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OUTER TRANSFER FUNCTIONS OF DIFFERENTIAL-ALGEBRAIC SYSTEMS

Achim Ilchmann[1] and Timo Reis[2]

Abstract. We consider differential-algebraic systems (DAEs) whose transfer function is *outer*: *i.e.*, it has full row rank and all transmission zeros lie in the closed left half complex plane. We characterize outer, with the aid of the Kronecker structure of the system pencil and the Smith–McMillan structure of the transfer function, as the following property of a behavioural stabilizable and detectable realization: each consistent initial value can be asymptotically controlled to zero while the output can be made arbitrarily small in the $\mathcal{L}^2$-norm. The zero dynamics of systems with outer transfer functions are analyzed. We further show that our characterizations of outer provide a simple and very structured analysis of the linear-quadratic optimal control problem.

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Nomenclature

$\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$	the field of real numbers or complex numbers, resp.
$\mathbb{N}$, $\mathbb{N}_0$	set of natural numbers $\mathbb{N}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, resp.
$\mathbb{R}_{\geq 0}$, $\mathbb{R}_{>0}$,	set of non-negative, positive real numbers, resp.
$\mathbb{C}_+$, $\mathbb{C}_-$	open set of complex numbers with positive real part, negative real part, resp.
$\mathbb{K}[s]$, $\mathbb{K}(s)$	the ring of polynomials with coefficients in $\mathbb{K}$, and the quotient field of $\mathbb{K}[s]$, resp.
$p(s) \mid q(s)$	$p(s) \in \mathbb{K}[s]$ is a divisor of $q(s) \in \mathbb{K}[s]$;
$R^{n \times m}$	the set of $n \times m$ matrices with entries in a ring R;
$\mathrm{Gl}_n(R)$	the group of invertible $n \times n$ matrices with coefficients in a ring R;
I_n	identity matrix of size n;
$0_{m \times n}$	the zero matrix of $m \times n$;

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[1] Institut für Mathematik, Technische Universität Ilmenau, Weimarer Straße 25, 98693 Ilmenau, Germany.
achim.ilchmann@tu-ilmenau.de

[2] Fachbereich Mathematik, Universität Hamburg, Bundesstraße 55, 20146 Hamburg, Germany.
timo.reis@math.uni-hamburg.de

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$M^\top$, M^*	the transpose of $M \in \mathbb{R}^{m\times n}$ and conjugate transpose of $M \in \mathbb{C}^{m\times n}$, resp.
$M \succ 0$, $M \succeq 0$	$M \in \mathbb{C}^{n\times n}$ is Hermitian and positive definite, positive semi-definite, resp.
$\sigma(M)$	the spectrum of $M \in \mathbb{C}^{n\times n}$;
$\operatorname{diag}(A_1,\ldots,A_k)$	the block diagonal matrix with $A_i \in \mathbb{C}^{n_i\times m_i}$, $m_i, n_i \in \mathbb{N}_0$, for $i = 1,\ldots,k$ (*i.e.*, $A \in \mathbb{C}^{m\times n}$ with $m = m_1 + \ldots + m_k$, $n = n_1 + \ldots + n_k$);
$\dot f$	the distributional derivative of $f : \mathcal{I} \to \mathbb{K}^n$ with $\mathcal{I} \subseteq \mathbb{R}$;
$\operatorname{ess\,sup}_{t\in\mathcal{I}} f(t)$	the essential supremum of $f : \mathcal{I} \to \mathbb{R}$ on the set $\mathcal{I} \subseteq \mathbb{R}$;
$\mathcal{L}^2(\mathcal{I} \to \mathbb{K}^n)$, ($\mathcal{L}^2_{\text{loc}}(\mathcal{I}\to \mathbb{K}^n)$)	the set of measurable and (locally) square integrable functions $f : \mathcal{I} \to \mathbb{K}^n$ on the set $\mathcal{I} \subseteq \mathbb{R}$;
$\mathcal{AC}(\mathcal{I}\to \mathbb{R}^n)$	the set of functions $f : \mathcal{I}\to \mathbb{R}^n$ which are absolutely continuous on each compact interval $I \subset \mathcal{I}$ (see [14], p. 87);
$\mathcal{H}^2(\mathbb{C}_+\to \mathbb{C}^p)$	the Hardy space of holomorphic functions $\widehat{f} : \mathbb{C}_+ \to \mathbb{C}^p$ which have a square integrable extension to $i\mathbb{R}$ (see [9], Sect. A.6.3);
$\mathcal{H}^2(\mathbb{C}_+\to \mathbb{R}^p)$	$= \left\{ \widehat{f} \in \mathcal{H}^2(\mathbb{C}_+\to \mathbb{C}^p) \;\middle|\; \widehat{f}(\mathbb{R}_{>0}) \subset \mathbb{R}^p \right\}$;
$\mathcal{H}^\infty(\mathbb{C}_+\to \mathbb{C}^{p\times m})$	the Hardy space of bounded holomorphic functions $G\colon \mathbb{C}_+ \to\mathbb{C}^{p\times m}$ (see [9], Sect. A.6.3);
$\mathcal{H}^\infty(\mathbb{C}_+\to \mathbb{R}^{p\times m})$	$= \{ G \in \mathcal{H}^\infty(\mathbb{C}_+\to \mathbb{C}^{p\times m}) \mid G(\mathbb{R}_{>0}) \subset \mathbb{R}^{p\times m} \}$

1. Introduction

We consider linear differential-algebraic control systems of the form

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t}Ex(t) &= Ax(t) + Bu(t),\\ y(t) &= Cx(t) + Du(t), \end{aligned} \tag{1.1}$$

where $E, A \in \mathbb{K}^{n\times n}$, $B \in \mathbb{K}^{n\times m}$, $C \in \mathbb{K}^{p\times n}$, $D \in \mathbb{K}^{p\times m}$ and the pencil $sE - A \in \mathbb{K}[s]^{n\times n}$ is *regular*, *i.e.* $\det(sE - A) \in \mathbb{K}[s]\setminus\{0\}$; the set of these systems is denoted by $\Sigma_{n,m,p}(\mathbb{K})$ and we write $[E, A, B, C, D] \in \Sigma_{n,m,p}(\mathbb{K})$. $\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{C}$.

The function $u(\cdot) : \mathbb{R} \to \mathbb{K}^m$ is called *input*, $y(\cdot) : \mathbb{R} \to \mathbb{K}^p$ is called *output* of the system; we call $x(t)$ the *state* of $[E, A, B, C, D]$ at time $t \in \mathbb{R}$. A trajectory $(x(\cdot), u(\cdot), y(\cdot)) : \mathbb{R} \to \mathbb{K}^n \times \mathbb{K}^m \times \mathbb{K}^p$ is said to be a *solution* of (1.1) if it belongs to the *behaviour* of (1.1):

$$\mathfrak{B}_{[E,A,B,C,D]} := \left\{ (x, u, y) \in \mathcal{L}^2_{\text{loc}}(\mathbb{R}_{\ge 0}\to \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p) \;\middle|\; \begin{array}{l} Ex \in \mathcal{AC}(\mathbb{R}\to \mathbb{R}^n)\\ \text{and } (x, u, y) \text{ solves (1.1)}\\ \text{for almost all } t \in \mathbb{R} \end{array} \right\}.$$

In this article, we investigate *outer transfer functions*. In the single-input single-output case, the transfer function $G(s) \in \mathbb{K}(s)$ is scalar and we define

$$G(s) = \frac{\varepsilon(s)}{\psi(s)} \;\text{ is }\; \textit{outer} \quad :\Longleftrightarrow \quad \forall\, \lambda \in \mathbb{C}_+ \;:\; \varepsilon(\lambda) \neq 0\,, \quad \text{where } \varepsilon(s) \in \mathbb{K}[s],\; \psi(s) \in \mathbb{K}[s]\setminus\{0\}.$$

This means a scalar rational function is outer if, and only if, it is nonzero and all zeros are in the closed left half complex plane. The notion will be extended to multi-input multi-output transfer functions in Definition 3.1 in terms of the Smith–McMillan form.

Some of our results are also new for systems described by *ordinary differential equations* of the form

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t}x(t) &= Ax(t) + Bu(t), \qquad x(0) = x^0, \\ y(t) &= Cx(t) + Du(t), \end{aligned} \tag{1.2}$$

with unique solution $x(\cdot\,; x^0, u)$ and output $y(\cdot\,; x^0, u)$.

If the system (1.2) is stabilizable and detectable, then we will show that the transfer function satisfies the frequency domain criterium outer if, and only if, the following two properties hold:

(P3') $\qquad \forall\, y^0 \in \mathbb{K}^p \setminus \{0\}\ \exists\, x^0 \in \mathbb{R}^n, u \in \mathcal{L}^2_{\text{loc}}(\mathbb{R}_{\geq 0} \to \mathbb{K}^m)\ :\ (y^0)^* y(\cdot\,; x^0, u) \neq 0.$

(P4') $\qquad \begin{aligned} &\forall\, \varepsilon > 0\ \forall\, x^0 \in \mathbb{K}^n\ \exists\, u \in \mathcal{L}^2(\mathbb{R}_{\geq 0} \to \mathbb{K}^m) : \\ &\lim_{t\to\infty} x(t; x^0, u(\cdot)) = 0 \quad \wedge \quad \|y(\cdot\,; x^0, u)\|_{\mathcal{L}^2} < \varepsilon. \end{aligned}$

Property (P3') is simply equivalent to $\operatorname{rk}[C, D] = p$, as we will prove in Corollary 7.3.

Property (P4') means that for any initial condition one may find an $\mathcal{L}^2$-input such that the state is asymptotically steered to zero and the $\mathcal{L}^2$-norm of the output is arbitrarily small.

However, our main focus is on differential-algebraic equations **(DAEs)**. The Properties (P3') and (P4') become slightly more technical for DAEs since one has to take care of consistency of the initial value. The set of solutions of (1.1) which satisfies the initial condition $Ex(0) = Ex^0$ is denoted by

$$\mathfrak{B}_{[E,A,B,C,D]}(x^0) := \left\{ (x, u, y) \in \mathfrak{B}_{[E,A,B,C,D]}\ \middle|\ Ex(0) = Ex^0 \right\}.$$

The vector space of *consistent initial differential variables* of $[E, A, B, C, D]$ is denoted by

$$\mathcal{V}^{\text{diff}}_{[E,A,B,C,D]} := \left\{ x^0 \in \mathbb{K}^n\ \middle|\ \mathfrak{B}_{[E,A,B,C,D]}(x^0) \neq \emptyset \right\}.$$

The *transfer function* of $[E, A, B, C, D] \in \Sigma_{n,m,p}(\mathbb{K})$ is the rational function

$$G(s) = C(sE - A)^{-1}B + D \in \mathbb{K}(s)^{p\times m}.$$

Now the generalization of the Properties (P3') and (P4') is as follows

(P3) $\qquad \forall\, y^0 \in \mathbb{K}^p \setminus \{0\}\ \exists\, (x, u, y) \in \mathfrak{B}_{[E,A,B,C,D]}\ :\ (y^0)^* y(\cdot) \neq 0.$

(P4) $\qquad \begin{aligned} &\forall\, \varepsilon > 0\ \forall\, x^0 \in \mathcal{V}^{\text{diff}}_{[E,A,B,C,D]}\ \exists\, (x, u, y) \in \mathfrak{B}_{[E,A,B,C,D]}(x^0)\ : \\ &u \in \mathcal{L}^2(\mathbb{R}_{\geq 0} \to \mathbb{K}^m) \quad \wedge \quad \lim_{t\to\infty} Ex(t) = 0 \quad \wedge \quad \|y\|_{\mathcal{L}^2} < \varepsilon. \end{aligned}$

In Theorem 6.6 we will show, apart from some technicalities, that the transfer function of a behavioural stabilizable and detectable system (1.1) is outer if, and only if, Properties (P3) and (P4) holds.

Next we report the *literature* about outer transfer functions of systems described by ordinary differential equations. This class plays a fundamental role *e.g.* in $\mathcal{H}^\infty$-control, spectral factorization and linear-quadratic optimal control [5–8, 11, 13, 28]. For instance, it follows from the results in [23, 27] that the difference between the actual and optimal cost can be expressed as the square of the $\mathcal{L}^2$-norm of the spectral factor system which has an outer transfer function (*cf*. Sect. 8).

There are many different definitions of outer in the literature: in ([28], p. 366), a system (1.2) is called outer, if its transfer function belongs to $\mathcal{H}^\infty(\mathbb{C}_+ \to \mathbb{K}^{p\times m})$ and has full row rank in $\mathbb{C}_+$.

In [8], outer systems are defined *via* the property that the transfer function belongs to $\mathcal{H}^\infty(\mathbb{C}_+ \to \mathbb{K}^{p\times m})$ and that there exists a right inverse of the transfer function which has no poles in $\mathbb{C}_+$. In [17], outer (they are also called *minimum phase*) for infinite-dimensional systems governed by ordinary differential equations is defined by the property that the input-output map from $\mathcal{L}^2$ to $\mathcal{L}^2$ is bounded and the range of the input-output map is dense in $\mathcal{L}^2$. This is, in the frequency domain, equivalent to $G(s) \in \mathcal{H}^\infty(\mathbb{C}_+ \to \mathbb{K}^{p\times m})$ and the multiplication operator induced by the transfer function has dense range in $\mathcal{H}^2(\mathbb{C}_+ \to \mathbb{K}^p)$. In [25], analytic operator-valued functions are studied, where outer is defined via the property that a multiplication operator with dense range in $\mathcal{H}^2$ is induced. It is stated in [17] that for the rational matrix-valued case (*i.e.*, transfer functions of finite-dimensional systems) this is exactly the class of transfer functions of systems being outer according to the definition in [28].

We note that there is a certain inconsistency in the definition of *minimum phase* (which has been identified with outer in [17]). In [15] and further article of this author, systems (1.2) with $p = m$ and $D = 0$ are called minimum phase, if the system pencil $R(s) = \left[\begin{smallmatrix} sI-A, & -B \\ C & 0 \end{smallmatrix}\right]$ is invertible and all generalized eigenvalues lie in $\mathbb{C}_-$. Note that no stability assumption is required for this definition; however, generalized eigenvalues on the imaginary axis are not allowed in contrast to the aforementioned references. It is known that this minimum phase notion is equivalent to the fact that the *zero dynamics* (*i.e.*, the dynamics of the system generating a trivial output) are asymptotically stable. For a justification of the notion *minimum phase* in terms of Bode plots, we refer to [16] and the bibliography therein. The equivalence to asymptotical stability of the zero dynamics allows to generalize minimum phase to nonlinear systems [4].

In the present article we investigate outer differential-algebraic systems. We allow for transfer functions which are improper and/or have poles in the closed right half complex plane. Therefore, many applications (such as linear-quadratic optimal control) where asymptotic stability of the systems would be a restrictive assumption, are captured.

The paper is organized as follows. In Section 2, we collect some system theoretic concepts of differential-algebraic systems, the Kronecker canonical form and its consequences is investigated.

In Section 3, we show for behavioural stabilizable and detectable DAE (1.1) that its transfer function is outer if, and only if, the system pencil $R(s) = \left[\begin{smallmatrix} sE-A, & -B \\ C & D \end{smallmatrix}\right]$ has full row rank on $\mathbb{C}_+$. Furthermore, and this relates our concept to the definition in [8], outer is equivalent to the existence of a right inverse which has no poles in $\mathbb{C}_+$.

In Section 4, the zero dynamics of the DAE system (1.1) are studied and it is shown that outer is equivalent to the two properties: the system pencil satisfies $\mathrm{rk}_{\mathbb{K}(s)} R(s) = n + p$ and the zero dynamics are *polynomial stabilizability* (that is, for each consistent initial value, there exists a polynomially bounded trajectory of the zero dynamics). This allows to relate the present notion of outer to the notion of minimum phase in [15, 16].

In Section 5, we characterize outer of the transfer function $G(s)$ of the DAE system (1.1) if it is in addition stable, *i.e.* it belongs to $\mathcal{H}^\infty(\mathbb{C}_+ \to \mathbb{K}^{p\times m})$. We show that $G(s)$ is outer if, and only if, the input-output operator has dense range in $\mathcal{L}^2$. This means that our notion of outer is, in the stable case, equivalent to that of [17, 25].

Section 6 is the main section of the present paper. We show that outer and behavioural stabilizable DAE systems (1.1) have the property that any consistent initial value can be asymptotically controlled to zero under arbitrarily small output (in the $\mathcal{L}^2$-sense). The opposite statement holds true in the sense: if each consistent initial value can be asymptotically controlled to zero under arbitrarily small output, then some linearly dependent output components can be removed, such that an outer system remains.

In Section 7 we discuss (new) consequences of the previous sections for ODE systems (1.2) and show simple characterizations of outer.

Finally, in Section 8 the previous results are applied to the optimal control problem for ODE systems (1.2). Feasibility of the optimal control problem is characterized in terms of the Kalman–Yakubovich–Popov inequality and the Lur'e equation. These results are not new, but the approach is new. It provides a simple and very structured analysis of the optimal control problem. It also shows that the zero dynamics are instrumental to understand when the infimum of the optimal control problem is a minimum.

2. Preliminaries

In this section we recall some well-known basic concepts of system theory as well as of matrix pencils needed in the following sections; some results on matrix pencils are new.

2.1. System theory

Definition 2.1 (Impulse controllable, behavioural stabilizable, behavioural detectable)**.**
The system $[E, A, B, C, D] \in \Sigma_{n,m,p}(\mathbb{K})$ is called

$$\begin{aligned} \textit{impulse controllable} &:\iff \mathcal{V}^{\text{diff}}_{[E,A,B,C,D]} = \mathbb{K}^n, \\ \textit{behavioural stabilizable} &:\iff \forall\, x^0 \in \mathcal{V}^{\text{diff}}_{[E,A,B,C,D]}\ \exists\,(x,u,y) \in \mathfrak{B}_{[E,A,B,C,D]} \\ &\qquad :\ Ex(0) = Ex^0 \ \wedge\ \lim_{t\to\infty} Ex(t) = 0, \\ \textit{behavioural detectable} &:\iff \forall\,(x_1,u_1,y_1),\ (x_2,u_2,y_2) \in \mathfrak{B}_{[E,A,B,C,D]} \quad \text{with} \\ &\qquad u_1 = u_2,\ y_1 = y_2\ :\ \lim_{t\to\infty} E(x_1(t) - x_2(t)) = 0. \end{aligned}$$

Well-known characterizations of these concepts are the following.

Proposition 2.2 (Characterizations of impulse controllable, behavioural stabilizable and detectable)**.** *The system $[E, A, B, C, D] \in \Sigma_{n,m,p}(\mathbb{K})$ is*

$$\begin{aligned} &\text{(a)} \quad \textit{impulse controllable} && \iff && \operatorname{im}[E, B] + A \cdot \ker E = \mathbb{K}^n, \\ &\text{(b)}\ \textit{behavioural stabilizable} && \iff && \forall\,\lambda \in \overline{\mathbb{C}}_+ :\ \operatorname{rk}\left[\lambda E - A,\ B\right] = n, \\ &\text{(c)} \quad \textit{behavioural detectable} && \iff && \forall\,\lambda \in \overline{\mathbb{C}}_+ :\ \operatorname{rk}\begin{bmatrix} \lambda E - A \\ C \end{bmatrix} = n. \end{aligned}$$

Proof.

(a) See [1] and ([10], Thm. 2-2.3).
(b) In ([1], Rem. 3.11 and Cor. 3.12) it is shown that the definition of behavioural stabilizable is independent if $\mathfrak{B}_{[E,A,B,C,D]}$ is considered or the solution space of infinitely many times differentiable functions. Therefore ([22], Thm. 5.2.30) may be applied.
(c) This can be concluded from ([22], Thm. 5.3.17). □

2.2. Matrix pencils

A fundamental tool is the *Kronecker canonical form* which is a canonical form with respect to the following equivalence relation.

Definition 2.3 (System equivalence)**.**
Two pencils $\left[\begin{smallmatrix} sE_i - A_i, & -B_i \\ C_i & D_i \end{smallmatrix}\right] \in \mathbb{K}^{(n+p)\times(n+m)}[s]$, $i = 1, 2$, with $[E_i, A_i, B_i, C_i, D_i] \in \Sigma_{n,m,p}(\mathbb{K})$ are called *system equivalent*, if

$$\exists\, S, T \in \mathrm{Gl}_n(\mathbb{K})\ :\ \begin{bmatrix} S & 0 \\ 0 & I_p \end{bmatrix} \begin{bmatrix} sE_1 - A_1 & -B_1 \\ C_1 & D_1 \end{bmatrix} \begin{bmatrix} T & 0 \\ 0 & I_m \end{bmatrix} = \begin{bmatrix} sE_2 - A_2 & -B_2 \\ C_2 & D_2 \end{bmatrix}.$$

It can be verified immediately that system equivalence is an equivalence relation on $\mathbb{K}^{(n+p)\times(n+m)}[s]$. A canonical form of this equivalence relation is the Kronecker canonical form (KCF). To state this, the following notation is necessary:

$$N_k := \begin{bmatrix} 0 & & & \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix} \in \mathbb{R}^{k\times k},$$
$$K_k := \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \end{bmatrix}, \quad L_k := \begin{bmatrix} 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix} \in \mathbb{R}^{(k-1)\times k}, \quad k \in \mathbb{N}. \tag{2.1}$$

We are finally in a position to define the Kronecker canonical form. For the sake of the presentation, we will consider pencils over $\mathbb{C}$ and not over $\mathbb{R}$ since the *real* Kronecker form canonical form is more cumbersome.

