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TIME DELAY IN OPTIMAL CONTROL LOOPS FOR WAVE EQUATIONS

Martin Gugat[1] and Günter Leugering[1]

Abstract. In optimal control loops delays can occur, for example through transmission *via* digital communication channels. Such delays influence the state that is generated by the implemented control. We study the effect of a delay in the implementation of L^2-norm minimal Neumann boundary controls for the wave equation. The optimal controls are computed as solutions of problems of exact optimal control, that is if they are implemented without delay, they steer the system to a position of rest in a given finite time T. We show that arbitrarily small delays $\delta > 0$ can have a destabilizing effect in the sense that we can find initial states such that if the optimal control u is implemented in the form $y_x(t,1) = u(t-\delta)$ for $t > \delta$, the energy of the system state at the terminal time T is almost twice as big as the initial energy. We also show that for more regular initial states, the effect of a delay in the implementation of the optimal control is bounded above in the sense that for initial positions with derivatives of BV-regularity and initial velocities with BV-regularity, the terminal energy is bounded above by the delay δ multiplied with a factor that depends on the BV-norm of the initial data. We show that for more general hyperbolic optimal exact control problems the situation is similar. For systems that have arbitrarily large eigenvalues, we can find terminal times T and arbitrarily small time delays δ, such that at the time $T+\delta$, in the optimal control loop with delay the norm of the state is twice as large as the corresponding norm for the initial state. Moreover, if the initial state satisfies an additional regularity condition, there is an upper bound for the effect of time delay of the order of the delay with a constant that depends on the initial state only.

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1. Introduction

The sensitivity of optimal control problems to time delay is important for the application of optimal controls, since in the implementation of the controls, a delay can occur. Such a delay can be caused by the transmission of information or by computing times that are necessary in order to determine the optimal control. There are some studies where the delay is given by a number that appears in the definition of the optimization problem and is known *a priori*, for example [5,11] where systems with retarded ode dynamics are considered. The sensitivity of the optimal value as a parameter of the delay for a problem with retarded ode dynamics is studied in [24] and a limit model is introduced. The optimal control of a delay PDE is studied in chapter 10 of [4] using the method of direct transcription (see also the references therein). A sensitivity analysis for optimal control problems for

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[1] Department of Mathematics, Friedrich-Alexander Universität Erlangen-Nürnberg, Cauerstr. 11, 91058 Erlangen, Germany.
martin.gugat@fau.de; guenter.leugering@fau.de

parabolic-hyperbolic systems in which time delays appear both in the state equations and in the Neumann boundary conditions is given in [9].

In this paper, we take a different point of view and study the effect of a delay that occurs as a perturbation in the implementation of an optimal control and is not known *a priori*. The optimal control is obtained as the solution of an optimal control problem where the delay does not appear. We analyze the effect of a delay in the implementation of this optimal control in the sense that the optimal control information is inserted into the system with a certain delay.

In the first part of the paper we consider a system that is governed by the 1-d wave equation. We study the problem of L^2-norm minimal exact Neumann boundary control to a position of rest in a given finite time. In this optimal control problem, the desired terminal state is prescribed exactly which is possible on account of the exact controllability of the system (for exact controllability results based upon trigonometric moment problems, see for example [20]). The resulting optimal control is implemented with a delay $\delta > 0$. To make this possible, on the first part $[0, \delta]$ of the time-interval the control is switched off.

It is easy to see that for a fixed initial state, the error in the terminal state tends to zero as the delay goes to zero. However we show that for arbitrarily small delays there exists initial states such that if the optimal control is implemented with such a delay, the corresponding terminal energy is larger than the initial energy. It is well–known that also for the corresponding system with Neumann feedback boundary action, arbitrarily small delays have disastrous effects and cause instability, see for example [7,23]. In [7], theorem (iii) it is stated that for $k > 0$ and time delays $\delta > 0$, for the closed loop system

$$(\mathbf{FL})(k, y_0, y_1, \delta) \begin{cases} (y(0,\cdot), y_t(0,\cdot)) = (y_0,\ y_1) \text{ in } H^1(0,1)\ \times\ L^2(0,1) \\ y(\cdot, 0) = 0,\ t > 0 \\ y_x(t,1) = 0,\ t \in (0,\ \delta);\ y_x(t,\ 1) = -k\ y_t(t-\delta,\ 1),\ \ t > \delta \\ y_{tt} = y_{xx} \text{ in } (0,\infty) \times (0,1) \end{cases} \tag{1.1}$$

there is a dense open set D in $(0, \infty)$ such that for every δ in D the system admits exponentially unstable solutions. Recently there have also been some results of a different type about certain delays that do not destabilize this system. For example in [15] it has been shown that for k in a certain subinterval of $(0, 2/3)$ and δ switching between 4 and 8, the system is exponentially stable (see also [29]). Sufficient conditions for stability for a linear combination of controls with and without delay are given in [10, 25–27]. The effects of time delay in the implementation of optimal exact Dirichlet controls for the vibrating string are considered in [17], see also [14].

In this paper we show that for more regular initial states that satisfy an additional regularity condition, a delay in the implementation of the optimal control does not have such a destabilizing effect. We show that for initial positions with derivatives of bounded variation and initial velocities with bounded variation, there is an upper bound for the terminal energy in terms of the BV-norms of the initial state and the delay.

In the second part of the paper we show that for more general hyperbolic optimal exact control problems, the situation is similar. We show that if the system has an unbounded sequence of eigenvalues, there exist arbitrarily small time delays, such that for suitably chosen initial states, the effect of time–delay is large in the sense that at the time that is given by the sum of the prescribed terminal time and the delay, a certain norm of the state has doubled compared with the same norm for the initial state. Moreover, we show that if the initial state satisfies an additional regularity condition, there is an upper bound for the effect of time delay of the order of the delay. Our regularity assumption essentially requires that the coefficients in the eigenfunction expansions of the initial data are contained in a certain weighted l^1-space.

This paper has the following structure. In Section 2 we define the optimal control problem for the vibrating string. In Section 2.1 we define the system with time delay. In Section 2.2 we give an example for the destabilizing effect of time delay. In Section 2.3 we present a bound for the quotient of the terminal and the initial energy $E(T)/E(0)$ under a BV regularity assumption for the initial states. To prove the statements, we analyze the initial-boundary value problem for the vibrating string in Section 2.4. In Section 2.5, we present the optimal control for the vibrating string. Section 2.6 contains the proofs of our results on the effect of time-delay in the implementation of the optimal control for the vibrating string. Section 3 contains the statement of a more general hyperbolic optimal exact control problem. The corresponding results on the effect of time-delay in the general case are given in Section 3.1.

2. The optimal control problem for the vibrating string

Let a terminal time $T \geq 2$ and an initial state (y_0, y_1) with $y_0 \in H^1(0,1)$, $y_0(0) = 0$ and $y_1 \in L^2(0,1)$ be given.

For a given initial state and terminal time the optimal control

$$u(\cdot, T, y_0, y_1) \in L^2(0,T)$$

is defined as the solution of the problem of optimal exact control

$$(\mathbf{EC})(y_0, y_1, T) \begin{cases} \text{minimize}_{u \in L^2(0,T)} \ \|u\|^2_{L^2(0,T)} \text{ subject to} \\ (y(0,\cdot), y_t(0,\cdot)) = (y_0,\ y_1) \text{ in } H^1(0,1) \ \times\ L^2(0,1) \\ y(\cdot,0) = 0,\ y_x(\cdot,1) = u \ \text{ in } L^2(0,T) \\ y_{tt}(t,x) = y_{xx}(t,x),\ (t,x) \in (0,T) \times (0,1) \\ (y(T,\cdot),\ y_t(T,\cdot)) = 0 \ \text{ in } H^1(0,1) \ \times\ L^2(0,1). \end{cases} \tag{2.1}$$

The objective in problem $(\mathbf{EC})$ is to steer the system to a position of rest with minimal control cost that is given by the L^2-norm of the control function.

2.1. Definition of the system with time-delay for the vibrating string

In this section, we determine the effect of time delay $\delta > 0$, if for $t \geq \delta$, the optimal boundary control $u(\cdot, T, y_0, y_1)$ is implemented in the from

$$y_x(t,1) = u(t - \delta, T, y_0, y_1). \tag{2.2}$$

For $t \in (0, \delta)$, we prescribe the boundary condition

$$y_x(t,1) = 0. \tag{2.3}$$

Then the corresponding state is generated as the solution of the initial boundary value problem

$$(\mathbf{IBVP})(y_0, y_1, T, \delta) \begin{cases} (y(0,\cdot), y_t(0,\cdot)) = (y_0,\ y_1) \text{ in } H^1(0,1) \ \times\ L^2(0,1) \\ y(\cdot,0) = 0 \ \text{ in } L^2(0,T) \\ y_x(\cdot,1) = 0 \text{ in } L^2(0,\delta) \\ y_x(\cdot,1) = u(\cdot - \delta, T, y_0, y_1) \ \text{ in } L^2(\delta,T) \\ y_{tt}(t,x) = y_{xx}(t,x),\ (t,x) \in (0,T) \times (0,1). \end{cases} \tag{2.4}$$

$(\mathbf{IBVP})(y_0, y_1, T, \delta)$ determines the state that is generated by an optimal control loop where the delay δ occurs in the implementation of the optimal control. Define the energy at the time t for the state y as

$$E(t) = \int_0^1 y_x(t,x)^2 + y_t(t,x)^2 \, \mathrm{d}x. \tag{2.5}$$

In order to analyze the effect of the time delay in the implementation of the optimal control, for initial states with non-zero energy we look at the quotient of the initial energy and the terminal energy, that is

$$\frac{E(T)}{E(0)}\cdot \tag{2.6}$$

2.2. An example for the destabilizing effect of time delay

Let $\delta \in (0, 1/2)$ be given. As an example for the destabilizing effect of time delay in the implementation of the optimal control, we consider the initial state that is given by the initial position

$$y_0(x) = \frac{2\delta}{\pi}\,\sin\left(\frac{\pi}{2\delta}x\right) \tag{2.7}$$

and the initial velocity $y_1(x) = 0$. Then we have $y_0(0) = 0$ and $y_0 \in H^1(0, 1)$. The energy of the initial state is

$$E(0) = \int_0^1 (y_0')^2(x)\,\mathrm{d}x = \int_0^1 \cos^2\left(\frac{\pi}{2\delta}x\right)\,\mathrm{d}x = \frac{1}{2}\left[1 + \frac{\delta}{\pi}\,\sin\left(\frac{\pi}{\delta}\right)\right]\cdot \tag{2.8}$$

Proposition 2.1. *Let $\delta \in (0, 1/2)$, $y_1 = 0$ and y_0 as in (2.7) be given. Let y denote the solution of* (**IBVP**)$(y_0,\, y_1,\, T,\, \delta)$ *for $T = 2K$ ($K \in \{1, 2, 3, \ldots\}$). Then for the corresponding energies we have the inequality*

$$\frac{E(T)}{E(0)} \geq \frac{2 - 4\,\delta}{1 + \frac{1}{\pi}\,\delta\,\sin\left(\frac{\pi}{\delta}\right)}\cdot \tag{2.9}$$

The proof of Proposition 2.1 is given in Section 2.6. It requires several intermediate results that are given starting from Section 2.4.

