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INTERPENETRATION OF MATTER IN PLATE THEORIES OBTAINED AS Γ-LIMITS *

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Abstract. We reconsider the derivation of plate theories as Γ-limits of 3-dimensional nonlinear elasticity and define a suitable notion for the interpenetration of matter in the limit configuration. This is done via the Brouwer degree. For the approximating maps, we adopt as definition of interpenetration of matter the notion of non-invertibility almost everywhere, see [J.M. Ball, *Proc. Roy. Soc. Edinburgh Sect. A* **88** (1981) 315–328]. Given a limit map satisfying the former interpenetration property, we show that any recovery sequence (in the sense of Γ-convergence) has to consist of maps that satisfy the latter interpenetration property except for finitely many sequence elements. Then we explain how our result is applied in the context of the derivation of plate theories.

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1. Introduction

In the mathematical theory of nonlinear elasticity, the elastic deformations of an elastic body are identified with (almost-) minimizers of some free elastic energy functional. This identification works as follows: The reference configuration of the elastic body is some domain $\Omega \subset \mathbb{R}^n$, the deformation is a map $y : \Omega \to \mathbb{R}^m$, and the associated energy $I : X \to \mathbb{R}$ has as domain the function space of deformations y. Of crucial importance is the right choice for the function space X. Unphysical deformations (*e.g.*, non-injective maps, which represent configurations displaying self-penetration of matter) should either be excluded from X, or the energy of these configurations should be infinite, signaling that it is not possible to observe them in the "real world". There exists a large amount of literature on how to choose the function space of elastic deformations in a manner that at the same time excludes unphysical configurations and ensures existence of energy minimizers. We do not attempt to give an exhaustive literature review here, and only mention [1–4, 6, 16, 17]. In [16], a framework has been introduced that allows for *cavitation*, *i.e.*, the free energy allows for the formation of holes in the elastic body. Cavity formation can be observed in experiments; the mathematical theory for radially symmetric cavities

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has been developed in [4]. In [16], the function space X is chosen such that cavities created at one point cannot be filled with matter from elsewhere. Clearly, this is another property that "physical" deformations of an elastic body should fulfill. The mathematical formulation of this condition (called "(INV)" in [16]) is rather technical.

An important question in nonlinear elasticity is the relation between models in three, two and one dimensions. Conceptually and mathematically, the most satisfying approach is the derivation of lower dimensional models from a 3-dimensional one by Γ-convergence [8]. In [11, 12], a hierarchy of 2-dimensional plate models has been derived from 3-dimensional nonlinear elasticity. These models can be classified by the assumed scaling of the energy per unit thickness I_h in the underlying 3d theory, where h denotes the thickness of the elastic sheet. Assuming $I_h \sim h^\beta$, where h is the thickness of the elastic plate, the Γ-limit for $\beta = 2$ is nonlinear bending theory [11]. The parameter choice $2 < \beta < 4$ results in "von-Kármán-like" plate theories, see [12].

The 3d models taken as a starting point for this hierarchy of Γ-limits do not require the condition (INV). In [16] it is shown that in general, if condition (INV) is not imposed, it is possible to construct sequences of (almost everywhere) invertible deformations of finite energy that weakly converge to a non-(a.e.) invertible one. We will give a slightly more detailed presentation of this construction in Section 2.2. What matters for us is that such a situation is potentially problematic for the derivation of plate theories by Γ-convergence: A weakly converging sequence of invertible (a.e.) functions might result in a non-invertible (a.e.) configuration with finite elastic energy in the 2d limit theory. The obvious cure would be, of course, to impose condition (INV) on the 3d theory. In the present contribution, we show that this is not necessary.

In contrast to the existing mathematical literature on interpenetration of matter that mainly focuses on finding sufficient conditions for invertibility of elastic deformations, we here identify sufficient conditions for non-invertibility. Questions related to the image of Sobolev functions are known to be a delicate issue, and these objects may display counter-intuitive features, *cf.* the pathological examples going back to Besicovitch [5, 15]. Here, such pathologies are not problematic, because we want to show that the image of the considered functions is sufficiently large.

This will be achieved in the main theorem of the present paper, Theorem 2.5. We will assume the typical conditions fulfilled by sequences of elastic deformations of thin films in the derivation of 3d-to-2d Γ-limits. Additionally, we will assume that the limit configuration is non-invertible in a suitable sense, see Definition 2.2. This definition is crucial for our method of proof to be workable. The statement of Theorem 2.5 is that under these assumptions, the considered sequence y_h of elastic deformations must consist of non-invertible functions for h small enough as $h \to 0$.

The structure of the present paper is as follows. In Section 2, we state our main result. In Section 3, we recall some results from the literature that we will use for its proof. The proof of the theorem (see Sect. 4) is based on a reduction to a 2-dimensional domain. The intersection on a sufficiently large set in the 2d-domain is proved by a homotopy argument, and the passage back to the 3-dimensional situation is performed with the help of the geometric rigidity result by Friesecke *et al.* [11]. In Section 5, we recall the derivation of plate theories as Γ-limits of 3d-nonlinear elasticity, and obtain some straightforward corollaries from the application of Theorem 2.5 to these settings.

Notation. The symbol C will be used as follows. A statement such as "$f \leq Cg$", where f, g are quantities that depends on a variable x, is to be read as: there exists a numerical constant $C > 0$ with the property that $f \leq Cg$ for all x. The value of C may change from one line to the next.

The d-dimensional Lebesgue measure is denoted by $\mathcal{L}^d$. Further we write $\omega(m) = \Gamma(1/2)^m/\Gamma(m/2+1)$; if $m \in \mathbb{N}$, then $\omega(m)$ is the volume of the m-dimensional ball.

2. Statement of results

2.1. Brouwer degree

First we need to recall the definition and some basic properties of the Brouwer degree. For $U \subset \mathbb{R}^n$ bounded, $f \in C^\infty(\bar{U}, \mathbb{R}^n)$, and $y \in \mathbb{R}^n \setminus f(\partial U)$ such that $\det \nabla f(x) \neq 0$ for all $x \in f^{-1}(y)$, the Brouwer degree is defined by

$$\deg(f,U,y)=\sum_{x\in f^{-1}(y)}\operatorname{sgn}(\det\nabla f(x)).$$

One can show that for $\varphi\in C_0^\infty(\mathbb{R}^n)$ with $\operatorname{supp}(\varphi)\cap f(\partial U)=\emptyset$, and any $y\in\mathbb{R}^n$ in the same connected component of $\mathbb{R}^n\setminus f(\partial U)$ as $\operatorname{supp}(\varphi)$,

$$\deg(f,U,y)\int_{\mathbb{R}^n}\varphi(z)\,\mathrm{d}z=\int_U\varphi(f(x))\det\nabla f(x)\,\mathrm{d}x.$$

By this formula and approximation by smooth functions, one can define the degree for any continuous $f\in C^0(\bar U,\mathbb{R}^n)$ and $y\notin f(\partial U)$. One can show that the degree only depends on $f|_{\partial U}$. Hence, from now on, we write $\deg(f,\partial U,y)\equiv\deg(f,U,y)$. Another basic property of the degree is

$$\deg(f,\partial U,y)\neq 0\quad\Rightarrow\quad y\in f(U).$$

On each connected component of $\mathbb{R}^n\setminus f(\partial U)$, $\deg(f,\partial U,\cdot)$ is constant. The latter yields the implication

$$y_0\in\partial\{y\in\mathbb{R}^n:\deg(f,\partial U,y)=k\}\quad\Rightarrow\quad y_0\in f(\partial U),\tag{2.1}$$

for any $k\in\mathbb{N}$. Finally, we will need the homotopy invariance of the degree: If $\gamma:[0,1]\to\mathbb{R}^n$ and $H:[0,1]\times\overline{U}\to\mathbb{R}^n$ are continuous, and $\gamma(t)\notin H(t,\partial U)$ for $t\in[0,1]$, then

$$\deg(H(0,\cdot),\partial U,\gamma(0))=\deg(H(1,\cdot),\partial U,\gamma(1)).\tag{2.2}$$

For the details of the definition and the proofs of the properties mentioned here, we refer to [9].

2.2. Invertibility almost everywhere and the example by Müller and Spector

Next we introduce appropriate notions of invertibility for Sobolev functions.

Definition 2.1 (Invertibility almost everywhere [3, 16])**.** Let $U\subset\mathbb{R}^n$, and let f be (a representative of an equivalence class) in $W^{1,1}(U,\mathbb{R}^n)$. We say f is invertible almost everywhere if there is a null set $N\subset U$ such that $f|_{U\setminus N}$ is injective.

