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SPECTRAL INEQUALITY AND OPTIMAL COST OF CONTROLLABILITY FOR THE STOKES SYSTEM *

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Abstract. In this paper we present a new proof of the null controllability property for the Stokes system. The proof is based on a new spectral inequality for the eigenfunctions of the Stokes operator. As a consequence, we obtain the cost of the null controllability for the Stokes system of order $e^{C/T}$, when T is small, *i.e.*, the same order in time as for the heat equation.

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1. Introduction

Let $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ be a bounded connected open set, whose boundary $\partial\Omega$ is smooth. Let $T > 0$ and let ω be a nonempty subset of Ω which will usually be referred to as a *control domain*. We will use the notation $Q := \Omega \times (0, T)$ and $\Sigma := \partial\Omega \times (0, T)$ and we will denote by $\nu(x)$ the outward unit normal to Ω at the point $x \in \partial\Omega$.

We introduce the following usual spaces in the context of fluid mechanics

$$\mathbf{V} = \{u \in H_0^1(\Omega)^N;\ \mathrm{div}\ u = 0\},$$

$$\mathbf{H} = \{u \in L^2(\Omega)^N;\ \mathrm{div}\ u = 0, u \cdot \nu = 0 \text{ on } \partial\Omega\}$$

and consider the controlled Stokes system

$$\left|\begin{array}{ll} y_t - \Delta y + \nabla p = f 1_\omega & \text{in } Q, \\ \mathrm{div}\ y = 0 & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0) = y_0 & \text{in } \Omega. \end{array}\right. \tag{1.1}$$

The goal of this paper is to present a proof of the following result.

Theorem 1.1. *Let ω be a nonempty subset of Ω. There exist constants $C_1 > 0, C_2 > 0$ depending only on Ω, ω, such that the following holds true.*

Keywords and phrases. Stokes system, null controllability, Carleman estimates, spectral inequality.

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For every $T > 0$ and every $y_0 \in \mathbf{H}$, there exists a control $f \in L^2(\omega \times (0,T))$ such that the associated solution of the Stokes system (1.1) *satisfies*

$$y(T) = 0, \tag{1.2}$$

and one has the following estimate on the cost of the control

$$\|f\|_{L^2(\omega\times(0,T))} \leq C_1 \mathrm{e}^{C_2/T} \|y_0\|_{\mathbf{H}}. \tag{1.3}$$

The problem of finding a control function f such that the solution of (1.1) satisfies (1.2) is known as the *null controllability problem* for the Stokes system, and has been proved by several authors in the past few years. Therefore what is new in our result is the bound (1.3) on the cost of the control.

Let us recall that in [8], a proof of null controllability is obtained by means of global Carleman inequalities for parabolic equations with homogeneous Dirichlet boundary conditions applied to the adjoint system of (1.1) (see also [2,9,11]). More recently, in [12,13], a slightly different proof is obtained by means of a global Carleman inequality for parabolic equations with non-homogeneous Dirichlet boundary conditions applied to the dual problem of the system satisfied by $w = \operatorname{curl} y$.

The smallest positive constant C_S for which one has

$$\|f\|_{L^2(\omega\times(0,T))} \leq C_S \|y_0\|_{\mathbf{H}}$$

is called the *cost of the null controllability* for the Stokes system at time T. Estimate (1.3) means that the cost of the null controllability for the Stokes system is at most of order $\mathrm{e}^{C/T}$ as $T \longrightarrow 0$, like the cost of null controllability for the heat equation (see, for instance, [7,22,24] and references therein). It is also important to mention that, although (1.3) gives a natural bound for the cost of the null controllability of the Stokes system, until now it was not known if, for any given control domain, estimate (1.3) was true or not. In fact, in all the previous results on the null controllability for the Stokes system by means of global Carleman estimates, the best upper bound one can obtain for C_S is of the form e^{C/T^4}. As far as we know, the only known attempt of optimizing the cost of the controllability for the Stokes system was performed by the first author in [1] where, using the Control Transmutation Method, it is shown that estimate (1.3) holds if the control domain ω satisfies some geometrical conditions.

Our proof of Theorem 1.1 follows the strategy introduced in [15] for the null controllability of the heat equation. As it is well-known, one of the key ingredients in [15] is the obtainment of an interpolation inequality for an appropriate elliptic equation. However, it is not possible to prove an interpolation inequality for our equivalent associated elliptic system because the system does not have local Carleman estimates (see Rem. 2.1). Therefore, instead of proving an interpolation inequality for any solution of the associated elliptic system, we consider only those given in terms of low modes of the Stokes operator and prove an "almost" interpolation inequality for its curl instead of for the solution itself (see Thm. 2.2 in Sect. 2 for the precise statement). This idea of proving an inequality for the curl can be seen as the spectral equivalent to the one developed in [12,13] and here the main difficulty is also to deal with the boundary terms, which are no longer zero. In Section 3, we deduce from Theorem 2.2 the spectral inequality given in Theorem 3.1. From this spectral inequality we then deduce in Section 4 a proof of the main Theorem 1.1; for this proof, we use an argument due to Seidman in [24] and revisited by Miller in [22]. The idea is to use the spectral inequality for low modes and the decay properties of the Stokes system to prove directly an observability inequality for the adjoint system. This strategy can be seen as dual analogous of the method developed in [15]. Finally, in Appendix A we give a proof of the Carleman inequality (2.14) following closely [15], and in Appendix B a proof of Lemma B.1, which gives a fundamental result on the spectral localization at the boundary; for the convenience of the reader, we recall also in Appendix C the proof of the interpolation estimate of Proposition 2.4.

Remark 1.2. Our proof of Theorem 1.1 applies as well in the more general case where Ω is a relatively compact connected open set in a Riemannian manifold M.

In all the paper, we will use semi-classical analysis in the formulation of Carleman estimates, as it is done in [14,15,17]. We refer to [3,18] for an introduction to semi-classical analysis.

2. Interpolation inequality

Let $\{e_j\}_{j=1}^{\infty}$ be an orthonormal basis of $\mathbf{H}$, given by the eigenvectors of the Stokes equation

$$\left| \begin{array}{ll} -\Delta e_j + \nabla p_j = \mu_j e_j & \text{in } \Omega, \\ \operatorname{div}\, e_j = 0 & \text{in } \Omega, \\ e_j = 0 & \text{on } \partial\Omega, \end{array} \right. \tag{2.1}$$

with the sequence of eigenvalues $\{\mu_j\}_{j=1}^{\infty}$ satisfying $0 < \mu_1 \le \mu_2 \le \ldots$ and $\lim_{j\to\infty} \mu_j = \infty$. Then $\{e_j\}_{j=1}^{\infty}$ is also an orthogonal basis of $\mathbf{V}$ and one has

$$\|\operatorname{curl}(e_j)\|_{L^2}^2 = \mu_j. \tag{2.2}$$

We introduce the sets

$$Z = (0,1) \times \Omega \text{ and } W = (1/4, 3/4) \times \Omega,$$

and as in [15], for $\Lambda \ge 1$, we introduce special solutions of the Stokes system of the form

$$(u,p) = \sum_{\mu_j \le \Lambda} a_j \frac{sinh(s\sqrt{\mu_j})}{\sqrt{\mu_j}} \big(e_j(x), p_j(x)\big), \tag{2.3}$$

where $(a_j)_j$ is a given sequence of complex numbers.

Remark 2.1. The pair (u,p) given by (2.3) is a solution of the elliptic system

$$\left| \begin{array}{ll} -\partial_{ss}^2 u - \Delta_x u + \nabla_x p = 0 & \text{in } Z, \\ \operatorname{div}_x\, u = 0 & \text{in } Z \end{array} \right. \tag{2.4}$$

with boundary conditions

$$\left| \begin{array}{ll} u(s,x) = 0 & \text{on } (0,1)\times\partial\Omega, \\ u(0,x) = 0 & \text{in } \Omega \\ \partial_s u(0,x) = \displaystyle\sum_{\mu_j \le \Lambda} a_j e_j(x) & \text{in } \Omega. \end{array} \right. \tag{2.5}$$

The system (2.4) does not have local unique continuation property. Indeed, consider a function $q = q(s,x)$ such that $\Delta_x q = 0$, then the pair

$$(u,p) = (\nabla_x q, \partial_{ss}^2 q)$$

is a solution of (2.4). Taking $q(s,x) = a(s)p(x)$ with $a \in C_0^\infty(\mathbb{R})$ and $\triangle p = 0$, we see that the local zeros of q do not propagate through the surfaces $s = s_0$. Therefore, we cannot obtain a local Carleman estimate for system (2.4).

Therefore, we will work with the 2-vector field $v = \operatorname{curl}(u)$, given by

$$v(s,x) = \sum_{\mu_j \le \Lambda} a_j \frac{\sinh(s\sqrt{\mu_j})}{\sqrt{\mu_j}} \operatorname{curl}(e_j)(x) \tag{2.6}$$

which satisfies

$$\left| \begin{array}{ll} -\partial_{ss}^2 v - \Delta_x v = 0 & \text{in } Z, \\ v(0,x) = 0 & \text{in } \Omega, \\ \partial_s v(0,x) = \displaystyle\sum_{\mu_j \le \Lambda} a_j \operatorname{curl}(e_j)(x) & \text{in } \Omega. \end{array} \right. \tag{2.7}$$

Recall that in $\mathbb{R}^N$, the 2-vector field $v = \operatorname{curl}(u)$ is given in coordinates by the 2-antisymetric tensor $v_{i,j} = \frac{\partial u_j}{\partial x_i} - \frac{\partial u_i}{\partial x_j}$. In the Riemannian setting, if the vector field u is identified with a 1-form, then $v = \operatorname{curl}(u)$ is identified with the 2-form $\mathrm{d}u$, where d denotes the exterior derivative. Observe that when u satisfies $u|_{\partial\Omega} = 0$, the only non zero components of $\mathrm{curl}(u)|_{\partial\Omega}$ are the tangential ones. Therefore, in that case, $\mathrm{curl}(u)|_{\partial\Omega}$ is a vector field tangent to the boundary, which is identified with the 1-form on the boundary $\mathrm{d}u \lrcorner \nu$, where $\lrcorner$ denotes the interior product of a differential form with a vector.

To overcome the lack of boundary condition for v on $(0,1) \times \partial\Omega$, we introduce a small parameter h related to Λ by

$$h = \frac{\delta}{\sqrt{\Lambda}} \quad \text{with} \quad \delta > 0 \quad \text{small.} \tag{2.8}$$

Informally, $x \mapsto v(s,x)$ is concentrated at frequencies $\leq \sqrt{\Lambda}$. Thus, for the semiclassical analysis with semi-classical parameter h, the spectrum of v will be concentrated in the set

$$|\xi| \leq h\sqrt{\Lambda} = \delta.$$

Therefore by taking $\delta > 0$ small, we will force localization near $\xi = 0$. The aim of this section is to prove the following result.

Theorem 2.2. *There exists $\Lambda_0 > 0$, $\mu > 0$, $c > 0$, $\alpha \in (0,1)$, and for $\delta > 0$ small enough $C_\delta > 0$, such that for all $\Lambda \geq \Lambda_0$, all sequence of complex numbers $(a_j)_j$, the 2-vector field v defined in (2.6) satisfies with $h = \delta\Lambda^{-1/2}$*

$$\|v\|_{H^1(W)} \leq C_\delta \Big(\mathrm{e}^{-\mu/h} \|v\|_{H^1(Z)} + \mathrm{e}^{c/h} \|v\|_{H^1(Z)}^{1-\alpha} \|\partial_s v(0,x)\|_{L^2(\omega)}^{\alpha} \Big). \tag{2.9}$$

Remark 2.3. If one want to minimize the right hand side of (2.9) with respect to h, we get that the minimum is achieved at h_* given by

$$\frac{c+\mu}{h_*} = \log(\frac{\mu}{c}) + \alpha \log\left(\frac{\|v\|_{H^1(Z)^N}}{\|\partial_s v(0,x)\|_{L^2(\omega)^N}} \right).$$

Therefore, in the case $\delta\Lambda^{-1/2} < h_*$, the minimum of the right hand side of (2.9) in the interval $h \in]0, \delta\Lambda^{-1/2}]$ will be achieved at $h = \delta\Lambda^{-1/2}$. That is why we do not replace (2.9) by a pure interpolation inequality for solutions of the system (2.7).

The proof of the inequality (2.9) is based on the use of elliptic Carleman inequalities, and an analytic deformation argument with respect to the s variable.

Proof of Theorem 2.2. In the proof, v will allways denote the 2-vector field defined in (2.6). We fix the notation $z = (s,x)$. Let $A := -\triangle_z = -\partial^2_{ss} - \Delta_x$. For $r > 0$ small, let W_r be the open set

$$W_r =]1/9, 9/10[\times \{x \in \Omega, \mathrm{dist}(x, \partial\Omega) > r/2\}. \tag{2.10}$$

While not stated explicitly in [14], the following proposition is contained in the proof of the interpolation estimates in paragraph 5.1 of [14], and also implicitly in [15]. For completeness, we recall its proof in Appendix C.

Proposition 2.4. *For all $r > 0$, there exists $D > 0$ and $\nu \in]0,1[$ such that for all $f \in H^2(Z)$ such that $f|_{s=0} = 0$, one has*

$$\|f\|_{H^1(W_r)} \leq D \|f\|_{H^1(Z)}^{1-\nu} \Big(\|A(f)\|_{L^2(Z)} + \|\partial_s f(0,x)\|_{L^2(\omega)} \Big)^{\nu}. \tag{2.11}$$

For $r > 0$, let K_r be the compact set

$$K_r = \{x \in \overline{\Omega},\ \mathrm{dist}(x, \partial\Omega) \leq r\}.$$

In the sequel, we choose r_0 small enough, such that the map from K_{r_0} into $[0, r_0] \times \partial\Omega$ defined by $x \mapsto (r,y)$, where $r = \mathrm{dist}(x, \partial\Omega)$ and $y \in \partial\Omega$ satisfies $\mathrm{dist}(x, \partial\Omega) = |x - y|$, is a C^∞ diffeomorphism. Let $s_0 \in [1/4, 3/4]$.