Remark 2.2. Consider the state of the system (**IBVP**)$(y_0,\, y_1,\, T,\, \delta)$ that is generated by the optimal control that is implemented with a delay $\delta > 0$. Proposition 2.1 states that for arbitrarily small time delays $\delta > 0$, we can find initial states (y_0, y_1) with $y_1 = 0$ and $y_0 \in H^1(0, 1)$, such that at the time $T = 2K$ for all $K \in \{1, 2, 3, \ldots\}$ the energy $E(T)$ is greater than the energy of the initial state. More precisely, inequality (2.9) implies that for

$$\delta \in \left(0,\ \frac{1}{4 + 1/\pi}\right),$$

we have $E(T) > E(0)$, that is the energy is increased by the control. Moreover, for all factors $\lambda \in (1,\, 2)$, if $\delta > 0$ is sufficiently small we get

$$E(T) \geq \lambda\, E(0).$$

Figures 1 and 2 illustrate the destabilizing effect of the time delay in the optimal control loop. Figure 1 shows for $\delta = 1/4$ and $T = 4$ the state that is generated by the optimal control $u(\cdot, T, y_0, y_1)$ that is implemented without delay and steers the system to a position of rest at $T = 4$. Figure 2 shows the state that is generated by the same control that is implemented with delay $\delta = 1/4$ as in (2.2) and (2.3).

2.3. A bound for $E(T)/E(0)$ for more regular initial states

For initial states (y_0, y_1) that satisfy more than the minimal regularity assumptions from Section 2, the corresponding terminal energies are robust against time-delay in the implementation of the optimal controls. More precisely for initial states that satisfy the additional assumption that the initial velocity and the derivative of the initial position are given by functions of bounded variation, that is

$$y_0',\ y_1 \in BV(0, 1), \tag{2.10}$$

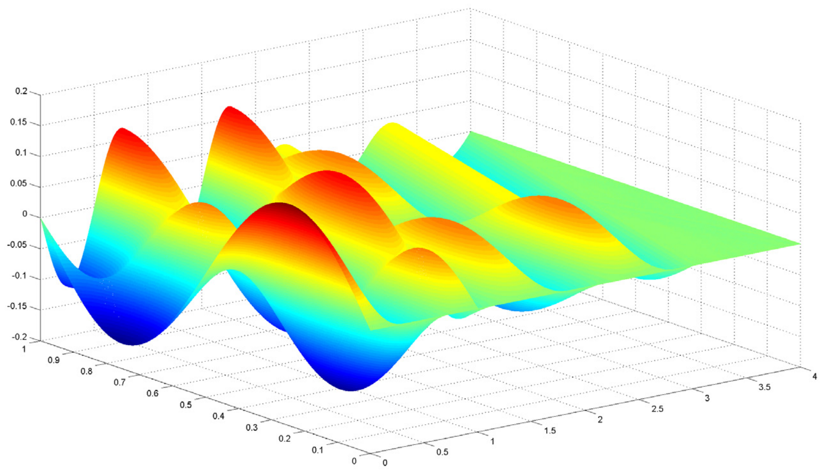

Figure 1. Implementation without delay.

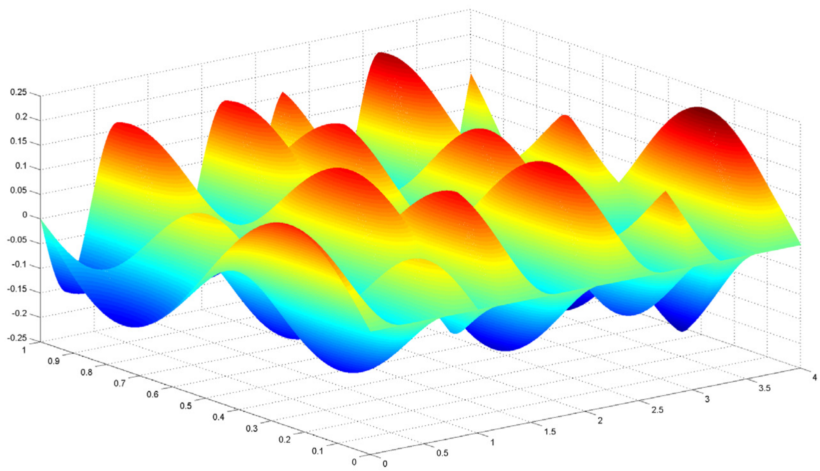

Figure 2. Implementation with delay $\delta = \frac{1}{4}$.

we give an upper bound for the terminal energy by the time delay multiplied with a quadratic term that depends on the initial state. The additional regularity requirements for the initial state that ensure robustness of the optimal controls against certain perturbations by time delay are related to the approach in [16], where a regularization term is added in the objective function in order to make the optimal controls robust against perturbations of the domain of the type that occurs in the process of discretization. In [13], the L^2-norm of the derivative of the control u is used as regularization term in the objective function. Note that in [13, 16], we have considered Dirichlet boundary control without delay whereas in this paper, we consider Neumann boundary control with time delay. For a function $y \in BV(0,1)$, define

$$D(y) = \lim_{h\to 0+} \int_0^{1-h} \frac{1}{h} \left|y(x+h) - y(x)\right| \, \mathrm{d}x.$$

Then Theorem 13.48 from [21] states that $D(y) < \infty$. Note that $BV(0,1)$ is embedded in $L^\infty(0,1)$ as stated in [1], Corollary 3.49.

Theorem 2.3. *Let $y_0 \in H^1(0,1)$ with $y_0(0) = 0$ and $y_0' \in BV(0,1)$ and $y_1 \in BV(0,1)$ be given. Let a terminal time $T = 2K$ with $K \in \{1, 2, 3, \dots\}$ be given.*

Then for the optimal control $u(\cdot, T, y_0, y_1) \in L^2(0,T)$ that solves $(\mathbf{EC})(y_0, y_1, T)$. we have

$$u(\cdot, T, y_0, y_1) \in BV(0,T).$$

If the optimal control is implemented with time-delay $\delta \in (0,1)$ as in (2.2) and (2.3), the energy at time T of the generated state that solves $(\mathbf{IBVP})(y_0,\, y_1,\, T, \delta)$ satisfies the inequality

$$E(T) = \int_0^1 (y_x(T,x))^2 + (y_t(T,x))^2 \, \mathrm{d}x$$

$$\leq 8\,\delta \left[2 \left(\|y_1\|_{L^\infty(0,1)} + \|y_0'\|_{L^\infty(0,1)}\right)^2 + \left(\|y_0'\|_{L^\infty(0,1)}\, D(y_0') + \|y_1\|_{L^\infty(0,1)}\, D(y_1)\right)\right]. \tag{2.11}$$

The proof of Theorem 2.3 is given in Section 2.6. As intermediate steps it requires results about the well-posedness and the solution of the optimal control problem, that are given in the next sections.

Remark 2.4. For the example considered in Section 2.2, that is y_0 as defined in (2.7), we have

$$|D(y_0')| \leq \frac{\pi}{2\delta}.$$

Thus with $y_1 = 0$ inequality (2.11) from Theorem 2.3 yields the upper bound

$$E(T) \leq 16\,\delta + 4\,\pi.$$

On the other hand inequality (2.9) from Proposition 2.1 and (2.8) provide the lower bound

$$E(T) \geq \frac{1 - 2\,\delta}{1 + \frac{1}{\pi}\,\delta\,\sin\left(\frac{\pi}{\delta}\right)} \left[1 + \frac{\delta}{\pi}\,\sin\left(\frac{\pi}{\delta}\right)\right].$$

The fact that for initial data with the BV-regularity (2.10) the optimal controls are in $BV(0,T)$ implies the following exact controllability result for the wave equation in the BV-framework.

Corollary 2.5. *Let $T \geq 2$ be a real number (not necessarily of the form $T = 2K$).*

For a given initial state (y_0, y_1) with $y_0 \in H^1(0,1)$, $y_0(0) = 0$, $y_0' \in BV(0,1)$ and $y_1 \in BV(0,1)$ there exists a control function $u \in BV(0,T)$ that steers the system

$$\begin{cases} (y(0,\cdot), y_t(0,\cdot)) = (y_0,\ y_1) \text{ in } H^1(0,1) \ \times\ L^2(0,1), \\ y(t,\,0) = 0 \text{ for } t \in (0,T), \\ y_x(t,\,1) = u(t) \ \text{ for } t \in (0,T), \\ y_{tt} = y_{xx} \text{ in } (0,T)\times(0,1) \end{cases} \tag{2.12}$$

to the position of rest at time T, that is $y(T,x) = 0$, $y_t(T,x) = 0$ for $x \in (0,L)$.

The proof of Corollary 2.5 is given in Section 2.6.

2.4. The initial-boundary value problem for the vibrating string

In order to determine the optimal controls, we first look at the solution of the initial-boundary value problem that appears in the optimal control problem $(\mathbf{EC})(y_0, y_1, T)$.

Let $y_0 \in H^1(0,1)$ with $y_0(0) = 0$ and $y_1 \in L^2(0,1)$ be given. Let a control time $T \geq 2$ be given. Let a control $u \in L^2(0,T)$ be given. We consider the initial boundary value problem

$$\begin{cases} y(0,x) = y_0(x),\ y_t(0,x) = y_1(x),\ x \in (0,1), \\ y(t,0) = 0,\ y_x(t,1) = u(t),\, t \in (0,T), \\ y_{tt}(t,x) = y_{xx}(t,x),\ (t,x) \in (0,T) \times (0,1). \end{cases} \tag{2.13}$$

We are looking for a solution of our initial boundary value problem of the form

$$y(t,x) = \alpha(x+t) + \beta(x-t),$$

where the functions α and β are determined from the initial data and the boundary data. For $t \in (0,1)$ the initial conditions imply the equations

$$\alpha(t) = \frac{1}{2}\left(y_0(t) + \int_0^t y_1(s)\,\mathrm{d}s\right) + C_0, \tag{2.14}$$

$$\beta(t) = \frac{1}{2}\left(y_0(t) - \int_0^t y_1(s)\,\mathrm{d}s\right) - C_0 \tag{2.15}$$

with a real constant C_0.

The boundary condition $y(t,0) = 0$ implies that for $t > 0$ we have
$0 = \alpha(t) + \beta(-t)$, thus for all $s < 0$ we have

$$\beta(s) = -\alpha(-s). \tag{2.16}$$

For the partial derivatives of y we get

$$y_x(t,x) = \alpha'(t+x) + \beta'(x-t), \tag{2.17}$$

$$y_t(t,x) = \alpha'(t+x) - \beta'(x-t). \tag{2.18}$$

Thus the boundary condition at $x = 1$ yields

$$y_x(t,1) = \alpha'(1+t) + \beta'(1-t) = u(t).$$

For α', this yields the equation

$$\alpha'(1+t) = u(t) - \beta'(1-t).$$

By integration from 0 to t we get

$$\alpha(1+t) - \alpha(1) = \beta(1-t) - \beta(1) + \int_0^t u(s)\,\mathrm{d}s.$$

Thus we have

$$\alpha(t+1) = \beta(1-t) + \int_0^t u(s)\,\mathrm{d}s + [\alpha(1) - \beta(1)].$$

Note that by (2.14) and (2.15) we have

$$\alpha(1) - \beta(1) = \int_0^1 y_1(s)\,\mathrm{d}s + 2\,C_0.$$

With the choice

$$C_0 = -\frac{1}{2}\int_0^1 y_1(s)\,\mathrm{d}s$$

we get $\alpha(1) - \beta(1) = 0$ and

$$\alpha(t+1) = \beta(1-t) + \int_0^t u(s)\,\mathrm{d}s. \tag{2.19}$$

With the values of α for $t \in (0,1)$ from (2.14), equation (2.16) yields $\beta|_{(-1,0)}$. For $t \in (0,1)$, we have the values of β for from (2.15). With the known values of $\beta|_{(-1,1)}$, equation (2.19) determines the values of α on the interval $(1,3)$.

