ESAIM: COCV 22 (2016) 1282–1324
DOI: 10.1051/cocv/2016039

ESAIM: Control, Optimisation and Calculus of Variations
www.esaim-cocv.org

A VISCOSITY METHOD FOR THE MIN-MAX CONSTRUCTION OF CLOSED GEODESICS *

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Abstract. We present a viscosity approach to the min-max construction of closed geodesics on compact Riemannian manifolds of arbitrary dimension. The existence is proved in the case of surfaces, and reduced to a topological condition in general. We also construct counter-examples in dimension 1 and 2 to the ε-regularity in the convergence procedure. Furthermore, we prove the lower semi-continuity of the index of our sequence of critical points converging towards a closed non-trivial geodesic.

Mathematics Subject Classification. 49J35, 58B20, 58E10.

Received June 6, 2016. Accepted June 7, 2016.

1. Introduction

1.1. General framework

This article intends at motivating the approach developed by the second author in [45] in the simpler case of the construction of closed geodesics. We present first the general framework of problems this method aims at tackling.

Suppose we want to construct a critical point of a C^1 function $f : X \to \mathbb{R}_+$, where X is a complete $C^{1,1}$ Finsler manifold, which we interpret as the energy of a geometric or physical problem. A critical point $x \in X$ of f is non-trivial if its energy is positive, *i.e.* if $Df(x) = 0$ and $f(x) > 0$. If $\inf f(X) = 0$, we cannot simply minimise f to search for a non-trivial critical point, so we use a so-called min-max method. Let us denote $\mathscr{P}^*(X) = \mathscr{P}(X) \setminus \{\varnothing\}$ the set of non-empty subsets of X and choose some $\mathscr{A} \subset \mathscr{P}^*(X)$. We define the min-max

$$\beta = \inf_{A\in\mathscr{A}} \sup_{x\in A} f(x).$$

Thanks of general theorem such as the “mountain pass” (see for example [47]), if $\beta < \infty$ and if the function f satisfies the Palais−Smale condition on X, under suitable assumptions on $\mathscr{A}$, β is a critical value of f. So if $\beta > 0$, we get a non-trivial critical point of f. We recall that f satisfies the Palais−Smale condition at $c \in \mathbb{R}$ if for all sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$, such that

$$f(x_n) \xrightarrow[n\to\infty]{} c, \quad \text{and} \quad Df(x_n) \xrightarrow[n\to\infty]{} 0,$$

Keywords and phrases. Geodesics, minimax problems, Finsler geometry.

* *Dedicated to Jean-Michel Coron at the occasion of his* 60th *anniversary.*

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there exists $x \in X$ and a subsequence of $\{x_n\}_{n\in\mathbb{N}}$ converging to x. In general, a lack of coerciveness can prevent the energy to verify the Palais−Smale condition. The viscosity method consists in approximating f by a function satisfying the Palais−Smale condition. If $g : X \to \mathbb{R}_+$ is C^1, we set, for all $\sigma \geq 0$,

$$f_\sigma(x) = f(x) + \sigma^2 g(x),$$

and define

$$\beta(\sigma) = \inf_{A\in\mathscr{A}} \sup_{x\in A} f_\sigma(x).$$

If for all $\sigma > 0$, f_σ verifies the Palais−Smale condition, and $\beta(\sigma) < \infty$, then we can get a critical point $x_\sigma \in X$ such that

$$f_\sigma(x_\sigma) = \beta(\sigma).$$

We can easily see that

$$\beta(\sigma) \underset{\sigma\to 0}{\longrightarrow} \beta(0) > 0,$$

and at this point if we can construct a sequence of positive numbers $\{\sigma_n\}_{n\in\mathbb{N}}$, and a sequence $\{x_{\sigma_n}\}_{n\in\mathbb{N}}$ of critical points associated to $\{f_{\sigma_n}\}_{n\in\mathbb{N}}$ such that

$$\sigma_n^2\, g(x_{\sigma_n}) \underset{n\to\infty}{\longrightarrow} 0,$$

if we manage to extract a subsequence of $\{x_{\sigma_n}\}_{n\in\mathbb{N}}$ converging in a sufficiently strong topology to an element $x \in X$, such that

$$f(x_{\sigma_n}) \underset{n\to\infty}{\longrightarrow} f(x), \quad \text{and} \quad Df(x_{\sigma_n}) \underset{n\to\infty}{\longrightarrow} Df(x),$$

then $x \in X$ will be critical point of f of non-trivial energy $\beta = \beta(0) > 0$.

One new feature of our work is the absence of ε-regularity, as the counter-examples shows in Section 10. The convergence is assured instead by the existence of a quasi-conservation law.

1.2. Construction of closed geodesics

The problem of the construction of closed geodesics on compact manifolds is an ancient problem which has stimulated great developments in the field of calculus of variations, dynamical systems [5, 21] and algebraic topology [10, 49]. After the pioneering work of Hadamard [22] and Poincaré [43], the first existence results on 2-dimensional spheres equipped with arbitrary metric were obtained by Birkhoff in 1917 [8] and in 1927 for the general case of spheres of higher dimension (we refer to [16] for a modern proof). We refer to [2, 11, 51] for historical developments, and to [10] for a more mathematical treatment of the subject.

Let (M^m, h) a compact Riemannian manifold of class C^ν ($\nu \geq 3$). Referring to the notations in the beginning of the introduction, we let $X = \mathrm{W}^{2,2}_\iota(S^1, M)$, where

$$\mathrm{W}^{2,2}_\iota(S^1, M) = \mathrm{W}^{2,2}(S^1, M) \cap \left\{u : u(t) \in M, \text{ and } \dot{u}(t) \neq 0 \text{ for all } t \in S^1\right\}.^{*} \tag{1.1}$$

Write $f = \mathfrak{L}$ the length of curve functional, such that for all $u \in \mathrm{W}^{2,2}_\iota(S^1, M)$,

$$\mathfrak{L}(u) = \int_{S^1} |\dot{u}|\, \mathrm{d}\mathscr{L}^1.$$

[*] Note that this makes sense thanks of the Sobolev imbedding $\mathrm{W}^{2,2}(S^1, M)$.

and $g=\kappa^2$, where

$$\kappa(u)=\left|\nabla_{\frac{\dot{u}}{|\dot{u}|}}\frac{\dot{u}}{|\dot{u}|}\right|$$

is the geodesic curvature of a curve $u\in \mathrm{W}^{2,2}_{\iota}(S^1,M)$. We then consider for all $\sigma\geq 0$, the energy E_σ: $\mathrm{W}^{2,2}_{\iota}(S^1,M)\to\mathbb{R}$, defined for all $u\in\mathrm{W}^{2,2}_{\iota}(S^1,M)$ by

$$E_\sigma(u)=\int_{S^1}(1+\sigma^2\kappa^2(u))|\dot{u}|\mathrm{d}\mathscr{L}^1.$$

Of course, we can replace f by the Dirichlet energy, which verifies the Palais−Smale condition: it is a classical way to construct a closed geodesic on compact manifolds (see for instance [47]). However, we are interested in the application of this method to the min-max construction of minimal surfaces, and the Dirichlet energy does not satisfy any more the Palais−Smale condition in dimension 2. Therefore it makes sense to consider first a simpler case, to see if the method works correctly, and where are the difficulties. Indeed, there are three issues that we might encounter.

Firstly, we need to construct an appropriate min-max method, giving a $\beta(0)>0$. Secondly, if $\{u_n\}_{n\in\mathbb{N}}$ is a sequence of critical points associated to $\{E_\sigma\}$ (where $\{\sigma_n\}_{n\in\mathbb{N}}$ is a sequence of positive numbers converging to 0)

$$\liminf_{n\to\infty}\int_{S^1}\sigma_n^2\kappa^2(u_n)\mathrm{d}\mathscr{L}^1\xrightarrow[n\to\infty]{}0.$$

Thirdly, passing to the limit in the Euler−Lagrange equation is delicate, as we loose the estimates on the second derivative.

The first problem can easily be solved, using basic properties of the injective radius of compact manifolds. For the second one, there exist indeed counter-examples, and we use a general technique coming from an article of Michael Struwe [46] to construct an "entropic" sequence of critical points, in the sense that

$$E_{\sigma_n}(u_n)=\beta(\sigma_n),\quad\text{and}\ \int_{S^1}\sigma_n^2\kappa^2(u_n)\mathrm{d}\mathscr{L}^1\leq\frac{1}{\log\frac{1}{\sigma_n}}\cdot$$

Finally, the limiting procedure depends on a quasi-conservation law of the Euler−Lagrange equation, corresponding to the general scheme of Noether theorem (see [25]).

We are almost in the position of stating our main result. We first recall that the index of a critical point $u\in\mathrm{W}^{2,2}_{\iota}(S^1,M)$ of E_σ ($\sigma\geq 0$ arbitrary) is the dimension of the maximal vector subspace of $\mathrm{W}^{2,2}_{u}(S^1,M)$ where the second derivative $D^2E_\sigma(u)$ is negative semi-definite.

For the definition of admissible sets and of the families of maps $\mathscr{A},\mathscr{A}_0$, we refer to Section 6.

Theorem 1.1. *Let (M^m,h) a compact Riemannian manifold of class C^ν, ($\nu\geq 3$). If there exists an admissible set $\mathscr{A}$ for $\mathrm{W}^{2,2}(S^1,M)$, then there exists a sequence of positive numbers $\{\sigma_n\}_{n\in\mathbb{N}}$ converging to 0 and a sequence of critical points $\{u_n\}_{n\in\mathbb{N}}$ associated to $\{E_{\sigma_n}\}_{n\in\mathbb{N}}$, such that*

$$\begin{aligned}
&\beta(\sigma_n)=\inf_{A_0\in\mathscr{A}_0}\sup_{u\in A_0}E_{\sigma_n}(u)<\infty,\quad \beta(0)=\inf_{A\in\mathscr{A}}\sup_{u\in A}\mathfrak{L}(u)>0,\\
&E_{\sigma_n}(u_n)=\beta(\sigma_n),\quad \sigma_n^2\int_{S^1}\kappa^2(u_n)|\dot{u}_n|\mathrm{d}\mathscr{L}^1\leq\frac{1}{\log\frac{1}{\sigma_n}},\\
&u_n\xrightarrow[n\to\infty]{\mathrm{L}^\infty}u,\quad\textit{and}\quad \dot{u}_n\xrightarrow[n\to\infty]{a.e.}\dot{u}
\end{aligned}\tag{1.2}$$

where u is a closed non-trivial C^ν geodesic of length $\beta(0)>0$. Furthermore, we have lower semi-continuity of the index, i.e.

$$\mathrm{Ind}(u)\leq\liminf_{n\to\infty}\mathrm{Ind}(u_n).$$

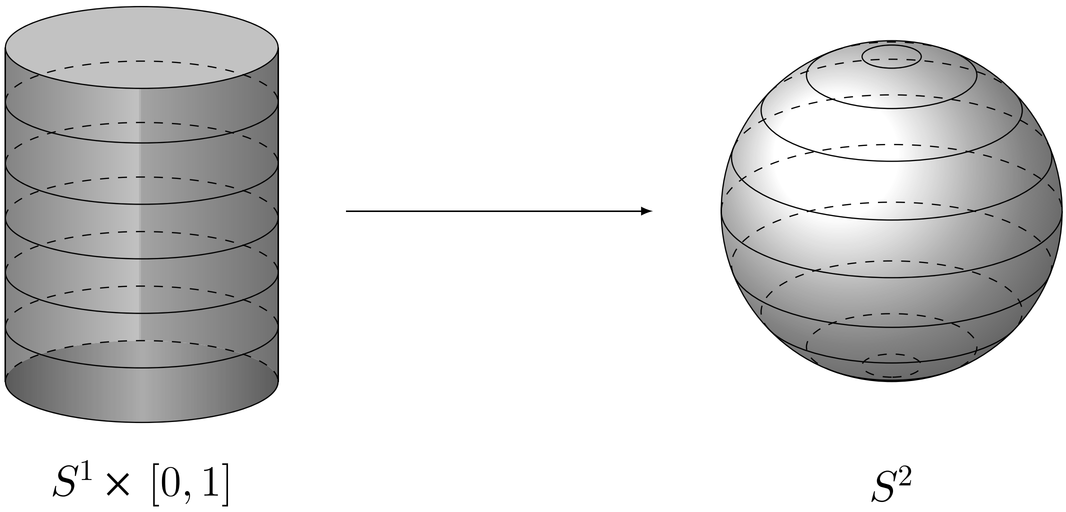

FIGURE 1. Canonical sweep-out of S^2.

