d\times[0,T])}.\end{aligned}$$

A further application of Young's inequality implies the result. □

7. Lipschitz regularity for the Hamilton–Jacobi equation

In what follows we combine the results of Section 4 with the arguments from Section 6 to obtain the Lipschitz regularity for the Hamilton–Jacobi equation.

Lemma 7.1. *Let (u^ϵ, m^ϵ) be a solution of (2.4)–(1.2). Suppose that A1–10 hold. Let $\theta, \tilde\theta, > 1$, $0 \le \upsilon, \tilde\upsilon \le 1$. Let $r = r_\theta$, $\tilde r = r_{\tilde\theta}$ be given by (2.5) and $p_{\upsilon,\theta}$, $p_{\tilde\upsilon,\tilde\theta}$ be given by (2.6). Suppose that $p_{\upsilon,\theta} > d$, $p_{\tilde\upsilon,\tilde\theta} > \frac{d}{2}$. Then,*

$$\|Du^\epsilon\|_{L^\infty(0,T;L^\infty(\mathbb{T}^d))} \le C + C\,\|Du^\epsilon\|_{L^\infty(0,T;L^\infty(\mathbb{T}^d))}^{\frac{2(1+\mu)\alpha}{(1-\mu)\beta_0}\left(\frac{r\upsilon}{\theta}+\frac{\tilde r\tilde\upsilon}{\tilde\theta}\right)}$$

Proof. For ease of presentation, we remove the ϵ. Theorem 2.3 implies that

$$\begin{aligned}\|Du\|_{L^\infty(0,T;L^\infty(\mathbb{T}^d))} \le& C + C\|g(m)\|_{L^\infty(0,T;L^p(\mathbb{T}^d))}^{\frac{1}{1-\mu}}\\ &+ C\|g(m)\|_{L^\infty(0,T;L^p(\mathbb{T}^d))}^{\frac{1}{1-\mu}}\|u\|_{L^\infty(0,T;L^\infty(\mathbb{T}^d))}^{\frac{1}{1-\mu}}.\end{aligned}$$

Because $\tilde p > \frac{d}{2}$, we have from Lemma 2.2 that

$$\|u\|_{L^\infty(0,T;L^\infty(\mathbb{T}^d))} \le C + C\,\|g(m)\|_{L^\infty(0,T;L^{\tilde p}(\mathbb{T}^d))}.$$

By combining these, we obtain

$$\begin{aligned}\|Du\|_{L^\infty(0,T;L^\infty(\mathbb{T}^d))} \le& C + C\,\|g(m)\|_{L^\infty(0,T;L^p(\mathbb{T}^d))}^{\frac{1}{1-\mu}}\\ &+ C\,\|g(m)\|_{L^\infty(0,T;L^p(\mathbb{T}^d))}^{\frac{1}{1-\mu}}\,\|g(m)\|_{L^\infty(0,T;L^{\tilde p}(\mathbb{T}^d))}^{\frac{1}{1-\mu}}.\end{aligned}$$

From Theorem 2.1, it follows that

$$\|Du\|_{L^\infty(0,T;L^\infty(\mathbb{T}^d))} \le C + C\,\|Du\|_{L^\infty(0,T;L^\infty(\mathbb{T}^d))}^{\frac{2(1+\mu)\alpha}{(1-\mu)\beta_0}\left(\frac{r\upsilon}{\theta}+\frac{\tilde r\tilde\upsilon}{\tilde\theta}\right)},$$

which establishes the result. □

Proposition 7.2. *Let (u^ϵ, m^ϵ) be a solution of (2.4)–(1.2). Assume that A1–10 hold. Let $\theta, \tilde\theta > 1$, $0 \le \upsilon, \tilde\upsilon \le 1$. Let $r = r_\theta$, $\tilde r = r_{\tilde\theta}$ be given by (2.5) and $p_{\upsilon,\theta}$, $p_{\tilde\upsilon,\tilde\theta}$ be given by (2.6). Suppose that $p_{\upsilon,\theta} > d$ and $p_{\tilde\upsilon,\tilde\theta} > \frac{d}{2}$ and that (7.2) is satisfied. Then, $Du^\epsilon \in L^\infty(\mathbb{T}^d \times [0,T])$.*

Proof. Lemma 7.1 ensures that

$$\|Du^\epsilon\|_{L^\infty(0,T;L^\infty(\mathbb{T}^d))} \le C + C\,\|Du^\epsilon\|_{L^\infty(0,T;L^\infty(\mathbb{T}^d))}^{\frac{2(1+\mu)\alpha}{(1-\mu)\beta_0}\left(\frac{r\upsilon}{\theta}+\frac{\tilde r\tilde\upsilon}{\tilde\theta}\right)}.$$

Because of (7.2), the result follows using Young's inequality. □

The results in this section strongly depend on several constraints for the parameters in the various estimates. It is critical to ensure that this set of constraints can be mutually satisfied. This is done in the following lemma:

