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BOUNDARY STABILIZATION OF A 2-D PERIODIC MHD CHANNEL FLOW, BY PROPORTIONAL FEEDBACKS

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Abstract. We consider an electrically conducting 2-D channel fluid flow affected by a transverse magnetic field. The governing equations are the magnetohydrodynamics equations. We design an explicit finite-dimensional exponentially stabilizing feedback, given in a very simple form, easily manageable from the computational point of view, for the Hartmann−Poiseuille profile. Moreover, the stability is assured independently of the value of the magnetic Reynolds number. The control acts on the normal components of both velocity and magnetic field, on the upper wall only.

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1. Introduction

We are concerned here with the problem of normal boundary feedback stabilization of the Hartmann−Poiseuille profile of a two-dimensional channel flow of an incompressible electrically conducting fluid, affected by a constant transverse magnetic field $\mathbb{B}_0$ (first results on this kind of flows, both experimental and theoretical, were obtained by Hartmann [8]). The governing equations are the magnetohydrodynamics equations (MHD, for short), which are a combination between the Navier−Stokes equations and the Maxwell equations (we refer to [22] for details). They are given by

$$\begin{cases} \rho(u_t - \nu\Delta u + uu_x + vu_y) + CC_x - CB_y = -p_x, \\ \rho(v_t - \nu\Delta v + uv_x + vv_y) + BB_y - BC_x = -p_y, \\ B_t - \frac{1}{\mu\sigma}\Delta B + uB_x + vB_y - Bu_x - Cu_y = 0, \\ C_t - \frac{1}{\mu\sigma}\Delta C + uC_x + vC_y - Bv_x - Cv_y = 0, \\ u_x + v_y = 0,\ B_x + C_y = 0,\ t \geq 0,\ x \in \mathbb{R},\ y \in (-L, L), \end{cases} \tag{1.1}$$

where (u, v) is the velocity vector field, p is the (scalar) pressure, and (B, C) is the magnetic field. The positive constants ρ, ν, μ and σ represent the fluid mass density, the kinematic viscosity, the magnetic permeability and the electrical conductivity, respectively; $2L$ is the distance between the walls. This model is considered a benchmark for applications such as liquid-metal cooling of nuclear reactors and supercomputers, plasma confinement, electromagnetic casting, hypersonic flight and propulsion.

Keywords and phrases. Magnetohydrodynamics equations, Hartmann−Poiseuille profile, stabilization, feedback controller, eigenvalues.

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For the sake of simplicity, we shall make some assumptions that do not alter the essential features of the behaviour of the flow, but reduce the complexity of the problem. More exactly, we shall assume that the velocity field, the magnetic field and the pressure of the fluid are 2π-periodic with respect to the x-coordinate. Moreover, the magnetic Prandtl number of the fluid, *i.e.*, $Pr_m := \nu\mu\sigma$, is assumed to be equal to one. Periodic MHD with different Prandtl number may be also considered by appealing to the so-called Elsasser variables (see for details [15], Rem. 2.1); however, periodic MHD channel flow, with magnetic Prandtl number equal to unity, is often studied as an approximation to torus devices of plasma controlled fusion, such as the Tokamak and the reversed field pinch, besides this, numerical simulations have shown that in the movement of this kind of flow may appear turbulences, that is, the flow may become unstable (see, for example, [7]).

The aforementioned Hartmann−Poiseuille profile, that we are going to stabilize, is given by (see for details [22]):

$$U^e(y^*) = \frac{1}{Ha}\frac{1}{\tanh(Ha)}\left[1 - \frac{\cosh(Hay^*)}{\cosh(Ha)}\right] \text{ and } V^e \equiv 0, \tag{1.2}$$

$$B^e(y^*) = -\frac{y^*}{Ha} + \frac{1}{Ha}\frac{\sinh(Hay^*)}{\sinh(Ha)} \text{ and } C^e \equiv \mathbb{B}_0, \tag{1.3}$$

where $y^* := \frac{y}{L}$ and $Ha := \mathbb{B}_0 L\sqrt{\frac{\sigma}{\rho\nu}}$ is the Hartmann number.

We define the dimensionless variables: $x^* := \frac{x}{L}$, $(u^*, v^*) := \frac{1}{\nu_0}(u, v)$, $t^* := \frac{\nu_0 t}{L}$, $(B^*, C^*) := \frac{1}{b_0}(B, C)$ with $\nu_0 := \frac{L^2}{\rho\nu}(-\partial_x p^e)$ and $b_0 := \mu L^2\sqrt{\frac{\sigma}{\rho\nu}}(-\partial_x p^e)$, where p^e is the pressure corresponding to the steady-state solution (1.2). Taking into account that the Prandtl number is equal to unity, re-denoting by u, v, B, C the fluctuation variables $u - U^e$, $v - V^e$, $B - B^e$ and $C - C^e$, respectively, the linearisation of (1.1), around the equilibrium profile (1.2)-(1.3), is given by (see for details [15], Eq. (5))

$$\begin{cases} u_t - \Delta u + U^e u_x + U^e_y v + B_0 C_x - B_0 B_y - B^e_y C = p_x, \\ v_t - \Delta v + U^e v_x + B^e_y B + B^e B_y - B^e C_x = p_y, \\ B_t - \Delta B + U^e B_x + B^e_y v - B^e u_x - B_0 u_y - (U^e)_y C = 0, \\ C_t - \Delta C + U^e C_x - B^e v_x - B_0 v_y = 0, \\ u_x + v_y = 0,\ B_x + C_y = 0,\ t \geq 0,\ x \in \mathbb{R},\ y \in (-1, 1), \\ u(t, x + 2\pi, y) = u(t, x, y),\ v(t, x + 2\pi, y) = v(t, x, y),\ p(t, x + 2\pi, y) = p(t, x, y), \\ B(t, x + 2\pi, y) = B(t, x, y),\ C(t, x + 2\pi, y) = C(t, x, y),\ t \geq 0,\ x \in \mathbb{R},\ y \in (-1, 1), \\ u(t, x, -1) = u(t, x, 1) = v(t, x, -1) = 0,\ v(t, x, 1) = \Psi(t, x), \\ B(t, x, -1) = B(t, x, 1) = C_y(t, x, -1) = C_y(t, x, 1) = 0,\ C(t, x, 1) = \Xi(t, x),\ t \geq 0,\ x \in \mathbb{R}, \end{cases} \tag{1.4}$$

and initial data u^0, v^0, B^0, C^0. (The star notation has been dropped for simplicity). Here, Ψ and Ξ are the boundary controllers.

We look to find functions Ψ, Ξ, in a feedback form (that is, depending on u, v, B, C), such that, for every initial data u^0, v^0, B^0, C^0 in $L^2((0, 2\pi) \times (-1, 1))$, once inserted into (1.4), the corresponding solution of the closed-loop system (1.4) satisfies the exponential decay

$$\begin{aligned} &\iint_{(0,2\pi)\times(-1,1)} \left(|u(t,x,y)|^2 + |v(t,x,y)|^2 + |B(t,x,y)|^2 + |C(t,x,y)|^2\right) \mathrm{d}x\mathrm{d}y \\ &\leq C\mathrm{e}^{-\eta t} \iint_{(0,2\pi)\times(-1,1)} \left(|u^0(x,y)|^2 + |v^0(x,y)|^2 + |B^0(x,y)|^2 + |C^0(x,y)|^2\right) \mathrm{d}x\mathrm{d}y,\ t \geq 0, \end{aligned}$$

for some positive constants C, η (these constants will be refereed as the constants of the exponential decay), that is, we globally exponentially stabilize the linearised system (1.4).