Definition 2.4 (Kronecker canonical form (KCF))**.** The pencil $sF - G \in \mathbb{C}^{g\times \ell}[s]$ is said to be in *Kronecker canonical form* (KCF) if

$$sF - G \;=\; \operatorname{diag}\big(sF_1 - G_1, \dots, sF_f - G_f\big), \tag{2.2}$$

where each of the pencils $sF_j - G_j$ is one of the types

$$(\textbf{UD}) \qquad sK_k - L_k \qquad = \begin{bmatrix} -1 & s & & \\ & \ddots & \ddots & \\ & & -1 & s \end{bmatrix} \in \mathbb{R}^{(k-1)\times k}[s],$$

$$(\textbf{ODE}) \qquad (s-\lambda)I_k - N_k = \begin{bmatrix} s-\lambda & & & \\ -1 & \ddots & & \\ & \ddots & \ddots & \\ & & -1 & s-\lambda \end{bmatrix} \in \mathbb{C}^{k\times k}[s],$$

$$(\textbf{AE}) \qquad sN_k - I_k \qquad = \begin{bmatrix} -1 & & & \\ s & \ddots & & \\ & \ddots & \ddots & \\ & & s & -1 \end{bmatrix} \in \mathbb{R}^{k\times k}[s],$$

$$(\textbf{OD}) \qquad sK_k^\top - L_k^\top \qquad = \begin{bmatrix} -1 & & \\ s & \ddots & \\ & \ddots & -1 \\ & & s \end{bmatrix} \in \mathbb{R}^{k\times (k-1)}[s].$$

The acronyms (UD), (ODE), (AE) and (OD) refer to the following meaning of the associated DAE $(\frac{\mathrm{d}}{\mathrm{d}t}F_i - G_i)(x) = 0$: under determined, ordinary differential equation, algebraic equation, over determined ([12], Chap. XII, Sect. 7).

Further note that a 0×1 (UD)-block (or a 1×0) (OD)-block) before or after some $sF_j - A_j$ block means that a column (or row) is attached to the $sF_j - A_j$ block.

Remark 2.5 (Kronecker canonical form)**.** Let $sF - G \in \mathbb{C}^{g\times \ell}[s]$ with Kronecker canonical form (2.2). Since the rank is invariant under system equivalence, the following facts hold:

(a) $g - \operatorname{rk}[F, G] = \#\{$(OD)-blocks of size $1 \times 0\} = \#\{$zero rows in (2.2)$\}$,
(b) $g - \operatorname{rk}\left[\begin{smallmatrix} F \\ G \end{smallmatrix}\right] = \#\{$(UD)-blocks of size $0 \times 1\} = \#\{$zero columns in (2.2)$\}$,
(c) $g - \operatorname{rk}_{\mathbb{C}(s)} sF - G = \#\{$(OD)-blocks in (2.2)$\}$,

(d) $\ell - \operatorname{rk}_{\mathbb{C}(s)} sF - G = \#\{\text{(UD)-blocks in (2.2)}\}$
(e) For $\lambda \in \mathbb{C}$ we have:

$$\operatorname{rk} \lambda F - G < \operatorname{rk}_{\mathbb{K}(s)} sF - G \quad \iff \quad \text{(2.2) contains an (ODE)-block } (s-\lambda)I_j - N_j.$$

As a consequence of these observations, we conclude, for the case $g = \ell$,

(f) $sF - G$ is regular $\iff$ $\#\{\text{(OD)-blocks in (2.2)}\} = \#\{\text{(UD)-blocks in (2.2)}\} = 0.$

Kronecker's celebrated result is that in each equivalence class of a pencil $sF - G \in \mathbb{C}^{m\times n}[s]$ there is a Kronecker canonical form.

Theorem 2.6 (Kronecker canonical form ([12], Chap. XII, Sects. 4 and 5)). *For every pencil $sF - G \in \mathbb{C}^{m\times n}[s]$ there exist $S \in \mathrm{Gl}_m(\mathbb{C})$ and $T \in \mathrm{Gl}_n(\mathbb{C})$ such that $sSFT - SGT = S(sF - G)T$ is in Kronecker canonical form.*

Definition 2.7 (Generalized eigenvalue, index). For a pencil $sF - G \in \mathbb{K}^{g\times \ell}[s]$ we define

$$\begin{aligned} \lambda \in \mathbb{C} \text{ a } \textit{generalized eigenvalue} &:\iff \operatorname{rk}(\lambda F - G) < \operatorname{rk}_{\mathbb{K}(s)}(sF - G), \\ \nu \in \mathbb{N}_0 \text{ the } \textit{index} &:\iff sF - G \text{ is regular and } \nu \text{ is the size of the} \\ &\qquad\ \text{largest (AE)-block in (2.2).} \end{aligned}$$

Remark 2.8 (Kronecker canonical form, generalized eigenvalue).

(a) Let $sF - G \in \mathbb{C}^{g\times \ell}[s]$ with Kronecker canonical form (2.2). We obtain from Remark 2.5 (e) that

$$\lambda \in \mathbb{C} \text{ is a generalized eigenvalue of } sF - G \iff \text{(2.2) contains an (ODE)-block } (s-\lambda)I_k - N_k.$$

(b) For any system $[E, A, B, C, D] \in \Sigma_{n,m,p}(\mathbb{K})$ we have, by Remark 2.5(a), (c), (e), that

$$\forall\, \lambda \in \mathbb{C}_+ : \quad \operatorname{rk} \left[\begin{smallmatrix} E & A & B \\ 0 & C & D \end{smallmatrix}\right] = \operatorname{rk} \left[\begin{smallmatrix} \lambda E - A, & -B \\ C & D \end{smallmatrix}\right]$$

$$\iff \quad \begin{cases} \#\{\text{(OD)-blocks of (2.2) of size } 1\times 0\} \;=\; \#\{\text{(OD)-blocks of (2.2)}\} \\ \text{and each (ODE)-block } (s-\lambda)I_k - N_k \text{ of (2.2) has } \lambda \in \overline{\mathbb{C}}_-. \end{cases}$$

Another useful system equivalence form is the following:

Proposition 2.9 (System equivalence form (SEF)).
Let $[E, A, B, C, D] \in \Sigma_{n,m,p}(\mathbb{K})$. Then

$$\exists S, T \in \mathrm{Gl}_n(\mathbb{K}) : \begin{bmatrix} S & 0 \\ 0 & I_p \end{bmatrix} \begin{bmatrix} sE - A, & -B \\ C & D \end{bmatrix} \begin{bmatrix} T & 0 \\ 0 & I_m \end{bmatrix} = \begin{bmatrix} sE_{11} - A_{11} & sE_{12} & -B_1 \\ 0 & sN - I_k & 0 \\ C_1 & C_2 & D \end{bmatrix}, \tag{2.3}$$

where $N \in \mathbb{K}^{k\times k}$ is nilpotent, and $[E_{11}, A_{11}, B_1, C_1]$ is impulse controllable.
Furthermore, the following statements hold true:

(a) $\quad C(sE - A)^{-1}B + D = C_1 \left(sE_{11} - A_{11}\right)^{-1} B_1 + D.$

(b) $\quad (x, u, y) \in \mathfrak{B}_{[E,A,B,C,D]} \quad \iff \quad (x_1, u, y) \in \mathfrak{B}_{[E_{11},A_{11},B_1,C_1,D]}$, *where* $x = T \left(\begin{smallmatrix} x_1 \\ 0 \end{smallmatrix}\right)$.

(c) $\quad \mathcal{V}^{\mathrm{diff}}_{[E,A,B,C,D]} = T \begin{pmatrix} \mathbb{K}^{n-k} \\ \ker \left[\begin{smallmatrix} E_{12} \\ N \end{smallmatrix}\right] \end{pmatrix}.$

(d) $$\forall\,\lambda\in\mathbb{C}:\ \operatorname{rk}\begin{bmatrix}\lambda E-A & -B\\ C & D\end{bmatrix}=k+\operatorname{rk}\begin{bmatrix}\lambda E_{11}-A_{11} & -B_1\\ C_1 & D\end{bmatrix}.$$

Proof. The existence of a form (2.3) is shown in ([3], Prop. 4.6) where it is also shown that $[E_{11},A_{11},B_1,C_1]$ is controllable at infinity. The latter yields impulse controllable (see [1], Cor. 4.3).

Assertion (a) and (b) follow from direct calculations, where the fact is used that nilpotency of N gives: $N\dot{x}_2=x_2$ implies $x_2=0$.

Now we prove assertion (c): the inclusion $\mathcal{V}^{\text{diff}}_{[E,A,B,C,D]}\supset T\left(\begin{smallmatrix}\mathbb{K}^{n-k}\\ \{0\}\end{smallmatrix}\right)$ is an immediate consequence of (b) and impulse controllability of the subsystem $[E_{11},A_{11},B_1,C_1,D]$. Since, further, the trivial trajectory satisfies $(0,0,0)\in\mathfrak{B}_{[E,A,B,C,D]}(x^0)$ for all $x^0\in T\left(\begin{smallmatrix}\{0\}\\ \ker\left[\begin{smallmatrix}E_{12}\\ N\end{smallmatrix}\right]\end{smallmatrix}\right)$, we obtain $\mathcal{V}^{\text{diff}}_{[E,A,B,C,D]}\supset T\left(\begin{smallmatrix}\{0\}\\ \ker\left[\begin{smallmatrix}E_{12}\\ N\end{smallmatrix}\right]\end{smallmatrix}\right)$. Altogether, this gives $\mathcal{V}^{\text{diff}}_{[E,A,B,C,D]}\supset T\left(\begin{smallmatrix}\mathbb{K}^{n-k}\\ \ker\left[\begin{smallmatrix}E_{12}\\ N\end{smallmatrix}\right]\end{smallmatrix}\right)$.

To prove the opposite inclusion, let $x^0\in\mathcal{V}^{\text{diff}}_{[E,A,B,C,D]}$, and define $\left(\begin{smallmatrix}x_1^0\\ x_2^0\end{smallmatrix}\right)=T^{-1}x^0$. Then there exists some $(x,u,y)\in\mathfrak{B}_{[E,A,B,C,D]}(x^0)$, whence, by (b), $x=T\left(\begin{smallmatrix}x_1\\ 0\end{smallmatrix}\right)$ for some $\mathbb{K}^{n_1}$-valued function x_1. Thus we have $(x,u,y)\in\mathfrak{B}_{[E,A,B,C,D]}\left(T\left(\begin{smallmatrix}x_1^0\\ 0\end{smallmatrix}\right)\right)$. This leads to

$$(0,0,0)=(x,u,y)-(x,u,y)\in\mathfrak{B}_{[E,A,B,C,D]}\left(x^0-T\left(\begin{smallmatrix}x_1^0\\ 0\end{smallmatrix}\right)\right)=\mathfrak{B}_{[E,A,B,C,D]}\left(T\left(\begin{smallmatrix}0\\ x_2^0\end{smallmatrix}\right)\right),$$

whence $T\left(\begin{smallmatrix}0\\ x_2^0\end{smallmatrix}\right)=0$. Thus we have

$$\begin{bmatrix}E_{12}\\ N\end{bmatrix}x_2^0=\begin{bmatrix}I_{n_1} & E_{12}\\ 0 & N\end{bmatrix}\begin{pmatrix}0\\ x_2^0\end{pmatrix}=SET\left(\begin{smallmatrix}0\\ x_2^0\end{smallmatrix}\right)=0.$$

This gives $x_2^0\in\ker\left[\begin{smallmatrix}E_{12}\\ N\end{smallmatrix}\right]$. Altogether, we obtain $x^0\in T\cdot\left(\begin{smallmatrix}\{0\}\\ \ker\left[\begin{smallmatrix}E_{12}\\ N\end{smallmatrix}\right]\end{smallmatrix}\right)$, and therefore the inclusion $\mathcal{V}^{\text{diff}}_{[E,A,B,C,D]}\subset T\left(\begin{smallmatrix}\{0\}\\ \ker\left[\begin{smallmatrix}E_{12}\\ N\end{smallmatrix}\right]\end{smallmatrix}\right)$ holds true.

Finally, we prove assertion (d): since N is nilpotent, we have $\operatorname{rk}(\lambda N-I_k)=k$ for all $\lambda\in\mathbb{C}$, and therefore

$$\forall\lambda\in\mathbb{C}:\ \operatorname{rk}\begin{bmatrix}\lambda E-A & -B\\ C & D\end{bmatrix}=\operatorname{rk}\begin{bmatrix}\lambda E_{11}-A_{11} & \lambda E_{12} & -B_1\\ 0 & \lambda N-I_k & 0\\ C_1 & C_2 & D\end{bmatrix}=k+\operatorname{rk}\begin{bmatrix}\lambda E_{11}-A_{11} & -B_1\\ C_1 & D\end{bmatrix}.$$

This completes the proof of the proposition. □

Definition 2.10 (Feedback equivalence)**.** The two pencils $\left[\begin{smallmatrix}sE_i-A_i, & -B_i\\ C_i & D_i\end{smallmatrix}\right]\in\mathbb{K}^{(n+p)\times(n+m)}[s]$, $i=1,2$, with $[E_i,A_i,B_i,C_i,D_i]\in\Sigma_{n,m,p}(\mathbb{K})$ are called *feedback equivalent*, if

$$\exists S,T\in\mathrm{Gl}_n(\mathbb{K})\ \exists F\in\mathbb{K}^{m\times n}:\ \begin{bmatrix}S & 0\\ 0 & I_p\end{bmatrix}\begin{bmatrix}sE_1-A_1 & -B_1\\ C_1 & D_1\end{bmatrix}\begin{bmatrix}T & 0\\ F & I_m\end{bmatrix}=\begin{bmatrix}sE_2-A_2 & -B_2\\ C_2 & D_2\end{bmatrix}.$$

Remark 2.11 (Invariance under feedback equivalence)**.** We collect, using the notation from Definition 2.10, the following observations:

(i) Feedback equivalence is an equivalence relation since

$$\begin{bmatrix}sE-A & -B\\ C & D\end{bmatrix}=\begin{bmatrix}S^{-1} & 0\\ 0 & I_p\end{bmatrix}\begin{bmatrix}sE_2-A_2 & -B_2\\ C_2 & D_2\end{bmatrix}\begin{bmatrix}T^{-1} & 0\\ -F & I_m\end{bmatrix}.$$

(ii) $(x, u, y) \in \mathfrak{B}_{[E_1,A_1,B_1,C_1,D_1]} \iff (T^{-1}x, u - Fx, y) \in \mathfrak{B}_{[E_2,A_2,B_2,C_2,D_2]}$.
(iii) By Proposition 2.2(a), (b) we have:
$[E_1, A_1, B_1, C_1, D_1]$ is behavioural stabilizable/detectable
$\iff$ $[E_2, A_2, B_2, C_2, D_2]$ is behavioural stabilizable/detectable, resp.
(iv) $T \cdot \mathcal{V}^{\text{diff}}_{[E_1,A_1,B_1,C_1,D_1]} = \mathcal{V}^{\text{diff}}_{[E_2,A_2,B_2,C_2,D_2]}$.
(v) $C_1(sE_1 - A_1)^{-1}B_1 + D_1 = C_2(sE_2 - A_2)^{-1}B_2 + D_2$.

The following feedback equivalence form (FEF) will we very useful as well.

Proposition 2.12 (Feedback equivalence form (FEF))**.** *Let $[E, A, B, C, D] \in \Sigma_{n,m,p}(\mathbb{K})$. Then there exist $S, T \in \mathrm{Gl}_n(\mathbb{K})$ and $F \in \mathbb{K}^{m\times n}$ such that*

$$\begin{bmatrix} S & 0 \\ 0 & I_p \end{bmatrix} \begin{bmatrix} sE - A & -B \\ C & D \end{bmatrix} \begin{bmatrix} T & 0 \\ FT & I_m \end{bmatrix} = \begin{bmatrix} sI_{n_1} - A_{11} & 0 & 0 & -B_1 \\ 0 & -I_{n_2} & sE_{23} & -B_2 \\ 0 & 0 & sN - I_k & 0 \\ C_1 & C_2 & C_3 & D \end{bmatrix}, \tag{2.4}$$

where $N \in \mathbb{K}^{k\times k}$ is nilpotent. Furthermore, the following statements hold true:

(a) $(x, u, y) \in \mathfrak{B}_{[E,A,B,C,D]} \iff (x_1, u - Fx, y) \in \mathfrak{B}_{[I_n,A_{11},B_1,C_1,D-C_2B_2]}$, *where* $x = T\left(\begin{smallmatrix} x_1 \\ B_2[u-Fx] \\ 0 \end{smallmatrix}\right)$.

(b) $\mathcal{V}^{\text{diff}}_{[E,A,B,C,D]} = T \begin{pmatrix} \mathbb{K}^{n-k} \\ \ker \left[\begin{smallmatrix} E_{23} \\ N \end{smallmatrix}\right] \end{pmatrix}$.

(c) $\forall \lambda \in \mathbb{C}: \ \mathrm{rk} \begin{bmatrix} \lambda E - A & -B \\ C & D \end{bmatrix} = k + \mathrm{rk} \begin{bmatrix} \lambda I_{n_1} - A_{11} & 0 & -B_1 \\ 0 & -I_{n_2} & -B_2 \\ C_1 & C_2 & D \end{bmatrix}$.

(d) *If $[E, A, B, C, D]$ is behavioural stabilizable, then S, T, F can be chosen such that $\sigma(A_{11}) \subset \mathbb{C}_-$.*
(e) *If $[E, A, B, C, D]$ is impulse controllable, then S, T, F can be chosen such that $k = 0$.*

Proof. By Proposition 2.9 there exist $S_1, T_1 \in \mathrm{Gl}_n(\mathbb{K})$, such that

$$S_1(sE - A)T_1 = \begin{bmatrix} s\widetilde{E}_{11} - \widetilde{A}_{11} & s\widetilde{E}_{12} \\ 0 & sN - I_k \end{bmatrix}, \quad SB = \begin{bmatrix} \widetilde{B}_1 \\ 0 \end{bmatrix}, \quad CT = \begin{bmatrix} \widetilde{C}_1 & \widetilde{C}_2 \end{bmatrix}, \tag{2.5}$$

where $[\widetilde{E}_{11}, \widetilde{A}_{11}, \widetilde{B}_1, \widetilde{C}_1, D]$ is impulse controllable and $N \in \mathbb{K}^{k\times k}$ is nilpotent. By ([1], Thm. 5.2(a)), there exists some $\widetilde{F}_1 \in \mathbb{K}^{n-k}$ such that the index of $s\widetilde{E}_{11} - (\widetilde{A}_{11} + \widetilde{B}_1\widetilde{F}_1)$ is at most one. Consequently, there exist some $\widetilde{S}_1, \widetilde{T}_1 \in \mathrm{Gl}_{n-k}(\mathbb{K})$ such that

$$\widetilde{S}_1(s\widetilde{E}_{11} - (\widetilde{A}_{11} + \widetilde{B}_1\widetilde{F}_1))\widetilde{T}_1 = \begin{bmatrix} sI_{n_1} - A_{11} & 0 \\ 0 & -I_{n_2} \end{bmatrix}.$$

Then, for

$$T_2 := T_1 \left[\begin{smallmatrix} \widetilde{T}_1 & 0 \\ 0 & I_{n_2} \end{smallmatrix}\right], \qquad S_2 := \left[\begin{smallmatrix} \widetilde{S}_1 & 0 \\ 0 & I_{n_2} \end{smallmatrix}\right] S_1, \qquad F_2 := \left[\begin{smallmatrix} \widetilde{F}_1 & 0 \end{smallmatrix}\right] T_1^{-1},$$

we obtain a form

$$\begin{bmatrix} S_2 & 0 \\ 0 & I_p \end{bmatrix} \begin{bmatrix} sE - A & -B \\ C & D \end{bmatrix} \begin{bmatrix} T & 0 \\ F_2T & I_m \end{bmatrix} = \begin{bmatrix} sI_{n_1} - A_{11} & 0 & sE_{13} & -B_1 \\ 0 & -I_{n_2} & sE_{23} & -B_2 \\ 0 & 0 & sN - I_k & 0 \\ C_1 & C_2 & C_3 & D \end{bmatrix}.$$

By using ([2], Cor. 2.3), there exist $W_3, T_3 \in \mathrm{Gl}_n(\mathbb{K})$ such that $W_3W_2B = W_2B$ and the matrix E_{13} is eliminated in $W_3W_2(sE-(A+BF_2))T_2T_3$. Consequently, the form (2.4) is achieved for $W = W_3W_2$, $T = T_3T_3$, and $F = F_2$.

Assertion (a) is a consequence of Remark 2.11(ii) and the fact that nilpotency of N yields that $N\dot{x}_2 = x_2$ implies $x_2 = 0$.

Then assertion (b) is an immediate consequence of (a).

The proof of statement (c) is analogous to the proof of Proposition 2.9(d) and omitted.

We prove assertion (d). Assume that $[E, A, B, C, D]$ is behavioural stabilizable. Then the system $[\widetilde{E}_{11}, \widetilde{A}_{11}, \widetilde{B}_1, \widetilde{C}_1, D]$ is strongly stabilizable in the sense of ([1], Def. 2.1(k)) and ([1], Thm. 5.2(c) & Rem. 5.3(i)) implies that $\widetilde{F}_1$ can be chosen so that the index of $s\widetilde{E}_{11} - (\widetilde{A}_{11} + \widetilde{B}_1\widetilde{F}_1)$ is at most one and, furthermore, all generalized eigenvalues of $s\widetilde{E}_{11} - (\widetilde{A}_{11} + \widetilde{B}_1\widetilde{F}_1)$ lie in $\mathbb{C}_-$.