Note that invertibility almost everywhere only depends on the equivalence class.

In [16], Müller and Spector gave an example of a sequence of a.e. invertible maps that weakly converge to a map that is 2-to-1 on a set of positive measure. (As in [16], by saying that a map u is 2-to-1 at a point x we mean that there exists exactly one point $\bar x\neq x$ such that $u(x)=u(\bar x)$.) Their examples were two-dimensional, but similar (slightly more complicated) constructions can be carried out in higher dimensions too. Crucial for their construction is the assumed regularity. The formation of cavities must be permitted, which is the case if the deformations are $W^{1,p}$ with $p<n$, where n is the dimension of the domain. We do not give the explicit formulas for the examples, but only give a qualitative explanation and refer to Figure 1, where the construction is sketched.

The domain of the example is a strip $\Omega\subset\mathbb{R}^2$. The deformations are in $W^{1,p}(\Omega,\mathbb{R}^2)$ for all $p<2$, and are constructed as follows: one starts off with the formation of one single cavity in a quadratic reference configuration, and subsequent continuous deformation. This is depicted in the upper left frame in Figure 1. In the upper right frame, this building block is scaled and periodically continued to a larger square. Two of these larger squares are the end parts of the deformed rectangular strip $u(\Omega)$. Then the strip is bent so that material from one end covers the voids from the other (see the lower left frame of Fig. 1). The map constructed in this way is invertible almost everywhere. Letting the period of the perforation tend to 0, the resulting sequence converges weakly in $W^{1,p}(\Omega,\mathbb{R}^2)$, for all $p<2$, to a deformation that is 2-to-1 on a set of positive measure (see the lower right frame of Fig. 1).

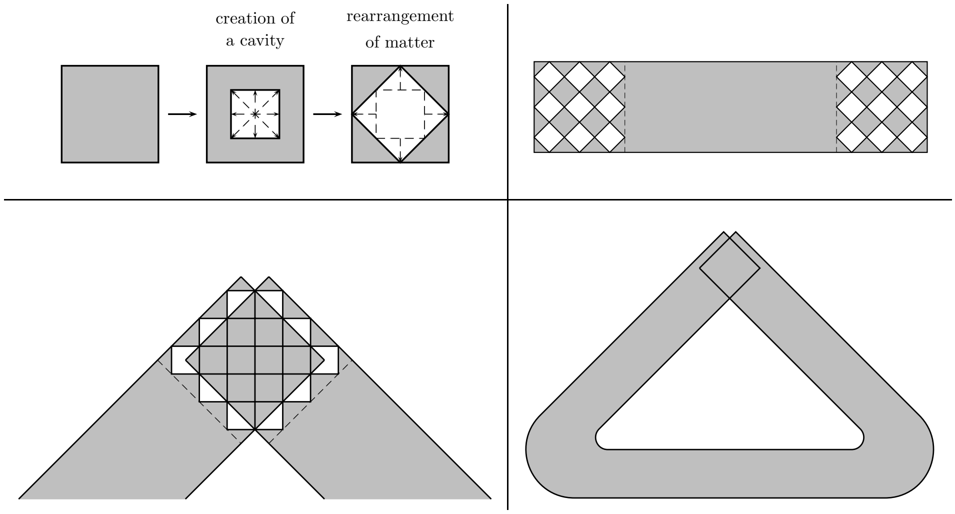

FIGURE 1. The pathological example by Müller and Spector.

2.3. Interpenetration for codimension one maps

For maps $\mathbb{R}^{n-1} \supset U \to \mathbb{R}^n$ as they occur in plate theories, the above definition of invertibility almost everywhere is not suitable. Here, modifications on sets of measure zero will be enough to make deformations with interpenetration of matter injective. We by-pass this problem by restricting ourselves to continuous deformations – in fact, we will even require Lipschitz continuity, since this is general enough for all applications to the derivation of plate theories.

For $U \subset \mathbb{R}^{n-1}$, we let $\hat{U} \subset \mathbb{R}^n$ denote the boundary of the cylinder over U:

$$\hat{U} := \partial\,(U \times [0,1]).$$

In the following, we will identify U with $U \times \{0\} \subset \hat{U}$.

Definition 2.2 (Interpenetration)**.** For $i \in \{1,2\}$, let $U_i \subset \mathbb{R}^{n-1}$ be simply connected Lipschitz domains and $u_i \in \mathrm{Lip}(U_i, \mathbb{R}^n)$. We say that u_2 interpenetrates u_1 if there exists a Lipschitz-continuous extension $\hat{u}_1 : \hat{U}_1 \to \mathbb{R}^n$ of u_1 with the following properties:

(i) The sets

$$\left\{x \in U_2 :\ u_2(x) \notin \hat{u}_1(\hat{U}_1),\ \deg\left(\hat{u}_1, \hat{U}_1, u_2(x)\right) = k\right\}, \quad k \in \mathbb{N}$$

have positive $\mathcal{L}^{n-1}$-measure for at least two different $k \in \mathbb{N}$.

(ii) The extension satisfies

$$\begin{aligned} \hat{u}_1(\hat{U}_1 \setminus U_1) \cap u_1(U_1) =& \emptyset, \\ \overline{\hat{u}_1(\hat{U}_1 \setminus U_1)} \cap \overline{u_2(U_2)} =& \emptyset. \end{aligned} \tag{2.3}$$

We have depicted the extension $\hat{u}_1 : \hat{U}_1 \to \mathbb{R}^n$ in Figure 2, and the typical situation of interpenetration in Figure 3.

Example 2.3. Let $U_1 = U_2 = [0,1]^2$ and $v_i : C^1(U_i)$ for $i = 1,2$. Moreover, suppose that the set $A_1 := \{x :\ v_1(x) < v_2(x)\}$ and that $A_2 := \{x :\ v_2(x) < v_1(x)\}$ are both open, non-empty and simply connected.

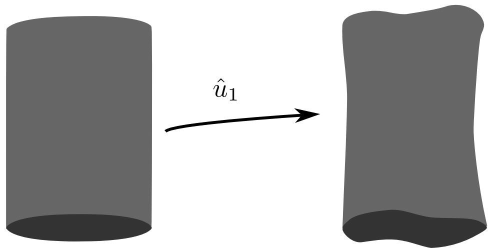

FIGURE 2. The extension $\hat{u}_1 : \hat{U}_1 \to \mathbb{R}^3$.

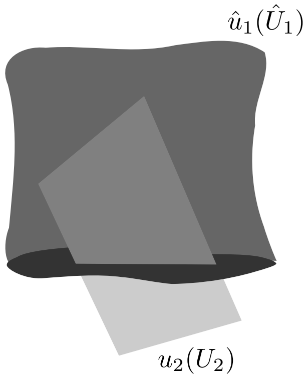

FIGURE 3. An example of interpenetration.

Set $u_i(x') = (x', v_i(x'))$ for $i = 1, 2$, and define the extension of u_1 by $\hat{u}_1 : \hat{U}_1 = \partial[0,1]^3 \to \mathbb{R}^3$, $(x', x_3) \mapsto (x', v_1(x') + x_3 M)$, where $M := \sup |v_2 - v_1|$. We will now check that $\deg(u_1, \hat{U}_1, z) = 0$ for $z \in A_2$ and that there exists $z \in A_1$ such that $\deg(u_1, \hat{U}_1, z) > 0$. Indeed, there is an obvious way to extend $\hat{u}_1$ to $[0,1]^3$:

$$\hat{u}_1^* : [0,1]^3 \to \mathbb{R}^3, \quad (x', x_3) \mapsto (x', v_1(x') + x_3 M).$$

Trivially, $\hat{u}_1 = \hat{u}_1^*$ on $\hat{U}_1$. Hence, $\deg(\hat{u}_1, \hat{U}_1, \cdot) = \deg(\hat{u}_1^*, \hat{U}_1, \cdot)$. For $\hat{u}_1^*$, we may compute the degree by (*cf.* Sect. 2.1)

$$\deg(\hat{u}_1^*, \hat{U}_1, z) = \sum_{x \in \hat{u}_1^{-1}(z)} \operatorname{sgn}(\det(\nabla \hat{u}_1^*(x))).$$

For every $z \in \mathbb{R}^3$, we have that $(\hat{u}_1^*)^{-1}(z)$ is either empty or has one element. In the latter case, it holds

$$\det(\nabla \hat{u}_1^*(x)) = \det \begin{pmatrix} \mathrm{id} & \mathbf{0} \\ (\nabla' v)^T & 1 \end{pmatrix} = 1,$$

where $\hat{u}_1^*(x) = z$. In particular for any $x \in A_2$, one has that $\deg(\hat{u}_1^*, \hat{U}_1, u_2(x)) = 0$. On the other side, one easily sees that for every $x \in A_1$ one has that

$$\deg(\hat{u}_1^*, \hat{U}_1, u_2(x)) = 1.$$

Thus u_2 interpenetrates u_1.