Concerning the stabilization of this kind of flows, we emphasize the results obtained in the works [15,26]. The first one provides stabilizing feedback controllers, *via* the backstepping method, that can be easily numerically computed, unlike to the results in [15], where, the Riccati approach designs feedbacks that are not easily manageable from the computational point of view. On the other hand, the result in [26] holds true only for flows with low magnetic Reynolds number, $R_m := \mu\sigma(L^2/\rho\nu)(-\partial p^e/\partial x)L$, while, in [15] the algorithm works equally well for any value of R_m. In the present paper, we design finite-dimensional stabilizing feedbacks Ψ, Ξ, (see (2.6) below) given in an explicit simple form, so-called of proportional type, being easy to manipulate from the computational point of view, without any require on the magnetic Reynolds number R_m. This represents an important step forward concerning this problem, since, usually in the literature, explicit results are obtained only under the require of a low magnetic Reynolds number, reducing so the MHD equations to the so-called SMHD (simplified MHD) equations, which are nothing but the Navier$-$Stokes system perturbed by $\mathcal{N}u$ in the first equation, where $\mathcal{N}$ is a positive constant. Therefore, each result of feedback stabilization for the Navier$-$Stokes system has high chances to work equally-well for the SMHD model too. This is totally not the case for the MHD equations.

Here, the ideas are based on the results in [3, 17], where similar feedbacks were constructed for stabilizing steady-states solutions corresponding to parabolic type equations. We aim to construct proportional-type stabilizing feedbacks in the same manner as in [3]. We stress that, in order to apply for our case such a design, the requirement of linear independence of the derivatives of order three of a certain system of functions, computed in $y = 1$ is needed(more precisely, the derivative of order three of the one variable eigenfunctions system $\left\{\phi_j^{k*} = \phi_j^{k*}(y)\right\}_j$ of the dual operator $-\mathbf{A}_k^*$ of $-\mathbf{A}_k$ defined in (2.5) below). Since these derivatives are just numbers, in order for this assumption to be full-filed the system must contain only one function. However, this is not applicable for what we need here. Instead, we shall use the results in [17], which overcome the problem of the linear independence assumption in [3].

Another important results on this subject are [1,10,12,16,21,23,25] and the references therein. For more details on the magnetohydrodynamic flows one may consult the books [6, 13, 14]. Finally, similar type of stabilizing feedbacks, as those in the present work based on the ideas in [3, 17], have been constructed for the phase field equations in [18], for the Navier$-$Stokes equations with fading memory in [19] and for parabolic equations with memory in [20].

The paper is organized as follows: in the next section, taking advantage of the periodicity assumption, we decompose system (1.4) into Fourier series, obtaining so an infinite parabolic system; then, after some notations, we give a priori the form of the stabilizing feedbacks Ψ, Ξ. Finally, in Theorem 3.1 we show that indeed they assure stability of (1.4) by showing the stability of the infinite parabolic system at each level $k \in \mathbb{Z}$.

2. Fourier decomposition of the system and the form of the stabilizing feedbacks

Let $L^2(Q)$, $Q = (0, 2\pi) \times (-1, 1)$, be the space of all functions $u \in L^2_{loc}(\mathbb{R} \times (-1, 1))$, that are $2\pi-$periodic in x. These functions are characterized by their Fourier series

$$u(x, y) = \sum_{k\in\mathbb{Z}} u_k(y) \exp(\mathrm{i}kx),\ u_k = \overline{u_{-k}},\ \forall k \in \mathbb{Z},$$

such that

$$\sum_{k\in\mathbb{Z}} \int_{-1}^{1} |u_k(y)|^2 \mathrm{d}y < \infty.$$

(Here, $\overline{z}$ stands for the complex conjugate of $z \in \mathbb{C}$.) The norm in $L^2(Q)$ is defined as $\|u\| := \left(2\pi \sum_{k\in\mathbb{Z}} \int_{-1}^{1} |u_k|^2 \mathrm{d}y\right)^{\frac{1}{2}}$.

We shall consider H the complexified space of $L^2(-1, 1)$. We denote also by $\|\cdot\|$ the norm in H and by $\langle\cdot,\cdot\rangle$, the scalar product. The product space $H \times H$ is defined as a complex Hilbert space, in the standard way. Finally, we shall denote by $H^m(-1, 1)$, $H_0^m(-1, 1)$, $m = 1, 2, \ldots$, the standard Sobolev spaces on $(-1, 1)$.

Now, we return to system (1.4) and decompose the velocity field, the magnetic field and the pressure in Fourier series, to get

$$\begin{cases}(u_k)_t-(-k^2u_k+u_k'')+\mathrm{i}kU^eu_k+(U^e)'v_k+\mathrm{i}kB_0c_k-B_0b_k'-(B^e)'c_k=\mathrm{i}kp_k,\\(v_k)_t-(-k^2v_k+v_k'')+\mathrm{i}kU^ev_k+(B^e)'b_k+B^eb_k'-\mathrm{i}kB^ec_k=p_k',\\(b_k)_t-(-k^2b_k+b_k'')+\mathrm{i}kU^eb_k+(B^e)'v_k-\mathrm{i}kB^eu_k-B_0u_k'-(U^e)'c_k=0,\\(c_k)_t-(-k^2c_k+c_k'')+\mathrm{i}kU^ec_k-\mathrm{i}kB^ev_k-B_0v_k'=0,\\\mathrm{i}ku_k+v_k'=0,\ \mathrm{i}kb_k+c_k'=0,\ t\geq 0,\ y\in(-1,1),\\b_k(-1)=b_k(1)=c_k(-1)=0,\ u_k(-1)=u_k(1)=v_k(-1)=0,\\v_k(1)=\psi_k,\ c_k(1)=\xi_k,\end{cases}\tag{2.1}$$

with initial data u_k^0, v_k^0, b_k^0, d_k^0. Where $\{u_k(t,y)\}_{k\in\mathbb{Z}}$, $\{v_k(t,y)\}_{k\in\mathbb{Z}}$, $\{b_k(t,y)\}_{k\in\mathbb{Z}}$, $\{c_k(t,y)\}_{k\in\mathbb{Z}}$, $\{u_k^0(y)\}_{k\in\mathbb{Z}}$, $\{v_k^0(y)\}_{k\in\mathbb{Z}}$, $\{b_k^0(y)\}_{k\in\mathbb{Z}}$, $\{c_k^0(y)\}_{k\in\mathbb{Z}}$, $\{\psi_k(t)\}_{k\in\mathbb{Z}}$, $\{\xi_k(t)\}_{k\in\mathbb{Z}}$ are the Fourier modes of $u,v,B,C,u^0,v^0,B^0,C^0,\Psi$ and Ξ, respectively. Here, $'$ denotes the derivative with respect to y, *i.e.*, $\frac{\partial}{\partial y}$.

Notice that, stabilizing system (1.4) is equivalent with stabilizing system (2.1), at each level $k\in\mathbb{Z}$.

For latter purpose, likewise in [15], for each $k\in\mathbb{Z}^*$, we introduce the operators

$$L_kv:=-v''+k^2v,\ \mathcal{D}(L_k)=H^2(-1,1)\cap H_0^1(-1,1),\tag{2.2}$$

$$\mathbf{L}_k(S\ D)^T:=(L_kS\ L_kD)^T,\ \mathcal{D}(\mathbf{L}_k)=\left(H^2(-1,1)\cap H_0^1(-1,1)\right)^2,\tag{2.3}$$

(here $(\cdot\ \cdot)^T$ means the transpose matrix) and

$$\begin{gathered}\mathbf{F}_k\begin{pmatrix}S\\D\end{pmatrix}:=\begin{pmatrix}S''''+B_0S'''-[2k^2+\mathrm{i}kD^e]S''-[\mathrm{i}k(D^e)'+k^2B_0]S'+[(k^4+\mathrm{i}k^3D^e]S+\mathrm{i}k[(S^e)'D]'\\D''''-B_0D'''-[2k^2+\mathrm{i}kS^e]D''-[\mathrm{i}k(S^e)'-k^2B_0]D'+[(k^4+\mathrm{i}k^3S^e]D+\mathrm{i}k[(D^e)'S]'\end{pmatrix},\\\mathcal{D}(\mathbf{F}_k)=\left(H^4(-1,1)\cap H_0^2(-1,1)\right)^2,\end{gathered}\tag{2.4}$$

respectively. We shall denote by $\mathcal{L}_k$ and by $\mathcal{F}_k$ the differential forms of the operators $\mathbf{L}_k$ and $\mathbf{F}_k$, respectively.