Assertion (e) follows since an impulse controllable system $[E, A, B, C, D]$ is already in a form (2.5) with $k = 0$. This completes the proof of the proposition. □

Many properties will be analyzed by means of the Smith–McMillan form; it is a canonical form on $\mathbb{K}(s)^{p\times m}$ under the group action of multiplication from the left and right with unimodular matrices (*i.e.*, units of the ring of square polynomial matrices).

Theorem 2.13 (Smith–McMillan form [18], Sect. 6.5.2)**.** *For $G(s) \in \mathbb{K}(s)^{p\times m}$ with $\mathrm{rk}_{\mathbb{K}(s)}\, G(s) = r$, there exist unimodular matrices $U(s) \in \mathrm{Gl}_m(\mathbb{K}[s])$ and $V(s) \in \mathrm{Gl}_m(\mathbb{K}[s])$ such that*

$$U^{-1}(s)G(s)V^{-1}(s) \;=\; \begin{bmatrix} D(s) & 0 \\ 0 & 0 \end{bmatrix}, \qquad \textit{where}\;\; D(s) = \mathrm{diag}\left(\frac{\varepsilon_1(s)}{\psi_1(s)}, \ldots, \frac{\varepsilon_r(s)}{\psi_r(s)}\right) \tag{2.6}$$

with unique monic and coprime polynomials $\varepsilon_i(s), \psi_i(s) \in \mathbb{R}[s]\backslash\{0\}$ satisfying $\varepsilon_i(s) \mid \varepsilon_{i+1}(s)$ and $\psi_{i+1}(s) \mid \psi_i(s)$ for all $i \in \{1, \ldots, r-1\}$.

Theorem 2.13 gives rise to the following (standard) definitions.

Definition 2.14 (Poles and zeros [28])**.** Let $G(s) \in \mathbb{K}(s)^{p\times m}$ with $\mathrm{rk}_{\mathbb{K}(s)}\, G(s) = r$ and use the notation from Theorem 2.13. Then

(a) $D(s)$ in (2.6) is called the *Smith–McMillan form of* $G(s)$;
(b) $\lambda \in \mathbb{C}$ is called a *transmission zero* of $G(s)$, if $\varepsilon_r(\lambda) = 0$;
(c) $\lambda \in \mathbb{C}$ is called an *invariant zero* of a realization $[E, A, B, C, D] \in \Sigma_{n,m,p}(\mathbb{K})$ of $G(s)$, if

$$\mathrm{rk} \begin{bmatrix} \lambda E - A, & -B \\ C & D \end{bmatrix} < \mathrm{rk}_{\mathbb{C}(s)} \begin{bmatrix} sE - A, & -B \\ C & D \end{bmatrix};$$

(d) $\lambda \in \mathbb{C}$ is called a *pole* of $G(s)$, if $\psi_1(\lambda) = 0$.

3. Outer transfer functions

The concept of an outer transfer function which will be defined next, is very closely related to the minimum phase property of ODE-systems. However, the term minimum phase is not treated consistently in the literature, in [16] this has been clarified. If minimum phase for a scalar strictly proper transfer function $G(s) \in \mathbb{K}(s)$ is understood in the sense that all zeros lie in the open left half plane, then the "only" difference to outer is that outer allows zeros on the imaginary axis as well. We are now ready to give the definition of outer for multivariable transfer functions.

Definition 3.1 (Outer transfer function)**.** A transfer function $G(s) \in \mathbb{K}(s)^{p\times m}$ with Smith–McMillan form (2.6) is called

$$\textit{outer} \qquad :\Longleftrightarrow \qquad \mathrm{rk}_{\mathbb{K}(s)}\, G(s) = p \quad \wedge \quad \forall\, \lambda \in \mathbb{C}_+ \;:\; \varepsilon_p(\lambda) \neq 0.$$

The following result relates outer transfer functions to the rank condition of an associated matrix pencil of the system $[E, A, B, C, D] \in \Sigma_{n,m,p}(\mathbb{K})$:

$$\textbf{(P1)} \qquad \forall\, \lambda \in \mathbb{C}_+ \;:\quad \operatorname{rk}\left[\begin{smallmatrix} \lambda E - A, & -B \\ C & D \end{smallmatrix}\right] = n + p.$$

Remark 3.2 (Property (P1))**.**

(a) Assume that $[E, A, B, C, D] \in \Sigma_{n,m,p}(\mathbb{K})$ fulfills (P1). Then $\operatorname{rk}[\lambda E - A,\ B] = n$ for all $\lambda \in \mathbb{C}_+$. Combining this fact with Proposition 2.2(b), we obtain

$$[E, A, B, C, D] \text{ is behavioural stabilizable} \quad\Longleftrightarrow\quad \forall \omega \in \mathbb{R}:\ \operatorname{rk}[i\omega E - A,\ B] = n.$$

(b) Consider a system $[E, A, B, C, D] \in \Sigma_{n,m,p}(\mathbb{K})$. In view of Remark 2.11(i), Property (P1) is invariant under feedback equivalence. Therefore, by Remark 2.5(c), (e) Property (P1) can be characterized in terms of the KCF as follows:

$$\text{(P1)} \quad\Longleftrightarrow\quad \begin{cases} \text{the blocks of the Kronecker canonical form of } \left[\begin{smallmatrix} sE - A, & -B \\ C & D \end{smallmatrix}\right] \text{ satisfy:} \\ \text{(OD)-blocks do not exist, and each (ODE)-block } (s-\lambda)I_k - N_k \text{ has } \lambda \in \overline{\mathbb{C}}_-. \end{cases}$$

Now we give the first "almost characterization" of outer transfer functions.

Theorem 3.3 (Characterization of (P1))**.** *For any $[E, A, B, C, D] \in \Sigma_{n,m,p}(\mathbb{K})$ with transfer function $G(s) = C(sE - A)^{-1}B + D \in \mathbb{K}(s)^{p\times m}$ we have:*

(a) (P1) $\Longrightarrow$ *$G(s)$ is outer.*

(b) (P1) $\Longleftarrow \begin{cases} G(s) \textit{ is outer and} \\ [E, A, B, C, D] \textit{ is behavioural stabilizable and detectable.} \end{cases}$

Proof. The proof follows from the observations in [24] specialized to systems of the form (1.1). □

We can furthermore characterize outer transfer functions by the structure of their right inverses.

Proposition 3.4 (Right inverses of outer functions)**.** *Let $[E, A, B, C, D] \in \Sigma_{n,m,p}(\mathbb{K})$ with transfer function $G(s) = C(sE - A)^{-1}B + D \in \mathbb{K}(s)^{p\times m}$. Then*

$$G(s) \text{ is outer} \quad\Longleftrightarrow\quad \begin{cases} \exists\, G^-(s) \in \mathbb{K}(s)^{m\times p} \;:\; G(s)G^-(s) = I_p \quad \text{and} \\ \lambda \in \mathbb{C} \text{ is a pole of } G^-(s) \;\Rightarrow\; \lambda \in \overline{\mathbb{C}}_-. \end{cases}$$

Proof. Suppose that $G(s)$ is in Smith–McMillan form (2.6). Then

$$G^-(s) = V^{-1}(s) \begin{bmatrix} D(s)^{-1} \\ 0 \end{bmatrix} U^{-1}(s)$$

is a right inverse and if $G(s)$ is outer, then $G^-(s)$ does not have any poles in $\mathbb{C}_+$.

To prove the converse, assume that $G^-(s) \in \mathbb{K}(s)^{m\times p}$ does not have any poles in $\mathbb{C}_+$ and is a right inverse: $G(s)G^-(s) = I_p$. Then, we have

$$V(s)G^-(s)U(s) = \begin{bmatrix} D(s)^{-1} \\ F(s) \end{bmatrix} \quad \text{for some } F(s) \in \mathbb{K}(s)^{(m-p)\times p}.$$

Since $G^-(s)$ has no poles in $\mathbb{C}_+$, this also holds true for $D(s)^{-1}$; and therefore $D(s)$ has no zeros in $\mathbb{C}_+$. This completes the proof of the proposition. □

Remark 3.5. The set of units in the ring $\mathcal{H}^\infty(\mathbb{C}_+ \to\ \mathbb{K}^{m\times m}) \cap \mathbb{K}(s)^{m\times m}$ is a subset of the class of outer function functions of dimension $m \times m$. The the set of outer functions in $\mathcal{H}^\infty(\mathbb{C}_+ \to \mathbb{K}^{m\times m}) \cap \mathbb{K}(s)^{m\times m}$ however further contains elements which might have inverses which are improper and/or have poles on the imaginary axis; this holds already true for $m = 1$.

4. Zero dynamics

An important time domain concept related to the pencil $\left[\begin{smallmatrix} sE-A, & -B \\ C & D \end{smallmatrix}\right]$ are the *zero dynamics*.

Definition 4.1 (Zero dynamics)**.** The *zero dynamics of* $[E,A,B,C,D] \in \Sigma_{n,m,p}(\mathbb{K})$ *is defined as*

$$\mathcal{ZD}_{[E,A,B,C,D]} := \left\{ (x,u) \in \mathcal{L}^2_{\rm loc}(\mathbb{R}_{\geq 0} \to \mathbb{K}^n \times \mathbb{K}^m) \;\middle|\; (x,u,0) \in \mathfrak{B}_{[E,A,B,C,D]} \right\}.$$

The set of zero dynamics initialized by the "initial state" $x^0 \in \mathbb{K}^n$ *is*

$$\mathcal{ZD}_{[E,A,B,C,D]}(x^0) := \left\{ (x,u) \in \mathcal{ZD}_{[E,A,B,C,D]} \;\middle|\; Ex^0 = Ex(0) \right\}.$$

The set of consistent initial differential variables for the zero dynamics *are*

$$\mathcal{ZD}^{\rm diff}_{[E,A,B,C,D]} := \left\{ x^0 \in \mathbb{K}^n \;\middle|\; \mathcal{ZD}_{[E,A,B,C,D]}(x^0) \neq \emptyset \right\}.$$

The zero dynamics $\mathcal{ZD}_{[E,A,B,C,D]}$ *are called*

$$\begin{aligned}
\text{polynomially bounded} :\Longleftrightarrow\ & \forall\, (x,u) \in \mathcal{ZD}_{[E,A,B,C,D]}\ \exists\, p(s) \in \mathbb{R}[s]\ \exists\, M \geq 0 \\
& \textit{for almost all } \tau \geq 0 : \ \left\| \big(x(\tau), u(\tau)\big) \right\| \leq M \cdot |p(\tau)|; \\
\text{asymptotically stable} :\Longleftrightarrow\ & \forall\, (x,u) \in \mathcal{ZD}_{[E,A,B,C,D]}\ :\ \lim_{t\to\infty} \operatorname{ess\,sup}_{\tau > t} \left\| \big(x(\tau), u(\tau)\big) \right\| = 0; \\
\text{polynomially stabilizable} :\Longleftrightarrow\ & \forall\, x^0 \in \mathcal{ZD}^{\rm diff}_{[E,A,B,C,D]} \\
& \exists\, p(s) \in \mathbb{R}[s]\ \exists\, M \geq 0\ \exists\, (x,u) \in \mathcal{ZD}_{[E,A,B,C,D]}(x^0) \\
& \textit{for almost all } \tau \geq 0 : \ \left\| \big(x(\tau), u(\tau)\big) \right\| \leq M \cdot |p(\tau)|; \\
\text{stabilizable} :\Longleftrightarrow\ & \forall\, x^0 \in \mathcal{ZD}^{\rm diff}_{[E,A,B,C,D]}\ \exists\, (x,u) \in \mathcal{ZD}_{[E,A,B,C,D]}(x^0) \\
& \lim_{t\to\infty} \operatorname{ess\,sup}_{\tau > t} \left\| \big(x(\tau), u(\tau)\big) \right\| = 0 \\
\text{autonomous} :\Longleftrightarrow\ & \forall\, x^0 \in \mathbb{K}^n \ :\quad \mathcal{ZD}_{[E,A,B,C,D]}(x^0) \textit{ contains at most one element.}
\end{aligned}$$

Alternatively, we may write the zero dynamics as

$$\mathcal{ZD}_{[E,A,B,C,D]} = \left\{ (x,u) \in \mathcal{L}^2_{\rm loc}(\mathbb{R}_{\geq 0} \to \mathbb{K}^n) \times \mathcal{L}^2_{\rm loc}(\mathbb{R}_{\geq 0} \to \mathbb{K}^m) \;\middle|\; \begin{array}{l} Ex \in \mathcal{AC}(\mathbb{R} \to \mathbb{R}^n) \quad \wedge \\ \begin{bmatrix} \frac{\mathrm{d}}{\mathrm{d}t}E-A, & -B \\ C & D \end{bmatrix} \begin{pmatrix} x \\ u \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{array} \right\}. \tag{4.1}$$

We will now show that the space of consistent initial differential variables is the whole $\mathbb{K}^n$. Namely, by using (4.1), performing the substitutions $E \rightsquigarrow \left[\begin{smallmatrix} E \\ 0 \end{smallmatrix}\right]$, $A \rightsquigarrow \left[\begin{smallmatrix} A \\ C \end{smallmatrix}\right]$ and $B \rightsquigarrow \left[\begin{smallmatrix} B \\ D \end{smallmatrix}\right]$ and invoking that $\ker \left[\begin{smallmatrix} E \\ 0 \end{smallmatrix}\right] = \ker E$, we can infer the subsequent statement from ([1], Cor 4.3):

Proposition 4.2. *For* $[E,A,B,C,D] \in \Sigma_{n,m,p}(\mathbb{K})$ *holds*

$$\mathcal{ZD}^{\rm diff}_{[E,A,B,C,D]} = \mathbb{K}^n \quad \Longleftrightarrow \quad \operatorname{im} \left[\begin{smallmatrix} E & B \\ 0 & D \end{smallmatrix}\right] + \left[\begin{smallmatrix} A \\ C \end{smallmatrix}\right] \cdot \ker E = \mathbb{K}^{n+p}.$$

Now all concepts in Definition 4.1 are characterized in terms of some algebraic properties of the pencil $R(s) = \left[\begin{smallmatrix} sE-A, & -B \\ C & D \end{smallmatrix}\right]$.

Proposition 4.3. *Let* $[E,A,B,C,D] \in \Sigma_{n,m,p}(\mathbb{K})$ *and set* $R(s) = \left[\begin{smallmatrix} sE-A, & -B \\ C & D \end{smallmatrix}\right]$. *Then the zero dynamics* $\mathcal{ZD}_{[E,A,B,C,D]}$ *are*

(a) autonomous $\iff$ $\operatorname{rk}_{\mathbb{R}(s)} R(s) = n+m$;
(b) polynomially bounded $\iff$ $\forall\, \lambda \in \mathbb{C}_+ \ : \ \operatorname{rk} R(\lambda) = n+m$;
(c) asymptotically stable $\iff$ $\forall\, \lambda \in \overline{\mathbb{C}}_+ \ : \ \operatorname{rk} R(\lambda) = n+m$;
(d) polynomially stabilizable $\iff$ $\forall\, \lambda \in \mathbb{C}_+ \ : \ \operatorname{rk}_{\mathbb{K}(s)} R(s) = \operatorname{rk} R(\lambda)$;
(e) stabilizable $\iff$ $\forall\, \lambda \in \overline{\mathbb{C}}_+ \ : \ \operatorname{rk}_{\mathbb{K}(s)} R(s) = \operatorname{rk} R(\lambda)$
$\iff$ $\forall\, x^0 \in \mathcal{ZD}^{\text{diff}}_{[E,A,B,C,D]}\ \exists\, (x,u) \in \mathcal{ZD}_{[E,A,B,C,D]}(x^0)$: $\lim_{t\to\infty} Ex(t) = 0$;
(f) polynomially bounded $\iff$ $[E,A,B,C,D]$ *is autonomous and polynomially stabilizable;*
(g) asymptotically stable $\iff$ $[E,A,B,C,D]$ *is autonomous and stabilizable*
$\iff$ $\forall\, x^0 \in \mathcal{ZD}^{\text{diff}}_{[E,A,B,C,D]}\ \exists!\, (x,u) \in \mathcal{ZD}_{[E,A,B,C,D]}(x^0)$: $\lim_{t\to\infty} Ex(t) = 0$.

Proof. By Theorem 2.6 we may choose $S \in \mathrm{Gl}_m(\mathbb{C})$ and $T \in \mathrm{Gl}_n(\mathbb{C})$ such that

$$SR(s)T \ = \ \operatorname{diag}\left(sF_1 - G_1, \ldots, sF_f - G_f\right) \tag{4.2}$$

is in Kronecker canonical form. Using (4.1), we see that

$$z = \begin{pmatrix} x \\ u \end{pmatrix} \in \mathcal{ZD}_{[E,A,B,C,D]} \quad\iff\quad (\tfrac{\mathrm{d}}{\mathrm{d}t}F_i - G_i)(z_i) = 0 \quad \forall\, i = 1, \ldots, f, \qquad \text{where } \begin{pmatrix} z_1 \\ \vdots \\ z_f \end{pmatrix} := T^{-1}z.$$

The representations of the solution sets of the DAEs $(\frac{\mathrm{d}}{\mathrm{d}t}F_i - G_i)(z_i) = 0$ in ([12], Chap. XII, Sect. 7) allow to conclude the following equivalences for the zero dynamics $\mathcal{ZD}_{[E,A,B,C,D]}$

(a') autonomous $\iff$ $\#\{\text{(UD)-blocks in (4.2)}\} = 0$;
(b') polynomially bounded $\iff$ $\#\{\text{(UD)-blocks in (4.2)}\} = 0$ and every (ODE)-block in (4.2) corresponds to a generalized eigenvalues in $\overline{\mathbb{C}}_-$;
(c') asymptotically stable $\iff$ $\#\{\text{(UD)-blocks in (4.2)}\} = 0$ and every (ODE)-block in (4.2) corresponds to a generalized eigenvalues in $\mathbb{C}_-$;
(d') polynomially stabilizable $\iff$ and every (ODE)-block in (4.2) corresponds to a generalized eigenvalues in $\overline{\mathbb{C}}_-$;
(e') stabilizable $\iff$ and every (ODE)-block in (4.2) corresponds to a generalized eigenvalues in $\mathbb{C}_-$.

Now by Remark 2.5(d) and (e) the assertions (a)–(e) follow from (a')–(e'), respectively.
Assertion (f) follows from a combination of assertions (a), (b), and (d).
Assertion (g) can be obtained by combining (a), (c), and (e). □

The above characterizations immediately give the following characterization of the rank condition (P1) which, in view of Theorem 3.3, is for stabilizable and detectable (in the behavioural sense) systems equivalent to the transfer function being outer.

Corollary 4.4 (Zero dynamics and (P1))**.** *Any* $[E,A,B,C,D] \in \Sigma_{n,m,p}(\mathbb{K})$ *with transfer function* $G(s) = C(sE-A)^{-1}B + D \in \mathbb{K}(s)^{p\times m}$ *satisfies:*

$$\textbf{(P1)} \qquad \iff \qquad \begin{cases} \operatorname{rk}_{\mathbb{K}(s)} \left[\begin{smallmatrix} sE-A, & -B \\ C & D \end{smallmatrix}\right] = n+p \quad \textit{and} \\ \mathcal{ZD}_{[E,A,B,C,D]} \ \textit{ is polynomable stabilizable.} \end{cases}$$

5. Stable outer transfer functions

We will now present a time domain characterization of outer transfer functions. Under the condition that the transfer function $G(s)$ belongs to $\mathcal{H}^\infty(\mathbb{C}_+ \to \mathbb{K}^{p\times m})$, *i.e.* all poles are in the open left half complex plane, the transfer function $G(s)$ is outer if, and only if, the time domain system $[E,A,B,C,D] \in \Sigma_{n,m,p}(\mathbb{K})$ satisfies the following Property (P2):

$$\textbf{(P2)}\quad \forall\, \varepsilon > 0 \quad \forall\, z \in \mathcal{L}^2(\mathbb{R}_{\geq 0} \to \mathbb{K}^p) \quad \exists\,(x,u,y) \in \mathfrak{B}_{[E,A,B,C,D]}(0)\ :\ \begin{cases} u \in \mathcal{L}^2(\mathbb{R}_{\geq 0} \to \mathbb{K}^m) \\ \wedge\ \ \|z-y\|_{\mathcal{L}^2} < \varepsilon. \end{cases}$$

Theorem 5.1 (Equivalence of outer and (P2))**.** *For any system* $[E,A,B,C,D] \in \Sigma_{n,m,p}(\mathbb{K})$ *with transfer function* $G(s) = C(sE-A)^{-1}B + D \in \mathbb{K}(s)^{p\times m} \cap \mathcal{H}^\infty(\mathbb{C}_+ \to \mathbb{K}^{m\times p})$ *we have*

$$G(s) \text{ is outer} \qquad \Longleftrightarrow \qquad \text{(P2)}.$$

Two technical lemmata are needed for the proof of the above theorem.