By (2.19) and(2.16) we get for $t \geq 1$

$$\alpha(t+1) = -\alpha(t-1) + \int_0^t u(s)\,\mathrm{d}s$$

or equivalently for $t \geq 0$

$$\alpha(t+2) = -\alpha(t) + \int_0^{t+1} u(s)\,\mathrm{d}s. \tag{2.20}$$

Equation (2.20) allows to determine α recursively: Starting with $\alpha|_{(1,3)}$, using the values of $u(t)$ we get $\alpha|_{(3,5)}$. Then in turn (2.20) yields $\alpha|_{(5,7)}$ *etc.*

Now we express the construction completely in terms of α only without using β. For this purpose, we extend the domain of α to include also $(-1,0)$. We define $\alpha|_{(-1,0)}$ by the equation $\alpha(t) = -\beta(-t)$ for $t \in (-1,0)$, with the values of $\beta|_{(0,1)}$ defined by (2.15). Note that this definition of $\alpha|_{(-1,0)}$ is compatible with (2.16). Then for $t \in (-1,\,0)$, equation (2.19) yields $\alpha(t+2) = -\alpha(t) + \int_0^{t+1} u(s)\,\mathrm{d}s$ and as above for larger values of t α is defined recursively by (2.20).

In the following Lemma, we summarize our construction of α.

Lemma 2.6. *Let $y_0 \in H^1(0,1)$ with $y_0(0) = 0$ and $y_1 \in L^2(0,1)$ be given. Let*

$$C_0 = -\frac{1}{2}\int_0^1 y_1(s)\,\mathrm{d}s. \tag{2.21}$$

Define $\alpha \in L^2(-1,1)$ as

$$\alpha(t) = \frac{1}{2}\left(-y_0(-t) + \int_0^{-t} y_1(s)\,\mathrm{d}s\right) + C_0 \text{ for } \ t \in (-1,0), \tag{2.22}$$

$$\alpha(t) = \frac{1}{2}\left(y_0(t) + \int_0^{t} y_1(s)\,\mathrm{d}s\right) + C_0 \text{ for } \ t \in [0,1). \tag{2.23}$$

Let a natural number $K \in \{1,2,3,\ldots\}$, $T = 2K$ and a control $u \in L^2(0,T)$ be given.

For a natural number $k \in \{1,2,3,\ldots\}$ and $t \in (-1,1)$ with $t < T+1-2k$ define the values of α recursively by the equation

$$\alpha(t+2k) = -\alpha(t+2(k-1)) + \int_0^{t+2k-1} u(s)\,\mathrm{d}s. \tag{2.24}$$

Then α is well-defined on the interval $(-1,T+1)$ and $\alpha \in H^1(-1,T+1)$.

Proof. From the construction it is clear that $\alpha|_{(k-1,k)} \in H^1(k,k+1)$ for all $k \in \{0,1,2,3,\ldots\}$. To show that $\alpha \in H^1(-1,T+1)$, it suffices to show that α is continuous. Since $\alpha(0+) = \alpha(0-) = C_0$, α is continuous at $t = 0$. Due to the definition of α by (2.22) and (2.23) this implies that α is continuous on $(-1,1)$.

At $t = 1$, due to the definition of the constant C_0 in (2.21) we have $\alpha(1+) = \alpha(1-) = \frac{1}{2}y_0(1)$. Thus α is continuous at $t = 1$. This yields the continuity of α on $(-1, 3)$.

We proceed by induction. Let $k \in \{1, 2, 3, \ldots\}$. Suppose that α is continuous on the interval $(-1, 1 + 2k)$. Then

$$\alpha((-1 + 2k)-) = \alpha((-1 + 2k)+).$$

By (2.24), we have

$$\begin{aligned}
\alpha((1 + 2k)-) &= -\alpha((1 + 2(k - 1))-) + \int_0^{2k} u(s)\,\mathrm{d}s \\
&= -\alpha((-1 + 2k)-) + \int_0^{2k} u(s)\,\mathrm{d}s \\
&= -\alpha((-1 + 2k)+) + \int_0^{2k} u(s)\,\mathrm{d}s \\
&= \alpha((-1 + 2(k + 1))+) \\
&= \alpha((1 + 2k)+).
\end{aligned}$$

Due to the definition of α by (2.22), (2.23) this implies that α is continuous on $(-1, 1 + 2(k + 1))$ as long as $k + 1 \leq K$. Thus we have proved Lemma 2.6.

Now we can use the function α as defined in Lemma 2.6 to obtain a representation of the initial boundary value problem (2.13).

Theorem 2.7. *Let $y_0 \in H^1(0, 1)$ with $y_0(0) = 0$ and $y_1 \in L^2(0, 1)$ be given. Let a natural number $K \in \{1, 2, 3, \ldots\}$, $T = 2K$ and a control $u \in L^2(0, T)$ be given.*

Let α be as defined in Lemma 2.6. Then the solution of the initial boundary value problem (2.13) is given by

$$y(t, x) = \alpha(t + x) - \alpha(t - x),\ (t, x) \in (0, T) \times (0, 1). \tag{2.25}$$

For $x \in (0, 1)$, the position at the time T is given by

$$\begin{aligned}
y(T,\, x) &= \alpha(T + x) - \alpha(T - x) &(2.26)\\
&= (-1)^K y_0(x) - \sum_{k=1}^{K} (-1)^k \int_{1-x}^{1+x} u(2(K - k) + s)\,\mathrm{d}s. &(2.27)
\end{aligned}$$

For the corresponding partial space derivative, we get

$$y_x(T,\, x) = (-1)^K y_0'(x) - \sum_{k=0}^{K-1} (-1)^{K-k} \left[u(2k + (1 + x)) + u(2k + (1 - x))\right]. \tag{2.28}$$

For $x \in (0, 1)$, the velocity at the time T is given by

$$y_t(T,\, x) = (-1)^K y_1(x) - \sum_{k=0}^{K-1} (-1)^{K-k} \left[u(2k + (1 + x)) - u(2k + (1 - x))\right]. \tag{2.29}$$

Proof. The construction of α implies that y as defined in (2.25) is a solution of the initial boundary value problem (2.13). For $t = 2k_0$ with $k_0 \in \{1, 2, 3, \ldots, K\}$ due to (2.24) we have

$$\begin{aligned}
y(2k_0,\, x) &= \alpha(2k_0 + x) - \alpha(2k_0 - x) \\
&= -\left[\alpha(2(k_0 - 1) + x) - \alpha(2(k_0 - 1) - x)\right] + \int_{2k_0-1-x}^{2k_0-1+x} u(s)\,\mathrm{d}s \\
&= -y(2(k_0 - 1), x) + \int_{2(k_0-1)+1-x}^{2(k_0-1)+1+x} u(s)\,\mathrm{d}s.
\end{aligned}$$

Thus for $k_0 \geq 2$ we get

$$y(2(k_0-1),x) = -y(2(k_0-2),x) + \int_{2(k_0-2)+1-x}^{2(k_0-2)+1+x} u(s)\,\mathrm{d}s$$

and

$$y(2k_0,\, x) = y(2(k_0-2),x) - \int_{2(k_0-2)+1-x}^{2(k_0-2)+1+x} u(s)\,\mathrm{d}s + \int_{2(k_0-1)+1-x}^{2(k_0-1)+1+x} u(s)\,\mathrm{d}s.$$

For $T = 2$ K this yields by induction

$$y(2K,\, x) = (-1)^K y_0(x) - \sum_{k=1}^{K} (-1)^k \int_{2(K-k)+1-x}^{2(K-k)+1+x} u(s)\,\mathrm{d}s$$

and (2.27) follows. Equation (2.28) follows by taking the derivative with respect to x and an index transformation in the sum. Using a similar proof by induction we get (2.29). □

2.5. The optimal control for the vibrating string

In this section we present an explicit representation of the solution of problem **(EC)** that is, we give the optimal control map that maps the initial state and the terminal time T to the optimal control. For general control times $T \geq 2$ and general L^p-norms of the control with $p \geq 2$ in the objective function, this result is proved in [12] using the method of moments.

Theorem 2.8. *Let $T \geq 2$ be given. Define $k = \max\{n \in \mathbb{N} : 2\,n \leq T\}$ and $\Delta = T - 2k$. For $t \in [0,2)$, define the function*

$$d(t) = \begin{cases} k+1, & t \in (0,\Delta], \\ k, & t \in (\Delta, 2). \end{cases} \tag{2.30}$$

Then the optimal control $u_0 = u(\cdot,\, T, y_0, y_1)$ that solves problem **(EC)** *is 4–periodic, with*

$$u_0(t) = \begin{cases} \frac{1}{2\,d(t)}\left[y_0'(1-t) - y_1(1-t)\right], & t \in (0,1), \\ \\ \frac{1}{2\,d(t)}\left[y_0'(t-1) + y_1(t-1)\right], & t \in (1,2). \end{cases} \tag{2.31}$$

For $l \in \{0,1,\ldots,k\}$ and $t \in (0,2)$ with $t + 2l \leq T$ we have

$$u_0(t+2l) = (-1)^l u_0(t). \tag{2.32}$$

Proof. For the convenience of the reader, for the case that $T = 2K$ where K is a natural number, we give a proof here that is based upon our representation (2.25) of the solution of the initial boundary value problem (2.13) of d'Alembert type. Based upon the traveling waves representation of the solution at time T given in Theorem 2.7, it is easy to see that the control u_0 is admissible in the sense that $y(T,\cdot) = 0$ and $y_t(T,\cdot) = 0$. For example, due to (2.27) for the position at time T we have for $x \in (0,1)$ almost everywhere

$$\begin{aligned} y(T,\, x) &= (-1)^K y_0(x) - \sum_{k=1}^{K} (-1)^k \int_{1-x}^{1+x} u_0(2(K-k)+s)\,\mathrm{d}s \\ &= (-1)^K y_0(x) - \sum_{k=1}^{K} (-1)^k\,(-1)^{K-k} \int_{1-x}^{1+x} u_0(s)\,\mathrm{d}s \\ &= (-1)^K y_0(x) - (-1)^K\, K \int_{1-x}^{1+x} u_0(s)\,\mathrm{d}s \\ &= (-1)^K y_0(x) - (-1)^K\, K \left[\frac{y_0(x)}{K}\right] \\ &= 0. \end{aligned}$$

For the velocity at time T, due to (2.32), (2.29) yields for $x \in (0,1)$ almost everywhere

$$\begin{aligned} y_t(T,\,x) &= (-1)^K y_1(x) - \sum_{k=1}^{K} (-1)^k\,(-1)^{K-k}\,[u_0(1+x) - u_0(1-x)] \\ &= (-1)^K\,[y_1(x) - K\,(u_0(1+x) - u_0(1-x))\,] \\ &= (-1)^K \left[y_1(x) - K\left(\frac{y_1(x)}{K}\right)\right] \\ &= 0. \end{aligned}$$

Thus the control u_0 defined in Theorem 2.8 is admissible for **(EC)**. It only remains to show that it is optimal, that is L^2-norm minimal in the set of admissible controls. This can be done using Lemma 2.7 from [18], for the case $p = 2$. For the convenience of the reader we present this lemma here:

Lemma 2.9. *Let $p \geq 2$, a natural number d and a real number g be given. Consider the optimization problem*

$$H(p,d,g): \quad \min_{(f_0,\ldots,f_d)\in\mathbb{R}^{d+1}} \sum_{j=0}^{d} |f_j|^p \text{ s.t. } \quad \sum_{j=0}^{d} (-1)^j f_j = g.$$

The unique solution of $H(p,d,g)$ has the components $f_j = (-1)^j g/(d+1)$ and the optimal value is $|g|^p/(d+1)^{p-1}$.