We introduce the function $\psi(s,r) = r - (s-s_0)^2$ and consider $\varphi = \mathrm{e}^{D\psi}$. It is well known that, for $D > 0$ large enough, φ satisfies Hormander's sub-ellipticity condition for the operator A in the variables (s,r,y) and one has

$$\partial_r \varphi(s,x) > 0 \quad \text{in} \quad [0,1]_s \times K_{r_0}. \tag{2.12}$$

We also consider a cut-off function $\chi(s,r) = \chi_0(s)\chi_1(r)$, with $\chi_0 \in C_0^\infty(]s_0 - 2s_*, s_0 + 2s_*[)$, $0 \le \chi_0 \le 1$, $\chi_0(s) = 1$ for $|s - s_0| \le 3s_*/2$, with $s_* > 0$ small, and $\chi_1 \in C_0^\infty([0, r_0[)$, $0 \le \chi_1 \le 1$, with $\chi_1 \equiv 1$ on $[0, r_0/2]$. Decreasing r_0, we will assume $r_0 < s_*^2$. In particular, we will use the following fact:

$$\text{There exists } \ \mu_0 > 0 \text{ such that } \{|s - s_0| \ge s_* \text{ and } r \le r_0\} \Rightarrow \{\varphi(s,r) \le \varphi(s_0,0) - \mu_0\}. \tag{2.13}$$

Applying the Carleman inequality given in Theorem A.5 in Appendix A to χv, there exist C and $h_1 > 0$ such that

$$\begin{aligned} &h||\mathrm{e}^{\varphi/h}\chi v||^2_{L^2(Z)} + h^3||\mathrm{e}^{\varphi/h}\nabla_{s,x}(\chi v)||^2_{L^2(Z)} \\ &\qquad \le C\Big(h||\mathrm{e}^{\varphi/h}\chi v||^2_{L^2(r=0)} + h^3||\mathrm{e}^{\varphi/h}\nabla_{s,y}(\chi v)||^2_{L^2(r=0)} + h^4||\mathrm{e}^{\varphi/h}A(\chi v)||^2_{L^2(Z)}\Big), \end{aligned} \tag{2.14}$$

for every $h \in (0, h_1]$. We now analyze each one of the terms on the right-hand side of (2.14). First, let us rewrite (2.14) in a more convenient way. We define the 2-vector field G by

$$G(s,r,y) = \chi_0(s)\chi_1(r)\mathrm{e}^{\varphi(s,r)/h}v(s,r,y) = \chi_0(s)\chi_1(r)\mathrm{e}^{\varphi(s,r)/h}\sum_{\mu_j \le \Lambda} a_j(s)\mathrm{curl}(\ e_j)(r,y) \tag{2.15}$$

and we denote by G_0 the trace of G on the boundary, which is the vector field tangent to the boundary

$$G_0(s,y) = \chi_0(s)\mathrm{e}^{\varphi_0(s)/h}v_0(s,y) = \chi_0(s)\mathrm{e}^{\varphi_0(s)/h}\sum_{\mu_j \le \Lambda} a_j(s)\mathrm{curl}(\ e_j)|_{\partial\Omega}(y),$$

where $a_j(s) = a_j \sinh(s\sqrt{\mu_j})/\sqrt{\mu_j}$, $v_0 = v|_{\partial\Omega}$ and $\varphi_0(s) = \varphi(s,0)$. Inequality (2.14) then reads (with an other constant C):

$$\|G\|^2_{1,sc} \le C\Big(\|G_0\|^2_{1,sc} + h^3||\mathrm{e}^{\varphi/h}A(\chi v)||^2_{L^2(Z)}\Big), \tag{2.16}$$

where we use the notations, with $X =]0,1[\times\partial\Omega$,

$$\|G\|^2_{1,sc} := \|G\|^2_{L^2(Z)} + \|h\nabla_{s,x}G\|^2_{L^2(Z)}, \quad \|G_0\|^2_{1,sc} := \|G_0\|^2_{L^2(X)} + \|h\nabla_{s,y}G_0\|^2_{L^2(X)}.$$

- *Estimate of the boundary terms:*

We consider a cut-off function $\theta \in C_0^\infty(]-2,2[)$ satisfying $0 \le \theta \le 1$, and $\theta \equiv 1$ in a neighborhood of $[-\sqrt{3}, \sqrt{3}]$. Let $\triangle_{\partial\Omega}$ be the Laplace operator on the boundary $\partial\Omega$ acting on vector fields. Let ε_j be an orthonormal basis of eigenfunctions of $\triangle_{\partial\Omega}$ with $-\triangle_{\partial\Omega}\varepsilon_j = \tau_j^2\varepsilon_j$. Let $\Theta = \theta(\sqrt{1 - \Lambda^{-1}\Delta_{\partial\Omega}})$ be the bounded operator acting on L^2 sections of the tangent bundle $T\partial\Omega$

$$\Theta(\sum_j f_j\varepsilon_j) = \sum_j \theta\Big(\sqrt{1 + \Lambda^{-1}\tau_j^2}\Big)f_j\varepsilon_j.$$

We know (see [3]) that Θ is a semi-classical pseudo-differential operator of degree 0 (with small parameter $\Lambda^{-1/2}$), with a scalar semi-classical principal symbol equal to

$$\sigma(\Theta) = \theta\Big(\sqrt{1 + |\eta|_y^2}\Big)Id$$

where $|\eta|_y^2$ denotes the square of Riemannian length of the covector $\eta \in T_y^*\partial\Omega$. From the properties of θ, we get that the essential support of Θ is contained in the set $|\eta|_y \leq \sqrt{3}$, and Θ is microlocally equal to Id on the set $|\eta|_y \leq \sqrt{2}$. We write

$$G_0(s,y) = G_1(s,y) + G_2(s,y),$$

with

$$G_1 = \Theta(G_0), \quad G_2 = (1-\Theta)(G_0).$$

Remark 2.5. Notice that the operator Θ acts only on the $y \in \partial\Omega$ variable, and that G_1 and G_2 are respectively the low and the high frequencies of G_0. More precisely, G_1 is concentrated in the set $|\partial_y| \leq \sqrt{3}\sqrt{\Lambda}$, and G_2 is concentrated in the set $|\partial_y| \geq \sqrt{2}\sqrt{\Lambda}$.

From (2.16), we get

$$\|G\|^2_{H^1_{sc}} \leq C\Big(h^3 ||\mathrm{e}^{\varphi/h} A(\chi v)||^2_{L^2(Z)} + 2\|G_1\|^2_{1,sc} + 2\|G_2\|^2_{1,sc}\Big). \tag{2.17}$$

To estimate the contribution of G_2 to the right hand side of (2.17), we first observe that Lemma B.1 in Appendix B implies that, for every $N \in \mathbb{N}$, there exists C_N such that for all $s \in [0,1]$ one has

$$\int_{\partial\Omega} |G_2(s,.)|^2 + |h\nabla_y G_2(s,.)|^2 \, \mathrm{d}\sigma(y) \leq C_N \Lambda^{-N} K(s), \tag{2.18}$$

where $K(s) = \sum_{\mu_j \leq \Lambda} \chi_0^2(s) \mathrm{e}^{2\varphi_0(s)/h} |a_j|^2 \frac{\sinh^2(s\sqrt{\mu_j})}{\mu_j}$. Next, we notice that

$$\int_\Omega |G|^2 \mathrm{d}x \ \geq \int_{r \leq r_0/2} |G|^2 \mathrm{d}x \ \geq \ \mathrm{e}^{2\varphi_0(s)/h} \chi_0^2(s) \sum_{\mu_j \leq \Lambda} |a_j|^2 \sinh^2(s\sqrt{\mu_j}) - \int_{r \geq r_0/2} \chi_0^2(s) \mathrm{e}^{2\varphi(s,r)/h} |v(s,.)|^2 \mathrm{d}x. \tag{2.19}$$

The proof of (2.19) is achieved taking into account that $\varphi(s,r) \geq \varphi_0(s)$ for every $r \in [0, r_0)$, $\chi_1(r) = 1$ for $r \in [0, r_0/2]$, curl $e_j \perp$ curl e_i in $L^2(\Omega)$ if $i \neq j$ and $||\text{curl } e_j||^2_{L^2(\Omega)} = \mu_j$ for all j. From $\mu_1 > 0$, we thus get from (2.19) that there exists $C > 0$ such that

$$K(s) \leq C \left(\int_\Omega |G|^2 \mathrm{d}x + \int_{r \geq r_0/2} \chi_0^2(s) \mathrm{e}^{2\varphi(s,r)/h} |v(s,.)|^2 \mathrm{d}x \right).$$

Integrating this last inequality with respect to s, using Proposition 2.4 and $A(v) = 0$, and also support$(K(s)) \subset$ support$(\chi_0(s)) \subset]1/9, 9/10[$, we get that there exists $\nu \in (0,1)$ and C_0 such that for all $h \in]0, h_1]$ the following inequality holds true

$$\int_0^1 K(s) \mathrm{d}s \leq C \left(\|G\|^2_{L^2(Z)} + \mathrm{e}^{C_0/h} \|v\|^{2(1-\nu)}_{H^1(Z)} \left(\int_\omega |\partial_s v(0,x)|^2 \mathrm{d}x \right)^{2\nu} \right). \tag{2.20}$$

Therefore, estimate (2.18) implies

$$\int_0^1 \int_{\partial\Omega} |G_2|^2 + |h\nabla_y G_2|^2 \, \mathrm{d}\sigma(y) \mathrm{d}s \leq C_N \Lambda^{-N} \left(\|G\|^2_{L^2(Z)} + \mathrm{e}^{C_0/h} \|v\|^{2(1-\nu)}_{H^1(Z)} \left(\int_\omega |\partial_s v(0,x)|^2 \mathrm{d}x \right)^{2\nu} \right). \tag{2.21}$$

By Lemma B.1, we also have

$$\int_{\partial\Omega} |h\partial_s G_2(s,.)|^2 \, \mathrm{d}\sigma(y) \leq C_N \Lambda^{-N} \sum_{\mu_j \leq \Lambda} \left| h\partial_s \left(\chi_0(s) \mathrm{e}^{\varphi_0(s)/h} a_j(s) \right) \right|^2. \tag{2.22}$$

Let $\tilde\chi_0 \in C_0^\infty(]s_0 - 2s_*, s_0 + 2s_*[)$, $0 \le \tilde\chi_0 \le 1$ and $\tilde\chi_0 = 1$ on the support of χ_0. Let

$$\tilde K(s) = \sum_{\mu_j \le \Lambda} \tilde\chi_0^2(s)\mathrm{e}^{2\varphi_0(s)/h}|a_j|^2\frac{\sinh^2(s\sqrt{\mu_j})}{\mu_j}.$$

For $s \ge 1/9$ and $\mu_j \le \Lambda$, one has $hcosh(s\sqrt{\mu_j}) \le c_0 h\sqrt{\Lambda}\frac{\sinh(s\sqrt{\mu_j})}{\sqrt{\mu}_j}$ with $c_0 = \sup_{x\ge\sqrt{\mu}_1/9}\frac{cosh(x)}{\sinh(x)}$. Thus, we get from $h\sqrt{\Lambda} = \delta \le 1$, that there exists C such that

$$|h\partial_s(\chi_0(s)\mathrm{e}^{\varphi_0(s)/h}a_j(s))|^2 \le$$

$$3|a_j|^2\mathrm{e}^{2\varphi_0(s)/h}\left(\left|h\partial_s\chi_0\ \frac{\sinh(s\sqrt{\mu_j})}{\sqrt{\mu}_j}|^2 + |\chi_0\frac{\sinh(s\sqrt{\mu_j})}{\sqrt{\mu}_j}\partial_s\varphi_0|^2 + |hcosh(s\sqrt{\mu_j})\chi_0\right|^2\right)$$

$$\le C\tilde\chi_0^2(s)\mathrm{e}^{2\varphi_0(s)/h}|a_j|^2\frac{\sinh^2(s\sqrt{\mu_j})}{\mu_j}.$$

Therefore, arguing as above and using (2.22), we get

$$\int_0^1\int_{\partial\Omega}|h\partial_s G_2|^2\ \mathrm{d}\sigma(y)\mathrm{d}s \le C_N\Lambda^{-N}\left(\|\tilde G\|^2_{L^2(Z)} + \mathrm{e}^{C_0/h}\|v\|^{2(1-\nu)}_{H^1(Z)^N}\left(\int_\omega|\partial_s v(0,x)|^2\mathrm{d}x\right)^{2\nu}\right). \tag{2.23}$$

where $\tilde G$ is defined as G in (2.15) with $\tilde\chi_0(s)$ instead of $\chi_0(s)$.
One has by construction $\tilde\chi_0 \le \chi_0 + \mathbf{1}_{\{s_*\le|s-s_0|\le 2s_*\}}$. This and (2.13)implies

$$|\tilde G| \le |G| + \mathrm{e}^{(\varphi(s_0,0)-\mu_0)/h}|v|.$$

Summing up, we get that the contribution of G_2 to the right hand side of (2.17) is estimate by

$$\|G_2\|^2_{H^1_{sc}} \le C_N\Lambda^{-N}\left(\|G\|^2_{L^2(Z)} + \mathrm{e}^{2(\varphi(s_0,0)-\mu_0)/h}\|v\|^2_{L^2(Z)} + \mathrm{e}^{C_0/h}\|v\|^{2(1-\nu)}_{H^1(Z)}\left(\int_\omega|\partial_s v(0,x)|^2\mathrm{d}x\right)^{2\nu}\right). \tag{2.24}$$

We now estimate the contribution of G_1. Let B be the h-pseudodifferential operator acting on the s variable defined by

$$B(f)(s) = \frac{1}{2\pi h}\int \mathrm{e}^{i(s-s')\sigma/h}b_1(\sigma)b_0(s'-s_0)f(s')\mathrm{d}s'\mathrm{d}\sigma \tag{2.25}$$

with $b_j \in C_0^\infty(]-\alpha_j,\alpha_j[)$ equal to 1 in $[-\alpha_j/2,\alpha_j/2]$, and $\alpha_j > 0$ small.

We have with $X =]0,1[\times\Omega$

$$\|G_1\|^2_{H^1_{sc}(X)} \le 2\|BG_1\|^2_{H^1_{sc}(X)} + 2\|(1-B)G_1\|^2_{H^1_{sc}(X)}. \tag{2.26}$$

For the first term on the right-hand side of (2.26), we prove the following.