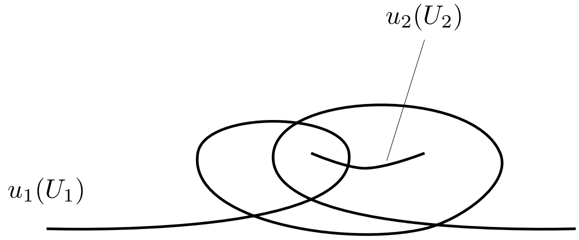

FIGURE 4. Interpenetrating curves with $\deg(\hat{u}_1, \hat{U}_1, u_2(x)) \neq 0$ for all extensions $\hat{u}_1$ of u_1 and $x \in U_2$.

Remark 2.4.

(1) Definition 2.2 is asymmetric with respect to u_1, u_2. This is done on purpose. It is always possible to reverse the roles by shrinking the domain of U_1, but we are neither going to prove nor use this fact.
(2) If U is closed and $u : U \to \mathbb{R}^3$ is an embedding, then there do not exist disjoint subsets $U_1, U_2 \subset U$ such that $u_2 := u|_{U_2}$ interpenetrates $u_1 := u|_{U_1}$. The converse is not true: there exist non-injective maps $u : U \to \mathbb{R}^3$ such that it is not possible to choose U_1, U_2 such that u_2 interpenetrates u_1 (defined as before). This is the case, *e.g.*, if the graphs $u(U_1)$ and $u(U_2)$ touch, but do not intersect. This is a desirable feature of a definition for interpenetration of matter. Indeed, two surfaces that are touching but can be separated via infinitesimal perturbations, should not be considered as interpenetrating, as they can be approximated by recovery sequences with disjoint graphs.
(3) Even though we chose the case of intersecting graphs to illustrate Definition 2.2 in Example 2.3, the definition is much more flexible than that. In particular, it is invariant under surface reparametrization.
(4) It might seem at first sight as if the requirement that the sets

$$\left\{x \in U_2 :\ u_2(x) \notin \hat{u}_1(\hat{U}_1),\ |\deg(\hat{u}_1, \hat{U}_1, u_2(x))| = k\right\}$$

have positive measure for $k \in \{0, 1\}$ would be equivalent to Definition 2.2 (i). However this would exclude cases such as the one depicted in Figure 4 (where $n = 2$), which are also covered by Definition 2.2 (i).

2.4. Statement of the main theorem

Let $S \subset \mathbb{R}^2$ be open and bounded, and let $\Omega_h = S \times (-h/2, h/2)$. We write $\Omega \equiv \Omega_1$. We will consider sequences of functions $z_h : \Omega_h \to \mathbb{R}^3$. It is convenient to define them on the same domain by introducing $y_h : \Omega \to \mathbb{R}^3$ *via* $y_h(x_1, x_2, x_3) = z_h(x_1, x_2, hx_3)$. Also, we introduce the scaled gradient

$$\nabla_h y = \left(\nabla' y, \frac{1}{h}\partial_3 y\right).$$

Theorem 2.5. *Let S, Ω_h and Ω be as above, and let $U_1, U_2 \subset S$ be disjoint simply connected Lipschitz sets. Let $u_1 : U_1 \to \mathbb{R}^3$, $u_2 : U_2 \to \mathbb{R}^3$ be Lipschitz and let u_2 interpenetrate u_1, let $\varepsilon > 0$ and y_h a sequence in $W^{1,2}(\Omega, \mathbb{R}^3)$ such that*

$$\|\mathrm{dist}\,(\nabla_h y_h, \mathrm{SO}(3))\|^2_{L^2(\Omega)} < Ch^{1+\varepsilon} \tag{2.4}$$

and

$$\fint_{-1/2}^{1/2} y_h(\cdot, x_3)\,\mathrm{d}x_3 \rightharpoonup u_i \quad \textit{in } W^{1,2}(U_i, \mathbb{R}^3) \textit{as } h \to 0 \textit{ for } i = 1, 2. \tag{2.5}$$

Then, for h small enough, y_h is not invertible almost everywhere.

Remark 2.6. The crucial assumption here is (2.4). This condition (or, more precisely, its 2-dimensional analog) is not fulfilled by the pathological examples from [16], whereas it does hold true for recovery sequences in the derivation of plate theories by Γ-convergence.

3. Preliminaries

For $A \subset \mathbb{R}^n$, we recall the definitions of m-dimensional Hausdorff and spherical Hausdorff pre-measures and of the "packing measure",

$$\mathcal{H}^m_\delta(A) = \inf\left\{\omega(m)\sum_j 2^{-m}\mathrm{diam}\,(A_j) : A \subset \cup_j A_j,\ \mathrm{diam}\,(A_j)/2 \le \delta\right\}$$

$$\mathcal{S}^m_\delta(A) = \inf\left\{\omega(m)\sum_j r_j^m : A \subset \cup_j B(x_j, r_j),\ r_j \le \delta\right\}$$

$$\mathcal{P}^m_\delta(A) = \omega(m)\delta^m \inf\left\{\#\{B(x_i,\delta)\} : \bigcup_i B(x_i,\delta) \supset A\right\}$$

where $m \in [0,\infty)$. In the above definition, we also allow $\delta = \infty$.

It is well known (see *e.g.* [10]) that the limits $\lim_{\delta\to 0}\mathcal{H}^m_\delta$, $\lim_{\delta\to 0}\mathcal{S}^m_\delta$ define Borel measures $\mathcal{H}^m, \mathcal{S}^m$ on $\mathbb{R}^n$, and that there exists a numerical constant $C = C(n)$ such that

$$C^{-1}\mathcal{S}^m_\delta(A) \le \mathcal{H}^m_\delta(A) \le C\mathcal{S}^m_\delta(A) \quad \text{and} \quad \mathcal{P}^m_\delta \ge \mathcal{S}^m_\delta(A) \ge \mathcal{H}^m_\infty$$

for every $A \subset \mathbb{R}^n$. Also, we recall the definition of the 1-capacity of a set $A \subset \mathbb{R}^n$,

$$\mathrm{cap}_1(A) = \inf\{\mathrm{Per}(E) :\ E \text{ is an open set of finite perimeter and } A \subset E\}.$$

From these definitions, it is easily seen that there exists a constant $C = C(n)$ with the property

$$\mathrm{cap}_1(A) \le C\mathcal{H}^1_\infty(A) \quad \text{for all } A \subset \mathbb{R}^n. \tag{3.1}$$

We cite the relative isoperimetric inequality for sets of finite perimeter. In the following statement, for a set of finite perimeter E, $\partial_* E$ denotes the reduced boundary of E (see [19]).

Theorem 3.1 ([19])**.** *Let $U \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary. Then there exists a constant $C = C(U)$ such that for every set $E \subset \mathbb{R}^n$ of finite perimeter,*

$$\min\{|E\cap U|, |U\setminus E|\}^{n-1/n} \le C\mathcal{H}^{n-1}(\partial_* E \cap U). \tag{3.2}$$

The same inequality holds true if one considers instead of U the whole $\mathbb{R}^n$. Namely, there exists a constant $C = C(n)$ such that

$$\min\{|E|, |\mathbb{R}^n\setminus E|\}^{n-1/n} \le C\mathcal{H}^{n-1}(\partial_* E). \tag{3.3}$$

Using the previous theorem, we will now prove a version of the isoperimetric inequality involving capacities instead of the Hausdorff measure. Note that due to $\mathrm{cap}_1 \le C\mathcal{H}^{d-1}$ the next lemma is stronger than Theorem 3.1.