Moreover, we define the operators

$$\mathbf{A}_k:=\mathbf{F}_k\mathbf{L}_k^{-1},\ \mathcal{D}(\mathbf{A}_k)=\left\{(S\ D)^T:\mathbf{L}_k^{-1}(S\ D)^T\in\mathcal{D}(\mathbf{F}_k)\right\}.\tag{2.5}$$

Regarding the operators $-\mathbf{A}_k$, we have the following two key results.

Lemma 2.1. *For each $k\in\mathbb{Z}^*$, the operator $-\boldsymbol{A}_k$ generates a C_0- analytic semi-group on $H\times H$ and for each $\lambda\in\rho(-\boldsymbol{A}_k)$ (the resolvent set of $-\boldsymbol{A}_k$), $(\lambda I+\boldsymbol{A}_k)^{-1}$ is compact. Moreover, there exists $M>0$ such that*

$$\sigma(-\boldsymbol{A}_k)\subset\{\lambda\in\mathbb{C}:\Re\lambda<0\},\forall|k|>M.$$

Here $\sigma(-\boldsymbol{A}_k)$ is the spectrum of $-\boldsymbol{A}_k$.

Proof. See the proof in ([15], Lem. 2.1). □

By Lemma 2.1, the operator $-\mathbf{A}_k$ has a countable set of eigenvalues, denoted by $\left\{\lambda_j^k\right\}_{j=1}^\infty$. Besides this, there is only a finite number N_k of eigenvalues for which $\Re\lambda_j^k\geq 0,\ j=1,\ldots,N_k$, the unstable eigenvalues. Let $\left\{\phi_j^k:=(\phi_{1j}^k\ \phi_{2j}^k)^T\right\}_{j=1}^\infty$ and $\left\{\phi_j^{k*}:=(\phi_{1j}^{k*}\ \phi_{2j}^{k*})^T\right\}_{j=1}^\infty$ denote the corresponding eigenfunctions of the operator $-\mathbf{A}_k$ and its dual $-\mathbf{A}_k^*$, respectively.

Using (eventually) the Gram−Schmidt procedure, we may assume that the systems $\left\{\phi_j^k\right\}_{j=1}^{N_k}$ and $\left\{\phi_j^{k*}\right\}_{j=1}^{N_k}$ are bi-orthogonal, that is

$$\left\langle\phi_i^k,\phi_j^{k*}\right\rangle=\delta_{ij},\ i,j=1,\ldots,N_k,$$

δ_{ij} being the Kronecker symbol.

The last key result is stated below.

Lemma 2.2. *Let any $0 < |k| \leq M$. Then, the eigenfunctions system $\left\{\phi_j^{k*}\right\}_{j=1}^{N_k}$ can be chosen in such a way that, for some $\mu_k \in \mathbb{C}$, it holds*

$$(\phi_{1j}^{k*})'''(1) + \mu_k(\phi_{2j}^{k*})'''(1) = 1,\ j = 1, \dots, N_k.$$

Proof. Due to ([15], Lem. 2.2), we can find some $\mu_k \in \mathbb{C}$ such that $(\phi_{1j}^{k*})'''(1)+\mu_k(\phi_{2j}^{k*})'''(1) > 0$. Then, replacing (eventually) ϕ_j^{k*} by $\frac{1}{(\phi_{1j}^{k*})'''(1)+\mu_k(\phi_{2j}^{k*})'''(1)}\phi_j^{k*}$, the claim follows immediately. It should be noticed that, in order to keep the bi-orthogonality, one should replace as-well ϕ_j^k by $[(\phi_{1j}^{k*})'''(1) + \mu_k(\phi_{2j}^{k*})'''(1)]\phi_j^k$. □

We shall work under the next assumption

$$(H1*) \qquad \textit{The unstable eigenvalues are simple},$$

that is, we have $\lambda_i^k \neq \lambda_j^k,\ \forall i, j = 1, \dots, N_k,\ i \neq j$. Even if this hypothesis is generic with respect to the coefficients of $-\mathbf{A}_k$, due to ([1], Thm. 3.16), the present algorithm can be equally-well applied to the general case of eigenvalues (see [17]). We shall not develop this subject here since the presentation is very similar with the simple eigenvalues case.

At this stage we may introduce the stabilizing forms. They are

$$\Psi(t,x) := \frac{1}{2}\sum_{0<|k|\leq M}(1+\overline{\mu_k})U^k(t)\mathrm{e}^{\mathrm{i}kx},\ \ \Xi(t,x) := \frac{1}{2}\sum_{0<|k|\leq M}(1-\overline{\mu_k})U^k(t)\mathrm{e}^{\mathrm{i}kx}. \tag{2.6}$$

Here, $\mu_k,\ 0 < |k| \leq M$, are the constants given in Lemma 2.2;

$$U^k(t) := \\ -\left\langle \Lambda_{sum}^k E^k \begin{pmatrix} \iint_Q \left[-(v_{yy}(t,x,y)+c_{yy}(t,x,y)) + k^2(v(t,x,y)+c(t,x,y))\right]\mathrm{e}^{-\mathrm{i}kx}\phi_{11}^{k*}(y) \\ +\left[-(v_{yy}(t,x,y)-c_{yy}(t,x,y)) + k^2(v(t,x,y)-c(t,x,y))\right]\mathrm{e}^{-\mathrm{i}kx}\phi_{21}^{k*}(y)\mathrm{d}x\mathrm{d}y \\ \iint_Q \left[-(v_{yy}(t,x,y)+c_{yy}(t,x,y)) + k^2(v(t,x,y)+c(t,x,y))\right]\mathrm{e}^{-\mathrm{i}kx}\phi_{12}^{k*}(y) \\ +\left[-(v_{yy}(t,x,y)-c_{yy}(t,x,y)) + k^2(v(t,x,y)-c(t,x,y))\right]\mathrm{e}^{-\mathrm{i}kx}\phi_{22}^{k*}(y)\mathrm{d}x\mathrm{d}y \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \iint_Q \left[-(v_{yy}(t,x,y)+c_{yy}(t,x,y)) + k^2(v(t,x,y)+c(t,x,y))\right]\mathrm{e}^{-\mathrm{i}kx}\phi_{1N_k}^{k*}(y) \\ +\left[-(v_{yy}(t,x,y)-c_{yy}(t,x,y)) + k^2(v(t,x,y)-c(t,x,y))\right]\mathrm{e}^{-\mathrm{i}kx}\phi_{2N_k}^{k*}(y)\mathrm{d}x\mathrm{d}y \end{pmatrix}, \begin{pmatrix} 1\\1\\1\\ \cdots\\ \cdots\\ \cdots\\ 1\\1\\1 \end{pmatrix}\right\rangle_{N_k}, \tag{2.7}$$

with $\Lambda_{sum}^k := \overline{\Lambda_{\gamma_1^k}^k + \ldots + \Lambda_{\gamma_{N_k}^k}^k}$, for

$$\Lambda_{\gamma_i^k}^k := \begin{pmatrix} \frac{1}{\gamma_i^k+\lambda_1^k} & 0 & \cdots & 0 \\ 0 & \frac{1}{\gamma_i^k+\lambda_2^k} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \frac{1}{\gamma_i^k+\lambda_{N_k}^k} \end{pmatrix},\ i = 1, \dots, N_k, \tag{2.8}$$

for some $0 < \gamma_1^k < \ldots < \gamma_{N_k}^k$, N_k real constants, sufficiently large such as relation (3.6) below holds true. Moreover,

$$E^k := (G_1^k + G_2^k + \ldots + G_{N_k}^k)^{-1}, \tag{2.9}$$

where

$$G_i^k := \overline{\Lambda_{\gamma_i^k}^k}G^k\Lambda_{\gamma_i^k}^k,\ i = 1, \dots, N_k, \tag{2.10}$$

G^k being the square matrix of order N_k with all the entries equal to one. In the Appendix, Lemma A.2, it is shown that the sum $G_1^k + \ldots + G_{N_k}^k$ is indeed invertible, consequently E^k is well-defined. Finally, $\langle \cdot, \cdot \rangle_{N_k}$ stands for the classical scalar product in $\mathbb{C}^{N_k}$.