Proof. The proof is made of the reunion of Theorems 6.4, 7.1, 8.1 and 9.3. In particular, the remark after Theorem 8.1 shows that the hypothesis is satisfied if M is two-dimensional. □

Methods of viscosity were already successfully used in the past in various contexts: in elliptic partial differential equations [46], hyperbolic partial differential equations [52, 53] harmonic maps from surfaces [29, 30, 48], and recently by the second author for free boundary problems [17], Yang–Mills equations [54]. One general feature in these pieces is the ε-regularity that one can get independently of σ. For example, in [29], the authors consider immersions of a Riemannian surface (M^2, h) into spheres S^k ($k \in \mathbb{N}$, and $k \geq 2$), with

$$E_\sigma(u) = \int_M \left(|\nabla u|^2 + \sigma^2 |\Delta u|^2\right) \mathrm{dvol}_g,$$

then the ε-regularity means that there exists $\varepsilon > 0$, and $\delta > 0$, such that for all $x \in M$, and $r > 0$, there exists a constant $C = C(r, \varepsilon)$ such that for all $\sigma > 0$, for all critical point u_σ of E_σ, the inequality

$$\int_{B_r(x)} \left(|\nabla u_\sigma|^2 + \sigma^2 |\Delta u_\sigma|^2\right) \mathrm{dvol}_g < \varepsilon,$$

implies that for all $k \in \mathbb{N}$, for all $0 < \alpha < 1$,

$$\|u_\sigma\|_{C^{k,\alpha}(B_{\delta r}(x))} \leq C,$$

and this ensures that the limits of $\{u_\sigma\}_{\sigma>0}$ are smooth, using classical results on the resolution of singularities for harmonic maps (see the references cited in [29]). One new phenomena is the absence of ε-regularity in our construction, as the following counter-examples shows (see Sect. 10 for the proof). We first define the canonical sweep-out of S^2 by Figure 1 (see the proof of Thm. 8.1 for a precise definition).

Proposition 1.2. *On S^2 equipped with its standard metric, let $\mathscr{A}$ the admissible set of curves given by the canonical sweep-out on S^2. There exists a sequence $\{\sigma_n\}_{n\in\mathbb{N}}$ of positive real numbers converging to 0 and a sequence of critical points $\{u_n\}_{n\in\mathbb{N}}$ of $\{E_{\sigma_n}\}_{n\in\mathbb{N}}$, and a curve $u \in \mathrm{W}^{1,2}(S^1, M)$, such that*

$$E_{\sigma_n}(u_n) \xrightarrow[n\to\infty]{} \beta(0) = \pi, \quad \mathfrak{L}(u_n) \xrightarrow[n\to\infty]{} \frac{\pi}{2}$$

and

$$u_n \xrightarrow[n\to\infty]{\mathrm{L}^\infty} u \quad \textit{strongly}, \quad \textit{and} \quad u_n \xrightarrow[n\to\infty]{\mathrm{W}^{1,2}} u \quad \textit{weakly}, \quad \dot{u}_n \not\xrightarrow[n\to\infty]{} \dot{u} \quad \textit{a.e.}$$

Furthermore, there exists a negligible subset $N \subset S^1$ *such that* $\{\dot{u}_n(t)\}_{n\in\mathbb{N}}$ *has no limit point for all* $t \in S^1 \setminus N$*, and for all open interval* $I \subset S^1$*,*

$$\mathfrak{L}(u|I) < \liminf_{n\to\infty} \mathfrak{L}(u_n|I).$$

Due to the absence of ε-regularity, we had to exploit quasi-conservation law issued from "almost Noether theorem", in order to apply technics from compensated compactness to get the strong convergence in (1.2).

Finally, we note that our approach can also be applied for the construction of non-compact manifolds admitting non-trivial closed geodesics thanks of the article of Benci and Giannoni [6].

2. Analytic and geometric preliminaries

Let (M^m, h) be a compact Riemannian manifold of dimension m greater than 2, and of class C^ν (where $\nu \geq 3$). We always assume that M is equipped with its Levi−Civita connection ∇ (we refer for definitions in Riemannian geometry to [26,31,40], and to [19] for the definitions and notations on measures). Let us recall the definition of Sobolev spaces used in the following. One possible construction is to embed isometrically M into an euclidean space $\mathbb{R}^q$ ($q \in \mathbb{N}$) thanks of Nash isometric embedding theorem, which we can apply here because M is a C^ν manifold and $\nu \geq 3$. Hence in the following, we can suppose that M is a submanifold of $\mathbb{R}^q$. Let us denote $S^1 = \mathbb{C} \cap \{z : |z| = 1\}$.

Definition 2.1. The Sobolev space $\mathrm{W}^{2,2}(S^1, M)$ is defined as follow

$$\mathrm{W}^{2,2}(S^1, M) = \mathrm{W}^{2,2}(S^1, M) \cap \left\{u : u(t) \in M \text{ for all } t \in S^1\right\}.$$

The space of Sobolev immersions $\mathrm{W}^{2,2}_\iota(S^1, M)$ is

$$\mathrm{W}^{2,2}_\iota(S^1, M) = \mathrm{W}^{2,2}(S^1, M) \cap \left\{u : \dot{u}(t) \neq 0 \text{ for all } t \in S^1\right\}.$$

Finally, the vector space of tangent vector fields along an immersion $u \in \mathrm{W}^{2,2}_\iota(S^1, M)$ is denoted by

$$\mathrm{W}^{2,2}_u(S^1, TM) = \mathrm{W}^{2,2}(S^1, TM) \cap \left\{v : v(t) \in T_{u(t)}M \text{ for all } t \in S^1\right\}.$$

Remark 2.2. All these conditions make sense, because of the Sobolev embedding theorem, there is a continuous injection $\mathrm{W}^{2,2}(S^1, \mathbb{R}^q)$ into the space $C^{1,\frac{1}{2}}(S^1, \mathbb{R}^q)$ of differentiable mappings with $\frac{1}{2}$-HÃ𝒫lder continuous derivative.

We first remark that $\mathrm{W}^{2,2}_\iota(S^1, M)$ has a natural C^2 complete Finsler manifold structure, modelled on the Hilbert space $\mathrm{W}^{2,2}(S^1, \mathbb{R}^q)$, as it is an open set of the Hilbert manifold $\mathrm{W}^{2,2}(S^1, M)$. Furthermore for all $u \in \mathrm{W}^{2,2}_\iota(S^1, M)$, the tangent space of $\mathrm{W}^{2,2}_\iota(S^1, M)$ can be identified with $\mathrm{W}^{2,2}_u(S^1, TM)$. For more precisions about the properties of $\mathrm{W}^{2,2}_\iota(S^1, M)$, we refer to the Appendix.

Definition 2.3. The covariant derivative along an immersion $u \in \mathrm{W}^{2,2}_\iota(S^1, M)$ induced by the Levi−Civita connexion ∇ with be denoted D_t when there is no ambiguity on the curve.

We recall that an immersion $u : S^1 \to M$ is said to be a *geodesic* if

$$D_t \dot{u} = 0.$$

We now make some remarks on the geodesic curvature. We first recall Fenchel's inequality [20]: for all $u \in W_\iota^{2,2}(S^1, \mathbb{R}^q)$, we have

$$\int_{S^1} \kappa_{\mathbb{R}^q}(u)|\dot{u}|\,\mathrm{d}\mathscr{L}^1 \geq 2\pi.$$

Therefore, as we suppose (M^m, h) isometrically embedded into $\mathbb{R}^q$. In particular, if $\overline{\nabla}$ is the Levi−Civita connection of $\mathbb{R}^q$ equipped with its flat metric, recalling that $\nabla = \nabla^h$ is the Levi−Civita connection of (M^m, h), we have

$$\nabla = (\overline{\nabla})^\top$$

where $\top$ is the tangent part. Therefore, as the second fundamental $\mathbb{I} = \mathbb{I}_M$ form of the immersion $M^m \hookrightarrow \mathbb{R}^q$ is defined for two tangent vectors $X, Y \in \Gamma(TM)$ by

$$\mathbb{I}(X, Y) = \mathbb{I}(Y, X) = (\overline{\nabla}_X Y)^\perp,$$

we have if $u \in \mathrm{W}_\iota^{2,2}(S^1, M)$, and $|\dot{u}| = 1$

$$\begin{aligned}\kappa_{\mathbb{R}^q}^2(u) &= |\overline{\nabla}_{\dot{u}}\dot{u}|^2 = |(\overline{\nabla}_{\dot{u}}\dot{u})^\top + (\overline{\nabla}_{\dot{u}}\dot{u})^\perp|^2 = |\nabla_{\dot{u}}\dot{u} + \mathbb{I}(\dot{u}, \dot{u})|^2\\ &= \kappa_M^2(u) + |\mathbb{I}(\dot{u}, \dot{u})|^2 = \kappa^2(u) + |\mathbb{I}(\dot{u}, \dot{u})|^2.\end{aligned}$$

Then, if $\{e_1, \dots, e_m\}$ is a local orthonormal frame in M, we define the norm of the second fundamental form by

$$|\mathbb{I}|^2 = \sum_{i,j=1}^m |\mathbb{I}(e_i, e_j)|^2$$

so by compactness of M^m, there exists a constant $0 < A_M < \infty$ such that

$$\|\mathbb{I}_M\|_{\mathrm{L}^\infty(M)} \leq A_M.$$

Then, by Fenchel's inequality, we have for all $u \in \mathrm{W}_\iota^{2,2}(S^1, M)$

$$2\pi \leq \int_{S^1} \left(\kappa^2(u) + A_M^2\right)^{\frac{1}{2}} |\dot{u}|\mathrm{d}\mathscr{L}^1$$

and by Cauchy−Schwarz inequality,

$$4\pi^2 \leq \mathfrak{L}(u)\left(\int_{S^1} (\kappa^2(u) + A_M^2)|\dot{u}|\mathrm{d}\mathscr{L}^1\right) \leq (1 + A_M^2)\mathfrak{L}(u)\int_{S^1}(1 + \kappa^2(u))|\dot{u}|\mathrm{d}\mathscr{L}^1$$

therefore for all $u \in \mathrm{W}_\iota^{2,2}(S^1, M)$

$$\frac{4\pi^2}{\mathfrak{L}(u)} \leq (1 + A_M^2)\int_{S^1}(1 + \kappa^2(u))|\dot{u}|\mathrm{d}\mathscr{L}^1. \tag{2.1}$$

3. First variation of energy

For all $\sigma \geq 0$, let $E_\sigma : \mathrm{W}_\iota^{2,2}(S^1, M) \to \mathbb{R}$ be given by

$$E_\sigma(u) = \int_{S^1}(1 + \sigma^2\kappa(u)^2)|\dot{u}|\mathrm{d}\mathscr{L}^1 \tag{3.1}$$

for all $u \in \mathrm{W}_\iota^{2,2}(S^1, M)$, where κ is the geodesic curvature defined in the preceding section. If $\sigma = 0$, then E_0 coincides with the length of curve and we note

$$\mathfrak{L}(u) = E_0(u) = \int_{S^1} |\dot{u}| \mathrm{d}\mathscr{L}^1$$

for all $u \in \mathrm{W}_\iota^{2,2}(S^1, M)$.

We will state and prove some elementary lemmas before we proceed with the derivation of the first and second variations of the energy.

Lemma 3.1. *For all $\sigma \geq 0$, the energy $E_\sigma : \mathrm{W}_\iota^{2,2}(S^1, M) \to \mathbb{R}$ is a $C^{\nu-1}$ function.*

Proof. Indeed, if $P : M \times \mathbb{R}^q \to TM$ is the orthogonal projection, then is it a $C^{\nu-1}$ function. If $F_\sigma : M \times \mathbb{R}^q \setminus \{0\} \times \mathbb{R}^q$ is the mapping defined by

$$(x, y, z) \mapsto F_\sigma(x, y, z) = \left(1 + \sigma^2 \langle P_x(z), P_x(z) \rangle_x\right) \langle y, y \rangle_x^{\frac{1}{2}},$$

then F_σ is a $C^{\nu-1}$ function. The claim is therefore a simple consequence of Lebesgue's dominated convergence theorem, as

$$E_\sigma(u) = \int_{S^1} F_\sigma(u, \dot{u}, \ddot{u}) \mathrm{d}\mathscr{L}^1.$$

This concludes the proof of the first lemma. □

We will now derive formulae for the derivatives of the curvature and other geometric quantities. A variation of a curve $u \in \mathrm{W}_\iota^{2,2}(S^1, M) \cap C^3(S^1, M)$ is a map $\gamma \in C^3(I \times S^1, M)$, such that I is an open interval of $\mathbb{R}$ containing 0, and $\gamma(0, \cdot) = u$, and for all $s \in I$, $\gamma(s, \cdot) \in \mathrm{W}_\iota^{2,2}(S^1, M)$. The variation vector field $v \in \mathrm{W}_u^{2,2}(S^1, TM) \cap C^2(S^1, M)$ is defined as

$$v = \partial_s \gamma(s, \cdot)|_{s=0}.$$

As a consequence, if $X = \mathrm{W}_\iota^{2,2}(S^1, M)$, we have

$$DE_\sigma(u) \cdot v = \langle DE_\sigma(u), v \rangle_{T^*X, TX} = \frac{\mathrm{d}}{\mathrm{d}s} E_\sigma(\gamma(s, \cdot))|_{s=0}.$$

We denote D_t (resp. D_s) the covariant derivative along the curve $t \mapsto \gamma(\cdot, t)$ (resp. $s \mapsto \gamma(s, \cdot)$). We have the following commutation result.