Lemma 3.2. *Let U be a bounded open set with Lipschitz boundary. Then there exists a constant $C = C(U)$ such that for every bounded set E of finite perimeter,*

$$(\min(|E\cap U|, |U\setminus E|))^{(n-1)/n} \le C\,\mathrm{cap}_1(\partial_* E\cap U). \tag{3.4}$$

Proof. Suppose that the claim of the lemma were not true. Then there exists a sequence of sets $E_k \subset \mathbb{R}^n$ such that

$$(\min(|E_k \cap U|, |U \setminus E_k|))^{(n-1)/n} \geq k\,\mathrm{cap}_1(\partial_* E_k \cap U). \tag{3.5}$$

Let us split the proof in two cases: either there exists an M and a sequence $\{E_k\}$ such that

$$\min(|E_k \cap U|, |U \setminus E_k|)^{(n-1)/n} \geq M$$

or for every sequence such that (3.5) holds, one has that

$$\min(|E_k \cap U|, |U \setminus E_k|)^{(n-1)/n} \downarrow 0 \text{ as } k \uparrow \infty.$$

To deal with the first case, we will show that it is not possible to have $\min(|E \cap U|, |U \setminus E|) > M$ and $\mathrm{cap}_1(\partial_* E \cap U) < \varepsilon$, with ε suitably small. To show that also the second case leads to a contradiction, we will use a spatial scaling and basically reduce ourselves to having

$$\min(|E \cap U|, |U \setminus E|) = 1.$$

Case 1. There exists an M and subsequence $\{E_k\}$ such that $\min(|E_k \cap U|, |U \setminus E_k|)^{(n-1)/n} \geq M$. The boundedness of U implies that $\mathrm{cap}_1(\partial_* E_k \cap U) \leq \frac{|U|^{n/(n-1)}}{k}$. In particular, one has that $\mathrm{cap}_1(\partial_* E_k \cap U) \downarrow 0$. Fix $\varepsilon > 0$ sufficiently small. By the definition of 1-capacity, there exists an open set of finite perimeter V_k such that $\partial_* E_k \cap U \subset V_k$ and $\mathrm{Per}(V_k) \leq \mathrm{cap}_1(\partial_* E_k \cap U) + \varepsilon \leq 2\varepsilon$. Using the second part of Theorem 3.1, one has that $|V_k| \leq C\varepsilon^{n/(n-1)}$. Let us define $\tilde{E}_k := E_k \cup V_k$. We claim that

$$\partial_* \tilde{E}_k \cap U \subset \partial_* V_k. \tag{3.6}$$

Indeed, note that $\partial_*(E_k \cup V_k) \cap U \subset (\partial_* E_k \cup \partial_* V_k) \cap U$. By $\partial_* E_k \cap U \subset V_k$, one has that every $x \in \partial_* E_k$ is an interior point (and in particular a set of 1-density, see [19]), and thus $x \notin \partial_*(E_k \cup V_k) \cap U$ which proves (3.6).

Hence,

$$\begin{aligned}
\min(|E_k \cap U|, |U \setminus E_k|) &\leq C \min\left(|\tilde{E}_k \cap U|, |U \setminus \tilde{E}_k|\right) + C\varepsilon^{n/(n-1)} \\
&\leq C(U)\left(\left(\mathcal{H}^{n-1}(\partial_* \tilde{E}_k \cap U_k)\right)^{n/(n-1)} + \varepsilon^{n/(n-1)}\right) \\
&\leq C(U)\left((\mathrm{Per}(V_k))^{n/(n-1)} + \varepsilon^{n/(n-1)}\right) \\
&\leq C(U)\left(\varepsilon^{n/(n-1)}\right),
\end{aligned} \tag{3.7}$$

where in the second inequality above, we have used Theorem 3.1. By the arbitrariness of ε, we obtain a contradiction.

Case 2. Let us now suppose that for every sequence E_k such that (3.5) holds, one has that $\min(|E_k \cap U|, |U \setminus E_k|) \downarrow 0$. Without loss of generality we may assume that $\min(|E_k \cap U|, |U \setminus E_k|) = |E_k \cap U|$. Note that both sides of (3.4) have the same spatial scaling. Thus, there exists $\lambda_k > 0$ such that $|\lambda_k E_k| = 1$ and

$$|\lambda_k E_k|^{(n-1)/n} \geq k\,\mathrm{cap}_1((\lambda_k \partial_* E \setminus) \cap (\lambda_k U)).$$

Hence, by rescaling by λ_k one can assume without loss of generality that $|E_k| = 1$ and $U_k \uparrow \mathbb{R}^d$, where $U_k := \lambda_k U$.

After this observation the proof will proceed in a similar fashion as in the first case. As in the previous case, one has that $\mathrm{cap}_1(\partial_*(E_k \cap U_k)) \downarrow 0$. Using the definition of 1-capacity, there exists an open set of finite perimeter V_k such that $\partial_*(E_k \cap U_k) \subset V_k$ and $\mathrm{Per}(V_k) \leq \mathrm{cap}_1(\partial_*(E_k \cap U_k)) + \varepsilon \leq 2\varepsilon$. Without loss of generality, we may assume additionally that $|V_k| < +\infty$. Indeed, let $B \supset U$ be a ball, and $B_k := \lambda_k B$. Because B_k is convex one has that $\mathrm{Per}(B_k \cap V_k) \leq \mathrm{Per}(V_k)$ thus by taking $V_k \cap B_k$ instead of V_k, one has the additional requested property.

Using the second part of Theorem 3.1, one has that $|V_k| \leq C\varepsilon^{n/(n-1)}$. Denote by $\tilde{E}_k := E_k \cup V_k$ and notice as before that $\partial_* \tilde{E}_k \subset \partial_* V_k$. Hence, following exactly the same chain of inequalities as in (3.7), this gives a contradiction as before. □

3.1. Miscellaneous results from the literature

In the proof of our main theorem, we will use the following geometric rigidity result.

Theorem 3.3 ([11], Thm. 3.1)**.** *Let $U \subset \mathbb{R}^n$ be a bounded Lipschitz domain, with $n \geq 2$. Then there exists a constant $C = C(U)$ with the following property: For every $v \in W^{1,2}(\mathbb{R}^n)$, there is an associated rotation $R \in \mathrm{SO}(n)$ such that,*

$$\|\nabla v - R\|_{L^2(U)} \leq C \|\mathrm{dist}\,(\nabla v, \mathrm{SO}(n))\|_{L^2(U)}$$

The constant $C(U)$ is invariant under rescaling of the domain.

We will also use Zhang's Lemma [18]. An inspection of its proof in the latter reference shows that the following (slightly modified) statement holds true as well.

Theorem 3.4 ([18], Lem. 3.1)**.** *Let $K > 0$. There exist constants $C_1 = C_1(n, m)$, $C_2 = C_2(n, m, K)$ with the following property: If $U \subset \mathbb{R}^n$ is open and bounded, $f \in W^{1,1}(U, \mathbb{R}^m)$ and $\varepsilon > 0$ such that*

$$\int_{U \cap \{|\nabla f| \geq K\}} |\nabla f| \, \mathrm{d}x < \varepsilon,$$

then there exists $\tilde{f} \in W^{1,\infty}(U, \mathbb{R}^m)$ such that

$$\begin{aligned} \|\nabla \tilde{f}\|_{L^\infty(U)} &\leq C_1 K, \\ \mathcal{L}^n\left(\{x : f(x) \neq \tilde{f}(x)\}\right) &\leq C_2 \varepsilon. \end{aligned}$$

4. Proof of Theorem 2.5

Let S, Ω and Ω_h be as defined in Section 2.4. Our strategy is as follows. In Proposition 4.2 below, we will consider maps y_h on a 3-dimensional domain. We reduce the domain to 2 dimensions and assume that the thus obtained maps are 2-to-1 on a "large set" in terms of capacity, and we will show that this is sufficient to conclude that the maps y_h are 2-to-1 on a set whose $\mathcal{L}^3$-measure is of order h^2. This will be the main step in the proof of Theorem 2.5. In the proof of the proposition, we will need the following lemma:

Lemma 4.1. *Let $\varepsilon > 0$, $x_h \in S$, $\alpha \in (0, 1/2]$, and $y_h \in W^{1,\infty}(\Omega_h; \mathbb{R}^3)$ with*

$$\begin{aligned} \int_{B(x_h, \alpha h)} \mathrm{dist}^2(\nabla y_h, \mathrm{SO}(3)) \, \mathrm{d}x &\leq C h^{3+\varepsilon}, \\ \|\nabla y_h\|_{L^\infty} &\leq C. \end{aligned}$$

Then there exist rigid motions $A_h : \mathbb{R}^3 \to \mathbb{R}^3$ such that

$$\sup_{x' \in B(x_h, \alpha h)} |y_h(x') - A_h(x')| \leq C h^{1+\bar{\varepsilon}}. \tag{4.1}$$

where $\bar{\varepsilon} = \varepsilon/(3 + \varepsilon)$.