Claim 1. There exist $\alpha_j > 0$ and $\delta_1 > 0$ such that for all $\delta \in]0,\delta_1]$, BG_1 satisfies

$$\forall N,\ \exists C_N \text{ such that } \|BG_1\|^2_{H^1_{sc}(X)} \le C_N h^N\|G\|^2_{L^2(Z)}. \tag{2.27}$$

Proof of Claim 1. One has

$$BG_1 = B\Theta(G)|_{\partial\Omega}. \tag{2.28}$$

The operator $B\Theta$ is a h-pseudodifferential tangential operator with essential support contained in the set

$$K = \{(s, y; \sigma, \eta), \quad |s - s_0| \le \alpha_0,\ y \in \partial\Omega,\ |\sigma| \le \alpha_1,\ \|\eta\| \le \sqrt{3}\delta\}.$$

By construction, one has with $A_\varphi = h^2 \mathrm{e}^{\varphi/h}\ A\ \mathrm{e}^{-\varphi/h}$

$$A_\varphi G = h^2 e^{\varphi/h} A(\chi_0(s)\chi_1(r)v) = 0 \quad \text{on the open set} \quad |s - s_0| < 3s_*/2,\ r \in]0, r_0/2[. \tag{2.29}$$

The principal symbol of A_φ is given by

$$\big(\sigma + i\partial_s\varphi\big)^2 + \big(\xi + i\nabla_x\varphi\big)^2$$

where $\xi = (\tau, \eta)$ is the dual variable of $x = (r, y)$. Since φ is radial, the principal symbol of A_φ is given by

$$\big(\sigma + i\partial_s\varphi\big)^2 + \big(\tau + i\partial_r\varphi\big)^2 + \eta^2.$$

The roots of the principal symbol with respect to τ are given by

$$\tau_\pm(r, s, y; \sigma, \eta) = -i\partial_r\varphi \pm i\sqrt{\eta^2 + \big(\sigma + i\partial_s\varphi\big)^2}.$$

From $\varphi = \mathrm{e}^{D\psi}$ and $\psi(s, r) = r - (s - s_0)^2$, we get that there exists a constant C such that for all $(r, s, y; \sigma, \eta) \in [0, r_0/2] \times K$ one has

$$Im\ \tau_\pm(r, s, y; \sigma, \eta) \le -e^{D\psi}(1 - C\alpha_0) + C(\delta + \alpha_1).$$

Therefore, for δ and the α_j's small enough, we get that there exists $c_0 > 0$ such that

$$\forall (r, s, y; \sigma, \eta) \in [0, r_0/2] \times K, \quad Im\ \tau_\pm(r, s, y; \sigma, \eta) \le -c_0 < 0. \tag{2.30}$$

Then Claim 1 follows from (2.29), (2.28), and classical elliptic boundary estimates applied to the differential operator A_φ. □

For the second term on the right-hand side of (2.26), we prove:

Claim 2. There exist $C > 0, \delta_0 > 0, c_0 > 0$ such that

$$\forall \delta \in]0, \delta_0], \quad \|(1 - B)G_1\|^2_{H^1_{sc}(X)} \le C\mathrm{e}^{2(\varphi(s_0,0) - c_0)/h}\|v\|^2_{H^1(Z)}. \tag{2.31}$$

Proof of Claim 2:. We begin noticing that

$$G_0(s, y) = \sum_{\mu_j \le \Lambda} a_j \chi_0(s) \mathrm{e}^{\varphi_0(s)/h} \left((\mathrm{e}^{s\sqrt{\mu_j}} - \mathrm{e}^{-s\sqrt{\mu_j}})/2\right) \left(\frac{\text{curl } e_j\big|_{r=0}}{\sqrt{\mu_j}}\right).$$

Set $\nu_j = h^2\mu_j$. We have $\sqrt{\nu_j} \le h\sqrt{\Lambda} = \delta$ and

$$h^{-1}\varphi_0(s) \pm s\sqrt{\mu_j} = h^{-1}\left(\varphi_0(s) \pm s\sqrt{\nu_j}\right).$$

We also have

$$\varphi_0(s) = \varphi_0(s_0) - D(s - s_0)^2 + O((s - s_0)^4).$$

Let us now make the change of variable $s \mapsto s_0 + t$. Set $\chi_0(s_0 + t)b_0(t) = \tilde{\chi}(t)$ and $\theta(t) = \varphi_0(s_0) - \varphi_0(s_0 + t) = Dt^2 + O(t^4)$. The function $\tilde{\chi} \in C_0^\infty$ is equal to 1 in a neighborhood of $t = 0$, and the function $\theta(t)$ is real analytic. Let $\phi_j(t, \sigma)$ be the phase function

$$\phi_j(t, \sigma) = -t\sigma + i(\theta(t) \pm (t + s_0)\sqrt{\nu_j}).$$

and let g_j be the vectors fields on the boundary $\partial\Omega$

$$g_j(y) = \frac{\Theta(\text{curl } e_j\big|_{r=0})(y)}{\sqrt{\mu_j}}\cdot$$

Then one has

$$(1-B)G_1(t,y) = \frac{\mathrm{e}^{\varphi(s_0,0)/h}}{2\pi h}\sum_{\mu_j\leq\Lambda} a_j g_j(y)\int \mathrm{e}^{i\sigma t/h}(1-b_1(\sigma))\left(\int \mathrm{e}^{i\phi_j(t',\sigma)/h}\tilde{\chi}(t')\mathrm{d}t'\right)\mathrm{d}\sigma. \tag{2.32}$$

By classical trace theorem, there exists C', m' such that $\|g_j\|_{H^1(\partial\Omega)} \leq C'\Lambda^{m'}$ for all j such that $0<\mu_1\leq\mu_j\leq\Lambda$, and by (2.2), one gets easily that there exists C such that $\sum_{\mu_j\leq\Lambda}|a_j|^2 \leq C\|v\|^2_{L^2(\Omega)} \leq C\|v\|^2_{H^1(\Omega)}$. Since $b_1(\sigma)=1$ for $|\sigma|\leq\alpha_1/2$, we get from (2.32), using Cauchy−Schwartz inequality and Weyl formula to estimate $\sum_{\mu_j\leq\Lambda}|a_j|\|g_j\|_{H^1(\partial\Omega)}$, that there exists C, m such that, with $\langle\sigma\rangle = (1+\sigma^2)^{1/2}$

$$\|(1-B)G_1\|_{H^1_{sc}(X)}\mathrm{e}^{-\varphi(s_0,0)/h} \leq C\Lambda^m h^{-1}\sup_{j,\mu_j\leq\sqrt{\Lambda}}\int_{|\sigma|\geq\alpha_1/2}\langle\sigma\rangle\left|\left(\int \mathrm{e}^{i\phi_j(t',\sigma)/h}\tilde{\chi}(t')\mathrm{d}t'\right)\mathrm{d}\sigma\right|. \tag{2.33}$$

Therefore, Claim 2 is a consequence of (2.33), $h\sqrt{\Lambda}=\delta\leq 1$, and the following lemma.

Lemma 2.6. *There exist $c_0>0$ and $\delta_0>0$ such that for $0<\delta\leq\delta_0$, and uniformly in $|\sigma|\geq\alpha_1/2$ and $\sqrt{\nu}_j\leq\delta$, the following estimate holds true*

$$\forall N, \exists C_N, \qquad \left|\int \mathrm{e}^{i\phi_j(t,\sigma)/h}\tilde{\chi}(t)\mathrm{d}t\right| \leq C_N\langle\sigma\rangle^{-N}\mathrm{e}^{-c_0/h}. \tag{2.34}$$

Proof. To estimate (2.34), we consider a deformation of the real axis $t\mapsto g(t) = t - i\frac{\sigma}{\langle\sigma\rangle}\kappa(t)$. Here κ is a C_0^∞ function whose support is contained in the interior of the set $\{t;\ \tilde{\chi}(t)\equiv 1\}$, $0\leq\kappa\leq k_0$, $k_0>0$ small, and $\kappa\equiv k_0$ in some interval $[-\epsilon_0,\epsilon_0]$. On this curve, we have, with $\tilde{\phi}_j(t,\sigma)=\phi_j(g(t),\sigma)$

$$\begin{aligned}\text{Im }\tilde{\phi}_j &= -Im\ \sigma\left(t - i\frac{\sigma}{\langle\sigma\rangle}\kappa(t)\right) + \text{Re }\theta\left(t - i\frac{\sigma}{\langle\sigma\rangle}\kappa(t)\right) \pm \text{Re }\left(\left(t - i\frac{\sigma}{\langle\sigma\rangle}\kappa(t)\right)\sqrt{\nu_j}\right) \pm s_3\sqrt{\nu_j}\\ &= \frac{\sigma^2}{\langle\sigma\rangle}\kappa(t) + \text{Re }\theta\left(t - i\frac{\sigma}{\langle\sigma\rangle}\kappa(t)\right) \pm (t+s_3)\sqrt{\nu_j}\\ &= \frac{\sigma^2}{\langle\sigma\rangle}\kappa(t) + \theta(t) - \frac{\sigma^2}{\langle\sigma\rangle^2}\kappa^2(t)\theta''(t)/2 + O\left(\left(\frac{\sigma}{\langle\sigma\rangle}\right)^4\kappa^4(t)\right) \pm (t+s_3)\sqrt{\nu_j}),\end{aligned} \tag{2.35}$$

since $\theta(t+i\gamma) = \theta(t) + i\gamma\theta'(t) - \gamma^2\theta''(t)/2 - i\gamma^3\theta'''(t)/6 + O(\gamma^4)$. Hence

$$\text{Im }\tilde{\phi}_j(t,\sigma) = \frac{\sigma^2}{\langle\sigma\rangle}\kappa(t)\left(1 - \frac{\kappa}{\langle\sigma\rangle}\theta''(t)/2 + O\left(\frac{\sigma^2}{\langle\sigma\rangle^3}\kappa^3(t)\right)\right) + \theta(t) \pm (t+s_3)\sqrt{\nu_j}). \tag{2.36}$$

Let Γ be the integration contour in the complex plane $\Gamma = \{t - i\frac{\sigma}{\langle\sigma\rangle}\kappa(t),\ t\in\mathbb{R}\}$. Since $\phi(t)$ is a holomorphic function and support$(\kappa)\subset\{t;\tilde{\chi}(t)=1\}$, one has

$$\int_{\mathbb{R}} \mathrm{e}^{i\phi_j(t,\sigma)/h}\tilde{\chi}(t)\mathrm{d}t = \int_{\Gamma} \mathrm{e}^{i\phi_j(z,\sigma)/h}\tilde{\chi}(z)\mathrm{d}z = \int_{\mathbb{R}} \mathrm{e}^{i\tilde{\phi}_j(t,\sigma)/h}\tilde{\chi}(t)g'(t)\mathrm{d}t. \tag{2.37}$$

There exists $C_1>0$ such that for every $t\in\overline{\text{supp}(\tilde{\chi})}$, one has $|t+s_0|\leq C_1$, $|t|\leq sup(2s_*,\alpha_0)$, and there exists $C_2>0$ such that for $|\sigma|\geq\alpha_1/2$ one has $\frac{\sigma^2}{\langle\sigma\rangle}\geq C_2\langle\sigma\rangle$. Thus we get from (2.36), decreasing s_* and α_0 if necessary, for all $\delta\in]0,\delta_0]$, $t\in\overline{\text{supp}(\tilde{\chi})}$, $|\sigma|\geq\alpha_1/2$

$$\text{Im }\tilde{\phi}_j(t,\sigma) \geq C_2\langle\sigma\rangle\kappa(t)/4 + Dt^2 + O(t^4) - C_1\delta \geq C_2\langle\sigma\rangle\kappa(t)/4 + Dt^2/2 - C_1\delta_0.$$

Thus one has for all $\delta \in]0, \delta_0]$, $t \in \overline{\text{supp}(\tilde{\chi})}$, $|\sigma| \geq \alpha_1/2$

$$\text{Im } \tilde{\phi}_j(t,\sigma) \geq C_2\alpha_1 k_0/8 - C_1\delta_0 \quad \text{for} \quad |t| \leq \epsilon_0, \tag{2.38}$$
$$\text{Im } \tilde{\phi}_j(t,\sigma) \geq D\epsilon_0^2/2 - C_1\delta_0 \quad \text{for} \quad |t| \geq \epsilon_0. \tag{2.39}$$

For δ_0 small enough, we get that there exists $c_0 > 0$ such that for all $\delta \in]0, \delta_0]$, $t \in \overline{\text{supp}(\tilde{\chi})}$, $|\sigma| \geq \alpha_1/2$, one has Im $\tilde{\phi}_j(t,\sigma) \geq c_0$, and from (2.37), we get

$$\left| \int_{\mathbb{R}} \mathrm{e}^{i\phi_j(t,\sigma)/h} \tilde{\chi}(t)\mathrm{d}t \right| \leq C\mathrm{e}^{-c_0/h}. \tag{2.40}$$

Therefore, (2.35) is true for $N = 0$. The general case is proved by integration by parts in t. □

From Lemma 2.6 we conclude that Claim 2 holds true. □

From (2.17), (2.24), (2.27) and (2.31), we conclude that there exists Λ_0, C_0, $\mu > 0$ and $\nu \in]0, 1[$, such that for $\Lambda \geq \Lambda_0$, $\delta \in]0, \delta_0]$ and $h = \delta\Lambda^{-1/2}$, the following inequality holds true

$$\|G\|^2_{H^1_{sc}} \leq C\left(h^3 ||\mathrm{e}^{\varphi/h} A(\chi v)||^2_{L^2} + \mathrm{e}^{2(\varphi(s_0,0)-\mu)/h} \|v\|^2_{H^1(Z)} + \mathrm{e}^{2C_0/h} \|v\|^{2(1-\nu)}_{H^1(Z)} \left(\int_\omega |\partial_s v(0,x)|^2 \mathrm{d}x \right)^\nu \right). \tag{2.41}$$

• *Estimate of the term* $||\mathrm{e}^{\varphi/h} A(\chi v)||^2_{L^2}$:

We begin noticing that, since $Av = 0$, $A(\chi v) = \chi_0[A, \chi_1]v + [A, \chi_0]\chi_1 v$ and that $[A, \chi_0]$ and $[A, \chi_1]$ are first order operators whose support are contained in the set

$$V_1 \cup V_2 := \Big(\big([s_0 - 2s_*, s_0 - 3s_*/2] \cup [s_0 + 3s_*/2, s_0 + 2s_*]\big) \times [0, r_0] \Big) \cup \Big([s_0 - 2s_*, s_0 + 2s_*] \times [r_0/2, r_0] \Big).$$

One has from (2.13) $\varphi \leq \varphi(s_0, 0) - \mu_0$ on V_1 and for some $C_1 > 0$, $\varphi \leq C_1$ on V_2. Therefore, from (2.41), decreasing $\mu > 0$ if necessary, we get

$$\|G\|^2_{H^1_{sc}} \leq C\Big(e^{2C_1/h} ||v||^2_{H^1(V_2)} + e^{2(\varphi(s_0,0)-\mu)/h} \|v\|^2_{H^1(Z)} + e^{2C_0/h} \|v\|^{2(1-\nu)}_{H^1(Z)} \big(\int_\omega |\partial_s v(0,x)|^2 dx \big)^\nu \Big). \tag{2.42}$$

Let U be the open subset of Z defined by $U = \{\varphi(s,r) > \varphi(s_0, 0) - \mu/2,\ |s - s_0| < s_*,\ r < r_0/2\}$. From Proposition 2.4, one has for some $\nu_1 \in]0, 1[$

$$||v||_{H^1(V_2)} \leq C \|v\|^{1-\nu_1}_{H^1(Z)^N} \|\partial_s v(0,x)\|^{\nu_1}_{L^2(\omega)}.$$

Decreasing eventually ν and increasing C_0 we conclude from (2.42) that there exists Λ_0, C_0, $\mu > 0$ and $\nu \in]0, 1[$, such that for $\Lambda \geq \Lambda_0$, $\delta \in]0, \delta_0]$ and $h = \delta\Lambda^{-1/2}$, the following inequality holds true:

$$||v||_{H^1_{sc}(U)} \leq C\left(\mathrm{e}^{-\frac{\mu}{2h}} \|v\|_{H^1(Z)} + \mathrm{e}^{C_0/h} \|v\|^{1-\nu}_{H^1(Z)} \|\partial_s v(0,x)\|^\nu_{L^2(\omega)} \right), \tag{2.43}$$

Since the open set U is a neighborhood of $[1/4, 3/4] \times \partial\Omega$, we conclude from (2.43), $||v||_{H^1(U)} \leq h^{-1}||v||_{H^1_{sc}(U)}$, and Proposition 2.4 that Theorem 2.2 holds true. □

3. Spectral inequality

In this section, using the interpolating inequality (2.9), we show a spectral inequality for the low modes of the Stokes operator. This result will be used in the next section to prove Theorem 1.1.