Lemma 5.2. *Let* $G(s) \in \mathbb{K}(s)^{p\times m} \cap \mathcal{H}^\infty(\mathbb{C}_+ \to \mathbb{K}^{p\times m})$ *be outer. Then*

$$\exists G_2(s) \in \mathbb{K}(s)^{(m-p)\times m} \cap \mathcal{H}^\infty(\mathbb{C}_+ \to \mathbb{K}^{(m-p)\times m}) : \begin{bmatrix} G(s) \\ G_2(s) \end{bmatrix} \in \mathbb{K}(s)^{m\times m} \cap \mathcal{H}^\infty(\mathbb{C}_+ \to \mathbb{K}^{m\times m}) \quad \text{is outer.} \tag{5.1}$$

Proof. Consider the Smith–McMillan form of $G(s)$ as in (2.6). Then Proposition 3.4 yields

$$U^{-1}(s)G(s)V^{-1}(s) = \begin{bmatrix} D(s) & 0 \end{bmatrix},$$

where, by the fact that $G(s)$ is outer and belongs to $\mathcal{H}^\infty(\mathbb{C}_+ \to \mathbb{K}^{p\times m})$, $D(s)$ neither has poles in $\overline{\mathbb{C}}_+$ nor zeros in $\mathbb{C}_+$. Since $V(s)$ is a polynomial matrix, we may choose $k \in \mathbb{N}$ such that

$$G_2(s) := \frac{1}{(s+1)^k}\, V(s) \in \mathcal{H}^\infty(\mathbb{C}_+ \to \mathbb{K}^{m\times m}).$$

Then we obtain that (5.1) holds true, and by

$$U^{-1}(s)\begin{bmatrix} G(s) \\ G_2(s) \end{bmatrix}\begin{bmatrix} V^{-1}(s) & 0 \\ 0 & I_{m-p} \end{bmatrix} = \begin{bmatrix} D(s) & 0 \\ 0 & \frac{1}{(s+1)^k} I_{m-p} \end{bmatrix},$$

we see that

$$\begin{bmatrix} G(s) \\ G_2(s) \end{bmatrix}^{-1} = V^{-1}(s)\begin{bmatrix} D(s)^{-1} & 0 \\ 0 & (s+1)^k \cdot I_{m-p} \end{bmatrix} U^{-1}(s)$$

does not have any poles in $\mathbb{C}_+$. The augmented matrix in (5.1) is therefore invertible and does not have any zeros in $\mathbb{C}_+$, whence it is outer. □

In the following lemma we show that Property (P2) yields that the input-output map of stable and outer system has dense range in $\mathcal{L}^2$. Therefore, for $\mathcal{H}^\infty$-transfer functions, our definition of an outer transfer function is equivalent to the definition in [17, 25].

Lemma 5.3. *For any* $G(s) \in \mathbb{K}(s)^{p\times m} \cap \mathcal{H}^\infty(\mathbb{C}_+ \to \mathbb{K}^{p\times m})$ *we have*

$$G(s) \text{ is outer} \quad \Longleftrightarrow \quad \begin{cases} \text{the multiplication operator} \\ M_G : \mathcal{H}^2(\mathbb{C}_+ \to \mathbb{K}^m) \to \mathcal{H}^2(\mathbb{C}_+ \to \mathbb{K}^p),\ \widehat{u}(s) \mapsto G(s)\widehat{u}(s)\ \forall s \in \mathbb{C}_+ \\ \text{has dense range.} \end{cases}$$

Proof.

(a) We prove the statement for $\mathbb{K} = \mathbb{C}$:
$\Longleftarrow$ Seeking for a contradiction, assume that $G(s)$ is not outer and M_G has dense range:
The Smith–McMillan form (2.6) implies that there exists some $\lambda \in \mathbb{C}_+$ and some $\xi \in \mathbb{C}^m \setminus \{0\}$ such that $\xi^* G(\lambda) = 0_{1,m}$. Consider the function

$$\widehat{z}(s) = \frac{\xi}{s - \overline{\lambda}} \in \mathcal{H}^2(\mathbb{C}_+ \to \mathbb{C}^m) \cap \mathbb{C}(s)^m. \tag{5.2}$$

By the density of $\operatorname{im} M_G$ in $\mathcal{H}^2(\mathbb{C}_+ \to \mathbb{K}^p)$ we have

$$\exists\, \widehat{u} \in \mathcal{H}^2(\mathbb{C}_+ \to \mathbb{C}^m) \ : \quad \|\widehat{z} - M_G \widehat{u}\|_{\mathcal{H}^2} < \frac{1}{\sqrt{2\operatorname{Re}(\lambda)}}. \tag{5.3}$$

Then, by using Cauchy's integral formula we obtain

$$\begin{aligned}\langle \widehat{z}, M_G \widehat{u}\rangle_{\mathcal{H}^2} &= \tfrac{1}{2\pi} \int_{-\infty}^{\infty} \left(\tfrac{1}{\imath\omega - \overline{\lambda}}\xi\right)^* G(\imath\omega)\, \widehat{u}(\imath\omega)\, \mathrm{d}\omega \\ &= \tfrac{1}{2\pi} \int_{-\infty}^{\infty} \tfrac{-1}{\imath\omega - \lambda} \cdot \xi^* G(\imath\omega)\, \widehat{u}(\imath\omega)\, \mathrm{d}\omega \ = \ -\xi^* G(\lambda)\, \widehat{u}(\lambda) \ = \ 0.\end{aligned}$$

Therefore, the Pythagorean theorem yields

$$\|\widehat{z} - M_G \widehat{u}\|_{\mathcal{H}^2}^2 = \|\widehat{z}\|_{\mathcal{H}^2}^2 + \|M_G \widehat{u}\|_{\mathcal{H}^2}^2 \geq \|\widehat{z}\|_{\mathcal{H}^2}^2 = \frac{1}{2\operatorname{Re}(\lambda)},$$

which contradicts (5.3).
$\Longrightarrow$ **Step 1**. We prove the implication $\Longrightarrow$ for the case $p = m$:
Since $G(s)$ is outer, it follows from the Smith–McMillan form (2.6) that the scalar rational function $g(s) := \det G(s) \in \mathbb{C}(s)$ is outer as well. Then, by ([21], p. 1251), the multiplication operator $M_g : \mathcal{H}^2(\mathbb{C}_+ \to \mathbb{C}) \to \mathcal{H}^2(\mathbb{C}_+ \to \mathbb{C})$ has dense range. Then the Helson–Lowdenslager Theorem ([20], p. 22) implies that M_G has dense range.
Step 2. We show the implication $\Longrightarrow$ for the case $p \neq m$:
Let $\widehat{z} \in \mathcal{H}^2(\mathbb{C}_+ \to \mathbb{C}^p)$ and $\varepsilon > 0$ be given. Using Lemma 5.2, we obtain that there exists some $G_2(s) \in \mathbb{K}(s)^{(m-p)\times m} \cap \mathcal{H}^\infty(\mathbb{C}_+ \to \mathbb{K}^{(m-p)\times m})$ such that

$$\widetilde{G}(s) := \begin{bmatrix} G(s) \\ G_2(s) \end{bmatrix} \in \mathbb{K}(s)^{m\times m} \cap \mathcal{H}^\infty(\mathbb{C}_+ \to \mathbb{K}^{m\times m})$$

is outer. By the result in Step 1 for the case $p = m$, there exists some $\widehat{u} \in \mathcal{H}^2(\mathbb{C}_+ \to \mathbb{C}^m)$ with

$$\left\| \begin{pmatrix} \widehat{z} \\ 0 \end{pmatrix} - \begin{pmatrix} M_G \widehat{u} \\ M_{G_1} \widehat{u} \end{pmatrix} \right\|_{\mathcal{H}^2} = \left\| \left(\begin{smallmatrix} \widehat{z} \\ 0 \end{smallmatrix}\right) - M_{\widetilde{G}} \widehat{u} \right\|_{\mathcal{H}^2} < \varepsilon.$$

This implies $\|\widehat{z} - M_G \widehat{u}\|_{\mathcal{H}^2} < \varepsilon$, whence this implication is shown.

(b) We prove the statement for $\mathbb{K} = \mathbb{R}$:
$\Longleftarrow$ Again seeking for a contradiction, assume that $G(s) \in \mathbb{R}(s)^{p\times m} \cap \mathcal{H}^\infty(\mathbb{C}_+ \to \mathbb{R}^{p\times m})$ is not outer and M_G has dense range.
Assume that $G(s)$ has a zero in $\lambda \in \mathbb{C}_+$. Then there exists some $z \in \mathbb{C}^p \setminus \{0\}$ with $z^* G(\lambda) = 0$. Since $G(s) \in \mathbb{R}(s)^{p\times m}$, we have that the element-wise conjugate of z satisfies $\overline{z}^* G(\overline{\lambda}) = 0$. Define the function $\widehat{z} \in \mathcal{H}^2(\mathbb{C}_+ \to \mathbb{C}^p) \setminus \{0\}$ as in (5.2). Then at least one of the functions

$$\widehat{z}_1 := \widehat{z} + \overline{\widehat{z}}(\overline{\cdot}) \in \mathcal{H}^2(\mathbb{C}_+ \to \mathbb{R}^p), \qquad z_2 = \tfrac{1}{\imath} \cdot (z - \overline{\widehat{z}}(\overline{\cdot})) \in \mathcal{H}^2(\mathbb{C}_+ \to \mathbb{R}^p)$$

is non-zero. Then, by the results in the complex case, we obtain for all $\widehat{u} \in \mathcal{H}^2(\mathbb{C}_+ \to \mathbb{R}^m)$ that $M_G\widehat{u}$ is, in the $\mathcal{H}^2$-sense, orthogonal to both $\widehat{z}_1$ and $\widehat{z}_2$ as in (5.2). This leads to the same contradiction as in the complex case.

$\Longrightarrow$ Let $G(s) \in \mathbb{R}(s)^{p\times m} \cap \mathcal{H}^\infty(\mathbb{C}_+ \to \mathbb{R}^{p\times m})$ be outer, $\widehat{z} \in \mathcal{H}^2(\mathbb{C}_+ \to \mathbb{R}^p)$ and $\varepsilon > 0$. Then, by (a), there exists some $\widehat{u}_1 \in \mathcal{H}^2(\mathbb{C}_+ \to \mathbb{C}^m)$, such that $\|\widehat{z} - M_G\widehat{u}\|_{\mathcal{H}^2} < \varepsilon$. Define

$$\widehat{u} := \frac{\widehat{u}_1 + \overline{\widehat{u}_1(\bar{\cdot})}}{2} \in \mathcal{H}^2(\mathbb{C}_+ \to \mathbb{R}^m).$$

The realness of $G(s)$ implies $M_G\overline{\widehat{u}_1(\bar{\cdot})} = \overline{M_G\widehat{u}_1(\bar{\cdot})}$, and thus

$$\begin{aligned}\|\widehat{z} - M_G\widehat{u}\|_{\mathcal{H}^2} &\leq \tfrac{1}{2}\cdot\|\widehat{z} - M_G\widehat{u}_1\|_{\mathcal{H}^2} + \tfrac{1}{2}\cdot\left\|\widehat{z} - \overline{M_G\widehat{u}_1(\bar{\cdot})}\right\|_{\mathcal{H}^2}\\ &= \tfrac{1}{2}\cdot\|\widehat{z} - M_G\widehat{u}_1\|_{\mathcal{H}^2} + \tfrac{1}{2}\cdot\left\|\overline{\widehat{z}(\bar{\cdot}) - M_G\widehat{u}_1(\bar{\cdot})}\right\|_{\mathcal{H}^2} = \|\widehat{z} - M_G\widehat{u}\|_{\mathcal{H}^2} < \varepsilon.\end{aligned}$$ □

Proof of Theorem 5.1. First note that

(i) for $(x,u,y) \in \mathfrak{B}_{[E,A,B,C,D]}(0)$, the Laplace transforms of y and u are related by $\widehat{y}(s) = G(s)\widehat{u}(s)\ \forall s \in \mathbb{C}_+$;
(ii) by the Paley–Wiener Theorem ([9], Thm. A.6.21), Laplace transform defines an isometric mapping from $\mathcal{L}^2(\mathbb{R}_{\geq 0} \to \mathbb{K}^m)$ to $\mathcal{H}^2(\mathbb{C}_+ \to \mathbb{K}^m)$;
(iii) the norm of the multiplication operator $M_G : \mathcal{H}^2(\mathbb{C}_+ \to \mathbb{K}^m) \to \mathcal{H}^2(\mathbb{C}_+ \to \mathbb{K}^p)$ defined by $M_G(\widehat{u})(s) = G(s)\widehat{u}(s)\,\forall s \in \mathbb{C}_+$ equals to $\|G\|_{\mathcal{H}^\infty}$ ([9], Thm. A.6.26);
(iv) for all infinitely often differentiable $u \in \mathcal{L}^2(\mathbb{R}_{\geq 0} \to \mathbb{K}^m)$ with support contained in $(0,\infty)$, there exists some unique $(x,u,y) \in \mathfrak{B}_{[E,A,B,C,D]}(0)$.

$\Longrightarrow$ Assume that $G(s)$ is outer, $\varepsilon > 0$, and $z \in \mathcal{L}^2(\mathbb{R}_{\geq 0} \to \mathbb{K}^p)$. Then we have to show

$$\exists\,(x,u,y) \in \mathfrak{B}_{[E,A,B,C,D]}(0)\ :\quad u \in \mathcal{L}^2(\mathbb{R}_{\geq 0} \to \mathbb{K}^m)\quad\wedge\quad \|z - y\|_{\mathcal{L}^2} < \varepsilon.$$

Let $\widehat{z}$ be the Laplace transform of z. By Lemma 5.3, there exists some $\widehat{u}_1 \in \mathcal{H}^2(\mathbb{C}_+ \to \mathbb{K}^m)$ with

$$\|\widehat{z} - M_G\widehat{u}_1\|_{\mathcal{H}^2} < \varepsilon/2. \tag{5.4}$$

By a density argument, we see that there exists some infinitely often differentiable $u \in \mathcal{L}^2(\mathbb{R}_{\geq 0} \to \mathbb{K}^m)$ with support contained in $(0,\infty)$, such that

$$\|u - u_1\|_{\mathcal{L}^2} \cdot \|G\|_{\mathcal{H}^\infty} < \varepsilon/2. \tag{5.5}$$

By statement (iv), there exist $x \in \mathcal{L}^2(\mathbb{R}_{\geq 0} \to \mathbb{K}^n)$, $y \in \mathcal{L}^2(\mathbb{R}_{\geq 0} \to \mathbb{K}^p)$ with $(x,u,y) \in \mathfrak{B}_{[E,A,B,C,D]}(0)$. Then we obtain

$$\begin{aligned}\|z - y\|_{\mathcal{L}^2} &\overset{(i)}{=} \|\widehat{z} - \widehat{y}\|_{\mathcal{H}^2} \overset{(ii)}{=} \|\widehat{z} - M_G\widehat{u}\|_{\mathcal{H}^2} \leq \|\widehat{z} - M_G\widehat{u}_1\|_{\mathcal{H}^2} + \|M_G(\widehat{u} - \widehat{u}_1\|_{\mathcal{H}^2}\\ &\overset{(iii)\&(5.4)}{\leq} \varepsilon/2 + \|G\|_{\mathcal{H}^\infty}\|\widehat{u} - \widehat{u}_1\|_{\mathcal{H}^2} \overset{(i)}{=} \varepsilon/2 + \|G\|_{\mathcal{H}^\infty}\|u - u_1\|_{\mathcal{L}^2} \overset{(5.5)}{<} \varepsilon.\end{aligned}$$

$\Longleftarrow$ Seeking for a contradiction, assume that $G(s)$ is not outer and Property (P2) holds.
By Lemma 5.3, there exists some $\widehat{z} \in \mathcal{H}^2(\mathbb{C}_+ \to \mathbb{K}^p)$ and some $\varepsilon > 0$ such that for all $\widehat{u} \in \mathcal{H}^2(\mathbb{C}_+ \to \mathbb{K}^m)$ holds

$$\|\widehat{z} - M_G\widehat{u}\|_{\mathcal{H}^2} \geq \varepsilon.$$

Assume that $(x,u,y) \in \mathfrak{B}_{[E,A,B,C,D]}(0)$. Then

$$\|z - y\|_{\mathcal{L}^2} \overset{(i)}{=} \|\widehat{z} - \widehat{y}\|_{\mathcal{H}^2} \overset{(ii)}{=} \|\widehat{z} - M_G\widehat{u}\|_{\mathcal{H}^2} \geq \varepsilon.$$

This contradicts Property (P2). □

Remark 5.4 (Stable outer functions)**.** Note that in [25] a (possibly non-rational) transfer function $G \in \mathcal{H}^\infty(\mathbb{C}_+ \to \mathbb{K}^{p\times m})$ is defined to be outer, if the multiplication operator M_G as in Lemma 5.3 is surjective. In the possibly non-rational case, Cauchy's integral formula (*cf.* the proof of $\Longleftarrow$ in Lemma 5.3) can as well be used to infer that outer functions do not have zeros in $\mathbb{C}_+$. The converse direction $\Longrightarrow$ in Lemma 5.3 however does no longer hold true for non-rational functions. A counterexample is $G(s) = \mathrm{e}^{-s}$, see [17].

6. Outer transfer functions

Next we waive the $\mathcal{H}^\infty$-condition in Theorem 5.1. To this end the Property (P2) is strengthened to the following two properties.

$$\textbf{(P3)} \qquad \forall\, y^0 \in \mathbb{K}^p \setminus \{0\}\ \exists\, (x,u,y) \in \mathfrak{B}_{[E,A,B,C,D]}\ :\ (y^0)^* y(\cdot) \neq 0$$

and

$$\textbf{(P4)} \qquad \begin{array}{l} \forall\, \varepsilon > 0\ \forall\, x^0 \in \mathcal{V}^{\text{diff}}_{[E,A,B,C,D]}\ \exists\, (x,u,y) \in \mathfrak{B}_{[E,A,B,C,D]}(x^0)\ : \\ u \in \mathcal{L}^2(\mathbb{R}_{\geq 0} \to \mathbb{K}^m) \quad \wedge \quad \lim_{t\to\infty} Ex(t) = 0 \quad \wedge \quad \|y\|_{\mathcal{L}^2} < \varepsilon. \end{array}$$

Properties (P3) and (P4) mean for systems described by ordinary differential equations simply (P3') and (P4'), respectively; see page 393.

Note that in Property (P4) we allow for arbitrary initial data $x^0 \in \mathcal{V}^{\text{diff}}_{[E,A,B,C,D]}$ but the internal state $Ex(t)$ has to go to zero. This replaces in a sense the $\mathcal{H}^\infty$-condition of the transfer function in Property (P2).

Remark 6.1. If a system $[E,A,B,C,D] \in \Sigma_{n,m,p}(\mathbb{K})$ has stabilizable zero dynamics, then Property (P4) holds and we have

$$\begin{array}{l} \forall\, x^0 \in \mathcal{V}^{\text{diff}}_{[E,A,B,C,D]}\ \exists\, (x,u,y) \in \mathfrak{B}_{[E,A,B,C,D]}(x^0)\ : \\ u \in \mathcal{L}^2(\mathbb{R}_{\geq 0} \to \mathbb{K}^m) \quad \wedge \quad \lim_{t\to\infty} Ex(t) = 0 \quad \wedge \quad \|y\|_{\mathcal{L}^2} = 0. \end{array}$$

The main result of this section is Theorem 6.6 where we show that a system has an outer transfer function "almost if, and only if", the Properties (P3) and (P4) hold. We first give an"almost characterization" of Property (P3).

Proposition 6.2 (Characterization of (P3))**.** *For any system* $[E,A,B,C,D] \in \Sigma_{n,m,p}(\mathbb{K})$ *the following statements hold true:*

(a) (P3) $\Longrightarrow$ $\operatorname{rk}\left[\begin{smallmatrix} E & A & B \\ 0 & C & D \end{smallmatrix}\right] = n+p$; *the reverse implication does in general not hold true.*
(b) (P3) $\Longleftarrow$ $\operatorname{rk}\left[\begin{smallmatrix} E & A & B \\ 0 & C & D \end{smallmatrix}\right] = n+p$ *and* $[E,A,B,C,D]$ *is impulse controllable.*

Proof.

(a) Seeking for a contradiction, assume that $\operatorname{rk}\left[\begin{smallmatrix} E & A & B \\ 0 & C & D \end{smallmatrix}\right] < n+p$. Then

$$\exists (x^0, y^0) \in \mathbb{K}^n \times \mathbb{K}^p \setminus \{(0,0)\} : \begin{pmatrix} x^0 \\ y^0 \end{pmatrix}^* \begin{bmatrix} E & A & B \\ 0 & C & D \end{bmatrix} = 0.$$

If $y^0 = 0$, then $x^0 \neq 0$ and $(x^0)^* E = (x^0)^* A = 0$ contradicted the regularity of $sE - A$. Therefore, $y^0 \neq 0$. Now assume that $(x,u,y) \in \mathfrak{B}_{[E,A,B,C,D]}$. Then

$$(y^0)^* y(\cdot) = (y^0)^* Cx(\cdot) + (y^0)^* Du(\cdot) = -(x^0)^* Ax(\cdot) - (x^0)^* Bu(\cdot) = -(x^0)^* E\dot{x}(\cdot) = 0$$

contradicts (P3).