Lemma 3.2. *Under the afore mentioned hypothesis, we have*

$$D_t \partial_s \gamma(s, t) = D_s \partial_t \gamma(s, t)$$

for all $(s, t) \in I \times S^1$, and if $[\cdot\,, \cdot]$ is the Lie bracket, then

$$[\partial_s \gamma, \partial_t \gamma] = 0.$$

Proof. If $\gamma = (\gamma_1, \ldots, \gamma_m)$ be the local expression of γ in a local coordinates system,

$$\partial_s \gamma = \sum_{k=1}^m \partial_s \gamma_k \partial_k, \quad \partial_t \gamma = \sum_{k=1}^d \partial_t \gamma_k \partial_k$$

Thanks of the defining properties of a connexion, we have

$$D_t\partial_s\gamma = \sum_{i,j,k=1}^{m} \left(\partial^2_{ts}\,\gamma_k + \partial_s\gamma_i\,\partial_t\gamma_j\,\Gamma^k_{ij}\right)\partial_k$$
$$D_s\partial_t\gamma = \sum_{i,j,k=1}^{m} \left(\partial^2_{st}\,\gamma_k + \partial_t\gamma_i\,\partial_s\gamma_j\,\Gamma^k_{ij}\right)\partial_k$$

as the connection is symmetric, *i.e.* $\Gamma^k_{ij} = \Gamma^k_{ji}$, if suffices now to exchange the index i and j of one of the two preceding lines. And by Schwarz lemma, we have $\partial^2_{s,t}\gamma = \partial^2_{t,s}\gamma$, so the first part of the lemma is proved.

$$D_t\partial_s\gamma(s,t) = \nabla_{\partial_t\gamma(s,t)}\partial_s\gamma(s,t), \quad D_s\partial_t\gamma(s,t) = \nabla_{\partial_s\gamma(s,t)}\partial_t\gamma(s,t).$$

we deduce that

$$\begin{aligned}[\partial_s\gamma(s,t),\partial_t\gamma(s,t)] = D_t\partial_s\gamma(s,t) &= \nabla_{\partial_t\gamma(s,t)}\partial_s\gamma(s,t) - \nabla_{\partial_s\gamma(s,t)}\partial_t\gamma(s,t)\\ &= D_t\partial_s\gamma(s,t) - D_s\partial_t\gamma(s,t) = 0,\end{aligned}$$

which completes the proof of the lemma. □

We will denote in the following, if $(x,v)\in TM$,

$$|v| = \sqrt{g_x(v,v)} = \sqrt{\langle v,v\rangle_x}.$$

We now aim at calculating the first variation of the curvature.

$$\kappa(s,t) = \left|D_t\left(\frac{\partial_t\gamma(s,t)}{|\partial_t\gamma(s,t)|}\right)\frac{1}{|\partial_t\gamma(s,t)|}\right|$$

where D_t is the covariant derivative along the curve $t\mapsto\gamma(\cdot,t)$. As γ is extensive for s close enough to 0, we have

$$\kappa(s,t) = \left|\nabla_{\frac{\partial_t\gamma(s,t)}{|\partial_t\gamma(s,t)|}}\frac{\partial_t\gamma(s,t)}{|\partial_t\gamma(s,t)|}\right|.$$

To simplify notations, let us write $\gamma_t = \partial_t\gamma$, $\gamma_s = \partial_s\gamma$, and $\overline{\gamma_t} = \dfrac{\partial_t\gamma}{|\partial_t\gamma|}$.

Proposition 3.3. *Under the preceding hypothesis, we have the following identities*

1. $\partial_s|\gamma_t| = \langle\nabla_{\gamma_t}\gamma_s,\overline{\gamma_t}\rangle = \alpha|\gamma_t|$, *where* $\alpha = \left\langle\nabla_{\overline{\gamma}_t}\gamma_s,\overline{\gamma}_t\right\rangle$,
2. $[\gamma_s,\overline{\gamma_t}] = -\left\langle\nabla_{\overline{\gamma_t}}\gamma_s,\overline{\gamma_t}\right\rangle\overline{\gamma_t} = -\alpha\overline{\gamma_t}$,
3. $\partial_s\kappa^2 = 2\left\langle\nabla^2_{\overline{\gamma_t}}\gamma_s,\nabla_{\overline{\gamma_t}}\overline{\gamma_t}\right\rangle - 4\alpha\kappa^2 + 2\left\langle R(\gamma_s,\overline{\gamma_t})\overline{\gamma_t},\nabla_{\overline{\gamma_t}}\overline{\gamma_t}\right\rangle$ $(\alpha = \langle\nabla_{\overline{\gamma_t}}\gamma_s,\overline{\gamma_t}\rangle)$.

Proof.

1. We have

$$\partial_s|\gamma_t| = \frac{1}{|\gamma_t|}\langle D_s\gamma_t,\gamma_t\rangle = \langle D_t\gamma_s,\overline{\gamma}_t\rangle = \langle\nabla_{\gamma_t}\gamma_s,\overline{\gamma}_t\rangle.$$

2. Indeed, thanks of Lemma 3.2, we have

$$\begin{aligned}[\gamma_s,\overline{\gamma}_t] = \nabla_{\gamma_s}\overline{\gamma}_t - \nabla_{\overline{\gamma}_t}\gamma_s &= \nabla_{\partial_s\gamma}\frac{\partial_t\gamma}{|\partial_t\gamma|} - \nabla_{\frac{\partial_t\gamma}{|\partial_t\gamma|}}\partial_s\gamma = D_s\frac{\partial_t\gamma}{|\partial_t\gamma|} - \frac{1}{|\partial_t\gamma|}\nabla_{\partial_t\gamma}\partial_s\gamma\\ &= \frac{D_s\partial_t\gamma}{|\partial_t\gamma|} - \frac{\partial_s|\partial_t\gamma|}{|\partial_t\gamma|^2}\partial_t\gamma - \frac{D_t\partial_s\gamma}{|\partial_t\gamma|} = -\left\langle\nabla_{\overline{\gamma}_t}\gamma_s,\overline{\gamma}_t\right\rangle\overline{\gamma}_t\end{aligned}$$

3. Finally,

$$\partial_s\kappa^2 = 2\left\langle D_s\nabla_{\overline{\gamma}_t}\overline{\gamma}_t, \nabla_{\overline{\gamma}_t}\overline{\gamma}_t\right\rangle = 2\left\langle \nabla_{\gamma_s}\nabla_{\overline{\gamma}_t}\overline{\gamma}_t, \nabla_{\overline{\gamma}_t}\overline{\gamma}_t\right\rangle$$

and $\nabla_{\gamma_s}\overline{\gamma}_t = \nabla_{\overline{\gamma}_t}\gamma_s - \alpha\overline{\gamma}_t$, therefore

$$\begin{aligned}\partial_s\kappa^2 =& 2\left\langle \nabla_{\gamma_s}\nabla_{\overline{\gamma}_t}\overline{\gamma}_t - \nabla_{\overline{\gamma}_t}\nabla_{\gamma_s}\overline{\gamma}_t - \nabla_{[\gamma_s,\overline{\gamma}_t]}\overline{\gamma}_t, \nabla_{\overline{\gamma}_t}\overline{\gamma}_t\right\rangle + 2\left\langle \nabla_{\overline{\gamma}_t}\nabla_{\gamma_s}\overline{\gamma}_t, \nabla_{\overline{\gamma}_t}\overline{\gamma}_t\right\rangle \\ &+ 2\left\langle \nabla_{[\gamma_s,\overline{\gamma}_t]}\overline{\gamma}_t, \nabla_{\overline{\gamma}_t}\overline{\gamma}_t\right\rangle \\ =& 2\left\langle R(\gamma_s,\overline{\gamma}_t)\overline{\gamma}_t, \nabla_{\overline{\gamma}_t}\overline{\gamma}_t\right\rangle + 2\left\langle \nabla_{\overline{\gamma}_t}(\nabla_{\overline{\gamma}_t}\gamma_s - \alpha\overline{\gamma}_t), \nabla_{\overline{\gamma}_t}\overline{\gamma}_t\right\rangle + 2\left\langle \nabla_{-\alpha\overline{\gamma}_t}\overline{\gamma}_t, \nabla_{\overline{\gamma}_t}\overline{\gamma}_t\right\rangle \\ =& 2\left\langle R(\gamma_s,\overline{\gamma}_t)\overline{\gamma}_t, \nabla_{\overline{\gamma}_t}\overline{\gamma}_t\right\rangle - 4\alpha\kappa^2 + 2\left\langle \nabla_{\overline{\gamma}_t}\nabla_{\overline{\gamma}_t}\gamma_s, \nabla_{\overline{\gamma}_t}\overline{\gamma}_t\right\rangle - 2\left\langle dg(\overline{\gamma}_t)\overline{\gamma}_t, \nabla_{\overline{\gamma}_t}\overline{\gamma}_t\right\rangle \\ =& 2\left\langle R(\gamma_s,\overline{\gamma}_t)\overline{\gamma}_t, \nabla_{\overline{\gamma}_t}\overline{\gamma}_t\right\rangle - 4\alpha\kappa^2 + 2\left\langle \nabla_{\overline{\gamma}_t}\nabla_{\overline{\gamma}_t}\gamma_s, \nabla_{\overline{\gamma}_t}\overline{\gamma}_t\right\rangle\end{aligned}$$

as $|\overline{\gamma}_t| = 1$, we deduce that $0 = d\left\langle\overline{\gamma}_t,\overline{\gamma}_t\right\rangle\cdot\overline{\gamma}_t = 2\left\langle \nabla_{\overline{\gamma}_t}\overline{\gamma}_t, \overline{\gamma}_t\right\rangle$.

This calculation ends the proof of the proposition. □

Therefore, a standard approximation argument gives the following theorem.

Proposition 3.4. *If $u \in \mathrm{W}^{2,2}_\iota(S^1, M)$, $L = \mathfrak{L}(u)$, and $v \in \mathrm{W}^{2,2}_u(S^1, TM)$, then*

$$DE_\sigma(u)\cdot v = \int_0^L \langle D_t v, \dot{u}\rangle\,\mathrm{d}\mathscr{L}^1 + \sigma^2\int_0^L 2\left\langle D_t^2 v, D_t\dot{u}\right\rangle - 3\langle D_t v, \dot{u}\rangle\,\kappa^2(u) + 2\langle R(v,\dot{u})\dot{u}, D_t\dot{u}\rangle\,\mathrm{d}\mathscr{L}^1. \tag{3.2}$$

if R is the Riemannian curvature tensor on (M^m, g).

Proof. Thanks of the preceding lemmas, if γ is a variation of u such that $\partial_s\gamma|_{s=0} = v$, then we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}E_\sigma(\gamma(s,\cdot)) &= \int_{S^1}\sigma^2(\partial_s\kappa^2(s,t))|\partial_t\gamma(s,t)|\mathrm{d}t + \int_{S^1}(1+\sigma^2\kappa^2(s,t))\partial_s|\partial_t\gamma(s,t)|\mathrm{d}t \\ &= \sigma^2\int_0^L \left\langle \nabla_{\overline{\gamma}_t}\nabla_{\overline{\gamma}_t}\gamma_s, \nabla_{\overline{\gamma}_t}\overline{\gamma}_t\right\rangle - 4\alpha\kappa^2 + 2\langle R(\gamma_s,\gamma_t)\gamma_t, D_t\gamma_t\rangle\,\mathrm{d}\mathscr{L}^1 + \int_0^L(1+\sigma^2\kappa^2)\alpha\mathrm{d}\mathscr{L}^1 \\ &= \int_0^L\langle D_t v, \dot{u}\rangle + \sigma^2\int_0^L 2\left\langle D_t^2 v, D_t\dot{u}\right\rangle\mathrm{d}\mathscr{L}^1 - 3\langle D_t v, \dot{u}\rangle\,\kappa^2(u) + 2\langle R(v,\dot{u})\dot{u}, D_t\dot{u}\rangle\,\mathrm{d}\mathscr{L}^1.\end{aligned}$$

So we have the desired result. □

If u is a critical point of E_σ of at least class C^3, then

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}E_s(\gamma(s,\cdot)) &= \int_0^L -2\sigma^2\left\langle \nabla_{\overline{\gamma}_t}\gamma_s, \nabla^2_{\overline{\gamma}_t}\overline{\gamma}_t\right\rangle + \left\langle \nabla_{\overline{\gamma}_t}\gamma_s, (1-3\sigma^2\kappa^2)\overline{\gamma}_t\right\rangle + 2\sigma^2\left\langle U_s, R(\nabla_{\overline{\gamma}_t}\overline{\gamma}_t, \overline{\gamma}_t)\overline{\gamma}_t\right\rangle\mathrm{d}\tau \\ &= \int_0^L \left\langle 2\sigma^2\nabla^3_{\overline{\gamma}_t}\overline{\gamma}_t + \nabla_{\overline{\gamma}_t}((3\sigma^2\kappa^2-1)\overline{\gamma}_t) + 2\sigma^2 R(\nabla_{\overline{\gamma}_t}, \overline{\gamma}_t)\overline{\gamma}_t, \gamma_s\right\rangle\mathrm{d}\tau\end{aligned}$$

as $\left\langle R(\gamma_s,\overline{\gamma}_t)\overline{\gamma}_t, \nabla_{\overline{\gamma}_t}\overline{\gamma}_t\right\rangle = \left\langle \gamma_s, R(\nabla_{\overline{\gamma}_t}\overline{\gamma}_t, \overline{\gamma}_t)\overline{\gamma}_t\right\rangle$. As a consequence (3.2) is equivalent to the following Euler−Lagrange equation

$$D_t\dot{u} = \sigma^2\left\{D_t\left(2D_t^2\dot{u} + 3\kappa^2\dot{u}\right) + 2R(D_t\dot{u},\dot{u})\dot{u}\right\} \tag{3.3}$$

in the distributional sense. According to the forecoming part 7, this equation implies that u is a $C^{\nu-1}$ function.