Theorem 3.1. *Let $\omega \subset \Omega$ be a nonempty open set. There exist constants $M > 0$, $K > 0$ such that, for every sequence of complex numbers z_j and every real $\Lambda > 0$, we have*

$$\sum_{\mu_j \le \Lambda} |z_j|^2 = \int_\Omega \left| \sum_{\mu_j \le \Lambda} z_j e_j(x) \right|^2 \mathrm{d}x \le M e^{K\sqrt{\Lambda}} \int_\omega \left| \sum_{\mu_j \le \Lambda} z_j e_j(x) \right|^2 \mathrm{d}x. \tag{3.1}$$

Proof. Let $\tilde{\omega}$ be a nonempty set such that $\tilde{\omega} \subset\subset \omega$ and consider $u(s,x) = \sum_{\mu_j \le \Lambda} z_j \frac{\sinh(s\sqrt{\mu_j})}{\sqrt{\mu_j}} e_j(x)$ and $v = \mathrm{curl}\ u$. Using (2.2), the norm $\|v\|_{H^1(W)}$ can be estimated from below as follows

$$||v||^2_{H^1(W)} \ge ||v||^2_{L^2(W)} = \sum_{\mu_j \le \Lambda} \int_{1/4}^{3/4} |z_j|^2 \sinh^2(s\sqrt{\mu_j}) \mathrm{d}s \ge \sum_{\mu_j \le \Lambda} \mu_j |z_j|^2 \int_{1/4}^{3/4} s^2 \mathrm{d}s. \tag{3.2}$$

Next, from the fact that $\|v\|^2_{H^1(Z)} = \| \frac{\partial v}{\partial s} \|^2_{L^2(Z)} + \|\Delta_x u\|^2_{L^2(Z)}) + \|v\|^2_{L^2(Z)}$, we get

$$\|v\|^2_{H^1(Z)} \le \sum_{\mu_j \le \Lambda} (2\mu_j + 1) |z_j|^2 \int_0^1 \mathrm{e}^{2s\sqrt{\mu_j}} \mathrm{d}s \le \sum_{\mu_j \le \Lambda} (2\mu_j + 1) |z_j|^2 \mathrm{e}^{2\sqrt{\Lambda}}. \tag{3.3}$$

Since $\|\partial_s v(0,x)\|^2_{L^2(\tilde{\omega})} = \int_{\tilde{\omega}} \left| \sum_{\mu_j \le \Lambda} z_j \mathrm{curl}(e_j) \right|^2 \mathrm{d}x$, from Theorem 2.2, estimates (3.2) and (3.3), there exists $\alpha \in (0,1)$, C_0, $\delta_0 > 0$, $\mu > 0$, Λ_0, such that for all $\delta \in]0, \delta_0]$ and all $\Lambda \ge \Lambda_0$, one has

$$\sum_{\mu_j \le \Lambda} |z_j|^2 \le C_\delta \left(\mathrm{e}^{2C_0 \Lambda^{1/2}/\delta} \left(\sum_{\mu_j \le \Lambda} \Lambda |z_j|^2 \mathrm{e}^{2\sqrt{\Lambda}} \right)^{1-\alpha} \left(\int_{\tilde{\omega}} \left| \sum_{\mu_j \le \Lambda} z_j \mathrm{curl}(e_j) \right|^2 \mathrm{d}x \right)^\alpha + \mathrm{e}^{-2\mu\sqrt{\Lambda}/\delta} \sum_{\mu_j \le \Lambda} \Lambda |z_j|^2 \mathrm{e}^{2\sqrt{\Lambda}} \right). \tag{3.4}$$

Take $\delta_1 \in]0, \delta_0]$ such that $2\mu/\delta_1 - 2 > 1$ and Λ_1 such that one has $C_{\delta_1} \Lambda \mathrm{e}^{-\sqrt{\Lambda}} \le 1/2$ for all $\Lambda \ge \Lambda_1$. From (3.4), we get for all $\Lambda \ge \Lambda_1$ with $C = C_{\delta_1}$,

$$\sum_{\mu_j \le \Lambda} |z_j|^2 \le 2C \left(\mathrm{e}^{2C_0 \Lambda^{1/2}/\delta_1} \left(\sum_{\mu_j \le \Lambda} \Lambda |z_j|^2 \mathrm{e}^{2\sqrt{\Lambda}} \right)^{1-\alpha} \left(\int_{\tilde{\omega}} \left| \sum_{\mu_j \le \Lambda} z_j \mathrm{curl}(e_j) \right|^2 \mathrm{d}x \right)^\alpha \right)$$

which gives for some $\tilde{M}$ and $\tilde{K}$ positive

$$\sum_{\mu_j \le \Lambda} |z_j|^2 \le \tilde{M} \mathrm{e}^{\tilde{K}\sqrt{\Lambda}} \int_{\tilde{\omega}} \left| \sum_{\mu_j \le \Lambda} z_j \mathrm{curl}(e_j) \right|^2 \mathrm{d}x. \tag{3.5}$$

To finish the proof, we consider a function $\theta \in C_0^\infty(\omega)$ such that $\theta \geq 0$ and $\theta \equiv 1$ in $\tilde{\omega}$ and estimate the local integral in the right-hand side of (3.5) as follows

$$\begin{aligned}\int_{\tilde{\omega}} \left| \sum_{\mu_j \leq \Lambda} z_j \mathrm{curl}(e_j) \right|^2 \mathrm{d}x &\leq \int_\omega \theta \left| \sum_{\mu_j \leq \Lambda} z_j \mathrm{curl}(e_j) \right|^2 \mathrm{d}x \\ &= \int_\omega \theta \left(\sum_{\mu_j \leq \Lambda} z_j \mathrm{curl}(e_j) \right) \left(\sum_{\mu_k \leq \Lambda} \overline{z}_k \mathrm{curl}(e_k) \right) \mathrm{d}x \\ &= \int_\omega \left(\sum_{\mu_j \leq \Lambda} z_j e_j \right) P \left(\theta \sum_{\mu_k \leq \Lambda} \overline{z}_k \mathrm{curl}(e_k) \right) \mathrm{d}x, \end{aligned} \tag{3.6}$$

where P is a first order differential operator. Thus we get by Cauchy−Schwarz inequality

$$\begin{aligned}\int_{\tilde{\omega}} \left| \sum_{\mu_j \leq \Lambda} z_j \mathrm{curl}(e_j) \right|^2 \mathrm{d}x &\leq \left(\int_\omega \left| \sum_{\mu_j \leq \Lambda} z_j e_j \right|^2 \mathrm{d}x \right)^{1/2} \left(\int_\Omega \left| P \left(\theta \sum_{\mu_k \leq \Lambda} z_k curl(e_k) \right) \right|^2 \mathrm{d}x \right)^{1/2} \\ &\leq C\Lambda \left(\int_\omega \left| \sum_{\mu_j \leq \Lambda} z_j e_j \right|^2 \mathrm{d}x \right)^{1/2} \left(\sum_{\mu_j \leq \Lambda} |z_j|^2 \right)^{1/2}. \end{aligned} \tag{3.7}$$

Finally, from (3.5) and (3.7), we get that Theorem 3.1 holds true. □

4. Cost of the null controllability for the Stokes system

In this section we prove Theorem 1.1. This proof follows closely the ideas in [22, 24]. First, we introduce the adjoint system (with the change of orientation $t \mapsto T - t$) of the Stokes system (1.1):

$$\left| \begin{array}{ll} z_t - \Delta z + \nabla q = 0 & \text{in } Q, \\ \mathrm{div}\ z = 0 & \text{in } Q, \\ z = 0 & \text{on } \Sigma, \\ z(0) = z_0 & \text{in } \Omega. \end{array} \right. \tag{4.1}$$

It is well-know that the Stokes system 1.1 is null controllable at time T if and only if

$$\|z(T)\|_{\mathbf{H}}^2 \leq C_S^2 \int_0^T \int_\omega |z|^2 \,\mathrm{d}x\mathrm{d}t, \ \forall z_0 \in \mathbf{H}. \tag{4.2}$$

Moreover, we have that (4.2) holds if and only if (1.3) holds with $C_S = C_1 \mathrm{e}^{C_2/T}$.

To prove Theorem 1.1, we need the following two results:

Lemma 4.1. *Let $T' > 0$ and $m \in (0,1)$. If the approximate observability estimate*

$$f(t)\|z(t)\|^2 - f(mt)\|z_0\|^2 \leq \int_0^t \int_\omega |z|^2 \,\mathrm{d}x\mathrm{d}t, \ \forall z_0 \in H^2(\Omega)^N \cap \mathbf{V}, \ t \in (0, T'], \tag{4.3}$$

holds with $f(t) \longrightarrow 0$ as $t \longrightarrow 0^+$, then $C_S^2 \leq 1/f((1-m)T)$ for $T \in (0, T']$, i.e., *the cost does not grow more than the inverse of $\sqrt{f}$.*

Proof. Let $T < T'$, $\tau_k = m^k(1-m)T$ and consider a disjoint partition $\cup(T_{k+1}, T_k]$ of $(0, T]$, with $T_{k+1} = T_k - \tau_k$, $T_0 = T$, $k \in \mathbb{N}$. We apply (4.3) to $z_0 = z(T_{k+1})$ and $t = \tau_k$, we obtain

$$f(\tau_k)\|z(T_k)\|^2 - f(\tau_{k+1})\|z(T_{k+1})\|^2 \leq \int_{T_{k+1}}^{T_k} \int_\omega |z|^2 \,\mathrm{d}x\mathrm{d}t,\ k \in \mathbb{N}.$$

We add these inequalities to get

$$f(\tau_0)\|z(T)\|^2 - f(\tau_k)\|z(T_k)\|^2 \leq \int_{T_k}^{T} \int_\omega |z|^2 \,\mathrm{d}x\mathrm{d}t,\ k \in \mathbb{N}$$

and taking the limit $k \longrightarrow \infty$ completes the proof since $f(t) \longrightarrow 0$ and the function $t \mapsto \|z(t)\|$ is bounded in $[0, T]$. □

We use Lemma 4.1 to prove:

Lemma 4.2. *Let $T_0 > 0$ and $\beta > 0$. If the approximate observability estimate*

$$f(T)\|z(T)\|^2 - g(T)\|z_0\|^2 \leq \int_0^T \int_\omega |z|^2 \,\mathrm{d}x\mathrm{d}t,\ \forall z_0 \in H^2(\Omega)^N \cap \mathbf{V},\ t \in (0, T_0], \tag{4.4}$$

holds with $f(T) = f_0 \mathrm{e}^{-2/(d_2T)^\beta}$ and $g(T) = g_0 \mathrm{e}^{-2/(d_1T)^\beta}$, where f_0, g_0, $d_1 < d_2$ are positive, then for all $d \in (0, d_2 - d_1)$ there exists $T' \in (0, T_0]$ such that $C_S^2 \leq f_0^{-1}\mathrm{e}^{2/(dT)^\beta}$ for $T \in (0, T']$. Moreover, if $g_0 \leq f_0$ then we may take $d = d_1 - d_2$ and $T' = T_0$.

Proof. First, we compute the least m such that $g(T) \leq f(mT)$ for all $T \in (0, T']$. We find $m = \frac{d_1}{d_2}h(T')$, with $h(T') = \left(1 + \inf_{t\in(0,T')} t^\beta d_1^\beta \ln(f_0/g_0)/2\right)^{-1/\beta}$, where the parenthesis is 1 when $g_0 \leq f_0$ and positive for T' is small enough. Now $C_S^2 \leq 1/(f((1-m)T)) = \frac{1}{f_0}\mathrm{e}^{2/(d_3T)^\beta}$, with $d_3 = d_2 - d_1 h(T') \longrightarrow d_2 - d_1$ as $T' \longrightarrow 0$. □

Proof of Theorem 1.1. Let $T_0 > 0$, $T \in (0, T_0)$ and $z_0 \in \mathbf{H}$. Set $H_\Lambda = \mathrm{span}\{e_j; \mu_j \leq \Lambda\}$. For any Λ, the solution z of (4.1) can be split into $z = u + v$, where u and v are (together with some pressure) the solutions of (4.1) associated to $u_0 \in H_\Lambda$ and $v_0 \in H_\Lambda^\perp$, $z_0 = u_0 + v_0$, respectively. Moreover,

$$u(t) \in H_\Lambda \text{ and } \|v(t)\|_{\mathbf{H}} \leq \mathrm{e}^{-\Lambda t}\|v_0\|_{\mathbf{H}}, \tag{4.5}$$

for every $t > 0$. For every $M_1 > 0$, we have

$$\|z(T)\|_{\mathbf{H}}^2 \leq \frac{1}{T}\int_0^T \int_\Omega |z(t)|^2 \,\mathrm{d}x\mathrm{d}t \leq \frac{1}{M_1}\mathrm{e}^{\frac{M_1}{T}} \int_0^T \int_\Omega |z(t)|^2 \,\mathrm{d}x\mathrm{d}t,\ \forall T \in (0, T_0) \tag{4.6}$$

and

$$\int_0^T \int_\omega |z(t)|^2 \,\mathrm{d}x\mathrm{d}t \leq T\|z_0\|_{\mathbf{H}}^2. \tag{4.7}$$

From (4.6) and Theorem 3.1, we get

$$\|u(\tau)\|_{\mathbf{H}}^2 \leq \frac{M}{M_1}\mathrm{e}^{\frac{M_1}{\tau} + K\sqrt{\Lambda}} \int_0^\tau \int_\omega |u(t)|^2 \,\mathrm{d}x\mathrm{d}t,\ \forall \tau \in (0, T_0). \tag{4.8}$$

Let us now consider an observation time $\tau = \epsilon T$, with $\epsilon \in (0, 1)$ small enough and take $\sqrt{\Lambda} = \frac{1}{\tau}$. From (4.8), it follows that

$$\|u(T)\|_{\mathbf{H}}^2 \leq \frac{1}{f(T)}\int_{(1-\epsilon)T}^T \int_\omega |u(t)|^2 \,\mathrm{d}x\mathrm{d}t, \quad f(T) = \frac{M_1}{M}\mathrm{e}^{-\frac{M_1+K}{\epsilon T}}. \tag{4.9}$$

Using (4.9), the definition of z and (4.7), we get

$$f(T)\|u(T)\|_{\mathbf{H}}^2 \leq \int_{(1-\epsilon)T}^{T}\int_{\omega}|u(t)|^2\,\mathrm{d}x\mathrm{d}t \leq 2\int_{(1-\epsilon)T}^{T}\int_{\omega}|z(t)|^2\,\mathrm{d}x\mathrm{d}t + 2\epsilon T\|v((1-\epsilon)T)\|_{\mathbf{H}}^2. \tag{4.10}$$