To see that the reverse implication does not hold true in general, consider

$$[E,A,B,C,D] := \left[\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \end{bmatrix}, 0 \right] \in \Sigma_{2,1,1}. \tag{6.1}$$

In passing we note that the pencil $sE - A$ is regular. Since $\mathfrak{B}_{[E,A,B,C,D]} = \{0\} \times \mathcal{L}^2_{\text{loc}}(\mathbb{R}_{\geq 0} \to \mathbb{K}) \times \{0\}$, Property (P3) is not fulfilled. However,

$$\text{rk} \begin{bmatrix} E & A & B \\ 0 & C & D \end{bmatrix} = \text{rk} \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} = 3 = n + p.$$

(b) **Step 1**. We prove the assertion for the case $E = I_n$. Let $y^0 \in \mathbb{K}^p \setminus \{0\}$. Then

$$0 \neq \begin{pmatrix} 0 \\ y^0 \end{pmatrix}^* \begin{bmatrix} E & A & B \\ 0 & C & D \end{bmatrix}, \quad \text{and so} \quad (y^0)^* C \neq 0 \quad \vee \quad (y^0)^* D \neq 0.$$

Therefore,

$$\exists x^0 \in \mathbb{K}^n \ \exists u^0 \in \mathbb{K}^m : \ (y^0)^* C x^0 + (y^0)^* D u^0 \neq 0.$$

Define the trajectory

$$(x(\cdot), u(\cdot), y(\cdot)) = \left(\mathrm{e}^{A\cdot} x^0,\ u^0,\ C\mathrm{e}^{A\cdot} x^0 + C \int_0^\cdot \mathrm{e}^{A(\cdot - \tau)} B u^0 \, \mathrm{d}\tau + D u^0 \right) \in \mathfrak{B}_{[I_n, A, B, C, D]}. \tag{6.2}$$

Then (x, u, y) is continuous with $(x(0), u(0), y(0)) = (x^0, u^0, Cx^0 + Du^0)$. In particular we have $(y^0)^* y(0) = (y^0)^* C x^0 + (y^0)^* D u^0 \neq 0$, whence $(y^0)^* y(\cdot) \neq 0$.

Step 2. We prove the assertion for impulse controllable systems $[E, A, B, C, D] \in \Sigma_{n,m,p}(\mathbb{K})$: By Proposition 2.12 (e)

$$\exists S, T \in \text{Gl}_n(\mathbb{K}) \ \exists F \in \mathbb{K}^{m \times n} : \ \begin{bmatrix} S & 0 \\ 0 & I_p \end{bmatrix} \begin{bmatrix} E & A & B \\ 0 & C & D \end{bmatrix} \begin{bmatrix} T & 0 & 0 \\ 0 & T & 0 \\ 0 & FT & I_m \end{bmatrix} = \left[\begin{array}{cc|cc|c} I_{n_1} & 0 & A_{11} & 0 & B_1 \\ 0 & 0 & 0 & I_{n_2} & B_2 \\ \hline 0 & 0 & C_1 & C_2 & D \end{array}\right].$$

Since

$$\text{rk} \begin{bmatrix} E & A & B \\ 0 & C & D \end{bmatrix} = \text{rk} \left[\begin{array}{cc|cc|c} I_{n_1} & 0 & A_{11} & 0 & B_1 \\ 0 & 0 & 0 & I_{n_2} & B_2 \\ \hline 0 & 0 & C_1 & C_2 & D \end{array}\right]$$

and Property (P3) is invariant under feedback equivalence, it suffices to consider the DAE associated to the matrix on the right hand side, *i.e.*

$$\begin{aligned} \dot{x}_1 &= A_{11} x_1 + B_1 u \\ 0 &= x_2 + B_2 u \\ y &= C_1 x_1 + C_2 x_2 + D u \end{aligned}$$

or equivalently, $x_2 = -B_2 u$ together with

$$\begin{aligned} \dot{x}_1 &= A_{11} x_1 + B_1 u \\ y &= C_1 x_1 + (D - C_2 B_2) u. \end{aligned}$$

Now we may apply Step 1 to show Property (P3) for $\mathfrak{B}_{[I_{n_1}, A_{11}, B_1, C_1, D - C_2 B_2]}$, and it is easy to see that (P3) also holds for $\mathfrak{B}_{[E,A,B,C,D]}$.

This completes the proof of the proposition. □

Next we prove that, for any $[E, A, B, C, D] \in \Sigma_{n,m,p}$, the output space can be reduced to a system with Property (P3). This is a key result to provide an "almost characterization" of the Property (P4) in Proposition 6.4.

Proposition 6.3. *Let* $[E, A, B, C, D] \in \Sigma_{n,m,p}$*, define*

$$\mathcal{Y}_0 := \left\{ y^0 \in \mathbb{K}^p \;\middle|\; \forall (x, u, y) \in \mathfrak{B}_{[E,A,B,C,D]} \text{ and for almost all } t \in \mathbb{R} : \; (y^0)^* y(t) = 0 \; \right\}$$

and choose

$$Y \in \mathbb{K}^{p \times p_1} : \; \operatorname{im} Y = \mathcal{Y}_0^\perp \quad \wedge \quad Y^* Y = I_{p_1}. \tag{6.3}$$

Then we have

(i) $\mathfrak{B}_{[E,A,B,C,D]} = \left\{ (x, u, Y y_1) \;\middle|\; (x, u, y_1) \in \mathfrak{B}_{[E,A,B,Y^*C,Y^*D]} \right\}$;
(ii) $[E, A, B, Y^*C, Y^*D]$ *satisfies* (P3)*;*
(iii) $\forall \lambda \in \mathbb{C} : \quad \operatorname{rk} \begin{bmatrix} \lambda E - A & -B \\ C & D \end{bmatrix} = \operatorname{rk} \begin{bmatrix} \lambda E - A & -B \\ Y^*C & Y^*D \end{bmatrix}$.

Moreover, if (i) *and* (ii) *hold, instead for* Y*, for some* $\widehat{Y} \in \mathbb{K}^{p \times p_1}$ *with* $\widehat{Y}^* \widehat{Y} = I_{p_1}$*, then* $\operatorname{im} \widehat{Y} = \mathcal{Y}_0^\perp$ *and hence* Y *and* $\widehat{Y}$ *differ by a unitary factor from the right.*

Proof. Since $\mathcal{Y}_0 \subset \mathbb{K}^p$ is a linear subspace, the choice of Y is possible. We may also choose

$$Y_0 \in \mathbb{K}^{p \times (p - p_1)} : \; \operatorname{im} Y_0 = \mathcal{Y}_0 \quad \wedge \quad Y_0^* Y_0 = I_{p - p_1}. \tag{6.4}$$

Then we have $\operatorname{im} Y = \ker Y_0^*$ and

$$[Y, Y_0]\,[Y, Y_0]^* = I_p. \tag{6.5}$$

By the definition of $\mathcal{Y}_0$ we have

$$\forall (x, u, y) \in \mathfrak{B}_{[E,A,B,C,D]} \quad \Longrightarrow \quad \Big[\text{ for almost all } t \in \mathbb{R} : \; y(t) \in \mathcal{Y}_0^\perp \Big] \tag{6.6}$$

and $Y^* Y = I_{p_1}$ yields that

$$Y Y^* \in \mathbb{K}^{p \times p} \quad \text{is an orthogonal projector onto } \mathcal{Y}_0^\perp. \tag{6.7}$$

We now proceed in several steps.
(i), $\subset$: Let $(x, u, y) \in \mathfrak{B}_{[E,A,B,C,D]}$. Then

$$(x, u, y_1) \in \mathfrak{B}_{[E,A,B,Y^*C,Y^*D]} \qquad \text{for } y_1 := Y^* y = Y^* C x + Y^* D u$$

and (6.6) and (6.7) yield $y = Y Y^* y = Y y_1$.
(i), $\supset$: Let $(x, u, y_1) \in \mathfrak{B}_{[E,A,B,Y^*C,Y^*D]}$ and define $y = Cx + Du$. Then (6.6) and (6.7) yield

$$Y y_1 = Y[Y^* C x + Y^* D u] = Y Y^* y = y \qquad \text{and} \quad (x, u, y) \in \mathfrak{B}_{[E,A,B,C,D]}.$$

(ii): Seeking a contradiction, suppose that

$$\exists \widehat{y}^0 \in \mathbb{K}^{p_1} \setminus \{0\} \; \forall (x, u, y_1) \in \mathfrak{B}_{[E,A,B,Y^*C,Y^*D]} \text{ and for almost all } t \in \mathbb{R} : \; (\widehat{y}^0)^* y_1(t) = 0. \tag{6.8}$$

Fix $(x, u, y_1) \in \mathfrak{B}_{[E,A,B,Y^*C,Y^*D]}$ and define $y := Cx + Du$. Then

$$(x, u, y) \in \mathfrak{B}_{[E,A,B,C,D]} \quad \text{and} \quad y_1 = Y^*[Cx + Du] = Y^* y$$

and (6.6) and (6.7) yield that $Y y_1 = Y Y^* y = y$, and therefore

$$(Y \widehat{y}^0)^* y(t) = (\widehat{y}^0)^* \, Y^* Y \, y_1(t) \overset{(6.3)}{=} (\widehat{y}^0)^* y_1(t) \overset{(6.8)}{=} 0 \quad \text{for almost all } t \in \mathbb{R}.$$

This shows $Y \widehat{y}^0 \in \mathcal{Y}_0 \cap \operatorname{im} Y = \mathcal{Y}_0 \cap \mathcal{Y}_0^\perp = \{0\}$, whence, again by (6.3), $\widehat{y}^0 = 0$. This contradicts (6.8).

Before we show assertion (iii), we show the last statement of the proposition. Let $\widehat{Y} \in \mathbb{K}^{p\times p_1}$ with $\widehat{Y}^*\widehat{Y} = I_{p_1}$ such that (i) and (ii) hold for $\widehat{Y}$.

We show $\operatorname{im}\widehat{Y}^\perp \subset \mathcal{Y}_0$: since (i) holds for $\widehat{Y}$, we may choose

$$(x,u,y) \in \mathfrak{B}_{[E,A,B,C,D]}, (x,u,\widehat{y}_1) \in \mathfrak{B}_{[E,A,B,\widehat{Y}^*C,\widehat{Y}^*D]} : y = \widehat{Y}\widehat{y}_1.$$

Then we have, for $\widehat{y}^0 \in \operatorname{im}\widehat{Y}^\perp$,

$$(\widehat{y}^0)^* y(t) = (\widehat{y}^0)^*\widehat{Y}\widehat{y}_1(t) = 0 \qquad \text{for almost all } \ t \in \mathbb{R}$$

and thus $\widehat{y}^0 \in \mathcal{Y}_0$.

We show $\operatorname{im}\widehat{Y}^\perp = \mathcal{Y}_0$: seeking a contradiction, suppose there exists $y^0 \in \mathcal{Y}_0 \setminus \operatorname{im}\widehat{Y}^\perp$ and set $\widehat{y}^0 := \widehat{Y}^* y^0$. Since Property (P3) holds for $\widehat{Y}$, we have

$$\exists (x,u,\widehat{y}_1) \in \mathfrak{B}_{[E,A,B,\widehat{Y}^*C,\widehat{Y}^*D]} :\ (\widehat{y}^0)^*\widehat{y}_1 \neq 0.$$

Since (i) holds for $\widehat{Y}$, we have $(x,u,y) \in \mathfrak{B}_{[E,A,B,C,D]}$ for $y = \widehat{Y}\widehat{y}_1$ and we conclude

$$0 \neq (\widehat{y}^0)^*\widehat{y}_1 = (y^0)^*\widehat{Y}\widehat{y}_1 = (y^0)^* y$$

and this yields the contradiction $y^0 \notin \mathcal{Y}_0$.

(iii): First we show, for Y_0 as in (6.4),

$$E = I_n \qquad \Longrightarrow \qquad Y_0^*[C,D] = 0. \tag{6.9}$$

Let

$$y_k^0 := Y_0 e_k \in \operatorname{im} Y_0 \setminus \{0\} = \mathcal{Y}_0 \setminus \{0\} \qquad \text{for } k \in \{1,\ldots,p-p_1\}.$$

Then

$$\forall (x,u,y) \in \mathfrak{B}_{[I_n,A,B,C,D]} \text{ and for almost all } t \in \mathbb{R} :\ (y_k^0)^* y(t) = 0$$

and, for arbitrary $x_0 \in \mathbb{K}^n$ and $u_0 \in \mathbb{K}^m$ and any trajectory

$$(x(\cdot),u(\cdot),y(\cdot)) := \left(\mathrm{e}^{A\cdot}x^0,\ u^0,\ C\mathrm{e}^{A\cdot}x^0 + C\int_0^{\cdot} \mathrm{e}^{A(\cdot-\tau)}Bu^0\,\mathrm{d}\tau + Du^0\right) \in \mathfrak{B}_{[I_n,A,B,C,D]},$$

we conclude by continuity of y

$$(y_k^0)^* y(0) = (y_k^0)^*[C,D]\left(\begin{smallmatrix} x^0 \\ u^0 \end{smallmatrix}\right) = 0.$$

Since x_0 and u_0 are arbitrary, it follows that $(y_k^0)^*[C,D]$, and the claim follows since k is arbitrary.

Finally, we show (iii) for any $[E,A,B,C,D] \in \Sigma_{n,m,p}$.

In terms of the notion from Proposition 2.12 we have, for all $\lambda \in \mathbb{C}$,

$$\begin{aligned}
\operatorname{rk}\begin{bmatrix}\lambda E - A & -B\\ C & D\end{bmatrix} &\overset{(2.4)}{=} \operatorname{rk}\begin{bmatrix}\lambda I_{n_1} - A_{11} & 0 & 0 & -B_1\\ 0 & -I_{n_2} & \lambda E_{23} & -B_2\\ 0 & 0 & \lambda N - I_k & 0\\ C_1 & C_2 & C_3 & D\end{bmatrix}\\
&\overset{N \text{ nilpt.}}{=} k + \operatorname{rk}\begin{bmatrix}\lambda I_{n_1} - A_{11} & 0 & -B_1\\ 0 & -I_{n_2} & -B_2\\ C_1 & C_2 & D\end{bmatrix}\\
&\overset{(6.5)}{=} = k + \operatorname{rk}\begin{bmatrix}\lambda I_{n_1} - A_{11} & 0 & -B_1\\ 0 & -I_{n_2} & -B_2\\ Y^*C_1 & Y^*C_2 & Y^*D\\ Y_0^*C_1 & Y_0^*C_2 & Y_0^*D\end{bmatrix}.
\end{aligned}$$

Using Proposition 2.12(a), we further obtain from (6.9) that

$$Y_0^*C_1 = 0 \qquad \text{and} \qquad Y_0^*(D - C_2B_2) = 0 \tag{6.10}$$

and continue

$$\begin{aligned}
\operatorname{rk}\begin{bmatrix}\lambda E - A & -B\\ C & D\end{bmatrix} &\overset{(6.10)}{=} k + \operatorname{rk}\begin{bmatrix}\lambda I_{n_1} - A_{11} & 0 & -B_1\\ 0 & -I_{n_2} & -B_2\\ Y^*C_1 & Y^*C_2 & Y^*D\\ 0 & Y_0^*C_2 & Y_0^*C_2B_2\end{bmatrix}\\
&= k + \operatorname{rk}\begin{bmatrix}\lambda I_{n_1} - A_{11} & 0 & -B_1\\ 0 & -I_{n_2} & 0\\ Y^*C_1 & Y^*C_2 & Y^*(D - C_2B_2)\\ 0 & 0 & 0\end{bmatrix}\\
&= k + \operatorname{rk}\begin{bmatrix}\lambda I_{n_1} - A_{11} & 0 & -B_1\\ 0 & -I_{n_2} & -B_2\\ Y^*C_1 & Y^*C_2 & Y^*D\end{bmatrix}\\
&\overset{N \text{ nilpt.}}{=} \operatorname{rk}\begin{bmatrix}\lambda I_{n_1} - A_{11} & 0 & 0 & -B_1\\ 0 & -I_{n_2} & \lambda E_{23} & -B_2\\ 0 & 0 & \lambda N - I_k & 0\\ Y^*C_1 & Y^*C_2 & Y^*C_3 & Y^*D\end{bmatrix}\\
&\overset{(2.4)}{=} \operatorname{rk}\begin{bmatrix}\lambda E - A & -B\\ Y^*C & Y^*D\end{bmatrix}.
\end{aligned}$$

This completes the proof of the proposition. □

We are now in a position to show the first "almost characterization" of the Property (P4).

Proposition 6.4. *For any system* $[E,A,B,C,D] \in \Sigma_{n,m,p}(\mathbb{K})$ *the following statements hold true:*

(a) (P4) $\implies$ $\forall \lambda \in \mathbb{C}_+ :\ \operatorname{rk}\left[\begin{smallmatrix} E & A & B \\ 0 & C & D\end{smallmatrix}\right] = \operatorname{rk}\left[\begin{smallmatrix} \lambda E - A, & -B \\ C & D\end{smallmatrix}\right]$ *and* $[E,A,B,C,D]$ *is behavioural stabilizable; the reverse implication does in general not hold true.*

(b) (P4) *and* $[E,A,B,C,D]$ *is impulse controllable* $\implies$
$\forall \lambda \in \mathbb{C}_+ :\ \operatorname{rk}\left[\begin{smallmatrix} E & A & B \\ 0 & C & D\end{smallmatrix}\right] = \operatorname{rk}\left[\begin{smallmatrix} \lambda E - A, & -B \\ C & D\end{smallmatrix}\right]$ *and* $[E,A,B,C,D]$ *is behavioural stabilizable.*

Proof. We preface the proof with some basic observations needed in the steps of the proof. First note the following facts.

(O1) A real system $[E,A,B,C,D] \in \Sigma_{n,m,p}(\mathbb{R})$ satisfies Property (P4), if, and only if, it satisfies (P4) as a complex system $[E,A,B,C,D] \in \Sigma_{n,m,p}(\mathbb{C})$. The property of impulse controllability does also not depend on regarding $[E,A,B,C,D]$ as a real or as a complex system. Further, the transformation to Kronecker canonical form is complex, independent of the matrix pencil being real or complex. As a consequence, it suffices to prove the statements for the case $\mathbb{K} = \mathbb{C}$.

(O2) The property
$$\forall \lambda \in \mathbb{C}_+ :\ \operatorname{rk}\left[\begin{smallmatrix} E & A & B \\ 0 & C & D\end{smallmatrix}\right] = \operatorname{rk}\left[\begin{smallmatrix} \lambda E - A, & -B \\ C & D\end{smallmatrix}\right] \tag{6.11}$$
is equivalent to
$$\forall \lambda \in \mathbb{C}_+ :\ \operatorname{im}\begin{bmatrix} -E & -A & -B \\ 0 & C & D\end{bmatrix} = \operatorname{im}\left[\begin{smallmatrix} \lambda E - A, & -B \\ C & D\end{smallmatrix}\right] \tag{6.12}$$
since $\left[\begin{smallmatrix} -E & -A & -B \\ 0 & C & D\end{smallmatrix}\right] = \begin{bmatrix} -I_n & 0 \\ 0 & I_p\end{bmatrix}\left[\begin{smallmatrix} E & A & B \\ 0 & C & D\end{smallmatrix}\right]$ and
$$\forall \lambda \in \mathbb{C} :\ \operatorname{im}\begin{bmatrix} \lambda E - A & -B \\ C & D\end{bmatrix} = \operatorname{im}\left(\begin{bmatrix} -E & -A & -B \\ 0 & C & D\end{bmatrix}\begin{bmatrix} -\lambda I_n & 0 \\ I_n & 0 \\ 0 & I_p\end{bmatrix}\right) \subset \operatorname{im}\left[\begin{smallmatrix} -E & -A & -B \\ 0 & C & D\end{smallmatrix}\right].$$

(O3) Property (6.11) is invariant under feedback equivalence since
$$\begin{bmatrix} WET & W(A+BF)T & WB \\ 0 & CT & D\end{bmatrix} = \begin{bmatrix} W & 0 \\ 0 & I_p\end{bmatrix}\cdot\begin{bmatrix} E & A & B \\ 0 & C & D\end{bmatrix}\cdot\begin{bmatrix} T & 0 & 0 \\ 0 & T & 0 \\ 0 & FT & I_m\end{bmatrix}$$
and
$$\forall \lambda \in \mathbb{C} :\ \begin{bmatrix} \lambda WET - W(A+BF)T & -WB \\ CT & D\end{bmatrix} = \begin{bmatrix} W & 0 \\ 0 & I_p\end{bmatrix}\cdot\begin{bmatrix} \lambda E - A & -B \\ C & D\end{bmatrix}\cdot\begin{bmatrix} T & 0 \\ FT & I_m\end{bmatrix}.$$

We are now ready for the proof and proceed in several steps.