3. Main results

The main result is stated below.

Theorem 3.1. *Once plugged the feedbacks Ψ, Ξ, defined in (2.6), into the linear equation (1.4) it yields the asymptotic exponential decay of the corresponding solution to the closed-loop system (1.4). More precisely, for any initial data $(u^0, v^0, B^0, C^0) \in L^2_{loc}((0, 2\pi) \times (-1, 1))^4$, the corresponding solution of the closed-loop system*

$$\begin{cases} u_t - \Delta u + U^e u_x + U^e_y v + B_0 C_x - B_0 B_y - B^e_y C = p_x, \\ v_t - \Delta v + U^e v_x + B^e_y B + B^e B_y - B^e C_x = p_y, \\ B_t - \Delta B + U^e B_x + B^e_y v - B^e u_x - B_0 u_y - (U^e)_y C = 0, \\ C_t - \Delta C + U^e C_x - B^e v_x - B_0 v_y = 0, \\ u_x + v_y = 0,\ B_x + C_y = 0,\ t \geq 0,\ x \in \mathbb{R},\ y \in (-1,1), \\ u(t, x+2\pi, y) = u(t,x,y),\ v(t,x+2\pi,y) = v(t,x,y),\ p(t,x+2\pi,y) = p(t,x,y), \\ B(t,x+2\pi,y) = B(t,x,y),\ C(t,x+2\pi,y) = C(t,x,y),\ t \geq 0,\ x \in \mathbb{R},\ y \in (-1,1), \\ u(t,x,-1) = u(t,x,1) = v(t,x,-1) = 0,\ v(t,x,1) = \frac{1}{2}\sum_{0<|k|\leq M}(1+\overline{\mu_k})U^k(t)\mathrm{e}^{ikx}, \\ B(t,x,-1) = B(t,x,1) = C_y(t,x,-1) = C_y(t,x,1) = 0,\ C(t,x,1) = \frac{1}{2}\sum_{0<|k|\leq M}(1-\overline{\mu_k})U^k(t)\mathrm{e}^{ikx}, \\ \hfill t \geq 0,\ x \in \mathbb{R}, \end{cases} \tag{3.1}$$

satisfies the exponential decay

$$\begin{aligned} &\iint_{(0,2\pi)\times(-1,1)} \left(|u(t,x,y)|^2 + |v(t,x,y)|^2 + |B(t,x,y)|^2 + |C(t,x,y)|^2\right) \mathrm{d}x\mathrm{d}y \\ &\leq C\mathrm{e}^{-\eta t} \iint_{(0,2\pi)\times(-1,1)} \left(|u^0(x,y)|^2 + |v^0(x,y)|^2 + |B^0(x,y)|^2 + |C^0(x,y)|^2\right) \mathrm{d}x\mathrm{d}y,\ t \geq 0, \end{aligned}$$

for some positive constants C, η. Where, the feedbacks $U_k,\ 0 < |k| \leq M$, are defined in (2.7).

Proof. The idea is to show the stability of (2.1), at each level $k \in \mathbb{Z}$, with the coefficients of the exponential decay independent of the level. By the definition, we have that $\psi_k = \xi_k = 0$ for $k = 0$ and all $|k| > M$, and $\psi_k = \frac{1}{2}(1 + \overline{\mu_k})U^k,\ \xi_k = \frac{1}{2}(1 - \overline{\mu_k})U^k$ for $0 < |k| \leq M$, U^k defined by (2.7).

Concerning $k = 0$, it is shown in ([15], Eqs. (8) and (9)) that at this level the system is stable. Furthermore, as in [15], after Lemma 2.1, we have that system (2.1) is stable for all $|k| > M$. Hence, from now on, we consider only $0 < |k| \leq M$.

As in [15], we set $S_{1k} := u_k + b_k,\ S_{2k} := v_k + c_k$ and $D_{1k} := u_k - b_k,\ D_{2k} := v_k - c_k$, and, of course, $S^0_{1k} := u^0_k + b^0_k,\ S^0_{2k} := v^0_k + c^0_k$ and $D^0_{1k} := u^0_k - b^0_k,\ D^0_{2k} := v^0_k - c^0_k$. Then, following the steps in ([15], Eq. (10)) we reduce the complexity of the problem (2.1) to the following boundary controlled system with just two unknowns, S_{2k} and D_{2k}, namely,

$$\begin{cases} (-S''_{2k} + k^2 S_{2k})_t + S''''_{2k} + B_0 S'''_{2k} - [2k^2 + \mathrm{i}kD^e]S''_{2k} - [\mathrm{i}k(D^e)' + k^2 B_0]S'_{2k} \\ \qquad + [(k^4 + \mathrm{i}k^3 D^e]S_{2k} + \mathrm{i}k[(S^e)' D_{2k}]' = 0,\ y \in (-1,1), \\ (-D''_{2k} + k^2 D_{2k})_t + D''''_{2k} - B_0 D'''_{2k} - [2k^2 + \mathrm{i}kS^e]D''_{2k} - [\mathrm{i}k(S^e)' - k^2 B_0]D'_{2k} \\ \qquad + [(k^4 + \mathrm{i}k^3 S^e]D_{2k} + \mathrm{i}k[(D^e)' S_{2k}]' = 0,\ y \in (-1,1), \\ S'_{2k}(-1) = S'_{2k}(1) = S_{2k}(-1) = 0,\ S_{2k}(1) = \psi^S_k, \\ D'_{2k}(-1) = D'_{2k}(1) = D_{2k}(-1) = 0,\ D_{2k}(1) = \psi^D_k, \end{cases} \tag{3.2}$$

and initial data $S^0_{2k},\ D^0_{2k},\ k\in\mathbb{Z}\setminus\{0\}$. Here $S^e:=U^e+B^e,\ D^e:=U^e-B^e$ and $\psi^S_k:=\psi_k+\xi_k=U^k,\ \psi^D_k:=\psi_k-\xi_k=\overline{\mu}_kU^k$.