For a control $u \in L^2(0,2K)$, $t \in (0,2)$ and $k \in \{1,2,\ldots,\,K\}$ we introduce the notation

$$u_k(t) = u(2(K-k)+t).$$

For $x \in (0,1)$ almost everywhere Lemma 2.7 from [18] gives the solutions of the two optimization problems

$$\begin{cases} \min \frac{1}{2}\sum_{k=1}^{K} [u_k(1+x) + u_k(1-x)]^2 \\ \text{subject to } \ \sum_{k=1}^{K} (-1)^k\,[u_k(1+x) + u_k(1-x)] = (-1)^K y_0'(x) \end{cases}$$

and

$$\begin{cases} \min \frac{1}{2}\sum_{k=1}^{K} [u_k(1+x) - u_k(1-x)]^2 \\ \text{subject to } \ \sum_{k=1}^{K} (-1)^k\,[u_k(1+x) - u_k(1-x)] = (-1)^K y_1(x). \end{cases}$$

The definition (2.31), (2.32) of the control function u_0 implies that in fact u_0 solves both optimization problems. This implies that for $x \in (0,1)$ almost everywhere, u_0 also solves the optimization problem where we add the two objective functions and impose both constraints of the two optimization problem above, that is

$$\begin{cases} \min \frac{1}{2}\sum_{k=1}^{K} [u_k(1+x) + u_k(1-x)]^2 + [u_k(1+x) - u_k(1-x)]^2 \\ \text{subject to } \ \sum_{k=1}^{K} (-1)^k\,[u_k(1+x) + u_k(1-x)] = (-1)^K y_0'(x) \\ \text{and } \ \sum_{k=1}^{K} (-1)^k\,[u_k(1+x) - u_k(1-x)] = (-1)^K y_1(x). \end{cases}$$

By the montonicity of integration, this implies that u_0 also solves the optimization problem where we integrate the above objective functions over all $x \in (0,1)$ and prescribe the constraints for $x \in (0,1)$ almost everywhere, that is

$$\begin{cases} \min\limits_{u\in L^2(0,2K)} \frac{1}{2}\int_0^1 \sum_{k=1}^{K} [u_k(1+x) + u_k(1-x)]^2 + [u_k(1+x) - u_k(1-x)]^2 \,\mathrm{d}x \\ \\ \text{subject to } \ \sum_{k=1}^{K} (-1)^k\,[u_k(1+z) + u_k(1-z)] = (-1)^K y_0'(z) \\ \text{and } \ \sum_{k=1}^{K} (-1)^k\,[u_k(1+z) - u_k(1-z)] = (-1)^K y_1(z) \\ \text{with the constraints for } \ z \in (0,1) \text{ almost everywhere.} \end{cases} \tag{2.33}$$

For the objective function we have

$$\frac{1}{2}\int_0^1 \sum_{k=1}^{K} \left[u_k(1+x)+u_k(1-x)\right]^2 + \left[u_k(1+x)-u_k(1-x)\right]^2 \,\mathrm{d}x = \int_0^{2K} u(t)^2\,\mathrm{d}t.$$

Due to (2.28) and (2.29) the end conditions

$$y(T,x)=0,\ y_t(T,x)=0,\ x\in(0,1)$$

are equivalent to the equality constraints in the optimization problem (2.33). This yields the optimality of u_0. Thus Theorem 2.8 is proved. □

Remark 2.10. For $T=2K$, the optimal value of problem **(EC)** is given by

$$\int_0^T u_0^2\,\mathrm{d}t = \frac{1}{2\,K}\int_0^1 [y_0'(s)]^2+[y_1(s)]^2\,\mathrm{d}s.$$

2.6. The effect of time-delay in the implementation of the optimal control for the vibrating string

In this section we want to analyse the effect of a time-delay $\delta>0$ in the implementation of the optimal control $u(\cdot,T,y_0,y_1)$ in the sense of (2.2), (2.3). For this purpose, we determine the generated state at time $T=2K$, that is $y_t(2K)$ using the representation of the terminal state from Theorem 2.7 and the representation of the optimal control u_0 from Theorem 2.8.

Let a delay $\delta\in(0,1)$ be given.

We compute the velocity $y_t(T,\cdot)$ that is generated by an optimal control that is implemented with time delay $\delta\in(0,1)$ as in (2.2), (2.3). For this purpose, the optimal control u_0 is extended to negative arguments by letting $u_0(s)=0$ for $s<0$. For $x\in(0,1)$ and $\delta\in(0,\min\{x,\,1-x\})$ we have

$$y_t(T,\,x) = (-1)^K y_1(x) - \sum_{k=1}^{K}(-1)^k\left[u_0(2(K-k)+(1+x-\delta)) - u_0(2(K-k)+1-x-\delta)\right]. \tag{2.34}$$

Due to (2.32) this yields

$$\begin{aligned} y_t(T,\,x) &= (-1)^K y_1(x) - \sum_{k=1}^{K}(-1)^k\,(-1)^{K-k}\left[u_0(1+x-\delta)-u_0(1-x-\delta)\right] \\ &= (-1)^K y_1(x) - (-1)^K\,K\left[u_0(1+x-\delta)-u_0(1-x-\delta)\right]. \end{aligned}$$

The definition (2.31) of u_0 implies

$$u_0(1+x-\delta)-u_0(1-x-\delta) = \frac{1}{2K}\left[y_0'(x-\delta)-y_0'(x+\delta)\right] + \frac{1}{2K}\left[y_1(x-\delta)+y_1(x+\delta)\right].$$

This yields for x almost everywhere in $(\delta,\,1-\delta)$

$$\begin{aligned} y_t(T,\,x) &= (-1)^K y_1(x) - (-1)^K\frac{1}{2}\left[(y_0'(x-\delta)-y_0'(x+\delta))+(y_1(x-\delta)+y_1(x+\delta))\right] \\ &= (-1)^K\left[y_1(x)-\frac{1}{2}\left(y_1(x-\delta)+y_1(x+\delta)\right)\right] - (-1)^K\,\frac{1}{2}\left[y_0'(x-\delta)-y_0'(x+\delta)\right]. \end{aligned} \tag{2.35}$$

Now we come to the space-derivative $y_x(T, \cdot)$. For $x \in (0, 1)$ and $\delta \in (0, \min\{x,\, 1-x\})$ we have

$$y_x(T,\, x) = (-1)^K y_0'(x) - \sum_{k=1}^{K} (-1)^k \left[u_0(2(K-k) + (1 + x - \delta)) + u_0(2(K-k) + 1 - x - \delta)\right]. \tag{2.36}$$

This yields for x almost everywhere in $(\delta,\, 1-\delta)$

$$\begin{aligned} y_x(T,\, x) &= (-1)^K y_0'(x) - (-1)^K \frac{1}{2} \left[(y_0'(x-\delta) + y_0'(x+\delta)) + (y_1(x-\delta) - y_1(x+\delta))\right] \\ &= (-1)^K \left[y_0'(x) - \frac{1}{2}\left(y_0'(x-\delta) + y_0'(x+\delta)\right)\right] - (-1)^K \frac{1}{2} \left[y_1(x-\delta) - y_1(x+\delta)\right]. \end{aligned} \tag{2.37}$$

In order to compute a lower bound for the corresponding energy at time T, we consider

$$\tilde{E}_1(T, \delta) = \int_\delta^{1-\delta} y_x(T, z)^2 + y_t(T, z)^2 \, dz. \tag{2.38}$$

From (2.35) and (2.37) we get the equation

$$\begin{aligned} \tilde{E}_1(T, \delta) = \int_\delta^{1-\delta} &\left[\tfrac{1}{2}\left[y_0'(x-\delta) - y_0'(x+\delta)\right] - \left[y_1(x) - \frac{1}{2}(y_1(x-\delta) + y_1(x+\delta))\right]\right]^2 \\ &+ \left[\tfrac{1}{2}\left[y_1(x-\delta) - y_1(x+\delta)\right] - \left[y_0'(x) - \frac{1}{2}(y_0'(x-\delta) + y_0'(x+\delta))\right]\right]^2 \mathrm{d}x. \end{aligned}$$

Due to the continuity in the mean, this implies for each fixed initial state (y_0, y_1)

$$\lim_{\delta \to 0+} \tilde{E}_1(T, \delta) = 0.$$

Proof of Proposition 2.1. The definition (2.38) implies the inequality

$$E(T) \geq \tilde{E}_1(T, \delta).$$

Thus we can compute a lower bound for the terminal energy $E(T)$ for the initial state that is given in Section 2.2, with $y_1 = 0$ and y_0 as defined in (2.7). The definition (2.7) of y_0 implies

$$y_0'(x+\delta) = -y_0'(x-\delta)$$

and

$$y_0'(x+\delta) = \cos\left(\pi\left(\tfrac{x}{2\delta} + \tfrac{\delta}{2\delta}\right)\right) = \cos\left(\pi\, \tfrac{x}{2\delta} + \tfrac{\pi}{2}\right) = -\sin\left(\pi \tfrac{x}{2\delta}\right).$$

This yields the equations

$$\tfrac{1}{2}\left[y_0'(x-\delta) - y_0'(x+\delta)\right] = -y_0'(x+\delta)$$

and

$$\left[y_0'(x) - \frac{1}{2}(y_0'(x-\delta) + y_0'(x+\delta))\right] = y_0'(x) = \cos\left(\pi \tfrac{x}{2\delta}\right).$$

Thus we get

$$\tilde{E}_1(T, \delta) = \int_\delta^{1-\delta} \sin^2\left(\pi \tfrac{x}{2\delta}\right) + \cos^2\left(\pi \tfrac{x}{2\delta}\right) \mathrm{d}x = 1 - 2\delta = \tfrac{1}{2}\left[2 - 4\delta\right].$$