4. Second variation of energy

We recall that the second variation or Hessian is defined as follows. Let u be a critical point of E_σ. For every $v \in \mathrm{W}^{2,2}_u(S^1, TM)$, if $\gamma : I \times S^1$ is a C^2 variation of u such that $\partial_s \gamma_{s=0} = v$, we define

$$D^2 E_\sigma(u)[v,v] = \frac{\partial^2}{\partial s^2} E_\sigma(\gamma(s,\cdot))\Big|_{s=0}$$

and this definition is independent of the variation.

Proposition 4.1. *If $u \in \mathrm{W}^{2,2}_\iota(S^1, M)$ is a critical point of E_σ, then for all $v \in \mathrm{W}^{2,2}_u(S^1, TM)$, we have*

$$\begin{aligned} D^2E_\sigma(u)[v,v] = {} & 2\sigma^2 \int_0^L |D_t^2 v|^2 + |R(v,\dot{u})\dot{u}|^2 + 2\left(4\langle D_t v, \dot{u}\rangle^2 + 2\langle D_t v, \dot{u}\rangle - |D_t v|^2 + \langle R(\dot{u}, v)v, \dot{u}\rangle\right)\kappa^2(u) \\ & - \left(\langle D_t^2 v, \dot{u}\rangle + \langle D_t v, D_t \dot{u}\rangle\right)^2 - 8\langle D_t v, \dot{u}\rangle \langle D_t^2 v, D_t \dot{u}\rangle + \langle \nabla_{\dot{u}} R(v,\dot{u})v, D_t\dot{u}\rangle \\ & + \langle \nabla_v R(v,\dot{u})\dot{u}, D_t \dot{u}\rangle + \langle R(D_t v), \dot{u})v, D_t\dot{u}\rangle - \langle R(D_t \dot{u}, v)v, D_t\dot{u}\rangle + 3\langle R(v,\dot{u})D_t v, D_t \dot{u}\rangle \\ & + \langle R(\dot{u}, D_t v)\dot{u}, D_t\dot{u}\rangle + 2\langle R(v,\dot{u})D_t v, D_t\dot{u}\rangle - 6\langle D_t v, \dot{u}\rangle \langle R(v,\dot{u})\dot{u}, D_t\dot{u}\rangle \, \mathrm{d}\mathscr{L}^1 \\ & + 4\sigma^2 \int_0^L \langle D_t v, \dot{u}\rangle \left(\langle D_t^2 v, D_t\dot{u}\rangle - 2\langle D_t v, \dot{u}\rangle \kappa^2(u) + \langle R(v,\dot{u})\dot{u}, D_t\dot{u}\rangle\right) \mathrm{d}\mathscr{L}^1 \\ & + \int_0^L \left(1 + \sigma^2\kappa^2(u)\right)\left(|D_t v|^2 - \langle D_t v, \dot{u}\rangle^2 - \langle R(\dot{u}, v)v, \dot{u}\rangle\right) \mathrm{d}\mathscr{L}^1 \end{aligned} \tag{4.1}$$

Proof. We may then choose a variation γ such that

$$\begin{cases} D_s \partial_s \gamma = 0 \\ \gamma(0,\cdot) = u \\ \partial_s \gamma|_{s=0} = v \end{cases} \tag{4.2}$$

Indeed, as u is critical point of E_σ, it is a C^2 function (as $\nu - 1 \geq 2$). The Cauchy−Lipschitz theorem asserts the existence of a local C^2 function defined on an open neighbourhood of $\{0\} \times S^1$ of this differential system.

Let us denote with a slight change in the notations $\gamma_t = \dfrac{\partial_t \gamma}{|\partial_t \gamma_t|}$, $\gamma_s = \partial_s \gamma$, $\alpha = \langle \nabla_{\gamma_t} \gamma_s, \gamma_t \rangle$. We will make constant use of the following identity

$$\nabla_{\gamma_s} \nabla_{\gamma_t} = \nabla_{\gamma_t} \nabla_{\gamma_s} + R(\gamma_s, \gamma_t) - \alpha \nabla_{\gamma_t}. \tag{4.3}$$

which is a direct consequence of 3.3, as R is defined such that

$$R(\gamma_s, \gamma_t) = \nabla_{\gamma_s}\nabla_{\gamma_t} - \nabla_{\gamma_t}\nabla_{\gamma_s} - \nabla_{[\gamma_s, \gamma_t]}.$$

As $[\gamma_s, \gamma_t] = -\alpha\gamma_t$, the preceding equation is equivalent to (4.3).

We shall also use the notations $D_t = \nabla_{\gamma_t}$, $D_s = \nabla_{\gamma_s}$. As a consequence (4.3) reads

$$D_s D_t = D_t D_s + R(\gamma_s, \gamma_t) - \langle D_t \gamma_s, \gamma_t \rangle D_t$$

with $\alpha = \langle D_t \gamma_s, \gamma_t \rangle$. Finally, one has $[\gamma_s, \gamma_t] = -\alpha \gamma_t$, so in our new notation, this gives

$$D_s \gamma_t = D_t \gamma_s - \langle D_t \gamma_s, \gamma_t \rangle \gamma_t = D_t \gamma_s - \alpha \gamma_t.$$

Recall that

$$\kappa^2 = \langle D_t \gamma_t, D_t \gamma_t \rangle .$$

We shall now proceed with the calculus of the second derivative of κ^2. By compatibility of the metric with ∇, we have

$$\partial_s^2\kappa^2 = 2\left\langle D_s^2 D_t\gamma_t, D_t\gamma_t\right\rangle + 2\left\langle D_s D_t\gamma_t, D_s D_t\gamma_t\right\rangle = 2\left\{(1)+(2)\right\}.$$

now

$$\begin{aligned}D_s D_t\gamma_t &= D_t D_s\gamma_t + R(\gamma_s,\gamma_t)\gamma_t - \left\langle D_t\gamma_s,\gamma_t\right\rangle D_t\gamma_t\\ &= D_t\left(D_t\gamma_s - \left\langle D_t\gamma_s,\gamma_t\right\rangle\gamma_t\right) + R(\gamma_s,\gamma_t)\gamma_t - \left\langle D_t\gamma_s,\gamma_t\right\rangle D_t\gamma_t\\ &= D_t^2\gamma_s - (\left\langle D_t^2\gamma_s,\gamma_t\right\rangle + \left\langle D_t\gamma_s, D_t\gamma_t\right\rangle)\gamma_t - 2\left\langle D_t\gamma_s,\gamma_t\right\rangle D_t\gamma_t + R(\gamma_s,\gamma_t)\gamma_t\\ &= (\mathrm{I}) - (\mathrm{II}) - 2(\mathrm{III}) + (\mathrm{IV}).\end{aligned}$$

We split the computation into four parts.

$$\begin{aligned}D_s(\mathrm{I}) = D_s D_t^2\gamma_s &= D_t D_s D_t\gamma_s + R(\gamma_s,\gamma_t)D_t\gamma_s - \alpha D_t^2\gamma_s\\ &= D_t\left(D_t D_s\gamma_s + R(\gamma_s,\gamma_t)\gamma_s - \alpha D_t\gamma_s\right) + R(\gamma_s,\gamma_t)D_t\gamma_s - \alpha D_t^2\gamma_s\\ &= D_t R(\gamma_s,\gamma_t)\gamma_s + R(D_t\gamma_s,\gamma_t)\gamma_s + R(\gamma_s,D_t\gamma_t)\gamma_s + R(\gamma_s,\gamma_t)D_t\gamma_s\\ &\quad - (\partial_t\alpha)D_t\gamma_s - \alpha D_t^2\gamma_s + R(\gamma_s,\gamma_t)D_t\gamma_s - \alpha D_t^2\gamma_s\\ &= D_t R(\gamma_s,\gamma_t)\gamma_s + R(D_t\gamma_s,\gamma_t)\gamma_s + R(\gamma_s,D_t\gamma_t)\gamma_s + 2R(\gamma_s,\gamma_t)D_t\gamma_s\\ &\quad - (\partial_t\alpha)D_t\gamma_s - 2\alpha D_t^2\gamma_s.\end{aligned}$$

We recall the notation $\alpha = \left\langle D_t\gamma_s,\gamma_t\right\rangle$. As $(\mathrm{II}) = \partial_t\alpha\gamma_t$,

$$\begin{aligned}D_s(\mathrm{II}) &= \partial_s\partial_t\alpha\gamma_t + \partial_t\alpha D_s\gamma_t\\ &= \partial_s\partial_t\alpha\gamma_t + \partial_t\alpha D_t\gamma_s - \alpha\partial_t\alpha\gamma_t\end{aligned}$$

Furthermore,

$$\begin{aligned}\partial_s\alpha &= \left\langle D_s D_t\gamma_s,\gamma_t\right\rangle + \left\langle D_t\gamma_s, D_s\gamma_t\right\rangle\\ &= \left\langle D_t D_s\gamma_s + R(\gamma_s,\gamma_t)\gamma_s - \alpha D_t\gamma_s,\gamma_t\right\rangle + \left\langle D_t\gamma_s, D_t\gamma_s - \alpha\gamma_t\right\rangle\\ &= |D_t\gamma_s|^2 - 2\alpha^2 - \left\langle R(\gamma_t,\gamma_s)\gamma_s,\gamma_t\right\rangle\cdot\end{aligned}$$

Recalling that $(\mathrm{III}) = \alpha D_t\gamma_t$, one has

$$\begin{aligned}D_s(\mathrm{III}) &= \left(|D_t\gamma_s|^2 - 2\alpha^2 - \left\langle R(\gamma_t,\gamma_s)\gamma_s,\gamma_t\right\rangle\right)D_t\gamma_t + \alpha D_s D_t\gamma_t\\ &= \left(|D_t\gamma_s|^2 - 2\alpha^2 - \left\langle R(\gamma_t,\gamma_s)\gamma_s,\gamma_t\right\rangle\right)D_t\gamma_t + \alpha\left\{D_t^2\gamma_s - (\partial_t\alpha)\gamma_t - 2\alpha D_t\gamma_t + R(\gamma_s,\gamma_t)\gamma_t\right\}\\ &= \left(|D_t\gamma_s|^2 - 4\alpha^2 - \left\langle R(\gamma_t,\gamma_s)\gamma_s,\gamma_t\right\rangle\right)D_t\gamma_t + \alpha\left\{D_t^2\gamma_s - (\partial_t\alpha)\gamma_t + R(\gamma_s,\gamma_t)\gamma_t\right\}.\end{aligned}$$

According to the defining properties of the Riemannian curvature tensor R, we have

$$\begin{aligned}D_s(\mathrm{IV}) = D_s(R(\gamma_s,\gamma_t)\gamma_t) &= D_s R(\gamma_s,\gamma_t)\gamma_t + R(D_s\gamma_s,\gamma_t)\gamma_t + R(\gamma_s,D_s\gamma_t)\gamma_t + R(\gamma_s,\gamma_t)D_s\gamma_t\\ &= D_s R(\gamma_s,\gamma_t)\gamma_t + R(\gamma_s,D_t\gamma_s)\gamma_t + R(\gamma_s,\gamma_t)D_t\gamma_s - 2\alpha R(\gamma_s,\gamma_t)\gamma_t\end{aligned}$$

as $D_s\gamma_t = D_t\gamma_s - \alpha\gamma_t$. If $|\gamma_t| = 1$, we have

$$\left\langle\gamma_t, D_t\gamma_t\right\rangle = 0.$$

In $s = 0$, we have

$$\left\langle D_s(\mathrm{II}), D_t\gamma_t\right\rangle = \partial_t\alpha\left\langle D_t\gamma_s, D_t\gamma_t\right\rangle = \left(\left\langle D_t^2\gamma_s,\gamma_t\right\rangle + \left\langle D_t\gamma_s, D_t\gamma_t\right\rangle\right)\left\langle D_t\gamma_s, D_t\gamma_t\right\rangle$$

and

$$\langle D_s(\mathrm{III}), D_t\gamma_t\rangle = \left(|D_t\gamma_s|^2 - 2\alpha^2 - \langle R(\gamma_t,\gamma_s)\gamma_s,\gamma_t\rangle\right)\kappa^2 + \langle D_t\gamma_s,\gamma_t\rangle \left\langle D_t^2\gamma_s, D_t\gamma_t\right\rangle.$$