Hence,

$$f(T)\|z(T)\|_{\mathbf{H}}^2 \leq \int_{(1-\epsilon)T}^{T}\int_{\omega}|z(t)|^2\,\mathrm{d}x\mathrm{d}t + \epsilon T\|v((1-\epsilon)T)\ \|_{\mathbf{H}}^2 + 2f(T)\|v(T)\|_{\mathbf{H}}^2. \tag{4.11}$$

Using (4.5) in (4.11), we obtain

$$f(T)\|z(T)\|_{\mathbf{H}}^2 - \left(\epsilon T\mathrm{e}^{-2\Lambda(1-\epsilon)T} + 2f(T)\mathrm{e}^{-2\Lambda T}\right)\|v_0\|_{\mathbf{H}}^2 \leq \int_{(1-\epsilon)T}^{T}\int_{\omega}|z(t)|^2\,\mathrm{d}x\mathrm{d}t. \tag{4.12}$$

Since $\|v_0\|_{\mathbf{H}}^2 \leq \|z_0\|_{\mathbf{H}}^2$ and $f(T) \leq f(T_0)$, we obtain

$$f(T)\|z(T)\|_{\mathbf{H}}^2 - \left(T_0 + 2f(T_0)\right)\mathrm{e}^{-2\Lambda(1-\epsilon)T}\|z_0\|_{\mathbf{H}}^2 \leq \int_{(1-\epsilon)T}^{T}\int_{\omega}|z(t)|^2\,\mathrm{d}x\mathrm{d}t. \tag{4.13}$$

Therefore, since $\Lambda T = 1/(\epsilon^2 T)$, Lemma 4.2 holds with $\beta = 1$ and

$$f_0 = \frac{M_1}{M},\ g_0 = T_0 + 2f(T_0),\ d_2 = 2\epsilon/(M_1+K),\ d_1 = \epsilon^2/(1-\epsilon).$$

One has $g(T_0) \longrightarrow 0$ if $T_0 \longrightarrow 0$, and for ϵ small enough, one has $d_1 < d_2$. Thus we can apply Lemma 4.2 with $g_0 \leq f_0$ and T_0 small and conclude that there exists C_S for which (4.2) is true for all solutions of (4.1). Moreover, there exists $T' > 0$ and $C_1, C_2 > 0$ such that

$$C_S \leq C_1\mathrm{e}^{C_2/T}, \forall T \in (0, T']. \qquad \square$$

Remark 4.3. Arguing as in Theorem 2.2 of [22], it is possible to get some estimate on the size of C_2. Nevertheless, the obtainment of the optimal constant C_2 for which inequality (1.3) holds has an interest on its own and, as far as we know, it is open problem.

Appendix A. Carleman inequality for the Laplace operator

In this section we prove the Carleman inequality (2.14) that we have used in the proof of the Theorem 2.2. The proof follows closely the one given in [15], but we have to take care of the non zero Dirichlet data at the boundary. Since the notational distinction between the variables s and x plays here no role, we may replace the triple $((s,x),(0,s_0)\times\Omega, A)$ by the triple $(x,\Omega,-\triangle)$.

Remark A.1. Observe that since we work on $\mathbb{R}^N$, it is equivalent to prove the Carleman inequality (2.14) for vector fields or for functions. If one replace $\mathbb{R}^N$ by a Riemannian manifold (M,g), since the Laplace operator acting on vector fields has a scalar principal symbol equal to the principal symbol of the Laplace operator acting on functions, and since the Carleman inequality (2.14) is insensitive to lower order terms, it is still equivalent to prove the Carleman inequality (2.14) for vector fields or for functions.

Given $x_0 \in \partial\Omega$, we choose the normal geodesic coordinate system $x = (x', x_N)$, $x' \in \mathbb{R}^{N-1}$, where x' is a local coordinate system for $\partial\Omega$ so that $x_0 = 0$. In this new coordinate system, Ω is locally define by $x_N > 0$, the metric g is of the form $\mathrm{d}x_N^2 + \sum_{i,j\leq N-1} g_{i,j}\mathrm{d}x_i\mathrm{d}x_j$, and one has

$$\mathrm{dist}((x',x_N),\partial\Omega) = x_N, \tag{A.1}$$

where in (A.1), *dist* is the distance given by the Riemannian metric g. In this coordinate system, the Laplace operator is equal to

$$\Delta_g = \frac{1}{\sqrt{\det\, g}}\partial_{x_N}\left[\sqrt{\det\, g}\partial_{x_N}\right] + \sum_{i,j\leq N-1}\frac{1}{\sqrt{\det\, g}}\partial_{x'_i}\left[\sqrt{\det\, g}\; g^{i,j}\partial_{x'_j}\right], \tag{A.2}$$

where $(g^{i,j})$ is the inverse matrix of $g = (g_{i,j})$. Therefore, one has

$$-\Delta_g = -\partial^2_{x_N} - R(x', x_N, \partial_{x'}) + A_1(x, \partial_x), \tag{A.3}$$

where A_1 is a first order operator and

$$R(x', x_N, \partial_{x'}) = \sum_{i,j\leq N-1} g^{i,j}(x', x_N)\partial_{x_i}\partial_{x_j}.$$

Since A_1 is a first order operator, and first order terms do not affect the validity of the Carleman inequality of Theorem A.5, for the rest of this section, we will work with the operator

$$P = -\partial^2_{x_N} - R(x', x_N, \partial_{x'}). \tag{A.4}$$

In the sequel, we will use the notations of [15] for the class of tangential symbols S^j, and tangential operators $\mathcal{E}^j$ (see [15], Sect. 3, formulas (9) and (10)). We recall the following definition:

Definition A.2. Let V be an open set of $\mathbb{R}^N$ and $\varphi \in C^\infty(\mathbb{R}^N;\mathbb{R})$. We say that φ satisfies Hormander's sub-ellipticity condition on $\overline{V}$ for the operator P if, for every $x \in \overline{V}$, $\nabla\varphi(x) \neq 0$, and there exists $C > 0$ such that

$$\forall(x,\xi) \in \overline{V}\times\mathbb{R}^N,\; p_\varphi(x,\xi) = 0 \Longrightarrow \{\tilde{q}_1, \tilde{q}_2\}(x,\xi) \geq C,$$

where p_φ is the principal symbol of the conjugated operator

$$P_\varphi = h^2 \mathrm{e}^{\varphi(x)/h} P \mathrm{e}^{-\varphi(x)/h}, \tag{A.5}$$

i.e.,

$$p_\varphi = \tilde{q}_2 + i\tilde{q}_1,$$

with

$$\tilde{q}_2 = |\xi_N|^2 - |\partial_{x_N}\varphi|^2 + \sum_{i,j\leq N-1} g_{i,j}(x)(\xi_i\xi_j - \partial_{x_i}\varphi\partial_{x_j}\varphi)$$

and

$$\tilde{q}_1 = 2\left(\partial_{x_N}\varphi\xi_N + \sum_{i,j\leq N-1} g_{i,j}(x)\partial_{x_i}\varphi\xi_j\right).$$

Remark A.3. The set

$$Z = \{(x,\xi) \in \overline{V}\times\mathbb{R}^N;\; p_\varphi(x,\xi) = 0\} \tag{A.6}$$

is compact.

The following well known Lemma will be useful to construct functions satisfying Hormander's sub-ellipticity on a set $\overline{V}$ for the Lapace operator.

Lemma A.4. *Let $\psi \in C^\infty(\mathbb{R}^N;\mathbb{R})$ be such that $\nabla\psi(x) \neq 0$ for every $x \in \overline{V}$. Let $G \in C^\infty(\mathbb{R};\mathbb{R})$ be such that $G' > 0$ and $G'' > 0$. There exists a constant $A > 0$ such that if $G'' \geq AG'$, then $\varphi(x) = G(\psi(x))$ satisfies Hormander's sub-ellipticity condition on $\overline{V}$ for the Laplace operator operator $-\Delta$.*

Let $V = V' \times (-r, r)$ be a neighborhood of x_0 in $\mathbb{R}^N$ and $\varphi(x', x_N)$ a weight function satisfying Hormander's sub-ellipticity condition on $\overline{V}$ for P and such that

$$\frac{\partial\varphi}{\partial x_N}(x', x_N) \neq 0,\ \forall (x', x_N) \in \overline{V}.$$

We prove the following result.

Theorem A.5 (Carleman inequality)**.** *Let K be a compact set of V' and $r' < r$. There exist $h_1 > 0$ and $C > 0$ such that for every $h \in (0, h_1]$, and every $u(x', x_N) \in C^\infty(V' \times [0, r))$ satisfying $u(x', 0) = g(x')$ and supported in $K \times [0, r']$, the following inequality holds true*

$$h||\mathrm{e}^{\varphi/h}u||^2_{L^2} + h^3||\mathrm{e}^{\varphi/h}\nabla u||^2_{L^2} \leq C\bigg(h^4||\mathrm{e}^{\varphi/h}Pu||^2_{L^2} + h\int \mathrm{e}^{2\varphi(x',0)/h}|g(x')|^2\mathrm{d}x' + h^3\int \mathrm{e}^{2\varphi(x',0)/h}|\nabla_{x'}g(x')|^2\mathrm{d}x' + h^3\int \mathrm{e}^{2\varphi(x',0)/h}|\partial_{x_N}u(x',0)|^2\mathrm{d}x'\bigg). \tag{A.7}$$

Moreover, if $\partial_{x_N}\varphi > 0$ for every $(x', x_N) \in \overline{V}$, we have

$$h||\mathrm{e}^{\varphi/h}u||^2_{L^2} + h^3||\mathrm{e}^{\varphi/h}\nabla u||^2_{L^2} \leq C\bigg(h^4||\mathrm{e}^{\varphi/h}Pu||^2_{L^2} + h\int \mathrm{e}^{2\varphi(x',0)/h}|g(x')|^2\mathrm{d}x' + h^3\int \mathrm{e}^{2\varphi(x',0)/h}|\nabla_{x'}g(x')|^2\mathrm{d}x'\bigg). \tag{A.8}$$

Remark A.6. Since $-\Delta_g = P + A_1$, with A_1 being a first order operator, by taking h_1 smaller, inequalities (A.7) and (A.8) are still true if we replace P by $-\Delta_g$.

Proof. Taking $v = \mathrm{e}^{\varphi/h}u$, we see that v solves

$$\left|\begin{array}{ll} P_\varphi v = \tilde{f} & \text{in } V' \times (0, r), \\ v(x', 0) = \tilde{g}(x'), & \end{array}\right. \tag{A.9}$$

where, P_φ is given by (A.5), $\tilde{f} = h^2\mathrm{e}^{\varphi/h}f$ and $\tilde{g} = \mathrm{e}^{\varphi(x',0)/h}g$.

Noticing that $h\partial_{x_N}v = \mathrm{e}^{\varphi/h}(h\partial_{x_N}u + \varphi_{x_N}u)$ and using the fact that $u(x', 0) = g(x')$, we see that (A.7) is equivalent to (see [15] for the definition of the semi-classical tangential norms$\|.\|_{t,s}$)

$$h\bigg(\|v\|^2_{t,1} + \|h\partial_{x_N}v\|^2_{t,0}\bigg) \leq C\bigg(\|\ \tilde{f}\|^2_{t,0} + h\int |\tilde{g}(x')|^2\mathrm{d}x' + h^3\int |\nabla_{x'}\tilde{g}(x')|^2\mathrm{d}x' + h\int |h\partial_{x_N}v(x',0)|^2\mathrm{d}x'\bigg) \tag{A.10}$$

and (A.8) is equivalent to

$$h\bigg(\|v\|^2_{t,1} + \|h\partial_{x_N}v\|^2_{t,0}\bigg) \leq C\bigg(\|\ \tilde{f}\|^2_{t,0} + h\int |\tilde{g}(x')|^2\mathrm{d}x' + h^3\int |\nabla_{x'}\tilde{g}(x')|^2\mathrm{d}x'\bigg). \tag{A.11}$$

Next, we write $P_\varphi = \tilde{Q}_2 + i\tilde{Q}_1$, with $\tilde{Q}_2 = \mathrm{Re}(P_\varphi) = \frac{1}{2}(P_\varphi + P^*_\varphi)$ and $\tilde{Q}_1 = \mathrm{Im}(P_\varphi) = \frac{1}{2i}(P_\varphi - P^*_\varphi)$. Moreover, we separate the operators $\tilde{Q}_i$, $i = 1, 2$, in the derivatives in the normal variable x_N and the tangential ones. Indeed, we write

$$\tilde{Q}_2 = \left(\frac{h}{i}\partial_{x_N}\right)^2 + Q_2, \quad Q_2 \in \mathcal{E}^2$$
$$\sigma_2(Q_2) = q_2 = R(x', x_N, \xi') - R(x', x_N, \varphi_{x'}) - (\varphi_{x_N})^2$$

$$\tilde{Q}_1 = \frac{2h}{i}\varphi_{x_N}\partial_{x_N} + 2Q_1, \quad Q_1 \in \mathcal{E}^1$$
$$\sigma_1(Q_1) = q_1 = \hat{R}(x', x_N; \xi', \varphi_{x'})$$

where $\hat{R}(x', x_N; a, b) = \displaystyle\sum_{j,k\leq N-1} g_{j,k}(x', x_N)a_jb_k$.