In order to simplify the notations, since k is fixed, from now on we shall omit to write the index k, so, in what follows, we shall use the notations from (2.2) to (2.10), but with the index k dropped. The corresponding closed-loop system (3.2) writes as (see [15])

$$\begin{cases}\left(\mathcal{L}(S_2\ D_2)^T)\right)_t+\mathcal{F}(S_2\ D_2)^T=0,\ y\in(-1,1),\\ (S_2\ D_2)^T(1)=(U\ \overline{\mu}U)^T,\ (S_2\ D_2)^T(-1)=(S_2'\ D_2')^T(-1)=(S_2'\ D_2')^T(1)=0.\end{cases}\tag{3.3}$$

In order to lift the boundary conditions into the equations, aiming to use the spectral decomposition method, we introduce the Dirichlet operator as: let any $\alpha\in\mathbb{C}$, we denote by $\mathbb{D}_\gamma\alpha:=w$ the solution to the equation

$$\begin{cases}\mathcal{F}w+2\sum_{j=1}^N\lambda_j\left\langle\mathcal{L}w,\phi_j^*\right\rangle\phi_j+\gamma\mathcal{L}w=0,\ y\in(-1,1),\\ w(1)=(\alpha\ \overline{\mu}\alpha)^T,\ w(-1)=w'(-1)=w'(1)=0.\end{cases}\tag{3.4}$$

(It is known that for $\gamma>0$ large enough, the above equation has a unique solution in $(H^{\frac12}(-1,1))^2$, see [9], p. 6, line 16).

For latter purpose, let us compute $\langle\mathcal{L}\mathbb{D}_\gamma\alpha,\phi_m^*\rangle$, for some $1\le m\le N$. To this end, we have from (3.4) scalarly multiplied by ϕ_m^*, and by the bi-orthogonality of the eigenfunction systems, that

$$\begin{aligned}0&=\langle\mathcal{F}w,\phi_m^*\rangle+2\lambda_m\langle\mathcal{L}w,\phi_m^*\rangle+\gamma\langle\mathcal{L}w,\phi_m^*\rangle\\ &=-\alpha\overline{(\phi_{1m}^*)'''(1)+\mu(\phi_{2m}^*)'''(1)}+\langle w,\mathbf{F}^*\phi_m^*\rangle+(\gamma+2\lambda_m)\langle\mathcal{L}w,\phi_m^*\rangle\\ &\text{(by Lem. 2.2)}\\ &=-\alpha+\langle\mathcal{L}w,\mathbf{A}^*\phi_m^*\rangle+(\gamma+2\lambda_m)\langle\mathcal{L}w,\phi_m^*\rangle.\end{aligned}$$

It yields that

$$\langle\mathcal{L}\mathbb{D}_\gamma\alpha,\phi_m^*\rangle=\frac{\alpha}{\gamma+\lambda_m},\ 1\le m\le N.\tag{3.5}$$

Next, we choose N constants $0<\gamma_1<\gamma_2<\ldots<\gamma_N$, large enough, such that

$$\textit{Equation (3.4), corresponding to each } \gamma_i,\ i=1,\ldots,N, \textit{ has a solution},\tag{3.6}$$

and denote by $\mathbb{D}_{\gamma_i},\ i=1,\ldots,N$, the corresponding solutions.

It is clear that the feedback U, given in (2.7), can be equivalently written as

$$U(t):=-\left\langle\Lambda_{sum}E\begin{pmatrix}\left\langle\mathcal{L}(S_2\ D_2)^T(t),\phi_1^*\right\rangle\\ \left\langle\mathcal{L}(S_2\ D_2)^T(t),\phi_2^*\right\rangle\\ \ldots\\ \left\langle\mathcal{L}(S_2\ D_2)^T(t),\phi_N^*\right\rangle\end{pmatrix},\begin{pmatrix}1\\1\\ \ldots\\1\end{pmatrix}\right\rangle_N.\tag{3.7}$$

Now, let us introduce the feedbacks

$$\begin{aligned}U_i(t)&:=-\left\langle E\begin{pmatrix}\left\langle\mathcal{L}(S_2\ D_2)^T(t),\phi_1^*\right\rangle\\ \left\langle\mathcal{L}(S_2\ D_2)^T(t),\phi_2^*\right\rangle\\ \ldots\\ \left\langle\mathcal{L}(S_2\ D_2)^T(t),\phi_N^*\right\rangle\end{pmatrix},\begin{pmatrix}\frac{1}{\gamma_i+\lambda_1}\\ \frac{1}{\gamma_i+\lambda_2}\\ \ldots\\ \frac{1}{\gamma_i+\lambda_N}\end{pmatrix}\right\rangle_N\\ &=-\left\langle\overline{\Lambda_{\gamma_i}}E\begin{pmatrix}\left\langle\mathcal{L}(S_2\ D_2)^T(t),\phi_1^*\right\rangle\\ \left\langle\mathcal{L}(S_2\ D_2)^T(t),\phi_2^*\right\rangle\\ \ldots\\ \left\langle\mathcal{L}(S_2\ D_2)^T(t),\phi_N^*\right\rangle\end{pmatrix},\begin{pmatrix}1\\1\\ \ldots\\1\end{pmatrix}\right\rangle_N,\ t\ge0,\end{aligned}\tag{3.8}$$

for $i=1,2,\ldots,N$. Thus, $U=U_1+\ldots+U_N$.

For latter computations, we need to show that

$$\begin{pmatrix} \langle \mathcal{L}\mathbb{D}_{\gamma_i}U_i,\phi_1^*\rangle \\ \langle \mathcal{L}\mathbb{D}_{\gamma_i}U_i,\phi_2^*\rangle \\ \ldots \\ \langle \mathcal{L}\mathbb{D}_{\gamma_i}U_i,\phi_N^*\rangle \end{pmatrix} = -G_iE\begin{pmatrix} \left\langle \mathcal{L}(S_2\ D_2)^T,\phi_1^*\right\rangle \\ \left\langle \mathcal{L}(S_2\ D_2)^T,\phi_2^*\right\rangle \\ \ldots \\ \left\langle \mathcal{L}(S_2\ D_2)^T,\phi_N^*\right\rangle \end{pmatrix}, \tag{3.9}$$

where G_i are introduced in (2.10) above, for $i=1,\ldots,N$. This is indeed so. We have, *via* relation (3.5),

$$\langle \mathcal{L}\mathbb{D}_{\gamma_i}U_i,\phi_m^*\rangle = U_i\langle \mathcal{L}\mathbb{D}_{\gamma_i}1,\phi_m^*\rangle = -\left\langle E\begin{pmatrix} \left\langle \mathcal{L}(S_2\ D_2)^T,\phi_1^*\right\rangle \\ \left\langle \mathcal{L}(S_2\ D_2)^T,\phi_2^*\right\rangle \\ \ldots \\ \left\langle \mathcal{L}(S_2\ D_2)^T,\phi_N^*\right\rangle \end{pmatrix}, \begin{pmatrix} \frac{1}{(\gamma_i+\lambda_1)\overline{(\gamma_i+\lambda_m)}} \\ \frac{1}{(\gamma_i+\lambda_2)\overline{(\gamma_i+\lambda_m)}} \\ \ldots \\ \frac{1}{(\gamma_i+\lambda_N)\overline{(\gamma_i+\lambda_m)}} \end{pmatrix}\right\rangle_N,\ m=1,\ldots,N,$$

from where (3.9) follows immediately.