The first term of the second derivative is

$$\begin{aligned}
(1) = &\langle D_tR(\gamma_s,\gamma_t)\gamma_s + R(D_t\gamma_s,\gamma_t)\gamma_s + R(\gamma_s,D_t\gamma_t)\gamma_s + 2R(\gamma_s,\gamma_t)D_t\gamma_s, D_t\gamma_t\rangle && \text{(I)}\\
&- \left(\left\langle D_t^2\gamma_s,\gamma_t\right\rangle + \langle D_t\gamma_s, D_t\gamma_t\rangle\right)\langle D_t\gamma_s, D_t\gamma_t\rangle - 2\langle D_t\gamma_s,\gamma_t\rangle\left\langle D_t^2\gamma_s, D_t\gamma_t\right\rangle && \text{(I)}\\
&- \left(\left\langle D_t^2\gamma_s,\gamma_t\right\rangle + \langle D_t\gamma_s, D_t\gamma_t\rangle\right)\langle D_t\gamma_s, D_t\gamma_t\rangle && \text{(II)}\\
&- 2\left(|D_t\gamma_s|^2 - 4\langle D_t\gamma_s,\gamma_t\rangle^2 - \langle R(\gamma_t,\gamma_s)\gamma_s,\gamma_t\rangle\right)\kappa^2 - 2\langle D_t\gamma_s,\gamma_t\rangle\left\langle D_t^2\gamma_s, D_t\gamma_t\right\rangle && \text{(III)}\\
&+ \langle D_sR(\gamma_s,\gamma_t)\gamma_t, D_t\gamma_t\rangle + \langle R(\gamma_s,D_t\gamma_s)\gamma_t, D_t\gamma_t\rangle + \langle R(\gamma_s,\gamma_t)D_t\gamma_s, D_t\gamma_t\rangle && \text{(IV)}\\
&- 2\langle D_t\gamma_s,\gamma_t\rangle\langle R(\gamma_s,\gamma_t)\gamma_t, D_t\gamma_t\rangle && \text{(IV)}
\end{aligned}$$

while

$$\begin{aligned}
(2) &= \left|D_t^2\gamma_s - (\left\langle D_t^2\gamma_s,\gamma_t\right\rangle + \langle D_t\gamma_s, D_t\gamma_t\rangle)\gamma_t - 2\langle D_t\gamma_s,\gamma_t\rangle D_t\gamma_t + R(\gamma_s,\gamma_t)\gamma_t\right|^2\\
&= |D_t^2\gamma_s|^2 + \left(\left\langle D_t^2\gamma_s,\gamma_t\right\rangle + \langle D_t\gamma_s, D_t\gamma_t\rangle\right)^2 + 4\langle D_t\gamma_s,\gamma_t\rangle\kappa^2 + |R(\gamma_s,\gamma_t)\gamma_t|^2\\
&\quad - 2\left(\left\langle D_t^2\gamma_s,\gamma_t\right\rangle + \langle D_t\gamma_s, D_t\gamma_t\rangle\right)\left\langle D_t^2\gamma_s,\gamma_t\right\rangle - 4\langle D_t\gamma_s,\gamma_t\rangle\left\langle D_t^2\gamma_s, D_t\gamma_t\right\rangle + 2\left\langle R(\gamma_s,\gamma_t)\gamma_t, D_t^2\gamma_s\right\rangle\\
&\quad - 4\langle D_t\gamma_s,\gamma_t\rangle\langle R(\gamma_s,\gamma_t)\gamma_t, D_t\gamma_t\rangle
\end{aligned}$$

We deduce that in $s=0$, we have

$$\begin{aligned}
\partial_s^2\kappa^2 = &2|D_t^2\gamma_s|^2 + 2|R(\gamma_s,\gamma_t)\gamma_t|^2 + 4\left(4\langle D_t\gamma_s,\gamma_t\rangle^2 + 2\langle D_t\gamma_s,\gamma_t\rangle - |D_t\gamma_s|^2 + \langle R(\gamma_t,\gamma_s)\gamma_s,\gamma_t\rangle\right)\kappa^2\\
&- 2\left(\left\langle D_t^2\gamma_s,\gamma_t\right\rangle + \langle D_t\gamma_s, D_t\gamma_t\rangle\right)^2 - 16\langle D_t\gamma_s,\gamma_t\rangle\left\langle D_t^2\gamma_s, D_t\gamma_t\right\rangle\\
&+ 2\langle D_tR(\gamma_s,\gamma_t)\gamma_s, D_t\gamma_t\rangle + 2\langle D_sR(\gamma_s,\gamma_t)\gamma_t, D_t\gamma_t\rangle + 2\langle R(D_t\gamma_s,\gamma_t)\gamma_s, D_t\gamma_t\rangle\\
&+ 2\langle R(\gamma_s,D_t\gamma_t)\gamma_s, D_t\gamma_t\rangle + 6\langle R(\gamma_s,\gamma_t)D_t\gamma_s, D_t\gamma_t\rangle + 2\langle R(\gamma_t,D_t\gamma_s)\gamma_t, D_t\gamma_t\rangle\\
&+ 4\left\langle R(\gamma_s,\gamma_t)\gamma_t, D_t^2\gamma_s\right\rangle - 12\langle D_t\gamma_s,\gamma_t\rangle\langle R(\gamma_s,\gamma_t)\gamma_t, D_t\gamma_t\rangle
\end{aligned}$$

Now

$$\partial_s\kappa^2 = 2\left\langle D_t^2 v, D_t\dot{u}\right\rangle - 4\langle D_t v,\dot{u}\rangle\kappa^2 + 2\langle R(v,\dot{u})\dot{u}, D_t\dot{u}\rangle,$$

and

$$\begin{aligned}
&\int_{S^1}\left(1+\sigma^2\kappa^2\right)\partial_s^2|\gamma_t|\,\mathrm{d}\mathscr{L}^1 = \int_{S^1}\left(1+\sigma^2\right)\partial_s\left\langle\nabla_{\gamma_t}\gamma_s,\frac{\gamma_t}{|\gamma_t|}\right\rangle\mathrm{d}\mathscr{L}^1\\
&= \int_{S^1}\left(1+\sigma^2\kappa^2\right)\left\{\left\langle\nabla_{\gamma_s}D_{\gamma_t}\gamma_s,\frac{\gamma_t}{|\gamma_t|}\right\rangle + \left\langle\nabla_{\gamma_t}\gamma_s,\frac{\nabla_{\gamma_s}\gamma_t}{|\gamma_t|}\right\rangle - \left\langle\nabla_{\gamma_t}\gamma_s, -\langle\nabla_{\gamma_t}\gamma_s,\gamma_t\rangle\frac{\gamma_t}{|\gamma_t|^3}\right\rangle\right\}\mathrm{d}\mathscr{L}^1\\
&= \int_{S^1}(1+\sigma^2\kappa^2)\left\{\left\langle\nabla_{\gamma_t}\nabla_{\gamma_s}\gamma_s,\frac{\gamma_t}{|\gamma_t|}\right\rangle + \left\langle R(\gamma_s,\gamma_t)\gamma_s,\frac{\gamma_t}{|\gamma_t|}\right\rangle + |\nabla_{\frac{\gamma_t}{|\gamma_t|}}\gamma_s|^2|\gamma_t| - \left\langle\nabla_{\frac{\gamma_t}{|\gamma_t|}}\gamma_s,\gamma_t\right\rangle^2|\gamma_t|\right\}\mathrm{d}\mathscr{L}^1.
\end{aligned}$$

As $D_s\gamma_s = 0$, at $s=0$, the preceding equation is equal to

$$\int_0^L\left(1+\sigma^2\kappa^2(u)\right)\left\{|D_t v|^2 - \langle D_t v,\dot{u}\rangle^2 - \langle R(\dot{u},v)v,\dot{u}\rangle\right\}\mathrm{d}\mathscr{L}^1.$$

Furthermore

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2}E_\sigma(\gamma(s,\cdot)) = \int_{S^1}\sigma^2\partial_s^2\kappa^2|\gamma_t| + 2\sigma^2\partial_s\kappa^2\partial_s|\gamma_t| + (1+\sigma^2\kappa^2)\partial_s^2|\gamma_t|\,\mathrm{d}\mathscr{L}^1$$

Finally, we deduce that

$$\begin{aligned}D^2E_\sigma(u)[v,v] =& 2\sigma^2\int_0^L |D_t^2v|^2+|R(v,\dot{u})\dot{u}|^2+2\left(4\langle D_tv,\dot{u}\rangle^2+2\langle D_tv,\dot{u}\rangle-|D_tv|^2+\langle R(\dot{u},v)v,\dot{u}\rangle\right)\kappa^2(u)\\&-\left(\langle D_t^2v,\dot{u}\rangle+\langle D_tv,D_t\dot{u}\rangle\right)^2-8\langle D_tv,\dot{u}\rangle\langle D_t^2v,D_t\dot{u}\rangle+\langle\nabla_{\dot{u}}R(v,\dot{u})v,D_t\dot{u}\rangle\\&+\langle\nabla_vR(v,\dot{u})\dot{u},D_t\dot{u}\rangle+\langle R(D_tv),\dot{u})v,D_t\dot{u}\rangle-\langle R(D_t\dot{u},v)v,D_t\dot{u}\rangle+3\langle R(v,\dot{u})D_tv,D_t\dot{u}\rangle\\&+\langle R(\dot{u},D_tv)\dot{u},D_t\dot{u}\rangle+2\langle R(v,\dot{u})D_tv,D_t\dot{u}\rangle-6\langle D_tv,\dot{u}\rangle\langle R(v,\dot{u})\dot{u},D_t\dot{u}\rangle\,\mathrm{d}\mathscr{L}^1\\&+4\sigma^2\int_0^L\langle D_tv,\dot{u}\rangle\left(\langle D_t^2v,D_t\dot{u}\rangle-2\langle D_tv,\dot{u}\rangle\kappa^2(u)+\langle R(v,\dot{u})\dot{u},D_t\dot{u}\rangle\right)\mathrm{d}\mathscr{L}^1\\&+\int_0^L\left(1+\sigma^2\kappa^2(u)\right)\left(|D_tv|^2-\langle D_tv,\dot{u}\rangle^2-\langle R(\dot{u},v)v,\dot{u}\rangle\right)\mathrm{d}\mathscr{L}^1\end{aligned}\tag{4.4}$$

which concludes the proof of the proposition. □

We will use later the result to investigate the index of the curves in Section 9.

5. Palais−Smale condition

We recall the definition of the Palais−Smale condition.

Definition 5.1. Let X be a Finsler C^ν manifold ($\nu\in\mathbb{N}\cup\{\infty\}$), and $f\in C^1(X)$. We say that f satisfies the Palais−Smale condition at the level $c\in\mathbb{R}$ if for every sequence $\{x_n\}_{n\in\mathbb{N}}\subset X$, if

$$f(x_n)\xrightarrow[n\to\infty]{}c,\quad\text{and}\quad Df(x_n)\xrightarrow[n\to\infty]{}0$$

then there exist a subsequence of $\{x_n\}_{n\in\mathbb{N}}$ (strongly) converging towards an element $x\in X$.

However, as our Lagrangian E_σ is invariant under diffeomorphisms, we only have the Palais−Smale condition up to re-parametrisation (see also [34]).

Theorem 5.2. *Let $\sigma,c>0$ two positive real numbers, and $\{u_n\}_{n\in\mathbb{N}}$ a sequence such that*

$$E_\sigma(u_n)\xrightarrow[n\to\infty]{}c,\quad DE_\sigma(u_n)\xrightarrow[n\to\infty]{}0\tag{5.1}$$

Then there exists an immersion $u\in\mathrm{W}^{2,2}_\iota(S^1,M)$ and a subsequence of $\{u_n\}_{n\in\mathbb{N}}$ (still denotes $\{u_n\}_{n\in\mathbb{N}}$), and a sequence of orientation-preserving C^1-diffeomorphisms $\{\varphi_n\}_{n\in\mathbb{N}}\subset\mathrm{Diff}_+(S^1)$ such that $u_n\circ\varphi_n\xrightarrow[n\to\infty]{}u$ strongly in $\mathrm{W}^{2,2}(S^1,M)$.