Due to (2.8) this implies the inequality (2.9). $\square$

Proof of Theorem 2.3. The optimal control $u_0 = u(\cdot, T, y_0, y_1)$ that solves $(\mathbf{EC})(y_0, y_1, T)$ is given in (2.31) in Theorem 2.8. Due to our regularity assumptions y_0', $y_1 \in BV(0,1)$, (2.31) implies $u_0 \in BV(0,T)$. The definitions (2.5) and (2.38) imply

$$E(T) = \tilde{E}_1(T,\delta) + \int_0^\delta [y_x(T,z)]^2 + [y_t(T,z)]^2 \, dz + \int_{1-\delta}^1 [y_x(T,z)]^2 + [y_t(T,z)]^2 \, dz.$$

Due to (2.37) and (2.35), Young's inequality implies

$$\tilde{E}_1(T,\delta) \leq \int_\delta^{1-\delta} [y_0'(x-\delta) - y_0'(x+\delta)]^2 + [y_1(x-\delta) - y_1(x+\delta)]^2$$
$$+ 2\left[y_0'(x) - \frac{1}{2}(y_0'(x-\delta) + y_0'(x+\delta))\right]^2 + 2\left[y_1(x) - \frac{1}{2}(y_1(x-\delta) + y_1(x+\delta))\right]^2 \, \mathrm{d}x.$$

Using the inequality

$$\int_0^{1-\delta} |y(x+\delta) - y(x)| \, \mathrm{d}x \leq \delta \, D(y)$$

for $y \in BV(0,1)$ from Theorem 13.48 in [21] we get

$$\tilde{E}_1(T,\delta) \leq 8\left[\|y_0'\|_{L^\infty(0,1)} \, D(y_0') \, + \|y_1\|_{L^\infty(0,1)} \, D(y_1)\right] \delta. \tag{2.39}$$

We extend u_0 to the whole real line by letting $u_0(s) = 0$ for $s < 0$ and for $s > T$. By (2.31) we have

$$\|u_0\|_{L^\infty(0,T)} \leq \frac{1}{T}\left[\|y_0'\|_{L^\infty(0,1)} + \|y_1\|_{L^\infty(0,1)}\right].$$

By (2.34) we get the inequality

$$\|y_t(T,\cdot)\|_{L^\infty(0,1)} \leq 2\,\|y_1\|_{L^\infty(0,1)} + \|y_0'\|_{L^\infty(0,1)}.$$

By (2.36) we get the inequality

$$\|y_x(T,\cdot)\|_{L^\infty(0,1)} \leq \|y_1\|_{L^\infty(0,1)} + 2\,\|y_0'\|_{L^\infty(0,1)}.$$

Thus we obtain

$$\int_0^\delta [y_x(T,z)]^2 + [y_t(T,z)]^2 \, dz + \int_{1-\delta}^1 [y_x(T,z)]^2 + [y_t(T,z)]^2 \, dz \leq 16\,\delta\left[\|y_1\|_{L^\infty(0,1)} + \|y_0'\|_{L^\infty(0,1)}\right]^2.$$

Together with (2.39) this yields (2.11). □

Proof of Corollary 2.5. We define the control

$$u(t) = \begin{cases} u(t,\,2,\,y_0,\,y_1) & \text{for } t \in (0,2), \\ 0 & \text{for } t \in (2,T). \end{cases}$$

According to Theorem 2.3, due to the regularity of the initial state we have $u(t,\,2,\,y_0,\,y_1) \in BV(0,2)$. This implies $u \in BV(0,T)$. Moreover, since $y(2,x) = 0$, $y_t(2,x) = 0$ for $x \in (0,L)$ due to the definition of u the state remains at rest for $t > 2$, that is $y(t,x) = 0$ for $t > 2$, $x \in (0,L)$. In particular, this implies $y(T,x) = 0$, $y_t(T,x) = 0$ for $x \in (0,L)$. Thus Lemma 2.5 is proved. □

3. The optimal control problem: The general case

In this section we consider a more general optimal control problem for a hyperbolic system.

Let $\Omega \subset \mathbb{R}^n$ be bounded and connected with boundary Γ. Assume that the boundary Γ is of class C^2. We consider Dirichlet or Neumann boundary action. In order to describe the boundary action, we introduce a linear operator B that is well-defined for the boundary traces of the states. Then we can write the boundary conditions in the form

$$B\, y(t,x) = u(t,x) \text{ on } \Gamma_B$$

where $\Gamma_B \subset \Gamma$ is the part of the boundary where the control acts. On the remaining part of the boundary we prescribe homogeneous Dirichlet boundary conditions

$$y(t,x) = 0 \text{ for } x \in \Gamma,\ x \notin \Gamma_B.$$

In the Dirichlet case we have

$$B\, y(t,x) = y(t,x) \text{ on } \Gamma_B.$$

In the Neumann case we have

$$B\, y(t,x) = \partial_\nu\, y(t,x) \text{ on } \Gamma_B,$$

where ∂_ν denotes the normal derivative.

Let A be a linear operator that is defined for the elements of $H^2(\Omega)$ such that for all $k \in \{1,2,3,\ldots\}$ there exist real eigenvalues

$$\mu_k > k \tag{3.1}$$

and eigenfunctions $\varphi_k \in H^2(\Omega)$ such that

$$A\,\varphi_k = \mu_k\,\varphi_k,\ B\,\varphi_k(t,x) = 0 \text{ on } \Gamma_B,\ \varphi_k(t,x) = 0 \text{ on } \Gamma\backslash\Gamma_B. \tag{3.2}$$

Let X be a Banach space that contains the initial states of our system. Assume that for all $k \in \{1,2,3,\ldots\}$ we have

$$\begin{pmatrix} \varphi_k \\ 0 \end{pmatrix} \in X.$$

Let a time $T^* > 0$ be given.

We assume that the control to state map is continuous in the sense that there exists a constant $M_1 > 0$ such that for all $\delta \in (0, T^*]$, $u \in L^2((0,\,\delta);\, L^2(\Gamma))$ the solution y of the initial boundary value problem

$$\mathbf{L}(u,\,\delta) \begin{cases} \begin{pmatrix} y(0,\,\cdot) \\ y_t(0,\,\cdot) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \in X, \\ B\,y(t,x) = u(t,x) \text{ on } \Gamma_B,\ y(t,x) = 0 \text{ on } \Gamma\backslash\Gamma_B. \\ y_{tt} + Ay = 0 \text{ on } (0,\delta)\times\Omega \end{cases} \tag{3.3}$$

is well-defined and we have the inequality

$$\left\| \begin{pmatrix} y(\delta,\,\cdot) \\ y_t(\delta,\,\cdot) \end{pmatrix} \right\|_X^2 \le M_1 \int_0^\delta \int_\Gamma |u|^2 \,\mathrm{d}\Gamma\,\mathrm{d}t. \tag{3.4}$$

Assume that the system is *exactly controllable* to zero in the time $T^* > 0$ in the sense that for all $(y_0, y_1) \in X$ there exists a boundary control $u \in L^2((0,\,T^*);\, L^2(\Gamma))$ such that the solution y of the initial boundary value problem

$$\begin{cases} \begin{pmatrix} y(0,\cdot) \\ y_t(0,\cdot) \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \end{pmatrix} \\ B\,y(t,x) = u(t,x) \text{ for } t \in (0,T^*),\ x \in \Gamma_B,\ y(t,x) = 0 \text{ for } t \in (0,T^*),\ x \in \Gamma \setminus \Gamma_B \\ y_{tt} + Ay = 0 \text{ on } (0,T^*) \times \Omega \end{cases} \tag{3.5}$$

satisfies the end condition

$$\begin{pmatrix} y(T^*,\cdot) \\ y_t(T^*,\cdot) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \in X. \tag{3.6}$$

Let a terminal time $T \geq T^*$ be given

For a given initial state $(y_0, y_1) \in X$, consider the optimal boundary control problem

$$\mathbf{P}(y_0,y_1,T) \begin{cases} \min\limits_{u \in L^2((0,T);\,L^2(\Gamma))} J(u,y) = \int_0^T \int_\Gamma |u|^2 \,\mathrm{d}\Gamma\,\mathrm{d}t \\ \text{subject to} \\ y(0,\cdot) = y_0 \text{ in } \Omega \\ y_t(0,\cdot) = y_1 \text{ in } \Omega \\ y_{tt} + A\,y = 0 \text{ in } (0,T) \times \Omega \\ B\,y(t,x) = u(t,x) \text{ for } t \in (0,T),\ x \in \Gamma_B, \\ y(t,x) = 0 \text{ for } t \in (0,T),\ x \in \Gamma \setminus \Gamma_B \\ y(T,\cdot) = 0 \text{ in } \Omega \\ y_t(T,\cdot) = 0 \text{ in } \Omega. \end{cases} \tag{3.7}$$

Let $v(y_0, y_1, T)$ denote the optimal value of $\mathbf{P}(y_0, y_1, T)$.

Let $\bar{u} \in L^2(0,T; L^2(\Gamma))$ denote the solution of $\mathbf{P}(y_0, y_1, T)$ with the corresponding state $\bar{y}$.

Example 3.1. As an example for the linear operator A consider $A = -\Delta$ with homogenous Neumann boundary conditions. The corresponding optimal control problem is considered in [22].

Then (3.1) and (3.2) hold. In fact there is an orthonormal basis of $L^2(\Omega)$ that consists of Neumann eigenfunctions $\tilde{\varphi}_k \in H^2(\Omega)$ with Neumann eigenvalues $\tilde{\mu}_k$ that do not have an accumulation point and become arbitrarily large for sufficiently large k. Moreover, for $\Gamma_B = \Gamma$, (3.4) holds for the space

$$X = H^{1/2}(\Omega) \times (H^{1/2}(\Omega))'.$$

For the wave equation, the exact controllability property in time T^* is equivalent to the Geometric Control Condition, which asserts that all the rays of geometric optics in Ω enter the boundary Γ within the time interval $(0, T^*]$ see [2].

Example 3.2. Consider the 1-dimensional case with Neumann boundary control at $x = 1$ and homogeneous Dirichlet boundary conditions at $x = 0$. So let $n = 1$ and $\Omega = (0, 1)$, $\Gamma_B = \{1\}$, $A = -y_{xx}$ with homogeneous Dirichlet boundary conditions at $x = 0$ and Neumann boundary conditions at $x = 1$.

Then as in Example 3.1, (3.1) and (3.2) hold. Moreover, (3.4) holds for the space

$$X = H^1(\Omega) \times L^2(\Omega).$$

Example 3.3. Consider again the 1-dimensional case with Dirichlet boundary control at $x = 1$ and homogeneous Dirichlet boundary conditions at $x = 0$. So let $n = 1$ and $\Omega = (0, 1)$, $\Gamma_B = \{1\}$, $A = -y_{xx}$ with homogeneous Dirichlet boundary conditions at $x = 0$ and $x = 1$. Then (3.1) and (3.2) hold. Moreover, (3.4) holds for the space

$$X = L^2(\Omega) \times (H^1(\Omega))'.$$

Example 3.4. As in 10.9 in [28], consider $A = -\Delta$ with homogenous Dirichlet boundary conditions.