Lemma 7.3. *If $0 \le \mu < 1$, $d > 2$ and*

$$\alpha < \frac{2}{\mathrm{d}(1+\mu)-2}, \tag{7.1}$$

then there exist $\beta_0 \in \left[1, \frac{\mathrm{d}(1+\mu)}{\mathrm{d}(1+\mu)-2}\right)$, $1 < \theta, \tilde\theta$ and $0 \le \upsilon, \tilde\upsilon \le 1$ such that for $r = r_\theta$, $\tilde r = r_{\tilde\theta}$ given by (2.5) and $p = p_{\upsilon,\theta}$, $\tilde p = p_{\tilde\upsilon,\tilde\theta}$ given by (2.6), we have that $p > d$, $\tilde p > \frac{d}{2}$ and

$$\frac{2(1+\mu)\alpha}{(1-\mu)\beta_0}\left(\frac{r\upsilon}{\theta}+\frac{\tilde r\tilde\upsilon}{\tilde\theta}\right) < 1 \tag{7.2}$$

are satisfied.

Proof. To simplify, we introduce the variables $M = 1 + \mu$ and $w = \alpha(dM - 2)$. Because $0 < \mu < 1$, we have $1 \leq M < 2$. According to (7.1), $0 \leq w < 2$. Moreover, $\beta_0 = \frac{dM\lambda}{dM-2}$, for some $0 < \lambda < 1$.

We have

$$p = \frac{dM\theta\lambda}{(\theta + \upsilon - \theta\upsilon)w} \quad \text{and} \quad \tilde{p} = \frac{dM\tilde{\theta}\lambda}{(\tilde{\theta} + \tilde{\upsilon} - \tilde{\theta}\tilde{\upsilon})w}.$$

Since $2M \geq 2$ and $w < 2$, one easily verifies that $2M\tilde{\theta} > (\tilde{\theta} + \tilde{\upsilon} - \tilde{\theta}\tilde{\upsilon})w$; hence, for λ sufficiently close to 1, we have $\tilde{p} > \dfrac{d}{2}$.

Inequality (7.2) becomes

$$\left((2 + d(\theta - 1))\frac{\upsilon}{\theta} + (2 + d(\tilde{\theta} - 1))\frac{\tilde{\upsilon}}{\tilde{\theta}}\right) w < \lambda d(2 - M). \tag{7.3}$$

Because $d > 2$, we also have $2 + d(\theta - 1) \leq d\theta$, $2 + d(\tilde{\theta} - 1) \leq d\tilde{\theta}$. Therefore,

$$\Big((2 + d(\theta - 1))\frac{\upsilon}{\theta} + (2 + d(\tilde{\theta} - 1))\frac{\tilde{\upsilon}}{\tilde{\theta}}\Big) w < d(\upsilon + \tilde{\upsilon})w.$$

Consequently, if there are $\theta > 1$, υ, $\tilde{\upsilon} \in (0, 1)$ such that

$$M\theta > (\theta + \upsilon - \theta\upsilon)w, \tag{7.4}$$
$$(\upsilon + \tilde{\upsilon})w < 2 - M, \tag{7.5}$$

then, for λ sufficiently close to 1, we have $p > d$, $\tilde{p} > \frac{d}{2}$ and (7.3).

Since $2 - M \leq M < 2$, we consider 3 cases:

Case I. $w \leq 2 - M$**.** For $\upsilon < 1$

$$M + (\upsilon - 1)w \geq M + (\upsilon - 1)(2 - M) = \upsilon(2 - M) + 2(M - 1) \geq (2 - M)\upsilon.$$

For $\theta > 1$,

$$\theta(M + (\upsilon - 1)w) \geq \theta(2 - M)\upsilon > (2 - M)\upsilon \geq w\upsilon,$$

which gives (7.4).

For $\tilde{\upsilon} < 1 - \upsilon$, $(\upsilon + \tilde{\upsilon})w < w \leq 2 - M$.

Case II. $2 - M < w \leq M$**.** Fix $\nu > 0$ such that $0 < \upsilon w < 2 - M$. For $\theta > 1$,

$$\Big(\frac{M - w}{\upsilon w} + 1\Big)\theta > \Big(\frac{M - w}{2 - M} + 1\Big)\theta = \frac{(2 - w)\theta}{2 - M} \geq \theta > 1,$$

which gives (7.4).

Choosing $\tilde{\upsilon} < \dfrac{2 - M}{w} - \upsilon$, we get (7.5).

Case III. $M < w < 2$**.** For $w - M < \upsilon w < (2 - M)$, we get (7.4) by selecting

$$\theta > \frac{\upsilon w}{M - w + \upsilon w}.$$

Choosing $\tilde{\upsilon} < \dfrac{2 - M}{w} - \upsilon$, we get (7.5). □

Now, we close the paper with the Proof of Theorem 2.4:

Proof of Theorem 2.4. It remains to check that (2.5) as well as (2.6) and (7.2) hold simultaneously. In fact, under A10, this simultaneous condition follows from Lemma 7.3. □

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