Proof. Let $\{u_n\}_{n\in\mathbb{N}}\subset\mathrm{W}^{2,2}_\iota(S^1,M)$ a sequence such that

$$E_\sigma(u_n)\xrightarrow[n\to\infty]{}c,\quad\text{and}\quad DE_\sigma(u_n)\xrightarrow[n\to\infty]{}0$$

The second hypothesis should be interpreted as

$$\lim_{n\to\infty}\sup\left\{DE_\sigma(u_n)\cdot v,\ v\in\mathrm{W}^{2,2}_{u_n}(S^1,TM)\text{ and }\|v\|_{\mathrm{W}^{2,2}_{u_n}(S^1)}\leq 1\right\}=0\tag{5.2}$$

where we recall that for all $u\in\mathrm{W}^{2,2}_\iota(S^1,M)$ and $v\in\mathrm{W}^{2,2}_u(S^1,TM)$

$$DE_\sigma(u)\cdot v=\int_{S^1}\left(2\sigma^2\langle D_t\dot{u}_n,D_t^2v\rangle+(1-3\sigma^2\kappa^2(u_n))\langle\dot{u_n},D_tv\rangle+2\sigma^2\langle R(D_t\dot{u}_n,\dot{u}_n)\dot{u}_n,D_tv\rangle\right)|\dot{u}|\mathrm{d}\mathscr{L}^1\tag{5.3}$$

For all n large enough, we have

$$E_\sigma(u_n) \leq 2c$$

so we suppose that this assumption is realised for all $n \in \mathbb{N}$. Then thanks of (2.1)

$$\frac{2\pi^2\sigma^2}{(1+A_M^2)}\frac{1}{c} \leq \mathfrak{L}(u_n) \leq 2c \quad \text{for all } n \in \mathbb{N},$$

so the length cannot degenerate and there exists a constant $\varepsilon > 0$ such that

$$L_n = \mathfrak{L}(u_n) = \int_{S^1} |\dot{u}_n| \mathrm{d}\mathscr{L}^1 \geq \frac{\varepsilon}{c\sigma^2}, \text{ and } E_\sigma(u_n) \leq 2c$$

for all n great enough. We may then assume this property for all $n \in \mathbb{N}$, and that $L_n \xrightarrow[n\to\infty]{} L > 0$. As the manifold (M, g) is compact,

$$\sup_{n\in\mathbb{N}} \|u_n\|_{\mathrm{L}^\infty(S^1)} < \infty,$$

so obviously

$$\sup_{n\in\mathbb{N}} \|u_n\|_{\mathrm{L}^2(S^1)} < \infty.$$

Then, re-parametrising $\{u_n\}_{n\in\mathbb{N}}$ at constant speed (but keeping notations), *i.e.* $|\dot{u}_n| = \mathfrak{L}(u_n)$, we have

$$\|\dot{u}_n\|_{\mathrm{L}^\infty(S^1)} \leq 2c$$

so as $L_n \leq 2c$ for all $n \in \mathbb{N}$, we have

$$\|\dot{u}_n\|_{\mathrm{L}^2(S^1)} < \infty.$$

We deduce that

$$\sup_{n\in\mathbb{N}} \|u_n\|_{\mathrm{W}^{2,2}(S^1)} < \infty. \tag{5.4}$$

Then, by Cauchy−Schwarz inequality, for all $(x, y) \in S^1 \times S^1$,

$$|u_n(x) - u_n(y)| = \left| \int_{S^1} \dot{u}_n(t) \mathrm{d}\mathscr{L}^1 t \right| \leq \sqrt{|x-y|}\, \|\dot{u}_n\|_{\mathrm{L}^2(S^1)} \cdot$$

so the sequence $\{u_n\}_{n\in\mathbb{N}}$ is equicontinuous, and likewise

$$|\dot{u}_n(x) - \dot{u}_n(y)| \leq \sqrt{|x-y|}\, \|D_t \dot{u}_n\|_{\mathrm{L}^2(S^1)}, \tag{5.5}$$

and this gives the Sobolev embedding $\mathrm{W}^{2,2}(S^1, M) \subset \mathrm{C}^{0,\frac{1}{2}}(S^1, M)$. Therefore, by Rellich−Kondrachov (resp. Arzelà−Ascoli) theorem there exists $u \in \mathrm{W}^{2,2}(S^1, M)$ such that $\{u_n\}_{n\in\mathbb{N}}$ converges weakly (resp. strongly) in $\mathrm{W}^{2,2}(S^1, M)$ (resp. $\mathrm{W}^{1,\infty}(S^1, M)$) to u. So we have

$$\begin{aligned} u_n &\rightharpoonup u \quad \text{weakly in } \mathrm{W}^{2,2}(S^1), \\ u_n &\to u \quad \text{strongly in } \mathrm{L}^\infty(S^1), \\ \dot{u}_n &\to \dot{u} \quad \text{strongly in } \mathrm{L}^\infty(S^1). \end{aligned} \tag{5.6}$$

Furthermore, as $\{\dot{u}_n\}_{n\in\mathbb{N}}$ is at constant speed parametrisation and, by uniform convergence, if $L = \mathfrak{L}(u)$, we have

$$L_n \xrightarrow[n\to\infty]{} L$$

and we deduce that u is parametrised at constant speed on S^1. In particular, $u \in \mathrm{W}^{2,2}_\iota(S^1, M)$.

In particular,

$$\lim_{n\to\infty} \|u_n - u\|_{\mathrm{W}^{1,2}(S^1)} = 0. \tag{5.7}$$

By compactness of S^1, we deduce that $\{D_t\dot{u}_n\}_{n\in\mathbb{N}}$ converge in $\mathrm{L}^2(S^1, M)$ to $D_t\dot{u}$ if and only if

$$\int_{S^1} |P(u_n)(\ddot{u}_n - \ddot{u})|^2 |\dot{u}_n| \mathrm{d}\mathscr{L}^1 \xrightarrow[n\to\infty]{} 0.$$

As $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $\mathrm{W}^{2,2}_\iota(S^1, M)$, we deduce that if $v_n = P(u_n)(u_n - u)$, then

$$\lim_{n\to\infty} DE_\sigma(u_n)\cdot v_n = 0$$

and

$$DE_\sigma(u_n)\cdot v_n = \int_{S^1} \left(2\sigma^2 \left\langle D_t\dot{u}_n, P(u_n)(\ddot{u}_n - \ddot{u}) + 2DP(u_n)(\dot{u}_n)(\dot{u}_n - \dot{u}) + D^2P(u_n)(\dot{u}_n, \dot{u}_n)(u_n - u)\right)\right\rangle \tag{5.8}$$

$$+ (1 - 3\sigma^2\kappa^2(u_n)) \langle \dot{u}_n, P(u_n)(\dot{u_n} - \dot{u}) + DP(u_n)(\dot{u}_n)(u_n - u)\rangle \tag{5.9}$$

$$+2\langle R(D_t\dot{u}_n, \dot{u}_n)\dot{u_n}, P(u_n)(u_n - u)\rangle) |\dot{u}_n| \mathrm{d}\mathscr{L}^1. \tag{5.10}$$

As (M, g) is a C^ν compact manifold, and P is $C^{\nu-1}$ and $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $\mathrm{W}^{1,\infty}(S^1, M)$. This ensures the existence of a constant $c_2 = c_2(M) > 0$ independent of $n \in \mathbb{N}$ such that

$$\begin{aligned}
\|DP(u_n)(\dot{u}_n - \dot{u})\|_{\mathrm{L}^\infty(S^1)} &\leq c_2 \|\dot{u}_n\|_{\mathrm{L}^\infty(S^1)} \|\dot{u}_n - \dot{u}\|_{\mathrm{L}^\infty(S^1)}\\
\|DP(u_n)(\dot{u}_n)(u_n - u)\|_{\mathrm{L}^\infty(S^1)} &\leq c_2 \|\dot{u}_n\|_{\mathrm{L}^\infty(S^1)} \|u_n - u\|_{\mathrm{L}^\infty(S^1)}\\
\|DP(u_n)(\dot{u}_n)(\dot{u}_n - \dot{u})\|_{\mathrm{L}^2(S^1)} &\leq c_2 \|\dot{u}_n\|_{\mathrm{L}^\infty(S^1)} \|\dot{u}_n - \dot{u}\|_{\mathrm{L}^2(S^1)}\\
\left\|D^2P(u_n)(\dot{u}_n, \dot{u}_n)(u_n - u)\right\|_{\mathrm{L}^2(S^1)} &\leq c_2 \|\dot{u}_n\|^2_{\mathrm{L}^\infty(S^1)} \|u_n - u\|_{\mathrm{L}^2(S^1)}.
\end{aligned}$$

Now $\{D_t\dot{u}_n\}_{n\in\mathbb{N}}$ is bounded in $\mathrm{L}^2(S^1, TM)$ so by Cauchy−Schwarz inequality,

$$\begin{aligned}
&\left|\int_{S^1} \left\langle D_t\dot{u}_n, 2DP(u_n)(\dot{u}_n)(\dot{u}_n - \dot{u}) + D^2P(u_n)(\dot{u}_n, \dot{u}_n)(u_n - u)\right\rangle |\dot{u}_n| \mathrm{d}\mathscr{L}^1\right|\\
&\leq c_2 \|D_t\dot{u}_n\|_{\mathrm{L}^2(S^1)} \|\dot{u}_n\|^2_{\mathrm{L}^\infty(S^1)} \left(2\|\dot{u}_n - \dot{u}\|_{\mathrm{L}^2(S^1)} + \|\dot{u}_n\|_{\mathrm{L}^\infty(S^1)} \|u_n - u\|_{\mathrm{L}^2(S^1)}\right) \xrightarrow[n\to\infty]{} 0
\end{aligned}$$

We can estimate (5.9) as follows:

$$\begin{aligned}
&\left|\int_{S^1} \left(1 + 3\sigma^2\kappa^2(u_n)\right) \langle \dot{u}_n, P(u_n)(\dot{u_n})(u_n - u) + DP(u_n)(\dot{u}_n)(\dot{u}_n - \dot{u})\rangle |\dot{u}_n| \mathrm{d}\mathscr{L}^1\right|\\
&\leq c_2 \left(1 + 3\sigma^2 \|D_t\dot{u}_n\|^2_{\mathrm{L}^2(S^1)}\right) \|\dot{u}_n\|^3_{\mathrm{L}^2(S^1)} \left(\|\dot{u}_n - \dot{u}\|_{\mathrm{L}^2(S^1)} + \|u_n - u\|_{\mathrm{L}^2(S^1)}\right) \xrightarrow[n\to\infty]{} 0.
\end{aligned}$$

Finally, the metric g is C^ν, so the $(3,1)$-curvature tensor R is $C^{\nu-2}$, its components are bounded on the compact manifold (M, g) in the following sense: if we write

$$\int_{S^1} \langle R(D_t\dot{u}_n, \dot{u}_n)\dot{u_n}, P(u_n)(u_n - u)\rangle |\dot{u}_n| \mathrm{d}\mathscr{L}^1 = \int_{S^1} \sum_{i,j,k,l=1}^n R^l_{i,j,k} D_t\dot{u}^i_n \dot{u}^j_n \dot{u}^k_n (P(u_n)(u_n - u))^l |\dot{u}_n| \mathrm{d}\mathscr{L}^1.$$

and define $\|R\|_{\mathrm{L}^\infty(M)} = \sup_{1\leq i,j,k,l\leq n} \left\|R^l_{i,j,k}\right\|_{\mathrm{L}^\infty(M)}$, then $\|R\|_{\mathrm{L}^\infty(M)} < \infty$, and

$$\begin{aligned}
\left|\int_{S^1} \langle R(D_t\dot{u}_n, \dot{u}_n)\dot{u_n}, P(u_n)(u_n - u)\rangle |\dot{u}_n| \mathrm{d}\mathscr{L}^1\right| &\leq \|R\|_{\mathrm{L}^\infty(S^1)} \|D_t\dot{u}_n\|_{\mathrm{L}^2(S^1)} \left(\int_{S^1} |\dot{u}_n|^6 |P(u_n)(u_n - u)|^2 \mathrm{d}\mathscr{L}^1\right)^{\frac{1}{2}}\\
&\leq c_2 \|R\|_{\mathrm{L}^\infty(S^1)} \|D_t\dot{u}_n\|_{\mathrm{L}^2(S^1)} \|\dot{u}_n\|^3_{\mathrm{L}^\infty(S^1)} \|u_n - u\|_{\mathrm{L}^2(S^1)} \xrightarrow[n\to\infty]{} 0.
\end{aligned}$$

The first member of (5.8) is equal to

$$\int_{S^1} \langle D_t \dot{u}_n, P(u_n)(\ddot{u}_n - \ddot{u}) \rangle \, |\dot{u}_n| \mathrm{d}\mathscr{L}^1 = \int_{S^1} \left(|P(u_n)(\ddot{u}_n - \ddot{u})|^2 + \langle P(u_n)\ddot{u}, P(u_n)(\ddot{u}_n - \ddot{u}) \rangle \right) |\dot{u}_n| \mathrm{d}\mathscr{L}^1$$