(a) $\Leftarrow$ **Step 1**. We first additionally assume that all generalized eigenvalues of $sE - A$ are belonging to $\mathbb{C}_-$ and the index of $sE-A$ is at most one. The latter yields that $[E,A,B,C,D]$ is impulse controllable. Let $Y \in \mathbb{K}^{p\times p_1}$ be as in (6.3). Since (6.11) holds by assumption, we have, by using (O2),
$$\forall \lambda \in \mathbb{C}_+ :\ \operatorname{im}\begin{bmatrix} \lambda E - A & -B \\ Y^*C & Y^*D\end{bmatrix} = \begin{bmatrix} I_n & 0 \\ 0 & Y^*\end{bmatrix}\cdot\operatorname{im}\begin{bmatrix} \lambda E - A & -B \\ C & D\end{bmatrix}$$
$$\overset{(6.12)}{=} \begin{bmatrix} I_n & 0 \\ 0 & Y^*\end{bmatrix}\cdot\operatorname{im}\begin{bmatrix} -E & -A & -B \\ 0 & C & D\end{bmatrix} = \operatorname{im}\begin{bmatrix} -E & -A & -B \\ 0 & Y^*C & Y^*D\end{bmatrix},$$
and therefore
$$\forall \lambda \in \mathbb{C}_+ :\ \operatorname{rk}\begin{bmatrix} \lambda E - A & -B \\ Y^*C & Y^*D\end{bmatrix} = \operatorname{rk}\begin{bmatrix} -E & -A & -B \\ 0 & Y^*C & Y^*D\end{bmatrix} = \operatorname{rk}\begin{bmatrix} E & A & B \\ 0 & Y^*C & Y^*D\end{bmatrix} = n + p_1,$$

where the last equality follows from the fact that $[E, A, B, Y^*C, Y^*D]$ satisfies Property (P3) by Proposition 6.3(ii) and hence we may apply Proposition 6.2(a). Therefore, the system $[E, A, B, Y^*C, Y^*D]$ satisfies (P1). By Theorem 3.3(a), we obtain that the transfer function $G(s) = Y^*D + Y^*C(sE - A)^{-1}B \in \mathbb{C}(s)^{p_1 \times m}$ is outer. The assumption that all generalized eigenvalues of $sE - A$ are belonging to $\mathbb{C}_-$ and the index of $sE - A$ is at most one yields that, additionally, $G(s) \in \mathcal{H}^\infty(\mathbb{C}_+ \to \mathbb{K}^{p_1 \times m})$. Let $\varepsilon > 0$ and $x^0 \in \mathcal{V}^{\text{diff}}_{[E,A,B,C,D]} = \mathbb{R}^n$. The Kronecker canonical form allows to assume that the system is in the form

$$sE - A = s \begin{bmatrix} I_{n_1} & 0 \\ 0 & 0_{n_2} \end{bmatrix} - \begin{bmatrix} A_{11} & 0 \\ 0 & I_{n_2} \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad C = \begin{bmatrix} C_1 & C_2 \end{bmatrix}, \quad x^0 = \begin{pmatrix} x^{01} \\ x^{02} \end{pmatrix},$$

and, by the assumption that the set of generalized eigenvalues of $sE - A$ is contained in $\mathbb{C}_-$, we have $\sigma(A_{11}) \subset \mathbb{C}_-$. In these coordinates, a solution $(\left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right), u, y_1) \in \mathfrak{B}_{[E,A,B,Y^*C,Y^*D]}$ satisfies, for all $t \geq 0$,

$$\begin{aligned} x_1(t) &= \mathrm{e}^{A_{11}t} x^{01} + \int_0^t \mathrm{e}^{A_{11}(t-\tau)} B_1 u(\tau) \,\mathrm{d}\tau \\ x_2(t) &= -B_2 u(t) \\ y_1(t) &= Y^*C_1 x_1(t) + Y^*C_2 x_2(t) + Y^*D u(t). \end{aligned}$$

Apply

$$z(\cdot) \;:=\; C_1 \mathrm{e}^{A_{11}\cdot} x^{01} \in \mathcal{L}^2(\mathbb{R}_{\geq 0} \to \mathbb{K}^p)$$

to Theorem 5.1. Then

$$\exists \, (\left(\begin{smallmatrix} x_{21} \\ x_{22} \end{smallmatrix}\right), u, y_2) \in \mathfrak{B}_{[E,A,B,Y^*C,Y^*D]}(0) \;:\quad u \in \mathcal{L}^2(\mathbb{R}_{\geq 0} \to \mathbb{K}^m) \;\wedge\; \| -z - y_2 \|_{\mathcal{L}^2} < \varepsilon.$$

Since $\left(\left(\begin{smallmatrix} x_{21} + \mathrm{e}^{A_{11}\cdot} x^{01} \\ x_{22} \end{smallmatrix}\right), u, y_2 + z\right) \in \mathfrak{B}_{[E,A,B,Y^*C,Y^*D]}(x^0)$, we have, by linearity of the behaviour, that

$$(x, u, y_1) = \left(\left(\begin{smallmatrix} x_{21} + \mathrm{e}^{A_{11}\cdot} x^{01} \\ x_{22} \end{smallmatrix}\right), u, z + y_2\right) \in \mathfrak{B}_{[E,A,B,Y^*C,Y^*D]}(x^0).$$

By Proposition 6.3(i), we have $(x, u, y) := (x, u, Y y_1) \in \mathcal{B}_{[E,A,B,C,D]}(x^0)$, and the orthonormality of the columns of Y gives rise to

$$\|y\|_{\mathcal{L}^2} = \|Y y_1\|_{\mathcal{L}^2} = \|y_1\|_{\mathcal{L}^2} = \|z + y_2\|_{\mathcal{L}^2} < \varepsilon.$$

Moreover, since $\sigma(A_{11}) \subset \mathbb{C}_-$ and $u \in \mathcal{L}^2(\mathbb{R}_{\geq 0} \to \mathbb{K}^m)$, it can be shown (see *e.g.* [14], Rem. 2.3.11) that $x_1 \in \mathcal{L}^2(\mathbb{R}_{\geq 0} \to \mathbb{K}^{n_1})$, $x_2 \in \mathcal{L}^2(\mathbb{R}_{\geq 0} \to \mathbb{K}^{n_2})$, and $\lim_{t\to\infty} x_1(t) = 0$.

Step 2. We prove the implication $\Longleftarrow$ in the general case:
By Proposition 2.12, there exist $S, T \in \mathrm{Gl}_n(\mathbb{K})$ and $F \in \mathbb{K}^{m \times n}$, such that (2.4) holds true, where $N \in \mathbb{K}^{k \times k}$ is nilpotent and $\sigma(A_{11}) \subset \mathbb{C}_-$.

Step 2a. We prove that the system

$$[\widetilde{E}, \widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{D}] := \left[\begin{bmatrix} I_{n_1} & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A_{11} & 0 \\ 0 & I_{n_2} \end{bmatrix}, \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \begin{bmatrix} C_1 & C_2 \end{bmatrix}, D \right] \tag{6.13}$$

has property (6.11). It suffices to prove that

$$\forall \lambda \in \mathbb{C}_+ : \; \ker \begin{bmatrix} \lambda I_{n_1} - A_{11} & 0, & -B_1 \\ 0 & -I_{n_1} & -B_2 \\ C_1 & C_2 & D \end{bmatrix}^* \subset \ker \begin{bmatrix} I_{n_1} & 0 & A_{11} & 0 & B_1 \\ 0 & 0 & 0 & I_{n_2} & -B_2 \\ 0 & 0 & C_1 & C_2 & D \end{bmatrix}^*.$$

Assume that $\begin{pmatrix} \widetilde{x}_1 \\ \widetilde{x}_2 \\ \widetilde{u} \end{pmatrix} \in \ker \begin{bmatrix} \lambda I_{n_1} - A_{11} & 0, & -B_1 \\ 0 & -I_{n_1} & B_2 \\ C_1 & C_2 & D \end{bmatrix}^*$ with $\widetilde{x}_1 \in \mathbb{C}^{n_1}$, $\widetilde{x}_2 \in \mathbb{C}^{n_2}$. Then for

$$\widetilde{x}_3 := -(\lambda N - I)^{-*}\Big(\overline{\lambda} E_{23}^* \widetilde{x}_2 + C_3^* \widetilde{u}\Big)$$

we have

$$\begin{pmatrix} \widetilde{x}_1 \\ \widetilde{x}_2 \\ \widetilde{x}_3 \\ \widetilde{u} \end{pmatrix} \in \ker_{\mathbb{C}} \begin{bmatrix} \lambda I_{n_1} - A_{11} & 0 & 0, & -B_1 \\ 0 & -I_{n_2} & \lambda E_{23}, & -B_2 \\ 0 & 0 & \lambda N - I_k, & 0 \\ C_1 & C_2 & C_3 & D \end{bmatrix}^*.$$

Since, by Observation (O3), the property (6.11) is invariant under feedback equivalence, an application of Observation (O2) yields

$$\begin{pmatrix} \widetilde{x}_1 \\ \widetilde{x}_2 \\ \widetilde{x}_3 \\ \widetilde{u} \end{pmatrix} \in \ker_{\mathbb{C}} \begin{bmatrix} I_{n_1} & 0 & 0 & A_{11} & 0 & 0 & B_1 \\ 0 & 0 & E_{23} & 0 & -I_{n_2} & 0 & B_2 \\ 0 & 0 & N & 0 & 0 & I_k & 0 \\ 0 & 0 & 0 & C_1 & C_2 & C_3 & D \end{bmatrix}^*$$

and hence

$$\begin{pmatrix} \widetilde{x}_1 \\ \widetilde{x}_2 \\ \widetilde{u} \end{pmatrix} \in \ker_{\mathbb{C}} \begin{bmatrix} I_{n_1} & 0 & A_{11} & 0 & B_1 \\ 0 & 0 & 0 & -I_{n_2} & B_2 \\ 0 & 0 & C_1 & C_2 & D \end{bmatrix}^*.$$

Step 2b. We prove that $[E, A, B, C, D]$ has Property (P4):

Let $x^0 \in \mathcal{V}^{\text{diff}}_{[E,A,B,C,D]}$ and define $\begin{pmatrix} x_1^0 \\ x_2^0 \\ x_3^0 \end{pmatrix} = T^{-1} x^0$. By the results in Step 2a, we see that the system $[\widetilde{E}, \widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{D}]$ defined in (6.13) is impulse controllable and satisfies (6.11). Now Step 1 gives

$$\exists (\widetilde{x}, \widetilde{u}, \widetilde{y}) \in \mathfrak{B}_{[\widetilde{E},\widetilde{A},\widetilde{B},\widetilde{C},\widetilde{D}]}\left(\begin{pmatrix} x_1^0 \\ x_2^0 \end{pmatrix}\right) : \ \widetilde{u} \in \mathcal{L}^2(\mathbb{R}_{\geq 0} \to \mathbb{K}^m) \ \wedge \ \|\widetilde{y}\|_{\mathcal{L}^2} < \varepsilon.$$

Further, since all generalized eigenvalues of $s\widetilde{E} - \widetilde{A}$ belong to $\mathbb{C}_-$ and the index of $s\widetilde{E} - \widetilde{A}$ is at most one, the property $\widetilde{u} \in \mathcal{L}^2(\mathbb{R}_{\geq 0} \to \mathbb{K}^m)$ together with the latter statement in Step 1 implies that $\widetilde{x} \in \mathcal{L}^2(\mathbb{R}_{\geq 0} \to \mathbb{C}^{n_1+n_2})$. Proposition 2.12(a) gives

$$(x, u, y) = \left(T\begin{pmatrix} \widetilde{x} \\ 0 \end{pmatrix}, \widetilde{u} + FT\begin{pmatrix} \widetilde{x} \\ 0 \end{pmatrix}, y\right) \in \mathcal{B}_{[E,A,B,C,D]}.$$

The $\mathcal{L}^2$-norm of the output thus satisfies $\|y\|_{\mathcal{L}^2} = \|\widetilde{y}\|_{\mathcal{L}^2} < \varepsilon$. Since, further, Proposition 2.12(b) leads to $x_3^0 \in \ker \left[\begin{smallmatrix} E_{23} \\ N \end{smallmatrix}\right]$, we obtain

$$\begin{aligned} Ex(0) &= W^{-1}\begin{bmatrix} I_{n_1} & 0 & 0 \\ 0 & 0 & E_{23} \\ 0 & 0 & N \end{bmatrix}\begin{pmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{pmatrix} = W^{-1}\begin{bmatrix} \begin{bmatrix} I_{n_1} & 0 \\ 0 & 0 \end{bmatrix}\begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} \\ 0 \end{bmatrix} \\ &= W^{-1}\begin{bmatrix} \begin{bmatrix} I_{n_1} & 0 \\ 0 & 0 \end{bmatrix}\begin{pmatrix} x_1^0 \\ x_2^0 \end{pmatrix} \\ 0 \end{bmatrix} = W^{-1}\begin{bmatrix} I_{n_1} & 0 & 0 \\ 0 & 0 & E_{23} \\ 0 & 0 & N \end{bmatrix}\begin{pmatrix} x_1^0 \\ x_2^0 \\ x_3^0 \end{pmatrix} \\ &= W^{-1}\begin{bmatrix} I_{n_1} & 0 & 0 \\ 0 & 0 & E_{23} \\ 0 & 0 & N \end{bmatrix} T^{-1} x^0 = E x^0. \end{aligned}$$

Therefore, $(x, u, y) \in \mathfrak{B}_{[E,A,B,C,D]}(x^0)$, and summarizing we have shown that $[E, A, B, C, D]$ satisfies Property (P4).

(a) $\not\Leftarrow$ Consider the example $[E,A,B,C,D] \in \Sigma_{2,1,1}$ as in (6.1). Since the behaviour is $\mathfrak{B}_{[E,A,B,C,D]} = \{0\} \times \mathcal{L}^2_{\text{loc}}(\mathbb{R}_{\geq 0}\to\mathbb{K}) \times \{0\}$, it follows that (P4) is fulfilled. However,

$$\forall \lambda \in \mathbb{C}: \ \operatorname{rk} \begin{bmatrix} E & A & B \\ 0 & C & D \end{bmatrix} = \operatorname{rk} \begin{bmatrix} 0&1&1&0&0 \\ 0&0&0&1&0 \\ 0&0&1&0&0 \end{bmatrix} = 3 \neq 2 = \operatorname{rk} \begin{bmatrix} -1 & \lambda & 0 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \operatorname{rk} \begin{bmatrix} \lambda E - A & -B \\ C & D \end{bmatrix}.$$

(b) $\Rightarrow$ Since $[E,A,B,C,D]$ satisfies (P4), it follows that $[E,A,B,C,D]$ is behavioural stabilizable. Let $S \in \mathrm{Gl}_{n+p}(\mathbb{C})$, $T \in \mathrm{Gl}_{n+m}(\mathbb{C})$ such that

$$S \begin{bmatrix} \lambda E - A, & -B \\ C & D \end{bmatrix} T = \operatorname{diag}\left(sF_1 - G_1, \ldots, sF_f - G_f\right) \qquad \text{as in (2.2).}$$

By (ii) it suffices to prove that all (ODE)-blocks are corresponding to generalized eigenvalues in $\overline{\mathbb{C}}_-$, and all all (OD)-blocks are of size 1×0.

Step 1. We prove:

$$\begin{aligned} &\forall \varepsilon > 0 \ \forall j \in \{1,\ldots,f\} \ \forall z_j^0 \in \mathbb{C}^{k_j} \ \exists z_j \in \mathcal{L}^2(\mathbb{R}_{\geq 0}\to\mathbb{C}^{k_j}) \ : \\ &\quad F_j z_j(0) = F_j z_j^0 \ \wedge \ \lim_{t\to\infty} F_j z_j(t) = 0 \ \wedge \ F_j \dot z_j - G_j z_j \in \mathcal{L}^2(\mathbb{R}_{\geq 0}\to\mathbb{C}^{l_j}) \ \wedge \ \|F_j \dot z_j - G_j z_j\|_{\mathcal{L}^2} < \varepsilon. \end{aligned}$$

Since the blocks may be suitably reordered, it suffices to prove the statement for $j = 1$. Define, for $z_1^0 \in \mathbb{C}^{k_1}$,

$$\begin{pmatrix} x^0 \\ u^0 \end{pmatrix} = T^{-1} \begin{bmatrix} I_{k_1} \\ 0 \end{bmatrix} z_1^0 \qquad \text{where } x^0 \in \mathbb{C}^n,\ u^0 \in \mathbb{C}^m.$$

Then impulse controllability of $[E,A,B,C,D]$ and (P4) yields the existence of some $(x,u,y) \in \mathfrak{B}_{[E,A,B,C,D]}(x^0)$ such that $\lim_{t\to\infty} Ex(t) = 0$ and $\|y\|_{\mathcal{L}^2} < \|S\|^{-1} \cdot \varepsilon$. Equivalently, $\left(\begin{smallmatrix} x \\ u \end{smallmatrix}\right) = Tz$ satisfies

$$\begin{aligned} &\begin{pmatrix} 0 \\ y \end{pmatrix} = \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} \tfrac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} x \\ u \end{pmatrix} - \begin{bmatrix} A & B \\ -C & -D \end{bmatrix} \begin{pmatrix} x \\ u \end{pmatrix}, \quad && \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} x(0) \\ u(0) \end{pmatrix} = \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} x^0 \\ u^0 \end{pmatrix}, \\ &\lim\nolimits_{t\to\infty} \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} x(t) \\ u(t) \end{pmatrix} = 0\,, && \left\| \begin{pmatrix} 0 \\ y \end{pmatrix} \right\|_{\mathcal{L}^2} < \|S\|^{-1} \cdot \varepsilon. \end{aligned}$$

Then

$$\begin{aligned} F_1 z_1(0) &= [\, I_{l_1} \ 0 \,]\, S \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} T \begin{bmatrix} I_{k_1} \\ 0 \end{bmatrix} z_1(0) = [\, I_{l_1} \ 0 \,]\, S \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} x(0) \\ u(0) \end{pmatrix} \\ &= [\, I_{l_1} \ 0 \,]\, S \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} x^0 \\ u^0 \end{pmatrix} = [\, I_{l_1} \ 0 \,]\, S \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} T \begin{bmatrix} I_{k_1} \\ 0 \end{bmatrix} z_1^0 = F_1 z_1^0. \end{aligned}$$

and for $w_1 := [\, I_{l_1} \ 0 \,]\, S \left(\begin{smallmatrix} 0 \\ y \end{smallmatrix}\right) \in \mathcal{L}^2(\mathbb{R}_{\geq 0}\to\mathbb{C}^{l_1})$ we have

$$w_1 = F_1 \dot z_1 - G_1 z_1, \qquad \lim_{t\to\infty} F_1 z_1(t) = 0, \qquad \|F_1 \dot z_1 - G_1 z_1\|_{\mathcal{L}^2} = \|w_1\|_{\mathcal{L}^2} \leq \|[\, I_{l_j} \ 0 \,]\| \cdot \|S\| \cdot \|y\|_{\mathcal{L}^2} < \varepsilon.$$

This proves the claim in Step 1.

Step 2. We prove that if $sF_j - G_j = sI_{k_j} - (\lambda I_{k_j} + N_{k_j})$ is a (ODE)-block for some $j = 1,\ldots,k$, then $\lambda \in \overline{\mathbb{C}}_-$.
Seeking a contradiction, assume that $\lambda \in \mathbb{C}_+$. Again, it is no loss of generality to assume that $j = 1$. Then $\sigma(-G_1) \subset \mathbb{C}_-$, and by ([26], Thm. 3.28) there exists some $P \succ 0$ which solves the Lyapunov equation $(-G_1)P + P(-G_1)^* + I_{k_1} = 0$ or equivalently

$$G_1^* Q + Q G_1^* = Q^2 \qquad \text{for } Q := P^{-1}.$$

Let $z_1^0 \in \mathbb{C}^{k_1} \setminus \{0\}$ and set $\varepsilon := (z_1^0)^* Q z_1^0 = (P^{-1}z_1^0)^* P (P^{-1}z_1^0) > 0$. Then by Step 1 there exists $z_1 \in \mathcal{L}^2(\mathbb{R}_{\geq 0} \to \mathbb{C}^{k_1})$ such that

$$z_1(0) = z_1^0 \ \wedge \ \lim_{t\to\infty} z_1(t) = 0 \ \wedge \ w_1 := \dot z_1 - G_1 z_1 \in \mathcal{L}^2(\mathbb{R}_{\geq 0} \to \mathbb{C}^{k_1}) \ \wedge \ \|w_1\|_{\mathcal{L}^2}^2 < (z_1^0)^* Q z_1^0,$$

and we conclude, for all $t \geq 0$,

$$\begin{aligned}(z_1^0)^* Q z_1^0 - z_1(t)^* Q z_1(t) &= -\int_0^t \tfrac{\mathrm{d}}{\mathrm{d}\tau}\left(z_1(\tau)^* Q z_1(\tau)\right) \mathrm{d}\tau = -\int_0^t 2\, z_1(\tau)^* Q \dot z_1(\tau)\, \mathrm{d}\tau \\ &= -\int_0^t \left(z_1(\tau)^* Q^2 z_1(\tau) + 2\, z_1(\tau)^* Q w_1(\tau)\right) \mathrm{d}\tau \\ &= -\int_0^t \left(\|Q z_1(\tau) + w_1(\tau)\|^2 - \|w_1(\tau)\|^2\right) \mathrm{d}\tau \leq \int_0^t \|w_1(\tau)\|^2 \mathrm{d}\tau.\end{aligned}$$

Now taking the limit for $t \to \infty$ and invoking $\lim_{t\to\infty} z_j(t) = 0$ yields the contradiction

$$(z_1^0)^* Q z_1^0 \leq \|w_1\|_{\mathcal{L}^2}^2 < (z_1^0)^* Q z_1^0.$$

Step 3. We prove that if $sF_j - G_j = sK_{k_j}^\top - L_{k_j}^\top$ is an (OD)-block for some $j = 1, \ldots, k$, then its size is at most 1×0.