Returning to the linear equation (3.3), we denote by

$$z:=\mathbf{L}\left[(S_2\ D_2)^T-\mathbb{D}_{\gamma_1}U_1-\ldots-\mathbb{D}_{\gamma_N}U_N\right].$$

Obviously, $z\in\mathcal{D}(-\mathbf{A})$. Subtracting (3.3) and (3.4), corresponding to $\mathbb{D}_{\gamma_i},\ i=1,\ldots,N$, we arrive to

$$z_t=-\mathbf{A}z+2\sum_{i,j=1}^N\lambda_j\left\langle \mathcal{L}\mathbb{D}_{\gamma_i}U_i,\phi_j^*\right\rangle\phi_j+\sum_{i=1}^N\gamma_i\mathcal{L}\mathbb{D}_{\gamma_i}U_i-\left(\mathcal{L}\sum_{i=1}^N\mathbb{D}_{\gamma_i}U_i\right)_t. \tag{3.10}$$

In terms of the new variable z, the feedbacks $U_i,\ i=1,..,N$ have the form

$$U_i(t)=-\frac{1}{2}\left\langle E\begin{pmatrix} \langle z(t),\phi_1^*\rangle \\ \langle z(t),\phi_2^*\rangle \\ \ldots \\ \langle z(t),\phi_N^*\rangle \end{pmatrix}, \begin{pmatrix} \frac{1}{\gamma_i+\lambda_1} \\ \frac{1}{\gamma_i+\lambda_2} \\ \ldots \\ \frac{1}{\gamma_i+\lambda_N} \end{pmatrix}\right\rangle_N. \tag{3.11}$$

To see this, we do the following straightforward computations

$$\begin{aligned} &\frac{1}{2}\left\langle E\begin{pmatrix} \langle z(t),\phi_1^*\rangle \\ \langle z(t),\phi_2^*\rangle \\ \ldots \\ \langle z(t),\phi_N^*\rangle \end{pmatrix}, \begin{pmatrix} \frac{1}{\gamma_i+\lambda_1} \\ \frac{1}{\gamma_i+\lambda_2} \\ \ldots \\ \frac{1}{\gamma_i+\lambda_N} \end{pmatrix}\right\rangle_N = \frac{1}{2}\left\langle E\begin{pmatrix} \left\langle \mathcal{L}(S_2\ D_2)^T,\phi_1^*\right\rangle \\ \left\langle \mathcal{L}(S_2\ D_2)^T,\phi_2^*\right\rangle \\ \ldots \\ \left\langle \mathcal{L}(S_2\ D_2)^T,\phi_N^*\right\rangle \end{pmatrix}, \begin{pmatrix} \frac{1}{\gamma_i+\lambda_1} \\ \frac{1}{\gamma_i+\lambda_2} \\ \ldots \\ \frac{1}{\gamma_i+\lambda_N} \end{pmatrix}\right\rangle_N \\ &-\frac{1}{2}\sum_{j=1}^N\left\langle E\begin{pmatrix} \left\langle \mathcal{L}\mathbb{D}_{\gamma_j}U_j,\phi_1^*\right\rangle \\ \left\langle \mathcal{L}\mathbb{D}_{\gamma_j}U_j,\phi_2^*\right\rangle \\ \ldots \\ \left\langle \mathcal{L}\mathbb{D}_{\gamma_j}U_j,\phi_N^*\right\rangle \end{pmatrix}, \begin{pmatrix} \frac{1}{\gamma_i+\lambda_1} \\ \frac{1}{\gamma_i+\lambda_2} \\ \ldots \\ \frac{1}{\gamma_i+\lambda_N} \end{pmatrix}\right\rangle_N, \\ &\text{(taking into account relation (3.9))} \\ &=\frac{1}{2}\left\langle [I+E(G_1+\ldots+G_N)]\,E\begin{pmatrix} \left\langle \mathcal{L}(S_2\ D_2)^T,\phi_1^*\right\rangle \\ \left\langle \mathcal{L}(S_2\ D_2)^T,\phi_2^*\right\rangle \\ \ldots \\ \left\langle \mathcal{L}(S_2\ D_2)^T,\phi_N^*\right\rangle \end{pmatrix}, \begin{pmatrix} \frac{1}{\gamma_i+\lambda_1} \\ \frac{1}{\gamma_i+\lambda_2} \\ \ldots \\ \frac{1}{\gamma_i+\lambda_N} \end{pmatrix}\right\rangle_N \\ &=-U_i, \end{aligned}$$

since $E = (G_1 + \ldots + G_N)^{-1}$. Moreover, likewise in (3.9), we have now

$$\begin{pmatrix} \langle \mathcal{L}\mathbb{D}_{\gamma_i} U_i, \phi_1^* \rangle \\ \langle \mathcal{L}\mathbb{D}_{\gamma_i} U_i, \phi_2^* \rangle \\ \ldots \\ \langle \mathcal{L}\mathbb{D}_{\gamma_i} U_i, \phi_N^* \rangle \end{pmatrix} = -\frac{1}{2} G_i E \begin{pmatrix} \langle z(t), \phi_1^* \rangle \\ \langle z(t), \phi_2^* \rangle \\ \ldots \\ \langle z(t), \phi_N^* \rangle \end{pmatrix}, \ i = 1, \ldots, N. \tag{3.12}$$

Next, we decompose system (3.10) into its stable and unstable part. More precisely, we denote by $Z_N^u := \operatorname{span}\{\phi_j\}_{j=1}^N$, and $Z_N^s := \operatorname{span}\{\phi_j\}_{j=N+1}^\infty$. Then introduce the projections, P_N, and its dual P_N^*, defined by

$$P_N := -\frac{1}{2\pi \mathrm{i}} \int_\Gamma (\lambda I + \mathbf{A})^{-1} \mathrm{d}\lambda; \ P_N^* := -\frac{1}{2\pi \ \mathrm{i}} \int_{\bar{\Gamma}} (\lambda I + \mathbf{A}^*)^{-1} \mathrm{d}\lambda,$$

where Γ (its conjugate $\bar{\Gamma}$, respectively) separates the unstable spectrum from the stable one of $-\mathbf{A}$ ($-\mathbf{A}^*$, respectively). We set

$$-\mathbf{A}_N^u := P_N(-\mathbf{A}), \ -\mathbf{A}_N^s := (I - P_N)(-\mathbf{A}), \tag{3.13}$$

for the restrictions of $-\mathbf{A}$ to Z_N^u and Z_N^s, respectively. This projections commute with $-\mathbf{A}$. We then have that the spectra of $-\mathbf{A}$ on Z_N^u and Z_N^s coincide with $\{\lambda_j\}_{j=1}^N$ and $\{\lambda_j\}_{j=N+1}^\infty$, respectively.

Moreover, since $-\mathbf{A}$ generates a C_0-analytic semigroup on H, its restriction $-\mathbf{A}_N^s$ to Z_N^s generates likewise a C_0-analytic semigroup on Z_N^s. This implies that $-\mathbf{A}_N^s$ satisfies the spectrum determined growth condition on Z_N^s, and so, we have

$$\|\mathrm{e}^{-\mathbf{A}_N^s t}\|_{L(H,H)} \leq C_{\alpha_0} \mathrm{e}^{-\alpha_0 t}, \forall t \geq 0, \tag{3.14}$$

for some $\alpha_0 < |\Re \lambda_{N+1}|$.

The system (3.10) can accordingly be decomposed as

$$z = z_N + \zeta_N, z_N := P_N z, \zeta_N := (I - P_N) z,$$

where applying P_N and $(I - P_N)$ on (3.10), we obtain

$$\text{on } Z_N^u: \frac{\mathrm{d}}{\mathrm{d}t} z_N + \mathbf{A}_N^u z_{N_k} = P_N \left[2 \sum_{i,j=1}^N \lambda_j \left\langle \mathcal{L}\mathbb{D}_{\gamma_i} U_i, \phi_j^* \right\rangle \phi_j + \sum_{i=1}^N \gamma_i \mathcal{L}_k \mathbb{D}_{\gamma_i} U_i - \left(\mathcal{L} \sum_{i=1}^N \mathbb{D}_{\gamma_i} U_i \right)_t \right] \tag{3.15}$$

$$\text{on } Z_N^s: \frac{\mathrm{d}}{\mathrm{d}t} \zeta_N + \mathbf{A}_N^s \zeta_N = (I - P_N) \left[2 \sum_{i,j=1}^N \lambda_j \left\langle \mathcal{L}\mathbb{D}_{\gamma_i} U_i, \phi_j^* \right\rangle \phi_j + \sum_{i=1}^N \gamma_i \mathcal{L}\mathbb{D}_{\gamma_i} U_i - \left(\mathcal{L} \sum_{i=1}^N \mathbb{D}_{\gamma_i} U_i \right)_t \right] \tag{3.16}$$

respectively.