Let us denote $D_N = \frac{h}{i}\partial_{x_N}$ and $< z, w >_0 = \int z(x',0)\overline{w}(x',0)\mathrm{d}x'$ the scalar product of the trace of the functions z, w over the boundary $x_N = 0$. We have the following identities

$$(w_1, \tilde{Q}_2 w_2) = (\tilde{Q}_2 w_1, w_2) - ih\left(\langle w_1, D_N w_2\rangle_0 + \langle D_N w_1, w_2\rangle_0\right),$$

$$(w_1, \tilde{Q}_1 w_2) = (\tilde{Q}_1 w_1, w_2) - 2ih\langle \varphi_{x_N} w_1, w_2\rangle_0.$$

Then, we have

$$||\tilde{f}||^2_{L^2} = ||(\tilde{Q}_2 + i\tilde{Q}_1)v||^2_{L^2} = ||\tilde{Q}_2 v||^2_{L^2} + ||\tilde{Q}_1 v||^2_{L^2} + h(\frac{i}{h}[\tilde{Q}_2, \tilde{Q}_1]v, v) + hB_0(v),$$

with

$$B_0(v) = \langle (D_N\tilde{Q}_1 - 2\varphi_{x_N}\tilde{Q}_2)v, v\rangle_0 + \langle \tilde{Q}_1 v, D_N v\rangle_0. \tag{A.12}$$

One has

$$\begin{aligned} D_N\tilde{Q}_1 - 2\varphi_{x_N}\tilde{Q}_2 = D_N(2\varphi_{x_N}D_N + 2Q_1) - 2\varphi_{x_N}(D_N^2 + Q_2) &= 2D_N Q_1 - 2\varphi_{x_N}Q_2 + 2\frac{h}{i}(\partial^2_{x_N}\varphi)D_N \\ &= A_2 + A_1 D_N \end{aligned}$$

with $A_j \in \mathcal{E}^j$. Therefore, we get by Cauchy$-$Schwarz, with the notation $D_N v(x', 0) = w$

$$\left|\left\langle \left(D_N\tilde{Q}_1 - 2\varphi_{x_N}\tilde{Q}_2\right)v, v\right\rangle_0\right| \le C\Big(||\tilde{g}||_0^2 + h^2||\nabla_{x'}\tilde{g}||_0^2 + ||w||_0(||\tilde{g}||_0 + ||h\nabla_{x'}\tilde{g}||_0)\Big).$$

Since $\tilde{Q}_1 = 2\varphi_{x_N}D_N + 2Q_1$, we get from (A.12)

$$|B_0(v) - 2\langle \varphi_{x_N}(x',0)w, w\rangle_0| \le C\Big(||\tilde{g}||_0^2 + h^2||\nabla_{x'}\tilde{g}||_0^2 + ||w||_0(||\tilde{g}||_0 + ||h\nabla_{x'}\tilde{g}||_0)\Big). \tag{A.13}$$

Next, using that $\frac{i}{h}[\tilde{Q}_2, \tilde{Q}_1]$ is a differential operator of order 2, and the definition of $\tilde{Q}_2$ and $\tilde{Q}_1$, we see that

$$\frac{i}{h}[\tilde{Q}_2, \tilde{Q}_1] = H_0\tilde{Q}_2 + H_1\tilde{Q}_1 + H_2,$$

with $H_k \in \mathcal{E}^k$, $0 \le k \le 2$. Writing

$$\tilde{q}_2 = \xi_N^2 + q_2(x', x_N, \xi), \quad \tilde{q}_1 = 2(\varphi_{x_N}\xi_N + 2q_1(x', x_N, \xi))$$

and using the sub-ellipticity condition on φ, we have, for every $(x', x_N) \in [-r, r] \times \overline{V'}$ and every $\xi \in \mathbb{R}^N$,

$$\tilde{q}_2 = \tilde{q}_1 = 0 \Longrightarrow \{\tilde{q}_2, \tilde{q}_1\} \ge C. \tag{A.14}$$

To finish the proof, we use the following Lemma, which is proved in ([15], Sect. 3, Lem. 1):

Lemma A.7. *There exists* μ (*large enough*) *such that*

$$\frac{\mu}{\langle \xi'\rangle^2}(q_1^2 + \varphi_{x_N}^2 q_2) + \sigma_2(H_2) \ge C\langle \xi'\rangle^2. \tag{A.15}$$

To finish the proof of Theorem A.5, we consider

$$G = \frac{\mu}{\Lambda_t^2}(Q_1^2 + \varphi_{x_N}^2 Q_2) \in \mathcal{E}^0$$

and use Garding's inequality to see that there exists $h_1 > 0$ such that

$$\begin{aligned} ||\tilde{f}||^2_{L^2} - hB_0(v) &= ||\tilde{Q}_2 v||^2_{L^2} + ||\tilde{Q}_1 v||^2_{L^2} + h(\frac{i}{h}[\tilde{Q}_2, \tilde{Q}_1]v, v) \\ &\geq ||\tilde{Q}_2 v||^2_{L^2} + ||\tilde{Q}_1 v||^2_{L^2} + hRe(H_0\tilde{Q}_2 v, v) + hRe(H_1\tilde{Q}_1 v, v) \\ &- hRe(\tilde{Q}_1^2 + \varphi^2_{x_N}\tilde{Q}_2 v, Gv) - C_0 h||v||^2_{t,1}, \end{aligned} \tag{A.16}$$

for some $C_0 > 0$ and every $h \in (0, h_1)$. Since $H^k \in \mathcal{E}^k$, we have

$$|h\mathrm{Re}(H_0\tilde{Q}_2 v, v)| \leq Ch^{1/2}||\tilde{Q}_2 v||^2_{t,0} + Ch^{3/2}||v||^2_{t,0},$$

$$|h\mathrm{Re}(H_1\tilde{Q}_1 v, v)| \leq Ch^{1/2}||\tilde{Q}_1 v||^2_{t,0} + Ch^{3/2}||v||^2_{t,1}$$

and

$$||D_N v||^2_{t,0} \leq C\big(||\tilde{Q}_1 v||^2_{t,0} + ||v||^2_{t,1}\big).$$

Taking h_1 smaller, we obtain from (A.16) that

$$||\tilde{f}||^2_{L^2} - hB_0(v) + h\mathrm{Re}(\tilde{Q}_1^2 + \varphi^2_{x_N}\tilde{Q}_2 v, Gv) \geq \frac{1}{2}\big(||\tilde{Q}_2 v||^2_{L^2} + ||\tilde{Q}_1 v||^2_{L^2}\big) + Ch\big(||v||^2_{t,1} + ||D_N v||^2_{t,0}\big). \tag{A.17}$$

From the definition, we also have

$$\tilde{Q}_1^2 + \varphi^2_{x_N}\tilde{Q}_2 \in \varphi_{x_N}\tilde{Q}_2 - \frac{1}{2}D_N \circ \varphi_{x_N}\tilde{Q}_1 + \mathcal{E}^1\tilde{Q}_1 + h\mathcal{E}^0 D_N + h\mathcal{E}^1$$

and

$$h\mathrm{Re}(D_N \circ \varphi_{x_N}\tilde{Q}_1 v, Gv) = h\mathrm{Re}(\varphi_{x_N}\tilde{Q}_1 v, D_N Gv) - h\mathrm{Re}\left\langle \frac{h}{i}\varphi_{x_N}\tilde{Q}_1 v, Gv\right\rangle_0.$$

Taking h_1 smaller if necessary, and from the fact that $D_N G \in GD_N + h\mathcal{E}^0$, we deduce that

$$Ch\big(||v||^2_{t,1} + ||D_N v||^2_{t,0}\big) \leq ||\tilde{f}||^2_{L^2} - hB_0(v). \tag{A.18}$$

The proof of (A.10) and (A.11) then follows from (A.13). □

Appendix B. Spectral inequality for low frequencies at the boundary

This section is devoted to the proof of Lemma B.1, which gives an inequality for the traces on the boundary $\partial\Omega$ of vector fields of the form $u(x) = \sum_{\mu_j \leq \Lambda} a_j e_j(x)$. As in Appendix A, we choose on the open subset $\{x \in \Omega, \mathrm{dist}(x, \partial\Omega) < r_0\}$ of Ω, with r_0 small, the normal geodesic coordinate system $x = (x', x_N), x' \in \partial\Omega, x_N = \mathrm{dist}(x, \partial\Omega) \in]0, r_0[$.

Let $\mathbf{\Delta}$ be the Laplace operator acting on vector fields. Let us first recall the expression of $\mathbf{\Delta}$ in this coordinate system (see [6]). For a vector field u, we have the following decomposition

$$u = u_{||} + u_{\perp}\frac{\partial}{\partial_{x_N}}, \quad u_{||} = \sum_{j=1}^{N-1} u_{||,j}\frac{\partial}{\partial_{x'_j}}$$

and we write $u = (u_{||}, u_{\perp})$. We also have

$$\mathrm{div}_g\ u = \frac{1}{\sqrt{\det\ g}}\partial_{x_N}\left[\sqrt{\det\ g}u_{\perp}\right] + \mathrm{div}_{||}u_{||},$$

with

$$\mathrm{div}_{||}u_{||} = \frac{1}{\sqrt{\det g}} \sum_{j=1}^{N-1} \frac{\partial}{\partial_{x'_j}} \left(\sqrt{\det g} u_{||,j} \right).$$

For a given function p, we have

$$\nabla_g p = \left(\nabla_{x'} p, \frac{\partial p}{\partial_{x_N}} \right),$$

with

$$\nabla_{x'} p = \sum_{j=1}^{N-1} q_j \frac{\partial}{\partial_{x'_j}}, \quad q_j = \sum_{k=1}^{N-1} g^{j,k} \frac{\partial p}{\partial_{x'_k}}.$$

For $u = (u_{||}, u_\perp)$, the Laplace operator applied to u, $\mathbf{\Delta} u$, is the vector field

$$\mathbf{\Delta} \begin{pmatrix} u_{||} \\ u_\perp \end{pmatrix} = \begin{pmatrix} \Delta_{||} u_{||} \\ \Delta_g u_\perp \end{pmatrix} + M_1 \begin{pmatrix} u_{||} \\ u_\perp \end{pmatrix} + M_0 \begin{pmatrix} u_{||} \\ u_\perp \end{pmatrix}, \tag{B.1}$$

where Δ_g is the Laplace operator acting on functions, given by (A.2), and $\Delta_{||}$ the operator

$$\Delta_{||} = \frac{1}{\sqrt{\det g}} \partial_{x_N} [\sqrt{\det g} \partial_{x_N}] Id + P(x', x_N, \partial_{x'}).$$

Here P is a $(N-1) \times (N-1)$ matrix that contains only horizontal derivatives $\partial_{x'}$ of order at most 2. Moreover, the principal symbol of $\Delta_{||}$ is scalar, and one has $\sigma(\Delta_{||}) = \sigma(\Delta_g) Id$. In (B.1), M_1 is a differential operator of order 1 of the form

$$M_1 = \sum M_{1,j}(x', x_N) \frac{\partial}{\partial_{x'_j}}, \quad \sum M_{1,j} = \left(\begin{array}{c|c} * & * \\ \hline * & 0 \end{array} \right) \tag{B.2}$$

and M_0 is a $N \times N$ matrix (*i.e.*, a differential operator of order 0). Observe that M_1 does not contain derivatives in ∂_{x_N} and sends the normal vector ν into a horizontal vector.

As in Section 2, we consider a cut-off function $\theta \in C_0^\infty(]-2, 2[)$ satisfying $0 \leq \theta \leq 1$, and $\theta \equiv 1$ in a neighborhood of $[-\sqrt{3}, \sqrt{3}]$, and we denote by $\triangle_{\partial\Omega}$ the Laplace operator on the boundary $\partial\Omega$ acting on vector fields. Recall that Θ is the bounded operator acting on L^2 sections of the tangent bundle $T\partial\Omega$ defined by $\Theta = \theta(\sqrt{1 - \Lambda^{-1} \Delta_{\partial\Omega}})$. The aim of this section is to prove the following lemma.

Lemma B.1. *Let $Q(x, D_x)$ be a differential operator defined in a neighborhood of $\partial\Omega$. For all $M, N \in \mathbb{N}$ there exists constants C_M and $D_{M,N}$ such that for all $\Lambda > 0$ and $u(x) = \sum_{\mu_j \leq \Lambda} a_j e_j(x)$ with $a_j \in \mathbb{C}$, one has, with $v = Q(u)\Big|_{\partial\Omega}$ and $w = (Id - \Theta)v$*

$$||\Delta_{\partial\Omega}^M v||_{L^2(\partial\Omega)} \leq C_M \Lambda^{2M+1} \left(\sum_{\mu_j \leq \Lambda} |a_j|^2 \right)^{1/2} \text{ and } ||\Delta_{\partial\Omega}^M w||_{L^2(\partial\Omega)} \leq D_{M,N} \Lambda^{-N} \left(\sum_{\mu_j \leq \Lambda} |a_j|^2 \right)^{1/2}. \tag{B.3}$$

Proof. For notational simplicity, we write $\partial\Omega = Y$. The bound on $||\Delta_Y^M v||_{L^2(Y)}$ follows from classical trace theorems, thus we will just concentrated on the bound on $||\Delta_Y^M w||_{L^2(Y)}$. From

$$\Delta_Y^M w = \sum_{\mu_j \leq \Lambda} a_j \Delta_Y^M (1 - \theta)[\sqrt{1 - \Lambda^{-1} \Delta_Y}]((Q(x, D_x) e_j)\big|_Y),$$

we see that it is sufficient to show that

$$\sup_{\mu_j \le \Lambda} \|\Delta_Y^M (1-\theta)[\sqrt{1-\Lambda^{-1}\Delta_Y}]((Q(x,D_x)e_j)\big|_Y)\|_{L^2(Y)} \le D_{M,N}\Lambda^{-N}. \tag{B.4}$$

Indeed, if (B.4) holds true, by Cauchy−Schwartz, we get

$$\|\Delta_Y^M w\|_{L^2(Y)} \le \#\{j;\ \mu_j \le \Lambda\}^{1/2}\Big(\sum_{\mu_j\le\Lambda}|a_j|^2\Big)^{1/2} D_{M,N}\Lambda^{-N}$$

and the result will then follows from Weyl's formula (*i.e.*, $\#\{j;\ \mu_j \le \Lambda\}^{1/2} \le c\Lambda^c$). Let us now prove (B.4). We start with the following lemma.

Lemma B.2. *We have*

$$(Q(x,D_x)e_j)\big|_Y = Q_1(\mu_j,x',\partial_{x'})(\partial_{x_N}e_j)\big|_Y + Q_2(\mu_j,x',\partial_{x'})(\partial_{x_N}p_j)\big|_Y + Q_3(\mu_j,x',\partial_{x'})p_j\big|_Y,$$

where the $Q_i(\mu_j,x',\partial_{x'})$, $1\le i\le 3$, are differential operators at the boundary with polynomial dependence in μ.

Proof. We write $Q(x,D_x) = \sum_{k,\beta} a_{\alpha,\beta}(x',x_N)\partial_{x'}^\beta \partial_{x_N}^k$.

Since

$$\begin{aligned}\Delta_g &= \partial_{x_N}^2 + g_N(x',x_N)\partial_{x_N} + \tilde{R}(x',x_N,\partial_{x'})\\ \boldsymbol{\Delta} &= \partial_{x_N}^2 + M_N(x',x_N)\partial_{x_N} + \tilde{Q}(x',x_N,\partial_{x'}),\end{aligned}$$

we get from $\boldsymbol{\Delta} e_j = \nabla_g p_j - \mu_j e_j$

$$\partial_{x_N}^2 e_j = -M_N(x',x_N)\partial_{x_N}e_j - \tilde{Q}(x',x_N,\partial_{x'})e_j - \mu_j e_j + \nabla_g p_j.$$

We also have that $\Delta_g p = 0$, which gives

$$\partial_{x_N}^2 p_j = -g_N(x',x_N)\partial_{x_N}p_j - \tilde{R}(x',x_N,\partial_{x'})p_j.$$

We get easily from this identities by induction on $k\ge 2$,

$$\partial_{x_N}^k e_j = A_k^1(\mu_j,x,\partial_{x'})e_j + A_k^2(\mu_j,x,\partial_{x'})\partial_{x_N}e_j + A_k^3(\mu_j,x,\partial_{x'})p_j + A_k^4(\mu_j,x,\partial_{x'})\partial_{x_N}p_j.$$

where the $A_k^j(\mu,x,\partial_{x'})$ are tangential differential operators with polynomial dependence in μ. Then the result follows from $e_j|_{x_N=0} = 0$. □

Let us denote by WF_Λ the semi-classical wave front set of a family of functions $f(x',\hbar)$ on the boundary $\partial\Omega = Y$, with small semi-classical parameter $\hbar = \Lambda^{-1/2}$ (see [3,18]). Let us recall that the essential support of the operator $Id - \Theta$ is contained in the set $|\eta|_y \ge \sqrt{2}$. One has

$$\begin{aligned}\Delta_Y^M(1-\theta)[\sqrt{1-\Lambda^{-1}\Delta_Y}]\varphi(x')\partial_{x'}^\alpha &= \Lambda^{2M+|\alpha|}(\Lambda^{-2}\Delta_Y)^M(1-\theta)[\sqrt{1-\Lambda^{-1}\Delta_Y}]\varphi(x')\Lambda^{-|\alpha|}\partial_{x'}^\alpha\\ &= \Lambda^{2M+|\alpha|}p\left(x',\frac{\hbar}{i}\partial_{x'}\right),\end{aligned}$$

where p is an $\hbar$ pseudo-differential operator of degree $2M+|\alpha|$ and essential support contained in the set $\{|\eta|\ge\sqrt{2}\}$. Therefore, from Lemma B.2, the proof of (B.4) will be achieved if we show the existence of a constant $C<\sqrt{2}$, such that

$$WF_\Lambda(\partial_{x_N}e_j\big|_Y) \subset \{|\eta|\le C\}, \tag{B.5}$$

$$WF_\Lambda(p\big|_Y) \subset \{|\eta| \leq C\}, \tag{B.6}$$

$$WF_\Lambda(\partial_{x_N} p\big|_Y) \subset \{|\eta| \leq C\}. \tag{B.7}$$

We will get in fact that (B.5), (B.6) and (B.7) hold true with $C = 1$. Since p is a harmonic function, we have that (B.6) implies (B.7) and hence we just need to prove (B.5) and (B.6).