Again, it is no loss of generality to assume that $j = 1$. Seeking a contradiction, assume that $k_1 \geq 2$. Define $f_0, \ldots, f_{k_1} \in \mathbb{R}$ such that

$$(s-1)^{k_1} = f_0 + \ldots + f_{k_1-1} s^{k_1-1} + s^{k_1} \in \mathbb{R}[s] \qquad \text{and} \quad F := \begin{bmatrix} -f_{k_1-1} & \\ \vdots & I_{k_1} \\ -f_0 & \end{bmatrix} \in \mathbb{R}^{k_1 \times (k_1+1)}.$$

Then a straightforward calculation gives

$$sI_{k_1} - A_1 := F\left(sK_{k_1}^\top - L_{k_1}^\top\right) \qquad \text{satisfies } \det\left(sI_{k_1} - A_1\right) = (s-1)^{k_1}.$$

Let $z_1^0 \in \mathbb{C}^{k_1}$ and $\varepsilon > 0$. Then by Step 1

$$\begin{aligned}\exists z_1 \in \mathcal{L}^2(\mathbb{R}_{\geq 0} \to \mathbb{C}^{k_1}) :\ & K_{k_1} z_1(0) = K_{k_1} z_1^0 \ \wedge \ \lim_{t\to\infty} K_{k_1} z_1(t) = 0 \\ & \wedge \ w_1 := K_{k_1} \dot z_1 - L_{k_1} z_1 \in \mathcal{L}^2(\mathbb{R}_{\geq 0} \to \mathbb{C}^{k_1+1}) \ \wedge \ \|w_1\|_{\mathcal{L}^2}^2 < \|F\|^{-1} \cdot \varepsilon,\end{aligned}$$

and since K_{k_1} has full column rank, we see that $z_1(0) = z_1^0$ and $\lim_{t\to\infty} z_1(t) = 0$. Moreover, $Fw_1 = \dot z_1 - A_1 z_1 \in \mathcal{L}^2(\mathbb{R}_{\geq 0} \to \mathbb{C}^{k_1})$ and $\|Fw_1\|_{\mathcal{L}^2}^2 < \|F\| \cdot \|F\|^{-1} \cdot \varepsilon = \varepsilon$. This leads to the same contradiction as in Step 2. □

In the following we present a characterization of Property (P4) in terms of the reduced system in Proposition 6.3.

Proposition 6.5. *Let* $[E, A, B, C, D] \in \Sigma_{n,m,p}$ *and* $Y \in \mathbb{K}^{p \times p_1}$ *as in* (6.3)*. Then*

$$[E, A, B, C, D] \text{ satisfies (P4)} \quad \Longleftrightarrow \quad [E, A, B, Y^*C, Y^*D] \text{ satisfies (P1) and is behavioural stabilizable.}$$

Proof.

$\Leftarrow$ Let $\varepsilon > 0$ and $x^0 \in \mathcal{V}^{\text{diff}}_{[E,A,B,C,D]} = \mathcal{V}^{\text{diff}}_{[E,A,B,Y^*C,Y^*D]}$. Since $[E, A, B, Y^*C, Y^*D]$ satisfies (P4) by Proposition 6.4(a), behavioural stabilizability yields

$$\exists (x, u, y_1) \in \mathfrak{B}_{[E,A,B,Y^*C,Y^*D]}(x^0) :\ u \in \mathcal{L}^2(\mathbb{R}_{\geq 0} \to \mathbb{K}^m) \ \wedge \ \lim_{t\to\infty} Ex(t) = 0 \ \wedge \ \|y_1\|_{\mathcal{L}^2} < \varepsilon.$$

Now Proposition 6.3(i) together with the orthonormality of the columns of Y gives $(x, u, y) \in \mathfrak{B}_{[E,A,B,C,D]}(x^0)$ for $y = Yy_1$ and $\|y\|_{\mathcal{L}^2} = \|y_1\|_{\mathcal{L}^2} < \varepsilon$. Hence, $[E, A, B, C, D]$ satisfies Property (P4).

$\Rightarrow$ In view of $Y^*Y = I_{p_1}$ and Proposition 6.3(i) we see that $[E, A, B, Y^*C, Y^*D]$ satisfies (P4), Note also that behavioural stability of $[E, A, B, C, D]$ follows immediately from Property (P4).

Step 1. We prove, under the additional assumption that $[E, A, B, C, D]$ is impulse controllable, that $[E, A, B, Y^*C, Y^*D]$ satisfies (P1).
Since the assertion gives that $[E, A, B, Y^*C, Y^*D]$ is impulse controllable, we may conclude

$$\forall \lambda \in \mathbb{C}_+ : \ \operatorname{rk} \begin{bmatrix} \lambda E - A, & -B \\ Y^*C & Y^*D \end{bmatrix} \overset{\text{Pr. 6.4(b)}}{=} \operatorname{rk} \begin{bmatrix} E & A & B \\ 0 & Y^*C & Y^*D \end{bmatrix} \underset{\text{Pr. 6.2(b)}}{\overset{\text{Pr. 6.3(iii)}}{=}} n + p_1.$$

and so $[E, A, B, Y^*C, Y^*D]$ satisfies (P1).

Step 2. We prove the implication $\Longrightarrow$ in the general case:
Since the properties (P1) and (P4) are invariant under system equivalence, we can, in view of Proposition 2.9, assume that

$$\begin{bmatrix} sE - A, & -B \\ C & D \end{bmatrix} = \begin{bmatrix} sE_{11} - A_{11} & sE_{12} & -B_1 \\ 0 & sN - I_k & 0 \\ C_1 & C_2 & D \end{bmatrix}, \quad \begin{array}{l} \text{where } N \in \mathbb{K}^{k \times k} \text{ is nilpotent} \\ \text{and } [E_{11}, A_{11}, B_1, C_1] \\ \text{is impulse controllable.} \end{array} \tag{6.14}$$

Since $[E, A, B, C, D]$ satisfies (P4), an application of Proposition 2.9(b) yields that the subsystem $[E_{11}, A_{11}, B_1, C_1, D]$ satisfies (P4), too. Now we may apply Step 1 to conclude, for all $\lambda \in \mathbb{C}_+$,

$$\operatorname{rk} \begin{bmatrix} \lambda E - A & -B_1 \\ Y^*C & Y^*D \end{bmatrix} = \operatorname{rk} \begin{bmatrix} \lambda E_{11} - A_{11} & \lambda E_{12}, & -B_1 \\ 0 & \lambda N - I_k, & 0 \\ Y^*C_1 & Y^*C_2 & Y^*D \end{bmatrix} = k + \operatorname{rk} \begin{bmatrix} \lambda E_{11} - A_{11}, & -B_1 \\ Y^*C_1 & Y^*D \end{bmatrix} \overset{\text{Step 1}}{=} k + n_1 + p_1 = n + p_1.$$

Therefore, $[E, A, B, Y^*C, Y^*D]$ satisfies (P1).

This completes the proof of the proposition. □

Finally, we are in a position to "almost characterize" outer transfer functions in terms of Properties (P3) and (P4).

Theorem 6.6. *For any system $[E, A, B, C, D] \in \Sigma_{n,m,p}(\mathbb{K})$ with transfer function $G(s) = C(sE-A)^{-1}B + D \in \mathbb{K}(s)^{p \times m}$ the following statements hold true:*

(a) (P3) *&* (P4) $\Longleftrightarrow$ $\begin{cases} \text{(P1) and } [E, A, B, C, D] \\ \text{is behavioural stabilizable.} \end{cases}$

(b) (P3) *&* (P4) $\Longrightarrow$ *$G(s)$ is outer.*

(c) (P3) *&* (P4) $\Longleftarrow$ $\begin{cases} G(s) \text{ is outer and } [E, A, B, C, D] \text{ is} \\ \text{behavioural stabilizable and detectable.} \end{cases}$

Proof.

(a) $\Rightarrow$ Since $[E, A, B, C, D]$ satisfies (P3), we have $Y = I_p$ for Y as in (6.3). Now the implication is a consequence of Proposition 6.5.

(a) $\Leftarrow$ **Step 1**. We first additionally assume that $[E, A, B, C, D]$ is impulse controllable.
We have

$$\forall \lambda \in \mathbb{C}_+ : \ \operatorname{rk} \begin{bmatrix} E & A & B \\ 0 & C & D \end{bmatrix} \geq \operatorname{rk} \begin{bmatrix} \lambda E - A, & -B \\ C & D \end{bmatrix} \overset{\text{(P1)}}{=} n + p \geq \operatorname{rk} \begin{bmatrix} E & A & B \\ 0 & C & D \end{bmatrix},$$

and thus

$$\forall \lambda \in \mathbb{C}_+ : \ \mathrm{rk}\begin{bmatrix} \lambda E - A, & -B \\ C & D \end{bmatrix} = \mathrm{rk}\begin{bmatrix} E & A & B \\ 0 & C & D \end{bmatrix} = n + p.$$

Proposition 6.2(b) now yields Property (P3), and Proposition 6.4(a) implies Property (P4).

Step 2. We prove the implication for general $[E, A, B, C, D] \in \Sigma_{n,m,p}(\mathbb{K})$: since the Properties (P1), (P3), (P4) and behavioural stabilizability are invariant under system equivalence, we can again, by Proposition 2.9, assume that the system $[E, A, B, C, D]$ is in the form (6.14). Then

$$\forall \lambda \in \mathbb{C}_+ : n + p \overset{\text{(P1)}}{=} \mathrm{rk}\begin{bmatrix} \lambda E - A & -B \\ C & D \end{bmatrix} = k + \mathrm{rk}\begin{bmatrix} \lambda E_{11} - A_{11} & -B_1 \\ C_1 & D \end{bmatrix}$$

and so $[E_{11}, A_{11}, B_1, C_1, D]$ satisfies (P1) and is impulse controllable. We can immediately conclude from Proposition 2.9(b) that behavioural stabilizability of $[E, A, B, C, D]$ is equivalent to behavioural stabilizability of $[E_{11}, A_{11}, B_1, C_1, D]$. Therefore, we may apply the result of Step 1 to conclude that $[E_{11}, A_{11}, B_1, C_1, D]$ satisfies (P3) and (P4). Finally, Proposition 2.9(b) yields that $[E, A, B, C, D]$ satisfies (P3) and (P4).

(b) and (c) The implications in assertions (b) and (c) are a consequence of assertion (a) and Theorem 3.3.

This completes the proof of the theorem. □

7. Systems described by ordinary differential equations

Here we discuss consequences of the results in Sections 3–6 for systems described by ordinary differential equations

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t} x(t) &= Ax(t) + Bu(t), \qquad x(0) = x^0, \\ y(t) &= Cx(t) + Du(t). \end{aligned}$$

The essential additional feature of ordinary differential equations is that for any initial state $x^0 \in \mathbb{K}^n$ and input $u \in \mathcal{L}^2_{\mathrm{loc}}(\mathbb{R}_{\ge 0} \to \mathbb{R}^m)$, there exist unique functions $x = x(\cdot\,; x^0, u)$ and $y = y(\cdot\,; x^0, u)$ with $(x, u, y) \in \mathfrak{B}_{[I,A,B,C,D]}(x^0)$. The following conclusions can be drawn from this fact for any $[I, A, B, C, D] \in \Sigma_{n,m,p}(\mathbb{K})$:

(ODE 1) $\mathfrak{B}_{[I,A,B,C,D]} := \left\{ (x(\cdot\,; x^0, u), u, y(\cdot\,; x^0, u)) \;\middle|\; x^0 \in \mathbb{K}^n,\ u \in \mathcal{L}^2_{\mathrm{loc}}(\mathbb{R}_{\ge 0} \to \mathbb{R}^m) \right\};$
(ODE 2) $\mathcal{V}^{\mathrm{diff}}_{[I,A,B,C,D]} = \mathbb{K}^n$. In other words, any $[I, A, B, C, D] \in \Sigma_{n,m,p}(\mathbb{K})$ is impulse controllable;
(ODE 3) $[I, A, B, C, D]$ is behavioural stabilizable if, and only if, $[I, A, B, C, D]$ is stabilizable;
(ODE 4) $[I, A, B, C, D]$ is behavioural detectable if, and only if, $[I, A, B, C, D]$ is detectable.

For the notions of stabilizability and detectability of ordinary differential equations, we refer to ([26], Sects. 3.10 and 3.11).

Taking into account (ODE 1) and (ODE 2), we obtain that Properties (P1)–(P4) read as follows for ordinary differential equations:

(P1') $\forall \lambda \in \mathbb{C}_+ \ : \quad \mathrm{rk}\begin{bmatrix} \lambda I - A, & -B \\ C & D \end{bmatrix} = n + p.$

(P2') $\forall \varepsilon > 0 \quad \forall z \in \mathcal{L}^2(\mathbb{R}_{\ge 0} \to \mathbb{K}^p) \quad \exists u \in \mathcal{L}^2(\mathbb{R}_{\ge 0} \to \mathbb{K}^m) \ :$
$\|z - y(\cdot\,; 0, u)\|_{\mathcal{L}^2} < \varepsilon.$

(P3') $\forall y^0 \in \mathbb{K}^p \setminus \{0\} \ \exists x^0 \in \mathbb{R}^n, u \in \mathcal{L}^2_{\mathrm{loc}}(\mathbb{R}_{\ge 0} \to \mathbb{K}^m) \ : \ (y^0)^* y(\cdot\,; x^0, u) \ne 0.$

(P4') $\forall \varepsilon > 0 \ \forall x^0 \in \mathbb{K}^n \ \exists u \in \mathcal{L}^2(\mathbb{R}_{\ge 0} \to \mathbb{K}^m) :$
$\lim_{t \to \infty} x(t; x^0, u(\cdot)) = 0 \quad \wedge \quad \|y(\cdot\,; x^0, u)\|_{\mathcal{L}^2} < \varepsilon.$

Using Properties (ODE 3) & (ODE 4), we can formulate the following corollary of Theorem 3.3:

Corollary 7.1 (Equivalence of outer and (P1')). *For any* $[I, A, B, C, D] \in \Sigma_{n,m,p}(\mathbb{K})$ *with transfer function* $G(s) = C(sI - A)^{-1}B + D \in \mathbb{K}(s)^{p\times m}$ *we have:*

(a) (P1') $\Longrightarrow G(s)$ *is outer.*
(b) (P1') $\Longleftarrow G(s)$ *is outer and* $[I, A, B, C, D]$ *is stabilizable and detectable.*

It is straightforward that Theorem 5.1 becomes:

Corollary 7.2 (Characterization of (P2')). *For any system* $[I, A, B, C, D] \in \Sigma_{n,m,p}(\mathbb{K})$ *with transfer function* $G(s) = C(sI - A)^{-1}B + D \in \mathbb{K}(s)^{p\times m} \cap \mathcal{H}^\infty(\mathbb{C}_+ \to \mathbb{K}^{m\times p})$ *we have*

$$G(s) \text{ is outer} \qquad \Longleftrightarrow \qquad \text{(P2')}.$$

Using (ODE 2), Proposition 6.2 and

$$\operatorname{rk}\begin{bmatrix} I & A & B \\ 0 & C & D \end{bmatrix} = n + \operatorname{rk}[C,\ D], \tag{7.1}$$

the following characterization of Property (P3') can be made:

Corollary 7.3 (Characterization of (P3')).
For any system $[I, A, B, C, D] \in \Sigma_{n,m,p}(\mathbb{K})$ *the following statements hold true:*

$$\textit{(P3')} \Longleftrightarrow \qquad \operatorname{rk}[C,\ D] = p.$$

Property (ODE 1) implies that the space $\mathcal{Y}_0$ as defined in Proposition 6.3 reads as follows for an ordinary differential equation $[I, A, B, C, D] \in \Sigma_{n,m,p}$:

$$\mathcal{Y}_0 \ := \ \left\{ y^0 \in \mathbb{K}^p \ \middle|\ \begin{array}{l} \forall x^0 \in \mathbb{K}^n\ \forall u \in \mathcal{L}^2_{\text{loc}}(\mathbb{R}_{\ge 0} \to \mathbb{K}^m) \text{ and for almost all } t \in \mathbb{R}: \\ (y^0)^* y(\cdot\,; x^0, u) = 0 \end{array} \right\}. \tag{7.2}$$

Now we show that this space has a rather simple representation.

Proposition 7.4 (Representation of $\mathcal{Y}_0$). *For any system* $[I, A, B, C, D] \in \Sigma_{n,m,p}(\mathbb{K})$*, the space* $\mathcal{Y}_0$ *as in* (7.2) *is given by*

$$\mathcal{Y}_0 = (\operatorname{im}[C,\ D])^\perp.$$

Proof. $\supset$: Assume that $y^0 \in (\operatorname{im}[C,\ D])^\perp$. Then for all $x^0 \in \mathbb{K}^n$, $u \in \mathcal{L}^2_{\text{loc}}(\mathbb{R}_{\ge 0} \to \mathbb{K}^m)$ holds

$$(y^0)^* y(\cdot\,; x^0, u) = (y^0)^* [C,\ D] \left(\begin{smallmatrix} x(\cdot\,; x^0, u) \\ u \end{smallmatrix} \right) = 0,$$

and thus $y^0 \in \mathcal{Y}_0$.
$\subset$: Let $y^0 \in \mathcal{Y}_0$. Then for all $x^0 \in \mathbb{K}^n$ and $u^0 \in \mathbb{K}^m$, an application of the constant input $u(\cdot) = u^0$ gives

$$0 = (y^0)^* y(\cdot\,; x^0, u),$$

and we can conclude by continuity of y that

$$0 = (y^0)^* y(0) = (y^0)^* [C, D] \begin{pmatrix} x^0 \\ u^0 \end{pmatrix} \quad \forall x^0 \in \mathbb{K}^n, u^0 \in \mathbb{K}^m,$$

whence $y^0 \in (\operatorname{im}[C,\ D])^\perp$. □

An immediate consequence of Proposition 7.4 is that, for ordinary differential equations, the matrix $Y \in \mathbb{K}^{p\times p_1}$ as in (6.3) is equivalently characterized by

$$Y \in \mathbb{K}^{p\times p_1} : \ \operatorname{im} Y = [C,\ D] \quad \wedge \quad Y^*Y = I_{p_1}. \tag{7.3}$$

This representation of Y together with (ODE 2) and (7.1) allows to infer the subsequent characterization of (P4') from Propositions 6.4 and 6.5:

Corollary 7.5 (Characterization of (P4')). *Let $[I, A, B, C, D] \in \Sigma_{n,m,p}$ and $Y \in \mathbb{K}^{p\times p_1}$ as in (7.3). Then*

(P4') $\iff$ $n + \operatorname{rk}[C,\ D] = \operatorname{rk}\left[\begin{smallmatrix}\lambda I - A, & -B\\ C & D\end{smallmatrix}\right]$ $\forall \lambda \in \mathbb{C}_+$ *and $[I, A, B, C, D]$ is stabilizable,*

$\iff$ *$[I, A, B, Y^*C, Y^*D]$ satisfies (P1') and is stabilizable.*

Using Properties (ODE 3) and (ODE 4) we can conclude an equivalent characterization for (P3')& (P4') from Theorem 6.6.

Corollary 7.6. *For any system $[I, A, B, C, D] \in \Sigma_{n,m,p}(\mathbb{K})$ with transfer function $G(s) = C(sI - A)^{-1}B + D \in \mathbb{K}(s)^{p\times m}$ the following statements hold true:*

(a) (P3') & (P4') $\iff$ (P1') *and $[I, A, B, C, D]$ is stabilizable.*
(b) (P3') & (P4') $\implies$ *$G(s)$ is outer.*
(c) (P3') & (P4') $\impliedby$ *$G(s)$ is outer and $[I, A, B, C, D]$ is stabilizable and detectable.*

We finalize this section with an example where the generalized eigenvalues of the system pencil $R(s) = \left[\begin{smallmatrix}sI-A, & -B\\ C & D\end{smallmatrix}\right]$ lie on the imaginary axis.

Example 7.7.
(a) Consider the stabilizable and detectable system

$$\begin{aligned}\tfrac{\mathrm{d}}{\mathrm{d}t}x(t) &= -x(t) + u(t), & x(0) &= x^0,\\ y(t) &= \quad x(t) - u(t).\end{aligned} \tag{7.4}$$

Then the system pencil $R(s) = \left[\begin{smallmatrix}s+1, & -1\\ 1 & -1\end{smallmatrix}\right]$ has only the generalized eigenvalue $\lambda = 0$. Thus Property (P1') holds and, according to Corollary 7.1, the transfer function $G(s) = \frac{-s}{s+1}$ is outer. By Corollary 7.3, Property (P3') is valid and so, in view of Corollary 7.6, Property (P4') holds. This property says that for arbitrary $x^0 \in \mathbb{R}$ and arbitrarily small $\varepsilon > 0$, there exists some $u(\cdot) \in \mathcal{L}^2(\mathbb{R}_{\geq 0} \to \mathbb{R})$ such that $\lim_{t\to\infty} x(t; x^0, u(\cdot)) = 0$ and $\|y(\cdot\,; x^0, u)\|_{\mathcal{L}^2} < \varepsilon$.
For instance, choose $\delta > 0$ with $|x^0|^2\ \delta < 2\ \varepsilon^2$ and

$$u(\cdot) = (1-\delta)\ \mathrm{e}^{-\delta\cdot}\ x^0 \in \mathcal{L}^2(\mathbb{R}_{\geq 0}\!\to \mathbb{R}).$$

Then, by variation of constants, we obtain $x(\cdot\,; 1, u) = \mathrm{e}^{-\delta\cdot}\ x^0$, whence $y(\cdot) = \delta\ e^{-\delta\cdot}\ x^0$ and $\|y\|_{\mathcal{L}^2} = \sqrt{\delta/2}\ |x^0| < \varepsilon$.