With Theorem 10.9.3 from [28] we get (3.4) for the space

$$X = L^2(\Omega) \times (H^1(\Omega))'.$$

3.1. The effect of time-delay in the general case

In this section, we study the effect of a time delay $\delta > 0$, if for $t \in (\delta, T + \delta)$, the optimal boundary control $\bar{u}$ is implemented in the from

$$B\, y(t, x) = \bar{u}(t - \delta,\, x) \text{ for } \ x \in \Gamma_B. \tag{3.8}$$

For $t \in (0, \delta)$, we prescribe the boundary condition

$$B\, y(t, x) = 0 \text{ on } \ \Gamma_B. \tag{3.9}$$

The corresponding state is defined on the time interval $(0, T + \delta)$ as the solution of the initial boundary value problem

$$\mathbf{I}(y_0, y_1, \delta) \begin{cases} (y(0,\cdot), y_t(0,\cdot)) = (y_0,\ y_1) \in X \\ B\, y(t,x) = 0 \ \text{ for } \ t \in (0,\, \delta),\ x \in \Gamma_B, \\ B\, y(t,x) = \bar{u}(t-\delta,\, x) \ \text{ for } \ t \in (\delta,\ T+\delta),\ x \in \Gamma_B, \\ y(t,x) = 0 \ \text{ for } \ t \in (0,\ T+\delta),\ x \in \Gamma \backslash \Gamma_B \\ y_{tt} + Ay = 0 \text{ on } \ (0, T+\delta) \times \Omega. \end{cases} \tag{3.10}$$

Problem $\mathbf{I}(y_0, y_1, \delta)$ determines the state that is generated by an optimal control loop with the initial state $(y_0,\, y_1)$ where the delay δ occurs in the implementation of the optimal control. In order to analyze the effect of the time delay in the implementation of the optimal control, we compare the X-norm of the initial state and the X-norm of the terminal state.

The following Lemma on the behavior of the optimal value as a function of T shows that for certain initial states it decreases at least with a rate that depends on T/T^*.

Lemma 3.5. *If $n \in \{1, 2, 3, \ldots\}$ is such that*

$$n\,(T^* + 2\pi) \leq T,$$

for all $k \in \{1, 2, 3, \ldots\}$ we have

$$v(\varphi_k,\, 0,\, T) \leq \frac{1}{n} v(\varphi_k,\, 0,\, T^*). \tag{3.11}$$

Proof of Lemma 3.5. Let u^* denote the solution of $\mathbf{P}(\varphi_k,\, 0,\, T^*)$. Using u^* we construct a control u that is feasible for $\mathbf{P}(\varphi_k,\, 0,\, T)$. For $t \in (0,\, T^*)$ we define

$$u(t,\, \cdot) = \frac{1}{n}\, u^*(t,\, \cdot).$$

Then for the generated state we have

$$\begin{pmatrix} y(T^*,\, \cdot) \\ y_t(T^*,\, \cdot) \end{pmatrix} = \left(1 - \frac{1}{n}\right) \begin{pmatrix} \cos(\sqrt{\mu_k} T^*)\, \varphi_k(x) \\ -\sin(\sqrt{\mu_k} T^*)\, \varphi_k(x) \end{pmatrix}$$

Now we switch off the control and choose a number $\delta_1 \in [0, 2\pi]$ such that with the control $u|_{(T^*,\, T^*+\delta_1)} = 0$ we get

$$\begin{pmatrix} y(T^* + \delta_1,\, \cdot) \\ y_t(T^* + \delta_1,\, \cdot) \end{pmatrix} = \left(1 - \frac{1}{n}\right) \begin{pmatrix} \varphi_k(x) \\ 0 \end{pmatrix}.$$

This is possible, since with zero control our solution is periodic with period $\frac{2\pi}{\sqrt{\mu_k}} \leq 2\pi$ due to (3.1). For $t \in (T^* + \delta_1, 2\,T^* + \delta_1)$ we define

$$u(t,\,\cdot) = \frac{1}{n}\,u^*(t - (T^* + \delta_1),\,\cdot).$$

Then for the generated state we have

$$\begin{pmatrix} y(2\,T^* + \delta_1,\,\cdot) \\ y_t(2\,T^* + \delta_1,\,\cdot) \end{pmatrix} = \left(1 - \frac{2}{n}\right)\,\begin{pmatrix} \cos(\sqrt{\mu_k}(2\,T^* + \delta_1))\,\varphi_k(x) \\ -\sin(\sqrt{\mu_k}(2\,T^* + \delta_1))\,\varphi_k(x) \end{pmatrix}$$

Now we switch off the control again and choose a number $\delta_2 \in [0, 2\pi]$ such that with $u|_{(2T^*+\delta_1,\,2T^*+\delta_1+\delta_2)} = 0$ we get

$$\begin{pmatrix} y(2T^* + \delta_1 + \delta_2,\,\cdot) \\ y_t(2T^* + \delta_1 + \delta_2,\,\cdot) \end{pmatrix} = \left(1 - \frac{2}{n}\right)\,\begin{pmatrix} \varphi_k(x) \\ 0 \end{pmatrix}$$

By continuing the construction with suitably chosen numbers δ_i after n steps we obtain the state

$$\begin{pmatrix} y(n\,T^* + \sum_{i=1}^{n-1}\,\delta_i,\,\cdot) \\ y_t(n\,T^* + \sum_{i=1}^{n-1}\,\delta_i,\,\cdot) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

On the time interval that remains from $[0, T]$ we switch off the control again, that is we define $u = 0$. Thus we have constructed a control u that is admissible for $\mathbf{P}(\varphi_k,\,0,\,T)$. With the notation $\delta_0 = 0$ for the optimal value we get the inequality

$$\begin{aligned} v(\varphi_k,\,0,\,T) &\leq \int_0^T \int_\Gamma |u|^2\,\mathrm{d}\Gamma\,\mathrm{d}t \\ &= \sum_{i=0}^{n-1} \int_{\sum_{j=0}^{i}\,\delta_j + iT^*}^{\sum_{j=0}^{i}\,\delta_j + (i+1)\,T^*} \int_\Gamma |u|^2\,\mathrm{d}\Gamma\,\mathrm{d}t \\ &= \sum_{i=0}^{n-1} \frac{1}{n^2} \int_0^{T^*} \int_\Gamma |u^*|^2\,\mathrm{d}\Gamma\,\mathrm{d}t \\ &= \frac{1}{n}\,v(\varphi_k,\,0,\,T^*) \end{aligned}$$

and inequality (3.11) follows. □

3.2. An example for the destabilizing effect of time delay in the general case

In this section we give an example for the destabilizing effect of time delay in the implementation of the optimal control. In our analysis, we consider initial state of the form

$$\begin{pmatrix} y_0 \\ y_1 \end{pmatrix} = \begin{pmatrix} \varphi_k \\ 0 \end{pmatrix} \tag{3.12}$$

where the initial position is given by an eigenfunction and the initial velocity is zero. For such initial states with a natural number k that is chosen sufficiently large, we have the following Lemma.

Proposition 3.6. *Let $T \geq T^*$ be given. Let $k \in \{1, 2, 3, \ldots\}$ be such that*

$$\delta = \frac{\pi}{\sqrt{\mu_k}} \in (0,\,T^*).$$

Let $(y_0,\,y_1)$ be as in (3.12). Let y denote the solution of $\mathbf{I}(y_0, y_1, \delta)$. Then we have the equation

$$\left\|\begin{pmatrix} y(T + \delta,\,\cdot) \\ y_t(T + \delta,\,\cdot) \end{pmatrix}\right\|_X = 2\,\left\|\begin{pmatrix} -\cos(\sqrt{\mu_k}\,(T + \delta))\,\varphi_k(x) \\ \sqrt{\mu_k}\,\sin(\sqrt{\mu_k}\,(T + \delta))\,\varphi_k(x) \end{pmatrix}\right\|_X. \tag{3.13}$$

If

$$\frac{T}{\delta} \in \{1, 2, 3, \ldots\} \tag{3.14}$$

this yields

$$\left\| \begin{pmatrix} y(T+\delta,\,\cdot) \\ y_t(T+\delta,\,\cdot) \end{pmatrix} \right\|_X = 2\, \left\| \begin{pmatrix} y(0,\,\cdot) \\ y_t(0,\,\cdot) \end{pmatrix} \right\|_X . \tag{3.15}$$

Moreover, if $n \in \{1, 2, 3, \ldots\}$ *is such that*

$$n\,(T^* + 2\pi) \le T \tag{3.16}$$

we get the inequality

$$\left\| \begin{pmatrix} y(T,\,\cdot) \\ y_t(T,\,\cdot) \end{pmatrix} \right\|_X \ge 2\, \left\| \begin{pmatrix} -\cos(\sqrt{\mu_k}\,T)\,\varphi_k(x) \\ \sqrt{\mu_k}\,\sin(\sqrt{\mu_k}\,T)\,\varphi_k(x) \end{pmatrix} \right\|_X - \frac{\sqrt{M_1}}{\sqrt{n}} \sqrt{v(\varphi_k,\,0,\,T^*)}. \tag{3.17}$$

If in addition, we have

$$\frac{T}{\delta} \in \{1, 2, 3, \ldots\} \tag{3.18}$$

this yields

$$\left\| \begin{pmatrix} y(T,\,\cdot) \\ y_t(T,\,\cdot) \end{pmatrix} \right\|_X \ge 2\, \left\| \begin{pmatrix} y(0,\,\cdot) \\ y_t(0,\,\cdot) \end{pmatrix} \right\|_X - \sqrt{\frac{M_1}{n} v(\varphi_k,\,0,\,T^*)}. \tag{3.19}$$

If in addition, there exists a constant $M_2 > 0$ *such that for all* $(y_0, y_1) \in X$ *we have*

$$v(y_0,\,y_1,\,T^*) \le M_2 \left\| \begin{pmatrix} y_0 \\ y_1 \end{pmatrix} \right\|_X^2 \tag{3.20}$$

this implies

$$\frac{\left\| \begin{pmatrix} y(T,\,\cdot) \\ y_t(T,\,\cdot) \end{pmatrix} \right\|_X}{\left\| \begin{pmatrix} y(0,\,\cdot) \\ y_t(0,\,\cdot) \end{pmatrix} \right\|_X} \ge \left(2 - \sqrt{\frac{M_1 M_2}{n}} \right). \tag{3.21}$$

Remark 3.7. Inequality (3.20) means that for T^* the optimal control map that maps (y_0, y_1) to the optimal control is bounded.

Remark 3.8. Our example shows that there are arbitrarily small time delays $\delta > 0$, and initial states (y_0, y_1) such that for terminal times $T \ge T^*$ in the set

$$\{\delta,\, 2\delta,\, 3\delta,\, \ldots\}$$

at the time $T + \delta$, the norm $\|\cdot\|_X$ of the state is twice the norm $\|\cdot\|_X$ of the initial state.

Moreover, if T is sufficiently large, at the time T the norm $\|\cdot\|_X$ of the state is almost twice the norm $\|\cdot\|_X$ of the initial state.