Let us introduce $\tilde{e_j} = 1_{x_N \geq 0} e_j$. Using $e_j\big|_{x_N=0} = 0$, we get that $\tilde{e_j}$ satisfies

$$\left| \begin{array}{l} -\boldsymbol{\Delta}_g \tilde{e}_j + 1_{x_N \geq 0} \nabla_g p_j = \mu_j \tilde{e}_j + (k_{||}, k_\perp)\delta_0, \\ \text{div } \tilde{e}_j = 0, \end{array} \right. \tag{B.8}$$

where $(k_{||}, k_\perp) = (k_{||}, k_\perp)(x')$. Taking the divergence operator in $(\text{B.8})_1$, and using $(\text{B.8})_2$, we get

$$\text{div}_g(1_{x_N \geq 0} \nabla p_j) = \frac{1}{\sqrt{\det g}} \partial_{x_N} [\sqrt{\det g} k_\perp \delta_0] + \text{div}_{||}\ (k_{||}\delta_0).$$

Denoting by $p_j^0 := p_j\big|_{x_N=0}$, and $p_j^1 := \frac{\partial p_j}{\partial x_N}\big|_{x_N=0}$, and using $-\Delta_g p_j = 0$, we obtain

$$p_j^1 \delta_0 = k_\perp \delta_0' + \frac{1}{\sqrt{\det g}} \partial_{x_N} [\sqrt{\det g}] k_\perp \delta_0 + \text{div}_{||}\ (k_{||})\delta_0.$$

Therefore, we have that $k_\perp = 0$ and $p_j^1 = \text{div}_{||}\ (k_{||})$. Hence, from (B.8),

$$(-\mu_j - \boldsymbol{\Delta}_g)\tilde{e}_j + 1_{x_N \geq 0} \nabla_g p_j = (k_{||}, 0)\delta_0.$$

Now, since $\nabla_g \tilde{p}_j := \nabla_g(1_{x_N \geq 0} p_j) = 1_{x_N \geq 0} \nabla_g p_j + (0, p_j^0 \delta_0)$, we obtain

$$(-\mu - \boldsymbol{\Delta}_g)\tilde{e}_j + \nabla_g \tilde{p}_j = (k_{||}, p_j^0)\delta_0.$$

Noticing that

$$-\boldsymbol{\Delta}_g \tilde{e}_j = -1_{x_N \geq 0} \boldsymbol{\Delta}_g e_j - (\partial_{x_N} e_j)\big|_{x_N=0} \otimes \delta_0,$$

we get

$$(-\mu - \boldsymbol{\Delta}_g)\tilde{e}_j + \nabla_g \tilde{p}_j = (0, p_j^0)\delta_0 - \begin{pmatrix} \partial_{x_N} e_{j,||}\big|_Y \\ \partial_{x_N} e_{j,\perp}\big|_Y \end{pmatrix} \delta_0. \tag{B.9}$$

Moreover, we have that $\partial_{x_N} e_{j,\perp}\big|_Y = 0$ because $\text{div}_g\ e_j = 0$ and $\text{div}_{||}\ (k_{||})\big|_Y = 0$. Hence,

$$k_{||} = -\partial_{x_N} e_{j,||}\big|_Y \tag{B.10}$$

and we find that $\tilde{e}_j, \hbar \tilde{p}_j = \tilde{q}_j$ satisfy the following system:

$$\left| \begin{array}{l} (-\mu\hbar^2 - \hbar^2 \boldsymbol{\Delta}_g)\tilde{e}_j + \hbar \nabla_g \tilde{q}_j = (\hbar^2 k_{||}, \hbar q_j^0)\delta_0, \\ \hbar \text{div}_g\ \tilde{e}_j = 0. \end{array} \right. \tag{B.11}$$

The proof of Lemma B.1 will be achieved if we show that the system (B.11) is elliptic in $|\eta| > 1$ under the assumption $\mu\hbar^2 \leq 1$. In order to prove ellipticity, it is sufficient to verify that ellipticity holds true in the flat case, *i.e.*, when the boundary is $x_N = 0$ and $\boldsymbol{\Delta}$ is the constant coefficient Laplacian. This ellipticity property is supposed to be well known, but we will recall the proof for the convenience of the reader.

In order to prove ellipticity for the system (B.11), we will construct the parametrix (in the flat case) for the system

$$\left| \begin{array}{l} A(v,q) := (-\mu - \Delta)v + \nabla q = f = (f_{||}, f_{\perp}) \otimes \delta_0, \\ \text{div } v = 0 \\ v = 0 \text{ and } q = 0 \;\; \text{in } x_N < 0. \end{array} \right. \tag{B.12}$$

where $|\mu| \leq \Lambda$. Taking the Fourier transform in $\mathbb{R}^N$ in (B.12), we get

$$\left| \begin{array}{l} (|\zeta|^2 - \mu)\hat{v} + i\zeta\hat{q} = \hat{f}, \\ \zeta \cdot \hat{v} = 0. \end{array} \right. \tag{B.13}$$

From the fact that $\Delta q = \text{div} f$, we also have that

$$i|\zeta|^2\hat{q} = \zeta \cdot \hat{f}, \tag{B.14}$$

or equivalently for $\zeta \neq 0$

$$i\zeta\hat{q} = \frac{\zeta(\zeta \cdot \hat{f})}{|\zeta|^2}\cdot \tag{B.15}$$

From (B.13) and (B.15), we see that

$$\hat{v} = \frac{1}{|\zeta|^2 - \mu}\left[\hat{f} - \frac{\zeta(\zeta \cdot \hat{f})}{|\zeta|^2}\right],$$

for $|\zeta| > \Lambda$. Let us now consider a real valued function $\theta \in C^\infty$ such that $0 \leq \theta \leq 1$ and for $\varepsilon > 0$ small

$$\theta(\zeta) = 0, \text{ for } |\zeta| \leq (1+\varepsilon)\sqrt{\Lambda}$$

and

$$\theta(\zeta) = 1, \text{ for } |\zeta| \geq (1+2\varepsilon)\sqrt{\Lambda}$$

and introduce (recall that we have $|\mu| \leq \Lambda$, hence $\frac{\theta(\zeta)}{|\zeta|^2-\mu}$ is well defined)

$$\widehat{E(f)} = \left\{ \begin{array}{c} \hat{v}_\theta = \frac{\theta(\zeta)}{|\zeta|^2-\mu}\left[\hat{f} - \frac{\zeta(\zeta\cdot\hat{f})}{|\zeta|^2}\right], \\ \hat{q}_\theta = \theta(\zeta)\frac{(\zeta\cdot\hat{f})}{i|\zeta|^2}\cdot \end{array} \right. \tag{B.16}$$

It is easy to see that $(v_\theta, q_\theta) = E(f)$ is the solution of

$$(-\mu - \Delta)v_\theta + \nabla q_\theta = \theta(D)f.$$

Applying E to both sides of $(\text{B.12})_1$, we see that

$$EA(v,q) = (\theta(D)v, \theta(D)q) = E(f)$$

and then we get

$$E(f) = (v,q) + (\theta(D) - 1)(v,q).$$

Therefore

$$\lim_{x_N \to 0^-} E(f) = \lim_{x_N \to 0^-} (\theta(D) - 1)(v,q)$$

and hence

$$WF_\Lambda\left(\lim_{x_N\to 0^-} E(f)\right) = WF_\Lambda\left(\lim_{x_N\to 0^-} (\theta(D) - 1)(v,q)\right).$$

Writing $\zeta = (\xi, \eta)$, the dual variable of $(x_N, y) \in \mathbb{R} \times \mathbb{R}^{N-1}$, we have

$$\int \mathrm{e}^{-iy\cdot\eta}\big(\lim_{x_N\to 0^-}(\theta(D)-1)(v,q)\big)\mathrm{d}y = \frac{1}{2\pi}\lim_{x_N\to 0^-}\int \mathrm{e}^{ix_N\xi}(\theta(\xi,\eta)-1)(\hat{v},\hat{q})(\xi,\eta)\mathrm{d}\xi,$$
$$= 0, \quad \text{if } |\eta| \geq (1+2\varepsilon)\sqrt{\Lambda}, \tag{B.17}$$

which gives

$$WF_{\Lambda}(\lim_{x_N\to 0^-}(1-\theta(D))(v,q)) \subset \{\ |\eta| \leq (1+2\varepsilon)\sqrt{\Lambda},\}.$$

Using (B.11), we see that the proof will be consequence of the ellipticity for $|\eta| \geq (1+2\varepsilon)\Lambda$ of the map $(\hat{f}_{||}(\eta), \hat{f}_{\perp}(\eta)) \mapsto \lim_{x_N\to 0^-}\hat{E}(f)(\eta)$. Indeeed, for fixed $|\eta| \geq (1+2\varepsilon)\Lambda$, since $\theta(\xi,\eta) = 1$ for all ξ, we get that $\lim_{x_N\to 0^-}\hat{E}(f)(\eta)$ is equal to

$$\lim_{x_N\to 0^-}\frac{1}{2\pi}\int \mathrm{e}^{ix_N\xi}\frac{1}{\xi^2+\eta^2-\mu}\begin{bmatrix}\hat{f}_{\perp}(\eta) - \frac{\xi\left(\xi\hat{f}_{\perp}(\eta)+\eta\cdot\hat{f}_{||}(\eta)\right)}{\xi^2+\eta^2}\\ \hat{f}_{||}(\eta) - \frac{\eta\left(\xi\hat{f}_{\perp}(\eta)+\eta\cdot\hat{f}_{||}(\eta)\right)}{\xi^2+\eta^2}\end{bmatrix}\mathrm{d}\xi$$
$$= \frac{1}{2\sqrt{|\eta|^2-\mu}}\begin{bmatrix}\hat{f}_{\perp}(\eta) + i\frac{\sqrt{|\eta|^2-\mu}}{\mu}\left(-i\sqrt{|\eta|^2-\mu}\hat{f}_{\perp}(\eta)+\eta\cdot\hat{f}_{||}(\eta)\right)\\ \hat{f}_{||}(\eta) - \frac{\eta}{\mu}\left(-i\sqrt{|\eta|^2-\mu}\hat{f}_{\perp}(\eta)+\eta\cdot\hat{f}_{||}(\eta)\right)\end{bmatrix}$$
$$+\frac{i}{\mu}\begin{bmatrix}\frac{-i|\eta|\left(-i|\eta|\hat{f}_{\perp}(\eta)+\eta\cdot\hat{f}_{||}(\eta)\right)}{2i|\eta|}\\ \frac{\eta\left(-i|\eta|\hat{f}_{\perp}(\eta)+\eta\cdot\hat{f}_{||}(\eta)\right)}{2i|\eta|}\end{bmatrix} = \frac{1}{2\sqrt{|\eta|^2-\mu}}M_{\eta,\mu}(\hat{f}_{\perp}(\eta), \hat{f}_{||}(\eta)). \tag{B.18}$$

One find

$$M_{\eta,\mu}(\hat{f}_{\perp}, \hat{f}_{||}) = \big((1+g_1(\eta,\mu))\hat{f}_{\perp}, \hat{f}_{||} + g_2(\eta,\mu)\eta(\eta.\hat{f}_{||})\big)$$

with

$$g_1 = |\eta|\sqrt{\eta^2-\mu}\ g_2, \quad g_2(\eta,\mu) = \frac{\sqrt{\eta^2-\mu}-|\eta|}{\mu|\eta|} = \frac{-1}{|\eta|(|\eta|+\sqrt{\eta^2-\mu})}.$$

We also have

$$\lim_{x_N\to 0^-}\frac{1}{2\pi}\int \mathrm{e}^{ix_N\xi}\frac{\left(\xi\hat{f}_{\perp}(\eta)+\eta\cdot\hat{f}_{||}(\eta)\right)}{\xi^2+\eta^2}\mathrm{d}\xi = \frac{\left(-i|\eta|\hat{f}_{\perp}(\eta)+\eta\cdot\hat{f}_{||}(\eta)\right)}{2|\eta|}$$
$$= m_{\eta}(\hat{f}_{\perp}(\eta), \hat{f}_{||}(\eta)). \tag{B.19}$$

Let $a \in \mathbb{R}$, $b \in \mathbb{R}^{N-1}$, and define $h \in \mathbb{R}$ and $H = (H_1, H_2) \in \mathbb{R} \times \mathbb{R}^{N-1}$ by the system

$$\begin{cases} M_{\eta,\mu}(a,b) = H\\ m_{\eta}(a,b) = h.\end{cases} \tag{B.20}$$

It remains to prove that there exists C such that for $\Lambda \geq 1$, $|\eta| \geq (1+2\varepsilon)\Lambda$ and $|\mu| \leq \Lambda$ the following inequality holds true for solutions of (B.20)

$$|a| + |b| \leq C(|h| + |H|). \tag{B.21}$$

Since m_η and $M_{\eta,\mu}$ are homogeneous of degree zero under the action of $\mathbb{R}_+^*$ $(\eta,\mu) \mapsto (s\eta, s^2\mu)$, we may assume $|\eta| = 1$ and $|\mu| \leq c$ for some $c < 1$. Since the set $K = \{|\eta| = 1,\ |\mu| \leq c\}$ is compact, it remains to show that for any $(\eta,\mu) \in K$, the system (B.20) is injective, which is easy to verify. The proof of Lemma B.1 is complete. ☐

Appendix C. Proof of Proposition 2.4

In this section, we follow the arguments in [14]. It is sufficient to prove that for every $z \in \overline{W}_r$, there exist $D > 0$, $\nu \in]0,1[$ and a neighborhood U of z such that the following inequality holds true

$$\|v\|_{H^1(U)} \le D\, \|v\|^{1-\nu}_{H^1(Z)} \Big(\|Av\|_{L^2(Z)} + \|\partial_s v(0,x)\|_{L^2(\omega)} \Big)^{\nu}. \tag{C.1}$$

Let us begin explaining how we construct the weight function φ we need in order to apply Carleman inequalities. For a domain $U \subset \mathbb{R}^{N+1}$ and $z_0 \in U$, we will take our basic function $\psi \in C^\infty(U \setminus \{z_0\};\mathbb{R})$ verifying

$$\begin{array}{l} \lim_{z\to z_0} \psi(z) = +\infty \text{ and } \psi'(z) \neq 0 \ \forall z \in U \setminus \{z_0\}; \\ \exists c_0 > 0 \text{ such that } \{z \in U, c_0 \le \psi(z) \le c'\} \text{ is compact } \forall c' \ge c_0. \end{array} \tag{C.2}$$

We also choose a sequence of numbers satisfying

$$c_0 < c_1 < c_1' < c_2 < c_2' < c_3 < c_3' < \infty$$

and set

$$V = \{z \in U \setminus \{z_0\}, c_1 < \psi(z) < c_3'\},\ V' = \{z \in U \setminus \{z_0\}, c_1' < \psi(z) < c_3\},$$
$$V_j = \{z \in U \setminus \{z_0\}, c_j < \psi(z) < c_j'\} \subset V,\ j = 1,2,3.$$

We have $V' \subset V$ and the closure $\overline{V}$ of V in $\mathbb{R}^{N+1}$ is compact and contained in $U \setminus \{z_0\}$. Therefore, from Lemma A.4, there exists a constant $D > 0$ such that $\varphi = \mathrm{e}^{D\psi}$ satisfies Hormander's sub-ellipticity condition on $\overline{V}$ for the Laplace operator $A := -\partial^2_{ss} - \Delta$.