Proof. By Theorem 3.3, there exists a numerical constant $C = C(C_1, C_2)$ and $R_h \in \mathrm{SO}(3)$ such that

$$\int_{B(x_h,\alpha h)} |\nabla y_h - R_h|^2 \,\mathrm{d}x \le Ch^{3+\varepsilon}.$$

We set

$$b_h = \fint_{B(x_h,\alpha h)} (y_h(x) - R_h x)\,\mathrm{d}x.$$

By the Poincaré inequality, there exists $C = C(C_1, C_2)$ such that

$$\int_{B(x_h,\alpha h)} |y_h(x) - R_h x - b_h|^2 \,\mathrm{d}x \le Ch^{5+\varepsilon}. \tag{4.2}$$

Let A_h be the rigid motion

$$x \mapsto R_h x + b_h.$$

Since $\|y_h - A_h\|^2_{W^{1,2}(B(x_h,\alpha h))} \le Ch^{3+\varepsilon}$, and $\|y_h - A_h\|^2_{W^{1,\infty}(B(x_h,\alpha h))} \le C$, we also have

$$\|\nabla(y_h - A_h)\|^p_{L^p(B(x_h,\alpha h))} \le C(p)h^{3+\varepsilon}$$

for all $p \in [2,\infty)$.

Let $w_h = y_h - A_h$, and $B = B(x_h, \alpha h)$. Using (4.2) and Hölder's inequality, we have

$$\begin{aligned}\fint_B |w_h| &\le \frac{1}{\omega(3)(\alpha h)^3}\left(\int_B |w_h|^2\right)^{1/2} (\omega(3)(\alpha h)^3)^{1/2}\\ &\le Ch^{(2+\varepsilon)/2}.\end{aligned}$$

We set $p = 3 + \varepsilon$. For $x \in B$, we have the following estimate (which is used in a similar fashion in the proof of Morrey's Inequality, see *e.g.* the proof of the latter in [13])

$$\begin{aligned}\fint_B |w_h(x) - w_h(z)|\,\mathrm{d}z \le& C\int_B \frac{|\nabla w_h(z)|}{|x-z|^2}\,\mathrm{d}z\\ \le& C\left(\int_B |\nabla w_h|^p\right)^{1/p}\left(\int_B |x-z|^{-2p/(p-1)}\right)^{(p-1)/p}\\ \le& Ch^{(3+\varepsilon)/p}h^{1-3/p}\\ \le& Ch^{1+\varepsilon/(3+\varepsilon)}.\end{aligned}$$

Thus we get

$$\sup_{x\in B} |w_h(x)| \le \fint_B |w_h|\,\mathrm{d}z + \sup_{x\in B}\fint |w_h(x) - w_h(z)|\,\mathrm{d}z \le Ch^{1+\bar\varepsilon} \tag{4.3}$$

which proves (4.1). □

Proposition 4.2. *Let $y_h : \Omega_h \to \mathbb{R}^3$ be Lipschitz, and let $C^* \ge 1, \varepsilon > 0$ such that*

$$\begin{aligned}\int_{\Omega_h} \mathrm{dist}^2(\nabla y_h, \mathrm{SO}(3))\,\mathrm{d}x &\le C\,h^{2+\varepsilon},\\ \|\nabla y_h\|_{L^\infty} &\le C^*.\end{aligned}$$

Further, with $u_h(\cdot) = y_h(\cdot, 0)$, and

$$F_h := \{x :\ \textit{there exists } x' \in S \textit{ s.t. } u_h(x) = u_h(x') \textit{ and } |x - x'| > 2h\}$$

assume that

$$\mathrm{cap}_1(F_h) \geq C_1 \qquad \text{for all } h < h_0$$

for some constants $C_1, h_0 > 0$. *Then there exists* $c = c(C_1) > 0$ *such that for* h *small enough,*

$$\mathcal{L}^3\left(\{x : y_h \text{ is not 1-to-1 at } x\}\right) > ch^2. \tag{4.4}$$

Proof.

Step 1. Covering of F_h by balls of size h and definition of auxiliary partitions. For simplicity let us denote

$$E_h := \int_{\Omega_h} \mathrm{dist}^2(\nabla y_h, \mathrm{SO}(3))\,\mathrm{d}x.$$

Fix a set of points $X_h = \{x_i\}_{i\in I} \subset F_h$ such that $F_h \subset \cup_I B(x_i, h/2)$ and $B(x_i, h/10) \cap B(x_j, h/10) = \emptyset$ for $i \neq j$. Such a set X_h exists by Vitali's Covering Lemma. By definition of F_h, we may define a function $F_h \to F_h$, $x \mapsto \bar{x}$, such that $u_h(x) = u_h(\bar{x})$ and $B(x, h/2) \cap B(\bar{x}, h/2) = \emptyset$ for every $x \in F_h$.

In the following, we identify the points $x \in X_h \subset S$ with the points $(x, 0) \in S \times \{0\} \subset \Omega_h$, and whenever we speak of a ball around a point $x \subset X_h$, it is understood to be three-dimensional.

Now we introduce several useful partitions of X_h. First, we define the set of $x \in X_h$ with "low energy",

$$X_h^{\mathrm{low}} := \left\{ x \in X_h : \int_{B(x,h/10)} \mathrm{dist}^2(\nabla y_h, \mathrm{SO}(3)) \leq \frac{4hC_2}{C_1} E_h \right\}, \tag{4.5}$$

where C_2 is the constant from (3.1) with $n = 3$. The complement (the set of $x \in X_h$ with "high energy") is denoted by $X_h^{\mathrm{high}} = X_h \setminus X_h^{\mathrm{low}}$.

Secondly, for $x \in X_h$, we write $M(x) := B(x, h/(20C^*)) \cup B(\bar{x}, h/(20C^*))$. We define the set of x with "low pair-energy" as

$$\bar{X}_h^{\mathrm{low}} := \left\{ x \in X_h : \int_{M(x)} \mathrm{dist}^2(\nabla y_h, \mathrm{SO}(3)) \leq \frac{4hC_2}{C_1} E_h \right\}. \tag{4.6}$$

The complement (the set of $x \in X_h$ with "high pair-energy") is denoted by $\bar{X}_h^{\mathrm{high}} = X_h \setminus \bar{X}_h^{\mathrm{low}}$.

Finally, we introduce the partition $X_h = \mathcal{G}_h \cup \mathcal{B}_h$ where we call $\mathcal{G}_h$ the set of "good" points and $\mathcal{B}_h$ the set of "bad" points. We define the set of "good" points as the union $\mathcal{G}_h = \mathcal{G}_h^1 \cup \mathcal{G}_h^2$, where the latter are defined as follows,

$$\mathcal{G}_h^1 = \left\{ x \in X_h^{\mathrm{low}} : \exists x' \in X_h^{\mathrm{low}},\, x \neq x',\ \text{with } |y_h(x) - y_h(x')| \leq h/10 \right\}, \tag{4.7}$$

and

$$\mathcal{G}_h^2 = \bar{X}_h^{\mathrm{low}}. \tag{4.8}$$

Now we claim that there exists a constant $C_3 > 0$ such that for h small enough,

$$\#\mathcal{G}_h > C_3 h^{-1} \tag{4.9}$$

and

$$\mathcal{L}^3(\{x' \in B(x, h/10) : y_h \text{ is not 1-to-1 at } x'\}) > C_3 h^3 \qquad \text{for all } x \in \mathcal{G}_h. \tag{4.10}$$

This will be enough to prove the proposition since the balls of radius $h/10$ and centers in X_h are mutually disjoint.

Step 2. Proof of (4.9). Recalling the relations between capacities and Hausdorff pre-measures we have $\mathrm{cap}_1 \leq C_2\mathcal{H}^1_\infty \leq C_2\mathcal{P}^1_{h/2}$ for some numerical constant C_2. Hence $\mathcal{P}^1_{h/2}(F_h) \geq C_1 C_2^{-1}$, and in particular for every covering of F_h with balls $\{B_i\}$ of radius $h/2$ we have that

$$2\sum_i r(B_i) \geq C_1 C_2^{-1}, \tag{4.11}$$

where $r(B)$ denotes the radius of the ball B. Applying (4.11) to the cover by balls with centers in X_h constructed above, we get

$$\#X_h \geq \frac{1}{h}\mathcal{P}^1_h(F_h) \geq \frac{C_1}{hC_2}\cdot \tag{4.12}$$

By definition, $\mathcal{B}_h = X_h \setminus \mathcal{G}_h$, and hence

$$\mathcal{B}_h = \left\{ x \in \bar{X}_h^{\text{high}} : \text{Either } \left(x \in X_h^{\text{high}} \right) \text{ or } \left(\nexists x' \in X_h^{\text{low}},\, x \neq x',\text{ with } |y_h(x) - y_h(x')| \leq h/10 \right) \right\}. \tag{4.13}$$