Remark 3.9. Consider the application of Proposition 3.6 in the situation of Example 3.1. In this case we have

$$\begin{aligned} &\left\| \begin{pmatrix} -\cos(\sqrt{\mu_k}\,T)\,\varphi_k(x) \\ \sqrt{\mu_k}\,\sin(\sqrt{\mu_k}\,T)\,\varphi_k(x) \end{pmatrix} \right\|_X \\ &= \left(\|\cos(\sqrt{\mu_k}\,T)\,\varphi_k(x)\|^2_{H^{1/2}(\Omega)} + \|\sqrt{\mu_k}\,\sin(\sqrt{\mu_k}\,T)\,\varphi_k(x)\|^2_{(H^{1/2}(\Omega))'} \right)^{1/2} \\ &= \left(\cos^2(\sqrt{\mu_k}\,T)\,\|\,\varphi_k(x)\|^2_{H^{1/2}(\Omega)} + \mu_k\,\sin^2(\sqrt{\mu_k}\,T)\,\|\,\varphi_k(x)\|^2_{(H^{1/2}(\Omega))'} \right)^{1/2}. \end{aligned}$$

Proof of Proposition 3.6. For $t \in (0, \delta)$ we have

$$y(t,\,x) = \cos(\sqrt{\mu_k}\,t)\,\varphi_k(x).$$

This implies

$$y(\delta,\,x) = \cos(\sqrt{\mu_k}\,\delta)\,\varphi_k(x) = \cos(\pi)\,\varphi_k(x) = -\varphi_k(x).$$

For the corresponding velocity we get

$$y_t(\delta,\,x) = -\sqrt{\mu_k}\,\sin(\sqrt{\mu_k}\,\delta)\,\varphi_k(x) = 0.$$

Thus we have

$$\begin{pmatrix} y(\delta,\,\cdot) \\ y_t(\delta,\,\cdot) \end{pmatrix} = z_1 + z_2 \text{ with } \; z_1 = \begin{pmatrix} y_0 \\ y_1 \end{pmatrix},\; z_2 = -2\begin{pmatrix} y_0 \\ y_1 \end{pmatrix}.$$

We define f_1 as the solution of the initial boundary value problem

$$\begin{cases} \begin{pmatrix} y(\delta,\,\cdot) \\ y_t(\delta,\,\cdot) \end{pmatrix} = z_1 \\ B\,y(t+\delta, x) = \bar{u}(t,x) \text{ on } \;\Gamma_B,\; y(t,x) = 0 \text{ on } \;\Gamma\backslash\Gamma_B. \\ y_{tt} + Ay = 0 \text{ on } \;(\delta, T+\delta)\times\Omega. \end{cases} \tag{3.22}$$

Then we have

$$f_1(T+\delta,\,\cdot) = 0,\; \partial_t f_1(T+\delta,\,\cdot) = 0. \tag{3.23}$$

Now we want to derive an estimate for $f_1(T,\,\cdot)$ and $\partial_t f_1(T,\,\cdot)$. By (3.4) and reversal of the time (that is starting at $T+\delta$ and going backwards in time to T) we get

$$\left\| \begin{pmatrix} f_1(T,\,\cdot) \\ \partial_t f_1(T,\,\cdot) \end{pmatrix} \right\|_X^2 \le M_1 \int_{T-\delta}^{T} \int_\Gamma |\bar{u}|^2 \,\mathrm{d}\Gamma\,\mathrm{d}t. \tag{3.24}$$

If (3.16) holds, inequality (3.11) implies

$$\int_0^T \int_\Gamma |\bar{u}|^2 \,\mathrm{d}\Gamma\,\mathrm{d}t \le \frac{1}{n} v(\varphi_k,\,0,\,T^*).$$

Thus we get

$$\left\| \begin{pmatrix} f_1(T,\,\cdot) \\ \partial_t f_1(T,\,\cdot) \end{pmatrix} \right\|_X \le \sqrt{\frac{M_1}{n}}\sqrt{v(\varphi_k,\,0,\,T^*)}. \tag{3.25}$$

Now we look at the solution f_2 of the initial boundary value problem

$$\begin{cases} \begin{pmatrix} y(\delta,\,\cdot) \\ y_t(\delta,\,\cdot) \end{pmatrix} = z_2 \\ B\,y(t,x) = 0 \text{ on } \;\Gamma_B,\; y(t,x) = 0 \text{ on } \;\Gamma\backslash\Gamma_B. \\ y_{tt} + Ay = 0 \text{ on } \;(\delta, T+\delta)\times\Omega. \end{cases} \tag{3.26}$$

For $t \in [\delta,\, T+\delta]$ we get

$$f_2(t,\,x) = -2\;\cos(\sqrt{\mu_k}\,t)\,\varphi_k(x).$$

For $t \in [\delta,\, T+\delta]$ we have

$$y(t,\,\cdot) = f_1(t,\cdot) + f_2(t,\cdot).$$

With (3.23) we get

$$\left\| \begin{pmatrix} y(T+\delta,\,\cdot) \\ y_t(T+\delta,\,\cdot) \end{pmatrix} \right\|_X = \left\| \begin{pmatrix} f_2(T+\delta,\,\cdot) \\ \partial_t f_2(T+\delta,\,\cdot) \end{pmatrix} \right\|_X = \left\| \begin{pmatrix} -2\,\cos(\sqrt{\mu_k}\,T)\,\varphi_k(x) \\ 2\,\sqrt{\mu_k}\,\sin(\sqrt{\mu_k}\,T)\,\varphi_k(x) \end{pmatrix} \right\|_X .$$

Thus we have shown (3.13). If (3.16) holds, we get

$$\begin{aligned} \left\| \begin{pmatrix} y(T,\,\cdot) \\ y_t(T,\,\cdot) \end{pmatrix} \right\|_X &\geq \left\| \begin{pmatrix} f_2(T,\,\cdot) \\ \partial_t f_2(T,\,\cdot) \end{pmatrix} \right\|_X - \left\| \begin{pmatrix} f_1(T,\,\cdot) \\ \partial_t f_1(T,\,\cdot) \end{pmatrix} \right\|_X \\ &\geq \left\| \begin{pmatrix} -2\,\cos(\sqrt{\mu_k}\,T)\,\varphi_k(x) \\ 2\,\sqrt{\mu_k}\,\sin(\sqrt{\mu_k}\,T)\,\varphi_k(x) \end{pmatrix} \right\|_X - \sqrt{\frac{M_1}{n}}\sqrt{v(\varphi_k,\,0,\,T^*)}. \end{aligned}$$

Thus we have shown (3.17). If (3.18) holds, this yields

$$\begin{aligned} \left\| \begin{pmatrix} y(T,\,\cdot) \\ y_t(T,\,\cdot) \end{pmatrix} \right\|_X &\geq 2 \left\| \begin{pmatrix} \varphi_k(x) \\ 0 \end{pmatrix} \right\|_X - \sqrt{\frac{M_1}{n} v(\varphi_k,\,0,\,T^*)} \\ &= 2 \left\| \begin{pmatrix} y(0,\,\cdot) \\ y_t(0,\,\cdot) \end{pmatrix} \right\|_X - \sqrt{\frac{M_1}{n} v(\varphi_k,\,0,\,T^*)}. \end{aligned}$$

Thus we have shown (3.19). If (3.20) holds, this yields

$$\begin{aligned} \left\| \begin{pmatrix} y(T,\,\cdot) \\ y_t(T,\,\cdot) \end{pmatrix} \right\|_X &\geq 2 \left\| \begin{pmatrix} \varphi_k(x) \\ 0 \end{pmatrix} \right\|_X - \sqrt{\frac{M_1 M_2}{n}} \left\| \begin{pmatrix} \varphi_k(x) \\ 0 \end{pmatrix} \right\|_X \\ &= \left(2 - \sqrt{\frac{M_1 M_2}{n}}\right) \left\| \begin{pmatrix} \varphi_k(x) \\ 0 \end{pmatrix} \right\|_X . \end{aligned}$$

Thus we have shown (3.21) and Proposition 3.6 is proved. □

3.3. An upper bound for the destabilizing effect of time delay in the general case

In this section we consider the question: What is the worst effect that a delay δ in the optimal control loop can have? So we are looking for an upper bound for the X-norm of the state at time T in terms of the X-norm of the initial state. To achieve this goal, we have to assume that the initial states satisfies an additional regularity assumption.

Theorem 3.10. *Assume that there exists a complete sequence of eigenfunctions $(\varphi_k)_k$ with eigenvalues $\lambda_k > 0$ that is orthonormal in $L^2(\Omega)$ such that $(y_0,\,y_1) \in X$ can be represented as*

$$\begin{pmatrix} y_0 \\ y_1 \end{pmatrix} = \begin{pmatrix} \sum_{k=1}^\infty \alpha_k^0\,\varphi_k(\cdot) \\ \sum_{k=1}^\infty \alpha_k^1\,\varphi_k(\cdot) \end{pmatrix} .$$

Assume that the initial state satisfies the additional regularity condition

$$H_L = \sum_{k=1}^\infty \left[1 + \frac{1}{2}\,\sqrt{\lambda_k}\right] \left[\sqrt{\lambda_k}\,|\alpha_k^0| + |\alpha_k^1|\right] \left[\left\| \begin{pmatrix} \varphi_k \\ 0 \end{pmatrix} \right\|_X + \sqrt{\lambda_k}\,\left\| \begin{pmatrix} 0 \\ \varphi_k \end{pmatrix} \right\|_X\right] < \infty. \tag{3.27}$$

Let $\delta \in (0,\,\min\{\frac{1}{2}\,T,\,1\})$ be given.

Let y denote the solution of $\mathbf{I}(y_0, y_1, \delta)$. Then we have the inequality

$$\left\| \begin{pmatrix} y(T+\delta,\, \cdot) \\ y_t(T+\delta,\, \cdot) \end{pmatrix} \right\|_X \leq \delta\, H_L. \tag{3.28}$$

Moreover

$$\left\| \begin{pmatrix} y(T,\, \cdot) \\ y_t(T,\, \cdot) \end{pmatrix} \right\|_X \leq \delta\, H_L + \sqrt{M_1}\, \|\bar{u}(t,\cdot)\|_{L^2((T-\delta,\, T),\, L^2(\Gamma))}. \tag{3.29}$$

Remark 3.11. Theorem 3.10 shows that if for a given real number $r > 0$ the feasible initial states are contained in the set

$$B(r) = \left\{ \begin{pmatrix} y_0 \\ y_1 \end{pmatrix} = \begin{pmatrix} \sum_{k=1}^\infty \alpha_k^0\ \varphi_k(\cdot) \\ \sum_{k=1}^\infty \alpha_k^1\ \varphi_k(\cdot) \end{pmatrix} \in X : \right.$$

$$\left. \sum_{k=1}^\infty \left[1 + \frac{1}{2}\sqrt{\lambda_k}\right] \left[\sqrt{\lambda_k}\, |\alpha_k^0| + |\alpha_k^1|\right] \left[\left\| \begin{pmatrix} \varphi_k \\ 0 \end{pmatrix} \right\|_X + \sqrt{\lambda_k}\, \left\| \begin{pmatrix} 0 \\ \varphi_k \end{pmatrix} \right\|_X \right] \leq r \right\}$$

the effect of the time delay is uniformly bounded since

$$\left\| \begin{pmatrix} y(T+\delta,\, \cdot) \\ y_t(T+\delta,\, \cdot) \end{pmatrix} \right\|_X \leq \delta\, r. \tag{3.30}$$