(b) Consider the stabilizable and detectable system

$$\begin{aligned}\tfrac{\mathrm{d}}{\mathrm{d}t}x(t) &= -x(t) + u(t), & x(0) &= x^0,\\ y(t) &= \quad x(t).\end{aligned} \tag{7.5}$$

Then the system pencil $R(s) = \left[\begin{smallmatrix}s+1, & -1\\ 1 & 0\end{smallmatrix}\right]$ has only generalized eigenvalue at ∞, its Kronecker canonical form consists of only one 2×2 (AE)-block. It follows as above that the Properties (P1'), (P3'), and (P4') hold. For instance, choose $\delta > 0$ with $|x^0|^2\ \delta < 2\ \varepsilon^2$ and

$$u(\cdot) = (\delta - 1)/\delta\ \mathrm{e}^{-\cdot/\delta}\ x^0 \in \mathcal{L}^2(\mathbb{R}_{\geq 0}\!\to \mathbb{R}).$$

Then, by variation of constants, $y(\cdot) = x(\cdot\,; 1, u) = \mathrm{e}^{-\cdot/\delta}\ x^0$ and $\|y\|_{\mathcal{L}^2} = \sqrt{\delta/2}\ |x^0| < \varepsilon$.

8. The optimal control problem of ordinary differential equations

In this section we investigate the optimal control problem for stabilizable systems described by

$$\tfrac{\mathrm{d}}{\mathrm{d}t}x(t) = Ax(t) + Bu(t), \qquad x(0) = x^0, \tag{8.1}$$

where $A \in \mathbb{K}^{n\times n}$, $B \in \mathbb{K}^{n\times m}$, $x^0 \in \mathbb{K}^n$. The results of the present section are known; the novelty lies in the simple proofs. The concepts of outer transfer function as well as (stable) zero dynamics have a unifying power. This allows for simple and structurally interesting proofs of the relationships between the feasibility of the optimal control problem, Lur'e and Riccati matrix equations, the Kalman–Yakubovich–Popov (KYP) inequality, and – most importantly – of the zero dynamics and outer. For example, we will show that if u is an optimal control function, then (x, u) belongs to the zero dynamics of a certain system that will be constructed from a solution of Lur'e equations.

Moreover, we strongly believe that the approach of the present section is the right approach to solve the optimal control problem for differential-algebraic equations. This will be subject of future research.

Definition 8.1 (Feasibility of the optimal control problem, stabilizing solution of the Lur'e equation)**.** Consider a stabilizable system $[I, A, B, 0, 0] \in \Sigma_{n,m,0}(\mathbb{K})$ and

$$(Q, S, R) \in \mathbb{K}^{n\times n} \times \mathbb{K}^{n\times m} \times \mathbb{K}^{m\times m} \qquad \text{with } \; Q = Q^* \; \text{ and } \; R = R^*. \tag{8.2}$$

We say that the *optimal control problem for* $[I, A, B, 0, 0]$ *is feasible*, if the cost functional

$$V^+ : \mathbb{K}^n \to \mathbb{R} \cup \{-\infty\}, \qquad x^0 \mapsto \inf_{\substack{(x,u,y)\in\mathfrak{B}_{[I,A,B,0,0]}(x^0)\\ \lim_{t\to\infty} x(t)=0}} \int_0^\infty \begin{pmatrix} x(\tau)\\ u(\tau)\end{pmatrix}^* \begin{bmatrix} Q & S\\ S^* & R\end{bmatrix} \begin{pmatrix} x(\tau)\\ u(\tau)\end{pmatrix} \mathrm{d}\tau \tag{8.3}$$

satisfies

$$\forall x^0 \in \mathbb{K}^n : \; V^+(x^0) \in \mathbb{R}.$$

We call triple $(X, K, L) \in \mathbb{K}^{n\times n} \times \mathbb{K}^{p\times n} \times \mathbb{K}^{p\times m}$ with $X = X^*$ a *solution of the Lur'e equation*, if

$$\begin{aligned} A^*X + XA + Q &= K^*K,\\ XB + S &= K^*L,\\ R &= L^*L; \end{aligned} \tag{8.4}$$

and (X, K, L) is called a *stabilizing solution*, if additionally

$$\forall\, \lambda \in \mathbb{C}_+ \;\; : \quad \operatorname{rk} \begin{bmatrix} \lambda I - A, & -B\\ K & L\end{bmatrix} = n + p. \tag{8.5}$$

The reason why (8.5) leads to the notion of "stabilizing solution" is due to the fact that if for all $x^0 \in \mathbb{K}^n$ there exists a unique $(x, u, y) \in \mathfrak{B}_{[I,A,B,0,0]}(x^0)$ with $\lim_{t\to\infty} x(t) = 0$ and minimizing the cost functional (8.3), then the Lur'e equation is equivalent to an algebraic Riccati equation (see (8.13)), and its Hermitian solution solution leads to $\operatorname{rk}(\lambda I - (A - BR^{-1}(B^*X + S^*)) = n$ for all $\lambda \in \overline{\mathbb{C}}_+$. The latter is called *stabilizing solution* of algebraic Riccati equations (see [19], Sect. 9.3).

Algebraic criteria for the solvability of the Lur'e equation can be found in [23].

Remark 8.2 (Lur'e equation and Kalman–Yakubovich–Popov (KYP) inequality)**.**
We collect some important consequences of the Lur'e equation (8.4):

(i) The Lur'e equation (8.4) is equivalent to

$$\begin{bmatrix} A^*X + XA + Q & XB + S\\ B^*X + S^* & R\end{bmatrix} = \begin{bmatrix} K^*\\ L^*\end{bmatrix} \begin{bmatrix} K & L\end{bmatrix}. \tag{8.6}$$

(ii) If (X, K, L) solves the Lur'e equation (8.4), then by (8.6) the matrix X solves the *Kalman–Yakubovich–Popov (KYP) inequality*, i.e.,

$$\begin{bmatrix} A^*X + XA + Q & XB + S \\ B^*X + S^* & R \end{bmatrix} \succeq 0. \tag{8.7}$$

(iii) If X solves the Kalman–Yakubovich–Popov inequality (8.7), then we may choose $K \in \mathbb{K}^{n\times p}$ and $L \in \mathbb{K}^{m\times p}$ of full rank $p = \operatorname{rk} \left[\begin{smallmatrix} A^*X+XA+Q & XB+S \\ B^*X+S^* & R \end{smallmatrix}\right]$ so that (X, K, L) solves the Lur'e equation (8.6).

(iv) It is shown in [23] that if (X, K, L) is a stabilizing solution of Lur'e equation, then X is the maximal solution (with respect to the partial order $\succeq$) of the KYP inequality (8.7).

(v) If (X, K, L) solves the Lur'e equation (8.4), then we have, for every $(x, u, y) \in \mathfrak{B}_{[I,A,B,0,0]}$ and $0 \le t_1 \le t_2$, by the fundamental theorem of calculus, the product rule of differentiation, and omitting the arguments τ,

$$\begin{aligned} x(t_2)^*Xx(t_2) - x(t_1)^*Xx(t_1) &= \int_{t_1}^{t_2} \tfrac{\mathrm{d}}{\mathrm{d}\tau} x^*Xx \,\mathrm{d}\tau \;=\; \int_{t_1}^{t_2} 2x^*X\dot{x}\,\mathrm{d}\tau \;=\; \int_{t_1}^{t_2} 2x^*X(Ax+Bu)\,\mathrm{d}\tau \\ &\overset{(8.4)}{=} \int_{t_1}^{t_2} -x^*Qx + x^*K^*Kx - u^*S^*x + u^*L^*Kx - x^*Su + x^*K^*Lu - u^*Ru + u^*L^*Lu \,\mathrm{d}\tau \\ &= -\int_{t_1}^{t_2} \left(\begin{smallmatrix} x \\ u \end{smallmatrix}\right)^* \left[\begin{smallmatrix} Q & S \\ S^* & R \end{smallmatrix}\right] \left(\begin{smallmatrix} x \\ u \end{smallmatrix}\right) \mathrm{d}\tau + \int_{t_1}^{t_2} \|Kx + Lu\|^2 \,\mathrm{d}\tau. \end{aligned} \tag{8.8}$$

This yields that $V^*(x^0) := (x^0)^*Xx^0$ is a *dissipation function for* $[I, A, B, 0, 0]$, that is we have,

$$\forall (x,u,y) \in \mathfrak{B}_{[I,A,B,0,0]}\ \forall 0 \le t_1 \le t_2 :\ V^+(x(t_1)) - V^+(x(t_2)) \le \int_{t_1}^{t_2} \left(\begin{smallmatrix} x(\tau) \\ u(\tau) \end{smallmatrix}\right)^* \left[\begin{smallmatrix} Q & S \\ S^* & R \end{smallmatrix}\right] \left(\begin{smallmatrix} x(\tau) \\ u(\tau) \end{smallmatrix}\right) \mathrm{d}\tau. \tag{8.9}$$

(vi) If (X, K, L) solves the Lur'e equation (8.4), then (8.8) yields that for every $x^0 \in \mathbb{K}^n$ and $(x, u, y_s) \in \mathfrak{B}_{[I,A,B,K,L]}(x^0)$ with $\lim_{t\to\infty} x(t) = 0$ we have

$$\int_0^\infty \left(\begin{smallmatrix} x(\tau) \\ u(\tau) \end{smallmatrix}\right)^* \left[\begin{smallmatrix} Q & S \\ S^* & R \end{smallmatrix}\right] \left(\begin{smallmatrix} x(\tau) \\ u(\tau) \end{smallmatrix}\right) \mathrm{d}\tau \;=\; (x^0)^*Xx^0 + \int_{t_1}^{t_2} \|y_s(\tau)\|^2 \,\mathrm{d}\tau. \tag{8.10}$$

We are now in a position to state and to give a simple prove of the celebrated optimal control theorem.

Theorem 8.3 (Necessary and sufficient criteria for the optimal control problem)**.** *For any stabilizable system* $[I, A, B, 0, 0] \in \Sigma_{n,m,0}(\mathbb{K})$ *and* (Q, S, R) *as in* (8.2) *the following statements are equivalent:*

(a) *The optimal control problem is feasible.*
(b) $\exists X = X^* \in \mathbb{K}^{n\times n}\ \forall x^0 \in \mathbb{K}^n :\ V^+(x^0) = (x^0)^*Xx^0$. *This means, the cost functional is quadratic.*
(c) *There exists a stabilizing solution* (X, K, L) *of the Lur'e equation.*

Proof. (a) $\Leftrightarrow$ (b): This is stated in the proof of ([27], Thm. 3) where it is additionally assumed that the system $[I, A, B, 0, 0]$ is controllable. The claim can be proved, even without the assumption of controllability, by invoking the parallelogram law. The proof is omitted.

(b) $\Rightarrow$ (c): We proceed in several steps.

(i): Since V^+ is a *dissipation function for* $[I, A, B, 0, 0]$, Remark 8.2(ii) yields that (8.9) holds for $V^+(x^0) = (x^0)^*Xx^0$.

(ii): We show that X satisfies the KYP inequality (8.7).
Let $x^0 \in \mathbb{K}^n$, $u(\cdot) = u^0 \in \mathbb{K}^m$, and consider $(x, u, y) \in \mathfrak{B}_{[I,A,B,0,0]}(x^0)$. Then (8.9) yields, for all $h > 0$,

$$\frac{1}{h}\left((x^0)^*Xx^0 - x(h)^*Xx(h)\right) \le \frac{1}{h}\int_0^h \left(\begin{smallmatrix} x(\tau) \\ u(\tau) \end{smallmatrix}\right)^* \left[\begin{smallmatrix} Q & S \\ S^* & R \end{smallmatrix}\right] \left(\begin{smallmatrix} x(\tau) \\ u(\tau) \end{smallmatrix}\right) \mathrm{d}\tau,$$

and invoking continuity of u and x, and taking the limit $h \to 0$ gives

$$\begin{aligned}\begin{pmatrix} x^0 \\ u^0 \end{pmatrix}^* \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{pmatrix} x^0 \\ u^0 \end{pmatrix} \geq -\dot{x}(0)^* X x^0 - (x^0)^* X \dot{x}(0) &= -(Ax^0 + Bu^0)^* X x^0 - (x^0)^* X (Ax^0 + Bu^0) \\ &= \begin{pmatrix} x^0 \\ u^0 \end{pmatrix}^* \begin{bmatrix} -A^*X - XA & -XB \\ -B^*X & 0 \end{bmatrix} \begin{pmatrix} x^0 \\ u^0 \end{pmatrix}.\end{aligned}$$

Since x^0, u^0 are arbitrary, this proves (8.7).

(iii): Since (8.7) holds, it follows from Remark 8.2(iii) that (8.6) is valid. Therefore, the Lur'e equation (8.4) holds for (X, K, L) by Remark 8.2(i).

(iv): Since $\mathrm{rk}[K,\, L] = p$, Corollary 7.3 yields that $[I, A, B, K, L]$ satisfies the Property (P3').

(v): Equation (8.10) reads, for every $(x, u, y_s) \in \mathfrak{B}_{[I,A,B,K,L]}(x^0)$ with $\lim_{t\to\infty} x(t) = 0$,

$$\int_0^\infty \begin{pmatrix} x(\tau) \\ u(\tau) \end{pmatrix}^* \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{pmatrix} x(\tau) \\ u(\tau) \end{pmatrix} \mathrm{d}\tau \;=\; V^+(x^0) + \|y_s\|^2_{\mathcal{L}^2},$$

and hence the definition of V^+ yields that $[I, A, B, K, L]$ satisfies Property (P4').

(vi): Now it follows from Corollary 7.6(a) that $[I, A, B, K, L]$ satisfies Property (P1'). Therefore, (X, K, L) is a stabilizing solution.

(c) $\Rightarrow$ (b):

(vii): The inequality $V^+(x^0) \geq (x^0)^* X x^0$ for all $x^0 \in \mathbb{K}^n$ follows since we have, for all $(x, u, y) \in \mathfrak{B}_{[I,A,B,0,0]}(x^0)$ with $\lim_{t\to\infty} x(t) = 0$,

$$(x^0)^* X x^0 \leq (x^0)^* X x^0 + \int_0^\infty \|Kx(\tau) + Lu(\tau)\|^2 \mathrm{d}\tau \overset{(8.8)}{=} \int_0^\infty \begin{pmatrix} x(\tau) \\ u(\tau) \end{pmatrix}^* \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{pmatrix} x(\tau) \\ u(\tau) \end{pmatrix} \mathrm{d}\tau. \tag{8.11}$$

(viii): We show the inequality $V^+(x^0) \leq (x^0)^* X x^0$ for all $x^0 \in \mathbb{K}^n$.
For $x^0 \in \mathbb{K}^n$ and $(x, u, y_s) \in \mathfrak{B}_{[I,A,B,K,L]}(x^0)$, equation (8.8) reads

$$(x^0)^* X x^0 + \|y_s\|^2_{\mathcal{L}^2} = \int_0^\infty \begin{pmatrix} x(\tau) \\ u(\tau) \end{pmatrix}^* \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{pmatrix} x(\tau) \\ u(\tau) \end{pmatrix} \mathrm{d}\tau \;\geq\; V^+(x^0). \tag{8.12}$$

By (8.5), $[I, A, B, K, L]$ satisfies (P1'); and since $[I, A, B, 0, 0]$ is stabilizable by assumption, stabilizability of $[I, A, B, K, L]$ follows. Therefore, we may apply Corollary 7.6(a) to conclude that $[I, A, B, K, L]$ satisfies (P4'). Finally, (P4') applied to (8.12) shows $(x^0)^* X x^0 \geq V^+(x^0)$.

This completes the proof of the theorem. □

Remark 8.4 (Optimal control, Lur'e equations and outer)**.** Let $[I, A, B, 0, 0] \in \Sigma_{n,m,0}(\mathbb{K})$ and (Q, S, R) as in (8.2) and assume that (X, K, L) is a stabilizing solution of the Lur'e equation (8.4) (*i.e.*, the optimal control problem is feasible by Theorem 8.3). Then the following can be concluded from Theorem 7.6:

(a) $[I, A, B, K, L]$ is stabilizable.
(b) The transfer function of $[I, A, B, K, L]$ is outer.
(c) $[I, A, B, K, L]$ satisfies the Properties (P1'), (P3'), and (P4').

Next we characterize the existence of a minimizer (x, u, y) in (8.3); if it exists, then Willems [27] calls the corresponding input u the *optimal control*. We stress that this characterization shows that the concept of zero dynamics is an instrumental for the optimal control problem.

Proposition 8.5 (Characterizations of an infimum which is attained)**.**
Suppose $[I, A, B, 0, 0] \in \Sigma_{n,m,0}(\mathbb{K})$ *is stabilizable and the optimal control problem is feasible, where* (Q, S, R) *is as in* (8.2). *According to Theorem* 8.3(c), *we may choose a stabilizing solution* (X, K, L) *of the Lur'e equation. Then the following characterizations hold.*

(a) *The infimum in* (8.3) *is attained at* $(x,u,y) \in \mathfrak{B}_{[I,A,B,0,0]}(x^0)$ *for* $x^0 \in \mathbb{K}^n$ *if, and only if,*

$$(x,u,y) \in \mathcal{ZD}_{[I,A,B,K,L]}(x^0) \quad \text{and} \quad \lim_{t\to\infty} x(t) = 0.$$

(b) *The infimum in* (8.3) *is attained for all* $x^0 \in \mathbb{K}^n$ *if, and only if,*

$$\mathcal{ZD}^{\text{diff}}_{[I,A,B,K,L]} = \mathbb{K}^n \quad \text{and} \quad \mathcal{ZD}_{[I,A,B,K,L]} \text{ is stablilizable.}$$

(c) *The infimum in* (8.3) *is uniquely attained for all* $x^0 \in \mathbb{K}^n$ *if, and only if,*

$$\mathcal{ZD}^{\text{diff}}_{[I,A,B,K,L]} = \mathbb{K}^n \quad \text{and} \quad \mathcal{ZD}_{[I,A,B,K,L]} \text{ is asymptotically stable.}$$

Proof.

(a): The equivalence follows from (8.10).
(b): By (a), we see that the infimum in (8.3) is attained for all $x^0 \in \mathbb{K}^n$ if, and only if,

$$\forall x^0 \in \mathbb{K}^n \; \exists (x,u) \in \mathcal{ZD}_{[I,A,B,K,L]}(x^0) \; : \; \lim_{t\to\infty} x(t) = 0.$$

In view of the second characterization in Proposition 4.3(e), the above is equivalent to the second assertion in (b).
(c): Using again that, by (a), the infimum in (8.3) is uniquely attained for all $x^0 \in \mathbb{K}^n$ if, and only if,

$$\forall x^0 \in \mathbb{K}^n \; \exists!\, (x,u) \in \mathcal{ZD}_{[I,A,B,K,L]}(x^0) \; : \; \lim_{t\to\infty} x(t) = 0,$$

the second characterization in Proposition 4.3(g) implies that the above is equivalent to asymptotic stability of the zero dynamics of $[I,A,B,K,L]$. □

Proposition 8.5 allows to conclude the following necessary conditions for the uniquely attained infimum for all $x^0 \in \mathbb{K}^n$.

Corollary 8.6. *Let* $[I,A,B,0,0] \in \Sigma_{n,m,0}(\mathbb{K})$ *be stabilizable and suppose the optimal control problem is feasible, where* (Q,S,R) *is as in* (8.2). *According to Theorem* 8.3(c), *we may choose a stabilizing solution* (X,K,L) *of the Lur'e equation. If the infimum in* (8.3) *is uniquely attained for all* $x^0 \in \mathbb{K}^n$*, then we have:*

(a) *The zero dynamics* $\mathcal{ZD}_{[I,A,B,K,L]}$ *are autonomous.*
(b) *The pencil* $\begin{bmatrix} sI-A, & -B \\ K & L \end{bmatrix}$ *is regular, and hence* $p = m$.
(c) $L \in \mathrm{Gl}_m(\mathbb{K})$.
(d) $R \in \mathrm{Gl}_m(\mathbb{K})$.
(e) *The state feedback* $u(t) = -R^{-1}(B^*X + S^*)x(t)$ *applied to* (8.1) *yields an asymptotically stable closed loop system.*

Proof.
(a) Follow from Proposition 4.3 and Proposition 8.5.
(b): Proposition 4.3 yields that $\ker_{\mathbb{K}(s)} \begin{bmatrix} sI-A, & -B \\ K & L \end{bmatrix} = \{0\}$ and hence regularity follows from (8.5).
(c) is a consequence of Proposition 4.2.
(d) is a consequence of (8.6) and $R = L^*L$.
(e): Since $L^{-*}(B^*X + S^*) = K$, the Lur'e equation (8.4) can be written as the algebraic Riccati equation

$$A^*X + XA + Q - (B^*X + S^*)^* R^{-1} (B^*X + S^*) = 0. \tag{8.13}$$

Applying Proposition 4.3 again, we see that asymptotic stability of the zero dynamics implies that $\operatorname{rk}\left[\begin{smallmatrix}\lambda I-A, & -B\\ K & L\end{smallmatrix}\right] = n+m$ for all $\lambda \in \overline{\mathbb{C}}_+$. Therefore, the equations $B^*X + S^* = L^*K$ and $L^*L = R$ yield, for all $\lambda \in \overline{\mathbb{C}}_+$,

$$\begin{aligned} n+m = \operatorname{rk}\left[\begin{smallmatrix}\lambda I-A, & -B\\ K & L\end{smallmatrix}\right] &= \operatorname{rk}\left[\begin{smallmatrix}\lambda I-A, & -B\\ L^*K & L^*L\end{smallmatrix}\right] = \operatorname{rk}\left[\begin{smallmatrix}\lambda I-A, & -B\\ B^*X+S^* & R\end{smallmatrix}\right] \\ &= \operatorname{rk}\left[\begin{smallmatrix}\lambda I-(A-BR^{-1}(B^*X+S^*) & -B\\ 0 & I\end{smallmatrix}\right] = m + \operatorname{rk}(\lambda I - (A - BR^{-1}(B^*X+S^*)). \end{aligned}$$

This proves the assertion and completes the proof of the corollary. □

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