Hence we have $\mathcal{B}_h \subset \mathcal{B}^1_h \cup \mathcal{B}^2_h$ with

$$\mathcal{B}^1_h = X_h^{\text{high}} = \left\{ x \in X_h : \int_{B(x,h/10)} \operatorname{dist}^2(\nabla y_h, \mathrm{SO}(3)) > \frac{4hC_2}{C_1} E_h \right\}, \tag{4.14}$$

and

$$\mathcal{B}^2_h = \left\{ x \in X_h^{\text{low}} : \left(x \in \bar{X}_h^{\text{high}} \right) \text{ and } \left(\nexists x' \in X_h^{\text{low}},\, x' \neq x,\text{ with } |y_h(x) - y_h(x')| \leq h/10 \right) \right\}. \tag{4.15}$$

By (4.14) and the fact that the $h/10$-balls with centers in X_h are mutually disjoint, we have

$$\#\mathcal{B}^1_h \leq \frac{C_1}{4C_2 h}\cdot \tag{4.16}$$

For $x_1, x_2 \in \mathcal{B}^2_h$, we have

$$\begin{aligned} |\bar{x}_1 - \bar{x}_2| \geq& \frac{1}{C^*}|y_h(\bar{x}_1) - y_h(\bar{x}_2)| \\ =& \frac{1}{C^*}|y_h(x_1) - y_h(x_2)| \\ \geq& \frac{h}{10C^*}, \end{aligned} \tag{4.17}$$

and hence the balls $B(\bar{x}, h/(20C^*))$ with $x \in \mathcal{G}^2_h$ are mutually disjoint. By the definition of $\bar{X}_h^{\text{high}}$, this implies

$$\#\mathcal{B}^2_h \leq \frac{C_1}{2C_2 h}\cdot \tag{4.18}$$

Combining (4.12), (4.16) and (4.18), we have proved (4.9) for $C_3 \leq \frac{C_1}{4C_2}^c dot$

Step 3. Proof of (4.10) for $x \in \mathcal{G}^1_h$. Let $x \in \mathcal{G}^1_h$. By the definition of $\mathcal{G}^1_h$ in (4.7), there exists $x' \in \mathcal{G}^1_h$, $x' \neq x$, with $|y_h(x) - y_h(x')| \leq h/10$. Let $B^x_h = B(x, h/10)$, $B^{x'}_h = B(x', h/10)$. The conditions of Lemma 4.1 with $\alpha = 1/10$ are fulfilled for y_h on both of these balls, and hence we obtain the existence of rigid motions $A^x_h, A^{x'}_h$ (depending on x, x', h) that satisfy

$$\sup_{z \in B^x_h} |y_h(z) - A^x_h(z)| \leq Ch^{1+\bar{\varepsilon}}, \qquad \sup_{z \in B^{x'}_h} |y_h(z) - A^{x'}_h(z)| \leq Ch^{1+\bar{\varepsilon}}. \tag{4.19}$$

Note that C is independent of $x, x' \in \mathcal{G}^1_h$ and of h. The images of B^x_h and $B^{x'}_h$ under A^x_h and $A^{x'}_h$ respectively are balls of radius $h/10$ and centers $A^x_h(x), A^{x'}_h(x')$. By (4.19),

$$\sup_{h>0} \inf_{x \in \mathcal{G}^1_h} \frac{1}{h}|A^x_h(x) - A^{x'}_h(x')| \leq 1/10. \tag{4.20}$$

Set

$$c_0 = \frac{\mathcal{L}^3(B(0,1)\cap B(e_1,1))}{\mathcal{L}^3(B(0,1))}\cdot$$

By (4.20), given $\delta > 0$ sufficiently small, we may choose $h_1 = h_1(\delta)$ such that for all $h < h_1$, there exists a set $W_h \subset B_h^x$ with

$$\mathcal{L}^3(W_h) \geq (c_0 - \delta)\mathcal{L}^3(B_h^x) \quad (4.21)$$

$$A_h^x(W_h) \subset A_h^{x'}(B_h^{x'}) \quad (4.22)$$

$$\operatorname{dist}(A_h^x(W_h), A_h^{x'}(\partial B_h^{x'})) \geq \frac{\delta h}{C}\cdot \quad (4.23)$$

In particular, (4.22) implies

$$\deg(A_h^{x'}, \partial B_h^{x'}, A_h^x(z)) = 1 \quad \text{for } z \in W_h. \quad (4.24)$$

We define homotopies $H_h^x : [0,1]\times \overline{B_h^x} \to \mathbb{R}^3$, $H_h^{x'} : [0,1]\times \overline{B_h^{x'}} \to \mathbb{R}^3$ by

$$\begin{aligned} H_h^x(t,z) &= t y_h(z) + (1-t)A_h^x(z) \\ H_h^{x'}(t,z) &= t y_h(z) + (1-t)A_h^{x'}(z). \end{aligned} \quad (4.25)$$

By (4.19) and (4.23), we have (for h small enough)

$$H_h^x(t,z) \notin H_h^{x'}(\partial B_h^{x'}) \qquad \text{for } t \in [0,1],\ z \in W_h.$$

By (2.2) and (4.24), this yields

$$\begin{aligned} \deg(A_h^{x'}, \partial B_h^{x'}, A_h^x(z)) &= \deg(H_h^{x'}(0,\cdot), \partial B_h^{x'}, H_h^x(0,z)) \\ &= \deg(H_h^{x'}(1,\cdot), \partial B_h^{x'}, H_h^x(1,z)) \\ &= \deg(y_h|_{B_h^{x'}}, \partial B_h^{x'}, y_h(z)) = 1 \quad \text{for } z \in W_h. \end{aligned}$$

By (4.21) and the arbitrariness of δ, this implies

$$\liminf_{h\to 0} \inf_{x\in\mathcal{G}_h^1} \frac{\mathcal{L}^3\left(\left\{z \in B_h^x : \deg(y_h|_{B_h^{x'}}, \partial B_h^{x'}, y_h(z)) = 1\right\}\right)}{\mathcal{L}^3(B_h^x)} \geq c_0. \quad (4.26)$$

Note that $\deg(y_h|_{B_h^{x'}}, \partial B_h^{x'}, y_h(z)) = 1$ is sufficient to conclude that y_h is not 1-to-1 at $z \in B_h^x$. Hence, (4.26) proves (4.10) for $x \in \mathcal{G}_h^1$.

Step 4. Proof of (4.10) for $x \in \mathcal{G}_h^2$. This closely parallels the previous step, this time using the balls $B_h^x = B(x, h/(20C^*))$, $B_h^{\bar{x}} = B(\bar{x}, h/(20C^*))$. As in the last step, we use Lemma 4.1 to obtain rigid motions $A_h^x, A_h^{\bar{x}}$ that satisfy

$$\sup_{z\in B_h^x} |y_h(z) - A_h^x(z)| \leq Ch^{1+\bar{\varepsilon}}, \qquad \sup_{z\in B_h^{x'}} |y_h(z) - A_h^{\bar{x}}(z)| \leq Ch^{1+\bar{\varepsilon}}.$$

Here, we even have

$$\lim_{h\to 0} \inf_{x\in\mathcal{G}_h^2} \frac{1}{h}|A_h^x(x) - A_h^{\bar{x}}(\bar{x})| = 0,$$

and hence

$$\lim_{h\to 0} \inf_{x\in\mathcal{G}_h^2} \frac{\mathcal{L}^3\left(\{z \in B_h^x : \deg(A_h^{\bar{x}}, \partial B_h^{\bar{x}}, A_h^x(z)) = 1\}\right)}{\mathcal{L}^3(B_h^x)} = 1,$$

and

$$\lim_{h\to 0}\inf_{x\in\mathcal{G}_h^2}\frac{\mathcal{L}^3\left(\{z\in B_h^x:\deg\left(A_h^{\bar x},\partial B_h^{\bar x},A_h^x(z)\right)=1\}\right)}{\mathcal{L}^3(B_h^x)}=1.$$

This proves (4.10) for $x\in\mathcal{G}_h^2$ and completes the proof of the proposition. □

Proof of Theorem 2.5. Let $z_h\in W^{1,2}(\Omega_h,\mathbb{R}^3)$ be defined by

$$z_h(x',hx_3)=y_h(x',x_3)\quad\text{ for all } x'\in S, x_3\in[-1/2,1/2].$$

By (2.4),

$$\|\text{dist}\,(\nabla z_h,\text{SO}(3))\|^2_{L^2(\Omega_h)}\le Ch^{2+\varepsilon}. \tag{4.27}$$