In the sequel, we take $\rho_j = \mathrm{e}^{Dc_j}$, $\rho_j' = \mathrm{e}^{Dc_j'}$ and χ to be a cutt-off function in $C_0^\infty(V)$ such that $\chi \equiv 1$ in a neighborhood of $\overline{V'}$. We have $\rho_j \le \varphi \le \rho_j'$ in $\overline{V}_j$ for $j = 1,2,3$.

Proof of case 1: $z = (s, x_1) \in (0, s_0) \times \omega$, with $s > 0$ small.

Fix $z_1 = (-1, x_1)$ and consider $\psi(z) = (z - z_1)^{-1}$, with $U = \mathbb{R}^{N+1}$. We have that $\frac{\partial\psi}{\partial s}(s,x) \neq 0$ for $s > -1$. We begin choosing $c_0 < 1$ such that

$$\{x \in \mathbb{R}^N;\ 1 + |x - x_1|^2 \le c_0^{-2}\} \subset \omega. \tag{C.3}$$

Next, we choose $1 < c_3 < c_3'$ so that $\overline{V}_3 \subset \{s < 0\}$ and take $c_2' = 1$ and c_1, c_1', c_2 satisfying $c_0 < c_1 < c_1' < c_2 < 1$.

The vector function χv verifies $(\chi v)(0,x) = 0$ and $\partial_s(\chi v)(0,x) = \chi(0,x)\partial_s v(0,x)$. From Theorem A.5 (applied to each component), there exist $C, h_1 > 0$ such that, for every $h \in (0, h_1]$, we have

$$h||\mathrm{e}^{\varphi/h}\chi v||^2_{L^2} + h^3||\mathrm{e}^{\varphi/h}\nabla_{s,x}(\chi v)||^2_{L^2} \le C\left(h^4||\mathrm{e}^{\varphi/h}A(\chi v)||^2_{L^2} + h^3\int \mathrm{e}^{2\varphi(0,x)/h}|\chi\partial_s v(0,x)|^2\mathrm{d}x\right). \tag{C.4}$$

Noticing that $A(\chi v) = \chi(Av) + [A,\chi]v$, $[A,\chi]$ is a first order differential operator, $[A,\chi]v$ is supported in $V_1 \cup V_3$ and that $\chi = 1$ in $V_2 \subset V'$, from (C.3) and (C.4), we get

$$\mathrm{e}^{\rho_2/h}||v||_{H^1(V_2\cap Z)} \le C'\left(\mathrm{e}^{\rho_1'/h}||v||_{H^1(Z)} + \mathrm{e}^{\rho_3'/h}(||Av||_{L^2(Z)} + ||\partial_s v(0,x)||_{L^2(\omega)})\right), \tag{C.5}$$

for some $C' > 0$ and every $h \in (0, h_1]$. Then we use the following lemma due to Robbiano [23].

Lemma C.1. *Let C_1, C_2 be positive and M_0, M_1 and M_2 be nonnegative. Assume there exists $C_3 > 0$ such that $M_0 \le C_3 M_1$ and $\delta_0 > 0$ such that*

$$M_0 \le \mathrm{e}^{-C_1\delta}M_1 + \mathrm{e}^{C_2\delta}M_2$$

for every $\delta \ge \delta_0$. Then, there exits C_0 such that

$$M_0 \le C_0 M_1^{C_2/(C_1+C_2)} M_2^{C_1/(C_1+C_2)}.$$

Indeed, since $\rho_1' < \rho_2 < \rho_3'$, defining $C_1 = \rho_2 - \rho_1'$ and $C_2 = \rho_3' - \rho_2$, we obtain

$$||v||_{H^1(V_2\cap Z)} \leq C'\Big(\mathrm{e}^{-C_1/h}||v||_{H^1(Z)} + \mathrm{e}^{C_2/h}(||Av||_{L^2(Z)} + ||\partial_s v(0,x)||_{L^2(\omega)})||)\Big), \tag{C.6}$$

for every $h \in (0, h_1]$. Hence,

$$||v||_{H^1(V_2\cap Z)} \leq 2C'||v||^{1-\delta}_{H^1(Z)}(||Av||_{L^2(Z)} + ||\partial_s v(0,x)||_{L^2(\omega)})^{\delta}, \tag{C.7}$$

with $\delta = \frac{C_1}{C_1+C_2} \in (0,1)$. The proof is finished noticing that, since $c_2 < c_2' = 1$, the set $V_2 \cap Z$ contains the points of the form (s, x_1), with s small.

Proof of case 2: $z_2 = (s_2, x_2) \in (0, s_0) \times \Omega$.

From *case* 1, there exists $z_1 = (s_1, x_1)$, s_1 small and $x_1 \in \omega$ such that z_1 verifies (C.1). Since Ω is connected, Z is connected as well and there exists a path $\gamma \in C^\infty([0,1])$ within Z such that $\gamma(0) = z_1$ and $\gamma(1) = z$. Since $\gamma([0,1])$ is compact in Z, there exists an open set U such that $\gamma([0,1]) \subset U \subset Z$ and a C^∞ diffeomorphism $\Phi : U \longrightarrow \mathbb{R}^{N+1} = \{(x_1, x_2) \in \mathbb{R}\times\mathbb{R}^N\}$ such that $\Phi(\gamma(x_1)) = (x_1, 0)$ for every $x_1 \in [0,1]$. Taking $a = -1 + \sqrt{2}$, which gives $\frac{a}{1-a^2} = \frac{1}{2}$, and a C^∞ function F over $\mathbb{R}^{N+1} \setminus \{(0,0)\}$, defined by

$$F(x_1, x_2) = \sqrt{x_1^2 + x_2^2} - a x_1.$$

We have that $F > 0$ and $\nabla F \neq 0$ in $\mathbb{R}^{N+1} \setminus \{(0,0)\}$. Moreover, for every $d > 0$, the level sets of $F(x_1, x_2) = d$ are the ellipsoids with equation

$$2a(x_1 - d/2)^2 + |x_2|^2 = d^2(1 + a/2).$$

In particular, the open set $\{F(x_1,x_2) < 1\} \cup \{(0,0)\}$ contains the segment $x_1 \in [0,1], x_2 = 0$. Let us now define a C^∞ function ψ in $U \setminus \{z_1\}$ by

$$\psi(z) = \frac{1}{F(\Phi(z))}\cdot$$

The function ψ verifies assumption (C.2) (with $z_0 = z_1$ and any $c_0 > 0$). We denote by r_1, C_1 and δ_1 the constants for which interpolation (C.1) holds for the point z_1 and choose $c_2 = 1$, c_2' large enough such that $z_2 \in V_2$ and $c_3 > c_2'$ in such a way that $\overline{V}_3 \subset B_{r_1}(z_1) \cap Z$.

From ([17], Thm. 3.5), there exist $C, h_1 > 0$ such that

$$h||\mathrm{e}^{\varphi/h}\chi v||^2_{L^2} + h^3||\mathrm{e}^{\varphi/h}\nabla_{s,x}(\chi v)||^2_{L^2} \leq Ch^4||\mathrm{e}^{\varphi/h}A(\chi v)||^2_{L^2} \tag{C.8}$$

for every $h \in (0, h_1]$. Arguing as in *case* 1, we conclude that there exists C' such that

$$\mathrm{e}^{\rho_2/h}||v||_{H^1(V_2)} \leq C'\Big(\mathrm{e}^{\rho_1'/h}||v||_{H^1(Z)} + \mathrm{e}^{\rho_3'/h}||v||_{H^1(V_3)}\Big), \tag{C.9}$$

for every $h \in (0, h_1]$. From (C.1), and the choice of c_3, there exist C_1 and $\delta_1 \in (0,1)$ such that $||v||_{H^1(V_3)} \leq C_1||v||^{1-\delta_1}_{H^1(Z)}(||Av||_{L^2(Z)} + ||\partial_s v(0,x)||_{L^2(\omega)})^{\delta_1}$. Therefore, (C.9) implies

$$\mathrm{e}^{\rho_2/h}||v||_{H^1(V_2)} \leq C'\Big(\mathrm{e}^{\rho_1'/h}||v||_{H^1(Z)} + C_1\mathrm{e}^{\rho_3'/h}||v||^{1-\delta_1}_{H^1(Z)}(||Av||_{L^2(Z)} + ||\partial_s v(0,x)||_{L^2(\omega)})^{\delta_1}\Big), \tag{C.10}$$

for every $h \in (0, h_1]$. From Lemma C.1, there exist $C_2 > 0$ and $\delta \in (0,1)$ such that

$$||v||_{H^1(V_2)} \leq C_2||v||^{1-\delta\delta_1}_{H^1(Z)}||\partial_s v(0,x)||^{\delta\delta_1}_{L^2(\omega)}. \tag{C.11}$$

Since V_2 contains a neighborhood of z_2, (C.1) is a consequence of (C.11). This finishes the proof of *case* 2, hence the proof of Proposition 2.4.

References

[1] F.W. Chaves-Silva, A hyperbolic system and the cost of the null controllability for the Stokes system. *Comput. Appl. Math.* **34** (2015) 1057–1074.

[2] J.-M. Coron and S. Guerrero, Null controllability of the N-dimensional Stokes system with N-1 scalar controls. *J. Differ. Equ.* **246** (2009) 2908–2921.

[3] M. Dimassi and J. Sjöstrand, Spectral Asymptotics in the Semi-Classical Limit, Vol. 268 of *Lecture Note Series.* Cambridge University Press, London (1999) vii+226p.

[4] S. Ervedoza and E. Zuazua, Observability of heat processes by transmutation without geometric restrictions. *Math. Control Rel. Fields* **1** (2011) 177–187.

[5] S. Ervedoza and E. Zuazua, Sharp observability estimates for heat equations. *Arch. Rational Mech. Anal.* **202** (2011) 975–1017.

[6] C. Fabre and G. Lebeau, Régularité et unicité pour le problème de Stokes. *Commun. Partial Differ. Equ.* **27** (2002) 437–475.

[7] E. Fernández-Cara and E. Zuazua, The cost of approximate controllability for heat equations: the linear case. *Adv. Differ. Equ.* **5** (2000) 465–514.

[8] E. Fernández-Cara, S. Guerrero, O. Yu. Imanuvilov and J.-P. Puel, Local exact controllability of the Navier-Stokes system. *J. Math. Pures Appl.* **83** (2004) 1501–1542.

[9] E. Fernández-Cara, S. Guerrero, O. Yu. Imanuvilov and J.-P. Puel, Some controllability results for the N-dimensional Navier-Stokes and Boussinesq system. *SIAM J. Control Optim.* **45** (2006) 146–173.

[10] A.V. Fursikov and O. Yu. Imanuvilov, Controllability of Evolution Equations, Vol. 34 of *Lecture Notes Series*. Research Institute of Mathematics, Seoul National University, Seoul (1996).

[11] O. Yu. Imanuvilov, Remarks on exact controllability for the Navier−Stokes equations. *ESAIM: COCV* **6** (2001) 39–72.

[12] O. Yu. Imanuvilov, J.-P. Puel and M. Yamamoto, Carleman estimates for parabolic equations with nonhomogeneous boundary conditions. *Chin. Ann. Math. Ser. B* **30** (2009) 333–378.

[13] O. Yu. Imanuvilov, J.-P. Puel and M. Yamamoto, Carleman estimates for second order non homogeneous parabolic equations (preprint).

[14] G. Lebeau, Introduction aux inégalités de Carleman. Séminaires et Congrès. In vol. 29. *Publications SMF* (2015) 51–92.

[15] G. Lebeau and L. Robbiano, Contrôle exact de l'équation de la chaleur. *Commun. Partial Differ. Equ.* **20** (1995) 335–356.

[16] G. Lebeau and E. Zuazua, Null controllability of a system of linear thermoelasticity. *Arch. Ration. Mech. Anal.* **141** (1998) 297–329.

[17] J. Le Rousseau and G. Lebeau, On Carleman estimates for elliptic and parabolic operators. Applications to unique continuation and control of parabolic equations. *ESAIM: COCV* **18** (2012) 712–747.

[18] A. Martinez, *An introduction to Semiclassical and Microlocal Analysis.* Springer, New York (2002).

[19] G. Métivier, Valeurs propres d'opérateurs définis par la restriction de systèmes variationnels à des sous-espaces. *J. Math. Pures Appl.* **9** (1978) 133–156.

[20] L. Miller, Geometric bounds on the growth rate of null-controllability cost for the heat equation in small time. *J. Differ. Equ.* **204** (2004) 202–226.

[21] L. Miller, The Control Transmutation Method and the cost of fast controls. *SIAM J. Control Optim.* **45** (2006) 762–772.

[22] L. Miller, A direct Lebeau-Robbiano strategy for the observability of heat-like semigroups. *Discrete Contin. Dyn. Syst. Ser. B* **14** (2010) 1465–1485.

[23] L. Robbiano, Fonction de coût et contrôle des solutions des équations hyperboliques. *Asymptot. Anal.* **10** (1995) 95–115.

[24] T.I. Seidman, How violent are fast controls: III. *J. Math. Anal. Appl.* **339** (2008) 461–468.