As $\{D_t \dot{u}_n\}_{n\in\mathbb{N}}$ converges weakly towards $D_t \dot{u}$ in $\mathrm{L}^2(S^1, TM)$, and $\{P(u_n)\ddot{u}\}_{n\in\mathbb{N}}$ is bounded on $\mathrm{L}^2(S^1, TM)$, we have

$$\lim_{n\to\infty} \int_{S^1} \langle D_t u, P(u_n)(\ddot{u}_n - \ddot{u}) \rangle \, |\dot{u}_n| \mathrm{d}\mathscr{L}^1 = 0.$$

We finally deduce that

$$\lim_{n\to\infty} \int_{S^1} |D_t \dot{u}_n - P(u_n)\ddot{u}|^2 |\dot{u}_n| \mathrm{d}\mathscr{L}^1 = 0,$$

so thanks of (5.6)

$$\lim_{n\to\infty} \|u_n - u\|_{\mathrm{W}^{2,2}(S^1)} = 0.$$

This concludes the proof of the theorem, up to the following remark: as on compact manifolds, the Sobolev spaces do not depend on the Riemannian metric (see [3]), and $\{u_n\}_{n\in\mathbb{N}}$ converges in the $\mathrm{W}^{2,2}$ norm to an immersion $u \in \mathrm{W}^{2,2}_\iota(S^1, M)$, then $\{u_n\}_{n\in\mathbb{N}}$ also converges to u for the Finsler distance d on $\mathrm{W}^{2,2}_\iota(S^1, M)$ (see the appendix for the definitions). □

6. Min-Max construction of adapted sequence of critical points

We aim in this section as constructing a sequence of critical points $\{u_n\}_{n\in\mathbb{N}}$ associated to $\{\sigma_n\}_{n\in\mathbb{N}}$, where $\{\sigma_n\}_{n\in\mathbb{N}}$ is a sequence of positive numbers converging to 0, such that

$$\sigma_n^2 \int_{S^1} \kappa^2(u_n) |\dot{u}_n| \mathrm{d}\mathscr{L}^1 \xrightarrow[n\to\infty]{} 0.$$

The principle of proof is adapted from a result of Michael Struwe (see [46]).

Definition 6.1. A family of non-empty sets $\mathscr{A} \subset \mathscr{P}^*(\mathrm{W}^{2,2}(S^1, M))$ is admissible if the three following conditions are realised by $\mathscr{A}$:

(1) For all $A \in \mathscr{A}$, for all $u \in \mathscr{A}$, either u is a constant curve either $u \in \mathrm{W}^{2,2}_\iota(S^1, M)$,

(2) For every homeomorphism φ of $\mathrm{W}^{2,2}_\iota(S^1, M)$ with itself isotopic to the identity map, for all $A \in \mathscr{A}$, $\varphi(A) \in A$,

(3) There exists a positive integer $k \in \mathbb{N}$ such that for all $A \in \mathscr{A}$, we can write $A = \left\{u_t^A\right\}_{t\in[0,1]^k}$, and the map

$$\begin{aligned} [0,1]^k &\to \mathrm{W}^{2,2}(S^1, M) \\ t &\mapsto u_t^A \end{aligned}$$

is continuous.

We now fix an admissible set $\mathscr{A} \subset \mathscr{P}^*\left(\mathrm{W}^{2,2}(S^1, M)\right)$ such that

$$0 < \beta(0) = \inf_{A\in\mathscr{A}} \sup \mathfrak{L}(A) < \infty.$$

We then define the family $\mathscr{A}_0 \in \mathscr{P}^*(\mathrm{W}^{2,2}_\iota(S^1, M))$ from $\mathscr{A}$ by

$$\mathscr{A}_0 = \{A_0, A \in \mathscr{A} \text{ and } A_0 \neq \varnothing\},$$

where for all $A \in \mathscr{A}$,

$$A_0 = A \cap \left\{ u : \mathfrak{L}(u) \geq \frac{\beta(0)}{2} \right\} \cdot$$

We remark that if φ is an homeomorphism of $\mathrm{W}^{2,2}_\iota(S^1, M)$ isotopic to the identity, in general, $\varphi(A_0) \neq \varphi(A)_0$. For all $\sigma > 0$, we define

$$\beta(\sigma) = \inf_{A_0 \in \mathscr{A}_0} \sup E_\sigma(A_0) < \infty.$$

Indeed, the function $\mathfrak{L}$ is continuous on $\mathrm{W}^{2,2}(S^1, M)$, and for all $A_0 \in \mathscr{A}_0$, $A_0 = \{u_t\}_{t \in I}$, where I is a closed subset of $[0,1]^k$, so the application $I \to \mathrm{W}^{2,2}_\iota(S^1, M)$, $t \to u_t$ is continuous thus

$$\sup E_\sigma(A_0) = \sup_{t \in I} E_\sigma(u_t) < \infty.$$

and $\beta(\sigma) < \infty$.

We now observe that

$$\beta(\sigma) \xrightarrow[n \to \infty]{} \beta(0).$$

To prove this claim, remark that for all $\sigma > 0$, $\beta(\sigma) \geq \beta(0)$ so by definition of $\beta(0)$, for all $\varepsilon > 0$, there exists $A \in \mathscr{A}$, such that

$$\sup_{u \in A} \mathfrak{L}(u) < \beta(0) + \varepsilon.$$

Therefore

$$\sup_{u \in A_0} E_\sigma(u) \leq \beta(0) + \varepsilon + \sigma^2 \sup_{u \in A_0} \int_{S^1} \kappa^2(u) \mathrm{d}\mathscr{L}^1 \leq \beta(0) + \varepsilon + C\sigma^2$$

so for $C\sigma^2 \leq \varepsilon$, we have

$$\beta(0) \leq \beta(\sigma) \leq \beta(0) + 2\varepsilon$$

and β is increasing, so the claim is proved.

As β is monotone, Lebesgue theorem ensures that this real function is differentiable $\mathscr{L}^1$ almost everywhere. In particular,

$$\delta = \liminf_{\sigma \to 0} \left(\sigma \log \frac{1}{\sigma} \beta'(\sigma) \right) = 0. \tag{6.1}$$

Let us argue by contradiction. If $\delta > 0$, then for all $\sigma > 0$ small enough, we have

$$\beta(\sigma) - \beta(0) \geq \int_0^\sigma \beta'(s) \mathrm{d}s \geq \frac{1}{2} \delta \int_0^\sigma \frac{\mathrm{d}s}{s \log \frac{1}{s}} = \infty,$$

which gives the contradiction.

These observations allow to introduce the following definition.

Definition 6.2. Let $\sigma > 0$ a fixed positive real number. We say that the function β satisfies the entropy condition at a point σ if it is differentiable at σ and

$$\beta'(\sigma) \leq \frac{1}{\sigma \log \frac{1}{\sigma}} \cdot \tag{6.2}$$

A formal derivation under the min-max would give a sequence of positive $\{\sigma_n\}_{n\in\mathbb{N}}$ converging to 0, and a sequence of critical points $\{u_n\}_{n\in\mathbb{N}}$ associated to $\{\sigma_n\}_{n\in\mathbb{N}}$, such that

$$\sigma_n \log \frac{1}{\sigma_n} \frac{\mathrm{d}E_{\sigma_n}}{\mathrm{d}\sigma}(u_n) \to 0 \text{ when } n \to \infty,$$

which in turn would imply that

$$\lim_{n\to\infty} \sigma_n^2 \int_{S^1} \kappa^2(u_n)|\dot{u}_n|\mathrm{d}\mathscr{L}^1 = 0.$$

The preceding intuition can be made rigorous thanks of the following proposition.

Proposition 6.3. *There exists a constant $C = C(\beta(0))$, such that for all $0 < \sigma \leq C(\beta(0))$ for which β satisfies the entropy condition (6.2), there exists a critical point $u_\sigma \in \mathrm{W}^{2,2}_\iota(S^1, M)$ of E_σ such that*

$$E_\sigma(u_\sigma) = \beta(\sigma), \quad \textit{and} \quad \partial_\sigma E_\sigma(u_\sigma) \leq \beta'(\sigma) + \frac{1}{\sigma \log \frac{1}{\sigma}}. \tag{6.3}$$

Proof.
Step 1. Estimation of the derivative of E_σ.
Let $\varepsilon > 0$ a positive fixed constant. We consider a sequence $\{\sigma_n\}_{n\in\mathbb{N}}$ strictly decreasing to $\sigma > 0$. Let $A_0 \in \mathscr{A}_0$ and $u \in A_0$, such that

$$E_\sigma(u) \geq \beta(\sigma) - \varepsilon(\sigma_n - \sigma)$$

and

$$E_{\sigma_n}(u) \leq \beta(\sigma_n) + \varepsilon(\sigma_n - \sigma).$$

Such a pair (u, A_0) always exists, for n large enough. As β is differentiable at σ, we have

$$\beta(\sigma_n) \leq \beta(\sigma) + (\beta'(\sigma) + \varepsilon)(\sigma_n - \sigma)$$

for n large enough, from which we deduce that

$$\beta(\sigma) - \varepsilon(\sigma_n - \sigma) \leq E_\sigma(u) \leq E_{\sigma_n}(u) \leq \beta(\sigma) + (\beta'(\sigma) + 2\varepsilon)(\sigma_n - \sigma) \tag{6.4}$$

If u satisfies (6.4), then

$$\frac{E_{\sigma_n}(u) - E_\sigma(u)}{\sigma_n - \sigma} \leq \beta'(\sigma) + 3\varepsilon$$

so according to the mean value theorem, there exists $\sigma' \in [\sigma, \sigma_n]$, such that

$$\partial_\sigma E_{\sigma'}(u) \leq \beta'(\sigma) + 3\varepsilon.$$

But

$$\partial_\sigma E_{\sigma'}(u) = \int_{S^1} 2\sigma' \kappa^2(u)|\dot{u}|\mathrm{d}\mathscr{L}^1 = \frac{\sigma'}{\sigma}\partial_\sigma E_\sigma(u)$$

so for all $u \in \mathrm{W}^{2,2}_\iota(S^1, M)$ satisfying the inequalities (6.4),

$$\partial_\sigma E_\sigma(u) \leq \beta'(\sigma) + 3\varepsilon. \tag{6.5}$$

Step 2. Existence of almost Palais−Smale sequences.

We want to show that there exists a sequence $\{u_n\}_{n\in\mathbb{N}}$ satisfying (6.4) and such that

$$\|DE_{\sigma_n}(u_n)\| \underset{n\to\infty}{\longrightarrow} 0 \tag{6.6}$$

We shall be careful to distinguish this condition from the Palais−Smale condition for E_σ, but we will show in the next step that it implies Palais−Smale condition for E_σ.

We argue by contradiction, supposing the existence of a positive constant $\delta > 0$ such that for all immersion $u \in \mathrm{W}^{2,2}_\iota(S^1, M)$ satisfying (6.4), we have for n large enough

$$\|DE_{\sigma_n}(u)\| \geq \delta.$$

Let X_n^0 a pseudo-gradient vector field (see [47].) for E_{σ_n}, *i.e.* a locally Lipschitz bounded function $X_n^0 : \mathrm{W}^{2,2}_\iota(S^1, M) \to T\mathrm{W}^{2,2}_\iota(S^1, M)$, such that for all $w \in \mathrm{W}^{2,2}_\iota(S^1, M)$ such that $DE_{\sigma_n}(w) \neq 0$,

$$\|X_n^0(w)\| < 2\min\left\{\|DE_{\sigma_n}(w)\|, 1\right\}$$
$$DE_{\sigma_n}(w)\cdot X_n^0(w) > \min\left\{\|DE_{\sigma_n}(w)\|, 1\right\}\|DE_\sigma(w)\|.$$

Let $\psi \in \mathscr{D}(\mathbb{R})$ a positive non-decreasing cut-off function such that $0 \leq \psi \leq 1$, $\operatorname{supp}\psi \subset \mathbb{R}_+$, and $\psi = 1$ on $[1, \infty[$. We define * for all $n \in \mathbb{N}$,

$$\psi_n(u) = \psi\left(\frac{E_\sigma(u) - (\beta(\sigma) - \varepsilon(\sigma_n - \sigma))}{\varepsilon(\sigma_n - \sigma)}\right)\psi\left(\frac{4}{\beta(0)}\frac{\mathrm{e}^{-\frac{4}{\beta(0)}}}{\sigma}\left(\mathfrak{L}(u) - \frac{\beta(0)}{2}\right)\right).$$

Let φ_n the global flow associated to $-X_n = -\psi_n X_n^0$, defined by

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\varphi_n^t(u) = -X_n(\varphi_n^t(u)) \\ \quad\varphi_n^0(u) = u \end{cases} \tag{6.7}$$