Step 1. Approximation by Lipschitz functions. Using (4.27),

$$\begin{aligned}\int_{\{|\nabla z_h|>2\sqrt3\}}|\nabla z_h|\,\mathrm{d}x\le&\frac{1}{2\sqrt3}\int_{\{|\nabla z_h|>2\sqrt3\}}|\nabla z_h|^2\,\mathrm{d}x\\ \le&\frac{4}{2\sqrt3}\int_{\{|\nabla z_h|>2\sqrt3\}}\text{dist}^{\,2}(\nabla z_h,\text{SO}(3))\,\mathrm{d}x\\ \le&Ch^{2+\varepsilon}.\end{aligned}$$

We apply Theorem 3.4 (with $K\to2\sqrt3$, $f\to z_h$, $\varepsilon\to Ch^{2+\varepsilon}$) and obtain $\tilde z_h\in W^{1,\infty}(\Omega_h,\mathbb{R}^3)$ such that

$$|\{z_h\neq\tilde z_h\}|\le Ch^{2+\varepsilon} \tag{4.28}$$

$$\|\nabla\tilde z_h\|_{L^\infty(\Omega_h)}\le C. \tag{4.29}$$

Step 2. Extension to a sphere. By Definition 2.2, there exists an extension $\hat u_1:\hat U_1\to\mathbb{R}^3$ such that equation (2.3) is fulfilled. For $\delta>0$, let

$$U_{1,\delta}:=\{x\in U_1:\text{dist}\,(x,\partial U_1)<\delta\}.$$

Now we choose δ so that

$$\overline{u_1(U_{1,\delta})}\cap\overline{u_2(U_2)}=\emptyset.$$

Such a choice of δ is possible by the fact that u_2 interpenetrates u_1, *cf.* Definition 2.2. Set $\bar\delta$:= dist $(u_1(U_{1,\delta}),u_2(U_2))$. Next let $\chi_\delta\in C_0^\infty(U_1)$ with $\chi_\delta=1$ on $U_1\setminus U_{1,\delta}$ and $\|\nabla\chi_\delta\|_{L^\infty}<C\delta^{-1}$. Set

$$\hat u_{1,h}(x)=\begin{cases}\tilde z_h(x,0) & \text{if } x\in U_1\setminus U_{1,\delta}\\ \chi_\delta(x)\,(\tilde z_h(x,0))+(1-\chi_\delta(x))u(x) & \text{if } x\in U_{1,\delta}\\ \hat u_1(x) & \text{if } x\in\partial\hat U_1\setminus U_1\end{cases} \tag{4.30}$$

and

$$u_{2,h}=\tilde z_h(\cdot,0)|_{U_2}. \tag{4.31}$$

Step 3. Convergence of Brouwer degree in L^1.
Let

$$E:=\{x\in U_2:\ u_2(x)\in\hat u_1(\hat U_1)\}.$$

We claim that

$$\begin{aligned}\deg\,(\hat u_{1,h},&\hat U_1,u_{2,h}(\cdot))\to\deg\,(\hat u_1,\hat U_1,u_2(\cdot))\\ &\text{in } L^1(U_2\setminus E)\quad\text{as } h\to0.\end{aligned} \tag{4.32}$$

We prove this claim by a homotopy argument.

By definition of z_h and (2.5),

$$\fint_{[-h/2,h/2]} \mathrm{d}x_3 z_h(\cdot, x_3) \rightharpoonup u_i \text{ in } W^{1,2}(U_i, \mathbb{R}^3).$$

By definition of $\tilde{z}_h$, this holds also true if z_h is replaced by $\tilde{z}_h$. By the uniform Lipschitz bound (4.28) on $\tilde{z}_h$, we also have

$$\tilde{z}_h(\cdot, 0) \rightharpoonup u_i \text{ in } W^{1,2}(U_i, \mathbb{R}^3).$$

By the definitions of $\hat{u}_{1,h}, u_{2,h}$ in (4.30) and (4.31), we get

$$\hat{u}_{1,h} \rightharpoonup \hat{u}_1 \text{ in } W^{1,2}(\hat{U}_1, \mathbb{R}^3) \quad \text{and} \quad u_{2,h} \rightharpoonup u_2 \text{ in } W^{1,2}(U_2, \mathbb{R}^3). \tag{4.33}$$

Since the uniform Lipschitz bound holds for $\tilde{z}_h$, there also exist uniform Lipschitz bounds for $\hat{u}_{1,h}$ and $u_{2,h}$ by definition of the latter two. Hence the weak convergence in (4.33) is also true in $W^{1,p}$ for every $1 < p < \infty$. By the compact Sobolev embedding, we have $\hat{u}_{1,h} \to \hat{u}_1$ and $u_{2,h} \to u_2$ in $C^{0,\alpha}$ for every $0 < \alpha < 1$, and in particular, we have uniform convergence.

Set $E_\varepsilon := \{x \in U_2 \setminus E : \mathrm{dist}\,(x, U_2)$. Since E is relatively closed in U_2, we have $\mathcal{L}^2(E_\varepsilon) \to 0$ as $\varepsilon \to 0$. The claim (4.32) follows from the continuity of the degree function in the first and the third argument with respect to uniform convergence.

Step 4. Application of isocapacitary inequality and passage back to 3d. By the definition of interpenetration (Def. 2.2), there exist $k_1, k_2 \in \mathbb{N}$, $k_1 \neq k_2$ and some $C > 0$ such that

$$\left| \{x \in U_2 : \deg(\hat{u}_1, \hat{U}_1, u_2(x)) = k_i\} \right| > C \text{ for } i = 1, 2.$$

Hence by Step 3, there exists $h_0 > 0$ such that

$$\left| \{x \in U_2 : \deg(\hat{u}_{1,h}, \hat{U}_1, u_{2,h}(x)) = k_i\} \right| > C \,\text{for } i = 1, 2 \tag{4.34}$$

for $h < h_0$ (which we assume from now on). Let

$$A_h := \{x \in U_2 : \deg(\hat{u}_{1,h}, \hat{U}_1, u_{2,h}(x)) = k_1\}$$

and let U_2° denote the interior of U_2. Then by (4.34), $\min(|A_h \cap U_2^\circ|, |U_2^\circ \setminus A_h|) > C$. We apply Lemma 3.2 and obtain

$$\mathrm{cap}_1(\partial A_h \cap U_2^\circ) > C. \tag{4.35}$$

On the other hand, $x \in \partial A_h \cap U_2^\circ$ implies

$$u_{2,h}(x) \in \partial\{y \in \mathbb{R}^3 : \deg(\hat{u}_{1,h}, \hat{U}_1, y) = k_1\}$$

and hence by (2.1),

$$\partial A_h \cap U_2^\circ \subset \{x \in U_2 : \, u_{2,h}(x) \in \hat{u}_{1,h}(\hat{U}_1)\}.$$

By the definition of $\hat{u}_{1,h}$ in (4.30) and the uniform convergence $\hat{u}_{1,h} \to \hat{u}_1$, $u_{2,h} \to u_2$, we may assume that $\mathrm{dist}\,(x, \partial U_2) > \delta$ whenever $u_{2,h}(x) \in \hat{u}_{1,h}(\hat{U}_1)$, whence $\hat{u}_{1,h}(x) = u_h(x)$ for $x \in \partial A_h \cap U_2^\circ$ and

$$\partial A_h \cap U_2^\circ \subset F_h := \{x \in U_2 : \text{ there exists } \bar{x} \text{s.t. } u_h(x) = u_h(\bar{x}) \text{ and } |x - \bar{x}| > 2h\}.$$

By (4.35), we have proved

$$\mathrm{cap}_1(F_h) > C.$$

By this last inequality and the results from step 1, the conditions of Proposition 4.2 are fulfilled and we can apply it to $\tilde{z}_h$ and obtain that

$$\mathcal{L}^3\left(\{x:\ \tilde{z}_h \text{ is not 1-to-1}\}\right) > ch^2.$$

By (4.28) and the definition of z_h, one has that

$$\mathcal{L}^3\left(\{x:\ z_h \text{ is not 1-to-1}\}\right) = \mathcal{L}^3\left(\{x:\ y_h \text{ is not 1-to-1}\}\right) > ch^2,$$

which concludes the proof. □

5. Application to plate theories derived as Γ-limits

As before, let $S \subset \mathbb{R}^2$ be open and bounded and $\Omega = S \times [-1/2, 1/2]$. We define the elastic energy of a 3-dimensional body. Let the inhomogeneous stored energy $W : \Omega \times \mathbb{R}^{3\times 3} \to [0, \infty)$ satisfy:

(i) $W(x, FR) = W(x, F)$ for all $R \in SO(n)$.
(ii) $W(x, \mathrm{id}_{3\times 3}) = 0$.
(iii) $W(x, F) \geq c\,\mathrm{dist}^2(F, SO(3))$ for some uniform constant c.
(iv) $W \in C^2(S, \mathcal{T})$ where $\mathcal{T}$ is an ϵ-neighborhood of $SO(3)$.
(v) $W(x, F) = W(z, F)$ if $(x - z)\|e_3$.