Let us write z_N as

$$z_N(t, y) = \sum_{j=1}^N \left\langle z(t), \phi_j^* \right\rangle \phi_j(y).$$

We introduce that z_N in equation (3.15), multiply it successively by $\phi_j^*, \ j = 1, \ldots, N$, take account of the bi-orthogonality of the eigenfunctions systems, notice that we may assume that $P_N^* \phi_j^* = \phi_j^*$ (since P_N^* is idempotent) and take advantage of relation (3.12), to get that

$$\mathcal{Z}_t = \Lambda \mathcal{Z} - \sum_{i=1}^N \Lambda G_i E \mathcal{Z} - \frac{1}{2} \sum_{i=1}^N \gamma_i G_i E \mathcal{Z} + \frac{1}{2} \sum_{i=1}^N G_i E \mathcal{Z}_t, \ t \geq 0,$$

where $\mathcal{Z} := \begin{pmatrix} \langle z(t), \phi_1^* \rangle \\ \langle z(t), \phi_2^* \rangle \\ \dots \\ \langle z(t), \phi_N^* \rangle \end{pmatrix}$ and $\Lambda := \mathrm{diag}(\lambda_1, \lambda_2, \dots, \lambda_N)$.

Recalling that $E = (G_1 + \dots + G_N)^{-1}$, the above relation yields

$$\mathcal{Z}_t = -\gamma_1 \mathcal{Z} + \sum_{i=2}^{N} (\gamma_1 - \gamma_i) G_i E \mathcal{Z},\ t \geq 0. \tag{3.17}$$

Notice that $G_i,\ i = 1, \dots, N$, are positive semi-definite symmetric matrices (by the definition of G_i, Λ_{γ_i} and G; in fact, one can see in the proof of Lemma A.2 that this is true), therefore, $\langle G_i q, q \rangle_N \geq 0,\ \forall q \in \mathbb{C}^N,\ i = 1, \dots, N$. Consequently, $E = (G_1 + \dots + G_N)^{-1}$ is a positive definite symmetric matrix, thus one can define another positive definite symmetric matrix, denoted by $E^{\frac{1}{2}}$, such that $E^{\frac{1}{2}} E^{\frac{1}{2}} = E$ (the square root of E; for details see [5]). Let us scalarly multiply equation (3.17) by $E\mathcal{Z}$, to get

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \| E^{\frac{1}{2}} \mathcal{Z}(t) \|_N^2 = -\gamma_1 \| E^{\frac{1}{2}} \mathcal{Z}(t) \|_N^2 + \sum_{i=2}^{N} (\gamma_1 - \gamma_i) \left\langle G_i E \mathcal{Z}(t), E \mathcal{Z}(t) \right\rangle_N, \tag{3.18}$$

that leads to

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \| E^{\frac{1}{2}} \mathcal{Z}(t) \|_N^2 \leq -\gamma_1 \| E^{\frac{1}{2}} \mathcal{Z}(t) \|_N^2,\ t \geq 0,$$

since $\gamma_1 - \gamma_i < 0,\ i = 2, \dots, N$ (here, $\| \cdot \|_N$ stands for the euclidean norm in $\mathbb{C}^N$). The above relation implies the exponential decay of $\mathcal{Z}$ in the $\| E^{\frac{1}{2}} \cdot \|_N$-norm, *i.e.*,

$$\| E^{\frac{1}{2}} \mathcal{Z}(t) \|_N^2 \leq \mathrm{e}^{-2\gamma_1 t} \| E^{\frac{1}{2}} \mathcal{Z}_o \|_N^2,\ t \geq 0,$$

where using the fact that $E^{\frac{1}{2}}$ is a positive definite symmetric matrix, we finally arrive to

$$\| \mathcal{Z}(t) \|_N^2 \leq C \mathrm{e}^{-2\gamma_1 t} \| \mathcal{Z}_o \|_N^2,\ t \geq 0, \tag{3.19}$$

for some positive constant C. Thus, we obtain that the first N modes are exponentially stable. It is easy to deduce that this is enough to show that, in fact, the solution to (3.3) is exponentially stable, by appealing to classical arguments related to this spectral decomposition method, similarly as in [15]. Recalling the equivalence between (3.2), (2.1) and (1.4), the theorem follows immediately. The details are omitted. □

4. Conclusions

The main result of this paper provides simple finite-dimensional stabilizing feedbacks for the Hartmann−Poiseuille profile of a 2-D periodic MHD channel, with the stability assured independently on the value of the magnetic Reynolds number (see (2.6)). These feedback laws are easy to manipulate from the computational point of view. However, in practice, they require full-state knowledge of the normal components of the velocity field and of the magnetic field, together with their second order spatial derivatives. But, once we notice that

$$\iint_Q (v_{yy}(t,x,y) + c_{yy}(t,x)) \mathrm{e}^{-\mathrm{i}kx} \phi_{ij}^{k*}(y) \mathrm{d}x\mathrm{d}y = \iint_Q (v(t,x,y) + c(t,x,y)) \mathrm{e}^{\mathrm{i}kx} (\phi_{ij}^{k*}(y))_{yy} \mathrm{d}x\mathrm{d}y,$$

for all $i = 1, 2$ and $j = 1, \dots, N_k$, it follows that the second spatial derivatives of the velocity and magnetic field are, in fact, not needed to be known. But, of course, one has to solve the eigenfunction problem associated to the operator $-\mathbf{A}_k^*$, for $|k| \leq M$. We point out that, for each level $|k| \leq M$, these operators have similar form. That simplifies a lot this problem.

Thus, for implementation of these kind of feedbacks, it remains to measure the normal components of both the velocity field and the magnetic field in the whole channel. We stress that, all the results on stabilization of MHD channel flows, presented in the Introduction, also require full-state knowledge. The particle imaging velocimetry technique can be applied to this end. However, this way one obtains only local data about these targets. One way to overcome this problem, is to design an observer for estimation the velocity and electromagnetic fields of the Hartmann flow, based on boundary measurements of pressure, current and skin friction. Then, together with the feedback laws (2.6) one obtains an output feedback stabilizing boundary controller that only needs boundary measurements. This idea is due to the work [25], that treats the three-dimensional case of the SMHD channel, based on the observer designed in [24] (see Fig. 4 in [25]). In those papers, the backstepping technique is used. So, it remains, as a subsequent work, to design such an observer, this time, *via* the algorithm we presented here.

Another way to overcome this problem comes from the recent result [11]. There, a similar explicit stabilizing feedback, as here, is designed, this time for the Fischer's equation. The ideas are again based on those in [17]. Even if, for that case also, the form of the feedback requires full-state knowledge, the numerical simulations performed there show that, in fact, only knowledge on a part of the space is sufficient. More precisely, on $(a, 1)$, not the whole $(0, 1)$, where $a \leq 0.25$. It is possible that this a can be chosen nearer to 1. However, rigorous mathematical proof must be done for that case. So, it is possible that in our case the things are the same, namely, numerical simulations may show that only local knowledge of the state is needed. However, this subject remains also for a subsequent work, since it is not easy to deal with not even for the more simple Fischer's equation case.

Finally, one may suspect that global stability of the linearised system (which is what we proved here) implies local stability for the non-linear system. This is also not a simple task since, in order to prove this, usually one applies a fixed point argument. So, one is obliged to reduce the pressure from the system, and this is usually done by applying the Leray projector. Here, we can not do this because of the non tangential boundary conditions. This is also left for subsequent work, however, it should be noticed that, in practice, stability of the linear approximation is usually enough.

Appendix A.