Proof of Theorem 3.10. Since in the first part $[0, \delta]$ of the time interval the control is switched off (that is, it is zero), for $t \in (0, \delta]$ we have

$$y(t,\, x) = \sum_{k=1}^\infty \left[\alpha_k^0\ \cos(\sqrt{\lambda_k}\, t)\ + \alpha_k^1\, \frac{1}{\sqrt{\lambda_k}} \sin(\sqrt{\lambda_k}\, t)\right]\ \varphi_k(x).$$

For the corresponding velocity we get

$$y_t(t,\, x) = \sum_{k=1}^\infty \left[-\sqrt{\lambda_k}\alpha_k^0\ \sin(\sqrt{\lambda_k}\, t)\ + \alpha_k^1\ \cos(\sqrt{\lambda_k}\, t)\right]\ \varphi_k(x).$$

Thus we have

$$\begin{pmatrix} y(\delta,\, \cdot) \\ y_t(\delta,\, \cdot) \end{pmatrix} = z_1 + z_2 \text{ with } \ z_1 = \begin{pmatrix} y_0 \\ y_1 \end{pmatrix},$$

$$z_2 = \begin{pmatrix} \sum_{k=1}^\infty \left[\alpha_k^0\ \left[\cos(\sqrt{\lambda_k}\, \delta) - 1\right]\ + \alpha_k^1\, \frac{1}{\sqrt{\lambda_k}} \sin(\sqrt{\lambda_k}\, \delta)\right]\ \varphi_k(x) \\ \sum_{k=1}^\infty \left[-\alpha_k^0\, \sqrt{\lambda_k}\, \sin(\sqrt{\lambda_k}\, \delta)\ + \alpha_k^1\ \left[\cos(\sqrt{\lambda_k}\, \delta) - 1\right]\right]\ \varphi_k(x) \end{pmatrix}.$$

We define f_1 as the solution of the initial boundary value problem

$$\begin{cases} \begin{pmatrix} y(\delta,\, \cdot) \\ y_t(\delta,\, \cdot) \end{pmatrix} = z_1 \\ B\, y(t,x) = \bar{u}(t-\delta,\, x) \ \text{ for } \ t \in (\delta,\, T+\delta),\ x \in \Gamma_B,\ \ y(t,x) = 0 \text{ for } \ t \in (\delta,\, T+\delta),\ x \in \Gamma\backslash\Gamma_B \\ y_{tt} + Ay = 0 \text{ on } \ (\delta, T+\delta) \times \Omega. \end{cases} \tag{3.31}$$

Then we have

$$f_1(T+\delta,\, \cdot) = 0,\ \partial_t f_1(T+\delta,\, \cdot) = 0. \tag{3.32}$$

By (3.4) and reversal of the time this yields

$$\left\| \begin{pmatrix} f_1(T,\,\cdot) \\ \partial_t f_1(T,\,\cdot) \end{pmatrix} \right\|_X \leq \sqrt{M_1} \|\bar{u}(t,\cdot)\|_{L^2((T-\delta,\,T),\,L^2(\Gamma))}. \tag{3.33}$$

Now we look at the solution f_2 of the initial boundary value problem

$$\begin{cases} \begin{pmatrix} y(\delta,\,\cdot) \\ y_t(\delta,\,\cdot) \end{pmatrix} = z_2 \\ B\,y(t,x) = 0 \;\text{ for }\; t \in (\delta,\,T+\delta),\; x \in \Gamma_B,\;\; y(t,x) = 0 \text{ for }\; t \in (\delta,\,T+\delta),\; x \in \Gamma \backslash \Gamma_B \\ y_{tt} + Ay = 0 \text{ on }\; (\delta, T+\delta) \times \Omega. \end{cases} \tag{3.34}$$

Thus for $t \in [\delta,\,T+\delta]$ we get

$$\begin{aligned} f_2(t,\,x) = {} & \sum_{k=1}^{\infty} \left[\alpha_k^0 \,\left[\cos(\sqrt{\lambda_k}\,\delta) - 1 \right] + \alpha_k^1 \frac{1}{\sqrt{\lambda_k}} \sin(\sqrt{\lambda_k}\,\delta) \right] \cos(\sqrt{\lambda_k}\,(t-\delta))\varphi_k(x) \\ & + \sum_{k=1}^{\infty} \left[-\alpha_k^0 \sqrt{\lambda_k}\, \sin(\sqrt{\lambda_k}\,\delta) + \alpha_k^1 \left[\cos(\sqrt{\lambda_k}\,\delta) - 1 \right] \right] \frac{1}{\sqrt{\lambda_k}} \sin(\sqrt{\lambda_k}\,(t-\delta))\, \varphi_k(x). \\ \partial_t\, f_2(t,\,x) = {} & -\sqrt{\lambda_k} \sum_{k=1}^{\infty} \left[\alpha_k^0 \,\left[\cos(\sqrt{\lambda_k}\,\delta) - 1 \right] + \alpha_k^1 \frac{1}{\sqrt{\lambda_k}} \sin(\sqrt{\lambda_k}\,\delta) \right] \sin(\sqrt{\lambda_k}\,(t-\delta))\varphi_k(x) \\ & + \sum_{k=1}^{\infty} \left[-\alpha_k^0 \sqrt{\lambda_k}\, \sin(\sqrt{\lambda_k}\,\delta) + \alpha_k^1 \left[\cos(\sqrt{\lambda_k}\,\delta) - 1 \right] \right] \cos(\sqrt{\lambda_k}\,(t-\delta))\, \varphi_k(x). \end{aligned}$$

Since

$$\left| \cos(\sqrt{\lambda_k}\,\delta) - 1 \right| \leq \frac{\lambda_k\,\delta^2}{2}, \quad \left| \sin(\sqrt{\lambda_k}\,\delta) \right| \leq \sqrt{\lambda_k}\,\delta$$

we get

$$\begin{aligned} \left\| \begin{pmatrix} f_2(t,\,\cdot) \\ \partial_t f_2(t,\,\cdot) \end{pmatrix} \right\|_X & \leq \left\| \begin{pmatrix} f_2(t,\,\cdot) \\ 0 \end{pmatrix} \right\|_X + \left\| \begin{pmatrix} 0 \\ \partial_t f_2(t,\,\cdot) \end{pmatrix} \right\|_X \\ & \leq \sum_{k=1}^{\infty} \left[\delta + \sqrt{\lambda_k}\,\frac{\delta^2}{2} \right] \left[\sqrt{\lambda_k}\,|\alpha_k^0| + |\alpha_k^1| \right] \left\| \begin{pmatrix} \varphi_k \\ 0 \end{pmatrix} \right\|_X \\ & \quad + \sum_{k=1}^{\infty} \left[\delta + \sqrt{\lambda_k}\,\frac{\delta^2}{2} \right] \left[\lambda_k\,|\alpha_k^0| + \sqrt{\lambda_k}|\alpha_k^1| \right] \left\| \begin{pmatrix} 0 \\ \varphi_k \end{pmatrix} \right\|_X \\ & \leq \delta \sum_{k=1}^{\infty} \left[1 + \frac{1}{2}\,\sqrt{\lambda_k} \right] \left[\sqrt{\lambda_k}\,|\alpha_k^0| + |\alpha_k^1| \right] \left\| \begin{pmatrix} \varphi_k \\ 0 \end{pmatrix} \right\|_X \\ & \quad + \delta \sum_{k=1}^{\infty} \left[1 + \frac{1}{2}\,\sqrt{\lambda_k} \right] \left[\lambda_k\,|\alpha_k^0| + \sqrt{\lambda_k}|\alpha_k^1| \right] \left\| \begin{pmatrix} 0 \\ \varphi_k \end{pmatrix} \right\|_X \\ & = \delta\, H_L. \end{aligned}$$

For $t \in (\delta,\,T+\delta)$ we have

$$y(t,\,\cdot) = f_1(t,\cdot) + f_2(t,\cdot).$$

With (3.32) we get

$$\left\| \begin{pmatrix} y(T+\delta,\,\cdot) \\ y_t(T+\delta,\,\cdot) \end{pmatrix} \right\|_X \le \left\| \begin{pmatrix} f_1(T+\delta,\,\cdot) \\ \partial_t f_1(T+\delta,\,\cdot) \end{pmatrix} \right\|_X + \left\| \begin{pmatrix} f_2(T+\delta,\,\cdot) \\ \partial_t f_2(T+\delta,\,\cdot) \end{pmatrix} \right\|_X \le \delta\, H_L.$$

So we have shown (3.30). Moreover with (3.33) we get

$$\left\| \begin{pmatrix} y(T,\,\cdot) \\ y_t(T,\,\cdot) \end{pmatrix} \right\|_X \le \left\| \begin{pmatrix} f_1(T,\,\cdot) \\ \partial_t f_1(T,\,\cdot) \end{pmatrix} \right\|_X + \left\| \begin{pmatrix} f_2(T,\,\cdot) \\ \partial_t f_2(T,\,\cdot) \end{pmatrix} \right\|_X \le \sqrt{M_1}\|\bar{u}(t,\cdot)\|_{L^2((T-\delta,\,T),\,L^2(\Gamma))} + \delta\, H_L.$$

Thus we have shown (3.29) and Theorem 3.10 is proved. □

4. Conclusions

We have shown that similarly as in a feedback stabilization loop, also in an optimal control loop, time-delay in the implementation of the optimal control can have a destabilizing effect. For all sufficiently small delays $\delta > 0$ in the optimal control loop we have presented initial states such that the terminal energy is almost twice as big as the initial energy. Moreover, we have shown that this phenomenon can only occur if initial states that are dominated by arbitrarily high frequencies are feasible. If the initial state satisfies certain regularity conditions in addition to the conditions that are necessary to guarantee the existence of the optimal control, an upper bound for the effect of the time-delay on the terminal energy can be given. We have given a regularity condition for the initial states that requires its bounded variation and guarantees that the corresponding optimal controls are robust against a perturbation by time-delay in their implementation. Our results are an argument to use controls where the BV–norm is as small as possible in order to make the controls more robust against time delay. If derivatives of the control function exist, this can be achieved by making the norm of the derivatives small. A method to obtain smooth controls for smooth initial data, inspired by HUM, is given in [8]. In the future, we plan to extend our analysis to the case of nonconstant delays (see [3]).

For the minimal control time $T = 2$, the optimal control $u(\cdot,\,T,\,y_0,\,y_1)$ that is implemented without delay generates exactly the same state as the closed loop system $(\mathbf{FL})(1,\,y_0,\,y_1,\,0)$ that is defined in the introduction. This is a consequence of the facts that for $k = 1$, the system is finite-time stable (see [19]) and that for the minimal control time, the exact control is uniquely determined. However, it is not clear wether a delay in the implementation of the optimal control as in $(\mathbf{IBVP})(y_0,\,y_1,\,T,\delta)$ corresponds to the closed loop system $(\mathbf{FL})(1,\,y_0,\,y_1,\,\delta)$. This interesting question remains for future research.

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