We introduce the quadratic forms $Q_3 : \Omega \times \mathbb{R}^{3\times 3} \to \mathbb{R}$, $Q_2 : S \times \mathbb{R}^{2\times 2} \to \mathbb{R}$ by

$$\begin{aligned} Q_3(\bar{x}, \bar{F}) &= \frac{D^2 W(x, F)}{DF^2}\Big|_{x=\bar{x}, F=\mathrm{id}}(\bar{F}, \bar{F}) \\ Q_2(\bar{x}', \bar{F}') &= \min\left\{Q_3(\bar{x}, F' + a \otimes e_3 + e_3 \otimes a) : a \in \mathbb{R}^3\right\}. \end{aligned}$$

The integral of W satisfying properties (i) through (v) above is the (rescaled) elastic energy functional

$$\begin{aligned} I_h : W^{1,2}(\Omega, \mathbb{R}^3) &\to \mathbb{R} \\ y &\mapsto \int_\Omega W(x, \nabla_h y(x))\,\mathrm{d}x. \end{aligned}$$

The penalization of interpenetration of matter is expressed in a modification of the 3d energy functional I_h, assigning infinite energy to non-physical deformations. We define $\bar{I}_h : W^{1,2}(\Omega, \mathbb{R}^3) \to \mathbb{R} \cup \{+\infty\}$ by

$$\bar{I}_h(y) = \begin{cases} \int_\Omega W(x, \nabla_h y(x))\,\mathrm{d}x & \text{if } y \text{ is invertible a.e.} \\ +\infty & \text{else.} \end{cases} \tag{5.1}$$

5.1. Contractive maps

In [7], the Γ-limit of the functional $h^{-\beta} I_h$ for the scaling regime $0 < \beta < 5/3$ has been derived (using results from [14]). The result can be stated as follows: We say $y_h \in W^{1,2}(\Omega_h, \mathbb{R}^3)$ converges uniformly to $u \in W^{1,2}(S, \mathbb{R}^3)$ as $h \to 0$ if

$$\lim_{h\to 0} \text{ess sup}_{(x_1,x_2,x_3)\in\Omega_h} |y_h(x_1, x_2, x_3) - u(x_1, x_2)| = 0.$$

Further, we say that $u \in W^{1,\infty}(S, \mathbb{R}^3)$ is *short* if

$$\nabla u^T \nabla u \leq \mathrm{id}_{2\times 2} \quad \text{a.e.}$$

i.e., $\mathrm{id}_{2\times 2} - \nabla u^T \nabla u$ is positive semi-definite almost everywhere. The Γ-convergence result from [7] can be stated as saying that for $0 < \beta < 5/3$,

$$\left(\Gamma - \lim_{h\to 0} h^{-\beta} I_h\right)(u) = \begin{cases} 0 & \text{if } u \text{ is short} \\ +\infty & \text{else,} \end{cases}$$

where the Γ-limit is taken with respect to uniform convergence. In fact, it could just as well have been formulated for weak convergence in $W^{1,2}(\Omega, \mathbb{R}^3)$ (see the discussion in [7]). This result includes the trivial lower bound

$$\liminf_{h\to 0} h^{-\beta} I_h(y_h) \geq 0$$

for sequences y_h that converge towards a short map u. The application of Theorem 2.5 immediately yields the following corollary, that is a sharper lower bound for $h^{-\beta}\bar{I}_h$ for $1 < \beta < 5/3$.

Corollary 5.1 (To Thm. 2.5)**.** *Let $1 < \beta$, $u \in W^{1,\infty}(S, \mathbb{R}^3)$, and let $U_1, U_2 \subset S$ be disjoint simply connected Lipschitz domains such that with $u_1 := u|_{U_1}$, $u_2 := u|_{U_2}$, u_2 interpenetrates u_1. Further let $y_h \in W^{1,2}(\Omega_h)$ converge uniformly to u. Then*

$$\liminf_{h\to 0} h^{-\beta} \bar{I}_h(y_h) = +\infty.$$

5.2. Nonlinear bending theory

In [11], the nonlinear Kirchhoff plate theory was obtained as the Γ-limit of the scaled functional $h^{-2} I_h$. Nonlinear plate theory can be defined as follows:

Let the set of $W^{2,2}$-isometries of S into $\mathbb{R}^3$ be denoted by

$$\mathcal{A} = \{u \in W^{2,2}(S, \mathbb{R}^3) : \nabla u^T \nabla u = \mathrm{id}_{2\times 2}\}.$$

Further, the second fundamental form is given by

$$\mathrm{II}_{[u]} = \nabla u^T \cdot \nabla \nu,$$

where $\nu = u_{,1} \wedge u_{,2}$ is the normal of the isometry u. Nonlinear plate theory may be defined via the energy functional

$$\begin{aligned} I^{\mathrm{Kh.}} : W^{2,2}(S, \mathbb{R}^3) &\to \mathbb{R} \cup \{+\infty\} \\ u &\mapsto \begin{cases} \frac{1}{24} \int_S Q_2(x', \mathrm{II}_{[u]})\,\mathrm{d}x' & \text{if } u \in \mathcal{A} \\ +\infty & \text{else.} \end{cases} \end{aligned}$$

The limiting deformations with finite bending energy will be the set of $y \in W^{1,2}(\Omega, \mathbb{R}^3)$ such that there exists $u \in \mathcal{A}$ with

$$y(x', x_3) = u(x') \text{ for a.e. } x' \in S, x_3 \in [-1/2, 1/2]. \tag{5.2}$$

We define the auxiliary functional $I^{\mathrm{Kh.}}_{3d} : W^{1,2}(\Omega, \mathbb{R}^3) \to \mathbb{R} \cup \{+\infty\}$ by

$$I^{\mathrm{Kh.}}_{3d}(y) = \begin{cases} I^{\mathrm{Kh.}}(u) & \text{if } \exists u \in \mathcal{A} \text{ such that equation (5.2) holds} \\ \infty & \text{else.} \end{cases} \tag{5.3}$$

Theorem 5.2 ([11], $\Gamma - \liminf$-inequality)**.**

Let $y_h, y \in W^{1,2}(\Omega, \mathbb{R}^3)$, $y_h \rightharpoonup y$ in $W^{1,2}(\Omega, \mathbb{R}^3)$. Then

$$\liminf_{h\to 0} h^{-2} I_h(y_h) \geq I^{\mathrm{Kh.}}_{3d}(y).$$

The application of Theorem 2.5 to nonlinear bending theory yields the following sharper version of the lower bound for $h^{-2}\bar{I}_h$.

Corollary 5.3 (To Thm. 2.5)**.** *Let $u \in \mathcal{A}$, and let $U_1, U_2 \subset S$ be disjoint simply connected Lipschitz domains such that with $u_1 := u|_{U_1}$, $u_2 := u|_{U_2}$, u_2 interpenetrates u_1. Further, let $y_h \rightharpoonup y$ in $W^{1,2}(\Omega, \mathbb{R}^3)$, with*

$$\limsup_{h\to 0} h^{-2} \|\text{dist}\,(\nabla_h y_h, \text{SO}(3))\|^2_{L^2(\Omega)} < \infty$$

and

$$y(x', x_3) = u(x') \quad \textit{for a.e.} \quad x' \in S,\, x_3 \in [-1/2, 1/2].$$

Then

$$\liminf_{h\to 0} h^{-2} \bar{I}_h(y_h) = +\infty.$$

Remark 5.4. In the case $\beta > 2$, a consequence of the compactness part of the Γ-convergence result for $h^{-\beta} I_h$ in [12] is the following: whenever

$$\limsup_{h\to 0} h^{-\beta} I^h(y_h) < \infty \quad \text{and} \quad y_h \rightharpoonup y \text{ in } W^{1,2}(\Omega, \mathbb{R}^3)$$

then y is (up to a rigid motion) just the projection onto the first two components, $y(x) = x'$. This indicates that if S is connected, and $U_1, U_2 \subset S$ are disjoint subsets, it is impossible to state sufficient conditions for the limits that assure that $y_h|_{(U_1 \cup U_2)\times[-1/2,1/2]}$ is 2-to-1 on a set of positive measure. One can still create a setting in which our main result is applicable, considering reference sets S with more than one connected component. We refrain from doing so here for the sake of brevity.

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