Lemma A.1. *Under assumption $(H1*)$, for any $0 < \gamma_1 < \gamma_2 < \ldots < \gamma_N$, we have*

$$\begin{vmatrix} \dfrac{1}{\gamma_1+\lambda_1} & \dfrac{1}{\gamma_1+\lambda_2} & \cdots & \dfrac{1}{\gamma_1+\lambda_N} \\ \dfrac{1}{\gamma_2+\lambda_1} & \dfrac{1}{\gamma_2+\lambda_2} & \cdots & \dfrac{1}{\gamma_2+\lambda_N} \\ \cdots & \cdots & \cdots & \cdots \\ \dfrac{1}{\gamma_N+\lambda_1} & \dfrac{1}{\gamma_N+\lambda_2} & \cdots & \dfrac{1}{\gamma_N+\lambda_N} \end{vmatrix} \neq 0. \tag{A.1}$$

Proof. Let us prove this by mathematical induction over N.

Step 1. For $N = 2$, we have

$$\begin{vmatrix} \dfrac{1}{\gamma_1+\lambda_1} & \dfrac{1}{\gamma_1+\lambda_2} \\ \dfrac{1}{\gamma_2+\lambda_1} & \dfrac{1}{\gamma_2+\lambda_2} \end{vmatrix} = \frac{(\lambda_1-\lambda_2)(\gamma_1-\gamma_2)}{(\gamma_1+\lambda_1)(\gamma_2+\lambda_2)(\gamma_2+\lambda_1)(\gamma_1+\lambda_2)} \neq 0,$$

since $\lambda_1 \neq \lambda_2$ have positive real part, and $\gamma_1 < \gamma_2$.

Step 2. We assume that for $N-1$ the claim is true and prove it for N. To this end we have

$$\begin{vmatrix} \frac{1}{\gamma_1+\lambda_1} & \frac{1}{\gamma_1+\lambda_2} & \dots & \frac{1}{\gamma_1+\lambda_N} \\ \frac{1}{\gamma_2+\lambda_1} & \frac{1}{\gamma_2+\lambda_2} & \dots & \frac{1}{\gamma_2+\lambda_N} \\ \dots & \dots & \dots & \dots \\ \frac{1}{\gamma_N+\lambda_1} & \frac{1}{\gamma_N+\lambda_2} & \dots & \frac{1}{\gamma_N+\lambda_N} \end{vmatrix} =$$

(Subtracting from the first column the Nth one, …, from the $(N-1)$th column the Nth one)

$$= \begin{vmatrix} \frac{\lambda_N-\lambda_1}{(\gamma_1+\lambda_1)(\gamma_1+\lambda_N)} & \frac{\lambda_N-\lambda_2}{(\gamma_1+\lambda_2)(\gamma_1+\lambda_N)} & \dots & \frac{\lambda_N-\lambda_{N-1}}{(\gamma_1+\lambda_{N-1})(\gamma_1+\lambda_N)} & \frac{1}{\gamma_1+\lambda_N} \\ \frac{\lambda_N-\lambda_1}{(\gamma_2+\lambda_1)(\gamma_2+\lambda_N)} & \frac{\lambda_N-\lambda_2}{(\gamma_2+\lambda_2)(\gamma_2+\lambda_N)} & \dots & \frac{\lambda_N-\lambda_{N-1}}{(\gamma_2+\lambda_{N-1})(\gamma_2+\lambda_N)} & \frac{1}{\gamma_2+\lambda_N} \\ \dots & \dots & \dots & \dots & \\ \frac{\lambda_N-\lambda_1}{(\gamma_N+\lambda_1)(\gamma_N+\lambda_N)} & \frac{\lambda_N-\lambda_2}{(\gamma_N+\lambda_2)(\gamma_N+\lambda_N)} & \dots & \frac{\lambda_N-\lambda_{N-1}}{(\gamma_N+\lambda_{N-1})(\gamma_N+\lambda_N)} & \frac{1}{\gamma_N+\lambda_N} \end{vmatrix}$$

$$= \frac{1}{\gamma_N+\lambda_N}\prod_{k=1}^{N-1}\frac{\lambda_N-\lambda_k}{\gamma_k+\lambda_N} \begin{vmatrix} \frac{1}{\gamma_1+\lambda_1} & \frac{1}{\gamma_1+\lambda_2} & \dots & \frac{1}{\gamma_1+\lambda_{N-1}} & 1 \\ \frac{1}{\gamma_2+\lambda_1} & \frac{1}{\gamma_2+\lambda_2} & \dots & \frac{1}{\gamma_2+\lambda_{N-1}} & 1 \\ \dots & \dots & \dots & \dots & \\ \frac{1}{\gamma_N+\lambda_1} & \frac{1}{\gamma_N+\lambda_2} & \dots & \frac{1}{\gamma_N+\lambda_{N-1}} & 1 \end{vmatrix}$$

(Subtracting the Nth line from the first one, …, the Nth line from the $(N-1)$th one)

$$= \frac{1}{\gamma_N+\lambda_N}\prod_{k=1}^{N-1}\frac{(\lambda_N-\lambda_k)(\gamma_N-\gamma_k)}{(\gamma_k+\lambda_N)(\gamma_N+\lambda_k)} \begin{vmatrix} \frac{1}{\gamma_1+\lambda_1} & \frac{1}{\gamma_1+\lambda_2} & \dots & \frac{1}{\gamma_1+\lambda_{N-1}} \\ \frac{1}{\gamma_2+\lambda_1} & \frac{1}{\gamma_2+\lambda_2} & \dots & \frac{1}{\gamma_2+\lambda_{N-1}} \\ \dots & \dots & \dots & \dots \\ \frac{1}{\gamma_{N-1}+\lambda_1} & \frac{1}{\gamma_{N-1}+\lambda_2} & \dots & \frac{1}{\gamma_{N-1}+\lambda_{N-1}} \end{vmatrix} \neq 0$$

by the inductive hypothesis and the fact that $\lambda_i,\ i=1,\dots,N$ are mutually distinct, have positive real part, and $0<\gamma_1<\gamma_2<\dots<\gamma_N$. ☐

Lemma A.2. *The sum $G_1+G_2+\dots+G_N$ is an invertible matrix, where $G_m,\ m=1,\dots,N$, are introduced in relation* (2.10).

Proof. Arguing by contradiction, let us assume that there is $z=\begin{pmatrix} z_1 \\ z_2 \\ \dots \\ z_N \end{pmatrix}\in\mathbb{C}^N$, non-zero, such that $(G_1+\dots+G_N)z=0$. It follows that

$$\sum_{m=1}^{N}\langle G_m z, z\rangle_N = 0,$$

or, equivalently,

$$\sum_{m=1}^{N}\left|\sum_{i=1}^{N} z_i \frac{1}{\gamma_m+\lambda_i}\right|^2 = 0.$$

We arrive to the following homogeneous system

$$\begin{cases} \dfrac{1}{\gamma_1+\lambda_1}z_1 + \dfrac{1}{\gamma_1+\lambda_2}z_2 + \ldots + \dfrac{1}{\gamma_1+\lambda_N}z_N = 0 \\ \dfrac{1}{\gamma_2+\lambda_1}z_1 + \dfrac{1}{\gamma_2+\lambda_2}z_2 + \ldots + \dfrac{1}{\gamma_2+\lambda_N}z_N = 0 \\ \qquad\qquad\qquad\qquad \cdots \\ \dfrac{1}{\gamma_N+\lambda_1}z_1 + \dfrac{1}{\gamma_N+\lambda_2}z_2 + \ldots + \dfrac{1}{\gamma_N+\lambda_N}z_N = 0, \end{cases}$$

with the unknowns $z_1, \ldots, z_N$ and with non-zero determinant of the matrix of the system, by Lemma A.1. Hence, necessarily $z = 0$. This is in contradiction with our assumption. We conclude that the sum $G_1 + \ldots + G_N$ is indeed an invertible matrix. □

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