Note that φ_n is C^1 from respect of the first variable, and that for all $t \in \mathbb{R}_+$, $\varphi_n^t : \mathrm{W}^{2,2}_\iota(S^1, M) \to \mathrm{W}^{2,2}_\iota(S^1, M)$ is a locally Lipschitz homeomorphism. We remark that $\mathscr{A}_0$ is invariant under the action of φ_n, and that for all $A_0 \in \mathscr{A}_0$, for all $t \geq 0$, $\varphi_n^t(A_0) = \varphi_n^t(A_0)_0$. We have

$$\frac{\mathrm{d}}{\mathrm{d}t}E(\varphi_n^t(u)) = -\psi_n(u)DE_{\sigma_n}(u)\cdot X_n(u) \leq -\delta\psi_n(u) \leq 0$$

We would like to show that $t \mapsto E_\sigma(\varphi_n^t(u))$ is also decreasing. Consider, $u \in \mathrm{W}^{2,2}_\iota(S^1, M)$, $v \in \mathrm{W}^{2,2}_u(S^1, TM)$,

$$\begin{aligned} DE_{\sigma_n}(u)\cdot v &= \int_0^L \left(2\sigma_n^2\left\langle D_t\dot{u}, D_t^2 v\right\rangle - (3\sigma^2\kappa(u)^2 - 1)\left\langle \dot{u}, D_t v\right\rangle + 2\sigma_n^2\langle R(D_t\dot{u}, \dot{u})\dot{u}, v\rangle\right)\mathrm{d}\mathscr{L}^1 \\ &= DE_\sigma(u)\cdot v + (\sigma_n^2 - \sigma^2)\int_0^L 2\left\langle D_t\dot{u}, D_t^2 v\right\rangle - 3\kappa(u)^2\left\langle \dot{u}, D_t v\right\rangle + 2\langle R(D_t\dot{u}, \dot{u})\dot{u}, v\rangle\mathrm{d}\mathscr{L}^1 \end{aligned}$$

where $L = \mathfrak{L}(u)$ (recall that the arc-length parametrization where $|\dot{u}| = 1$ is possible because our Lagrangian is invariant under diffeomorphism). As a consequence, we have

$$\begin{aligned} |DE_{\sigma_n}(u)\cdot v - DE_\sigma(u)\cdot v| \leq (\sigma_n^2 - \sigma^2)\Big\{ & 2\left\|D_t\dot{u}\right\|_{\mathrm{L}^2([0,L])}\left\|D_t^2 v\right\|_{\mathrm{L}^2([0,L])} + 3\left\|D_t v\right\|_{\mathrm{L}^\infty([0,L])}\left\|D_t\dot{u}\right\|^2_{\mathrm{L}^2([0,L])} \\ & + 2\left\|R\right\|_{\mathrm{L}^\infty(M)}\left\|D_t\dot{u}\right\|_{\mathrm{L}^2([0,L])}\left\|v\right\|_{\mathrm{L}^\infty([0,L])}\Big\} \\ \leq C(u)(\sigma_n^2 - \sigma^2)\|v\|_{\mathrm{W}^{2,2}_u(S^1,TM)}. \end{aligned}$$

*Note that we use a different cut-off function from the original paper [46].

Furthermore, the proof of Theorem 5.2 (where we prove Palais−Smale condition), shows the existence of a continuous function $f_M : \mathbb{R}_+^* \times \mathbb{R}_+ \to \mathbb{R}_+$ increasing in each parameter, depending only on (M^m, g) such that

$$C(u) \leq f_M(\sigma_n, E_{\sigma_n}(u))$$

so for all u satisfying (6.4),

$$C(u) \leq f_M(\sigma_n, \beta(\sigma) + (\beta'(\sigma) + \varepsilon)(\sigma_n - \sigma)) \tag{6.8}$$

therefore $C(u)$ is uniformly bounded by a positive constant independent of u. We deduce that

$$\sup\left\{|(DE_{\sigma_n}(u) - DE_\sigma(u)) \cdot v|\,, \|v\|_{\mathrm{W}^{2,2}_u(S^1,TM)} \leq 1\right\} \underset{n\to\infty}{\longrightarrow} 0. \tag{6.9}$$

Now we can estimate the derivative of $t \mapsto E_\sigma(\varphi_n^t(u))$ as follows:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E_\sigma(\varphi_n^t(u)) &= DE_\sigma(\varphi_n^t(u)) \cdot X_n(\varphi_n^t(u))\\ &\leq -\psi_n(\varphi_n^t(u))DE_{\sigma_n}(\varphi_n^t(u)) \cdot X_n^0(\varphi_n^t(u)) + C(\varphi_n^t(u))(\sigma_n^2 - \sigma^2)\|X_n^0(\varphi_n^t(u))\|_{\mathrm{W}^{2,2}_{\varphi_n^t(u)}(S^1,TM)}\\ &\leq -2\psi_n(\varphi_n^t(u))\delta + 2C(\varphi_n^t(u))(\sigma_n^2 - \sigma^2)\end{aligned} \tag{6.10}$$

For all $n \in \mathbb{N}$, let us a fix an element $A_n \in \mathscr{A}_0$ such that

$$\sup_{u\in A_n} E_{\sigma_n}(u) \leq \beta(\sigma_n) + \varepsilon(\sigma_n - \sigma).$$

For all $u \in A_n$, the map $t \mapsto E_{\sigma_n}(\varphi_n^t(u))$ is decreasing, so for all $t \geq 0$, according to (6.8),

$$\begin{aligned}E_{\sigma_n}(\varphi_n^t(u)) &\leq E_{\sigma_n}(u) \leq \beta(\sigma_n) + \varepsilon(\sigma_n - \sigma)\\ C(\varphi_n^t(u)) &\leq f(\sigma_n, E_{\sigma_n}(\varphi_n^t(u)) \leq f(\sigma_0, \beta(\sigma) + (\beta'(\sigma) + 2\varepsilon)(\sigma_0 - \sigma))\end{aligned}$$

By invariance of $\mathscr{A}_0$ under the action of the semi-flow $\{\varphi_n^t\}_{t\geq 0}$, for all $t \geq 0$, we define

$$B_{A_n}(t) = \sup_{u\in A_n} E_\sigma(\varphi_n^t(u)) \geq \beta(\sigma)$$

and $B_{A_n}(t)$ is attained only at points $u_n^t = \varphi_n^t(u)$ satisfying (6.4), and for such a u_n^t, we have

$$\beta(\sigma) - \varepsilon(\sigma_n - \sigma) \leq E_\sigma(u_n^t) \leq E_{\sigma_n}(u_n^t) \leq \beta(\sigma) + (\beta'(\sigma) + 2\varepsilon)(\sigma_n - \sigma).$$

Furthermore,

$$\partial_\sigma E_\sigma(u_n^t) \leq \beta'(\sigma) + 3\varepsilon,$$

so if we choose $\varepsilon = \left(8\sigma \log\dfrac{1}{\sigma}\right)^{-1}$, as β satisfies the entropy condition (6.2) at σ, we have by (6.5)

$$\int_{S^1} \sigma^2\kappa^2(u_n^t)|\dot{u_n^t}|\mathrm{d}\mathscr{L}^1 \leq \frac{7}{8}\frac{1}{\log\frac{1}{\sigma}},$$

so for all $\sigma_n - \sigma \leq \dfrac{1}{8}$

$$\begin{aligned}\mathfrak{L}(u_n^t) &\geq \beta(\sigma) - \varepsilon(\sigma_n - \sigma) - \int_{S^1} \sigma^2 \kappa^2(u_n^t)|\dot{u_n^t}|\mathrm{d}\mathscr{L}^1 \\ &\geq \beta(0) - \frac{1}{\log \frac{1}{\sigma}} \\ &\geq \frac{3}{4}\beta(0)\end{aligned}$$

for $\sigma \leq C(\beta(0)) = \mathrm{e}^{-\frac{4}{\beta(0)}}$. Therefore, for all $\sigma \leq C(\beta(0))$, and n large enough such that $8(\sigma_n - \sigma) \leq 1$, we have $\psi_n(u_n^t) = 1$, so thanks of (6.10), if n is large enough,

$$\frac{\mathrm{d}}{\mathrm{d}t} B_{A_n}(t) \leq -\delta \tag{6.11}$$

thus

$$B_{A_n}(t) \leq \beta(\sigma_n) + \varepsilon(\sigma_n - \sigma) - \delta t$$

so for t large enough, $B_{A_n}(t) < \beta(\sigma)$, contradicting the definition of $\beta(\sigma)$.

Step 3. Convergence and conclusion.

Thanks of Step 2, we can choose a sequence $\{u_n\}_{n\in\mathbb{N}} \in \mathrm{W}_\iota^{2,2}(S^1, M)$ satisfying (6.4), and such that

$$\lim_{n\to\infty} \|DE_{\sigma_n}(u_n)\| = 0$$

Furthermore, we note by (6.5) and the proof of second step that for n large enough, we have

$$\mathfrak{L}(u_n) \geq \frac{3}{4}\beta(0).$$

So (6.9) gives

$$\sup_{n\in\mathbb{N}} E_\sigma(u_n) < \infty, \quad \text{and} \quad \|DE_\sigma(u_n)\| \underset{n\to\infty}{\longrightarrow} 0 \quad \inf_{n\in\mathbb{N}} \mathfrak{L}(u_n) > 0$$

As a consequence, $\{u_n\}_{n\in\mathbb{N}}$ is a Palais$-$Smale sequence for E_σ, we can suppose thanks of Theorem 5.2 that there exists $u \in \mathrm{W}_\iota^{2,2}(S^1, M)$ such that

$$u_n \underset{n\to\infty}{\longrightarrow} u \quad \text{strongly in } \mathrm{W}_\iota^{2,2}(S^1, M)$$

In particular, thanks of (6.4), we have

$$\beta(\sigma) = \lim_{n\to\infty} E_{\sigma_n}(u_n) = E_\sigma(u)$$

and

$$\partial_\sigma E_\sigma(u) \leq \liminf_{n\to\infty} \partial_\sigma E_{\sigma_n}(u) \leq \beta'(\sigma) + 3\varepsilon,$$

which concludes the proof of the proposition. □

We now come to the main result of this section.

Theorem 6.4. *There exists a sequence $\{\sigma_n\}_{n\in\mathbb{N}}$ of positive numbers converging to 0, and a sequence of critical points $\{u_n\}_{n\in\mathbb{N}}$ of $\{E_{\sigma_n}\}_{n\in\mathbb{N}}$ such that*

$$\beta(0) \leq E_{\sigma_n}(u_n) = \beta(\sigma_n), \quad \textit{and} \quad \lim_{n\to\infty} \sigma_n^2 \int_{S^1} \kappa^2(u_n)|\dot{u}_n|\mathrm{d}\mathscr{L}^1 = 0.$$

Proof. Choosing a sequence $\{\sigma_n\}_{n\in\mathbb{N}}$ converging to 0 such that for all $n \in \mathbb{N}$, the function β satisfies the entropy condition (6.2) at σ_n (which is possible as β is differentiable $\mathscr{L}^1$ almost everywhere and satisfies (6.1)), the theorem is now an easy consequence of the preceding proposition. □

7. Limiting procedure

Theorem 7.1. *Let (M^m,h) a Riemannian compact manifold of class C^ν ($\nu\geq 3$), such that there exist an admissible subset $\mathscr{A}\subset\mathscr{P}^*(\mathrm{W}^{2,2}(S^1,M))$, and define $\mathscr{A}_0$ as. For all $\sigma\geq 0$, we define*

$$\beta(\sigma)=\inf_{A_0\in\mathscr{A}_0}\sup_{u\in A}E_\sigma(u). \tag{7.1}$$

If for σ small enough, $\beta(\sigma)<\infty$ and $\beta(0)>0$, there exists a sequence $\{\sigma_n\}_{n\in\mathbb{N}}$ of positive numbers converging to 0, verifying

$$E_{\sigma_n}(u_n)=\beta(\sigma_n),\quad \sigma_n^2\int_{S^1}\kappa^2(u_n)\mathrm{d}\mathscr{L}^1\leq\frac{1}{\log\frac{1}{\sigma_n}}$$

and a closed non-trivial geodesic $u:S^1\to M$ such that $\{u_n\}_{n\in\mathbb{N}}$ converges to u strongly in $\mathrm{L}^\infty(S^1,M)$ and $\{\dot{u}_n\}_{n\in\mathbb{N}}$ converge to $\dot{u}$ almost everywhere.

Proof.
Step 1. Quasi-conservation law and length convergence.

Let $\{u_n\}_{n\in\mathbb{N}}$ a sequence given by the Theorem 6.4, in arc-length parametrization. We define, for all $n\in\mathbb{N}$, $L_n=\mathfrak{L}(u_n)$. Let $\{v_n\}_{n\in\mathbb{N}}$ defined by

$$v_n=\dot{u}_n-\sigma_n^2(2D_t^2\dot{u}_n+3\kappa^2(u_n)\dot{u}_n).$$

A priori, v_n belongs to the dual of $\mathrm{W}^{2,2}_{u_n}(S^1,TM)$. However, thanks of (3.3), we have

$$D_tv_n=2\sigma_n^2R(D_t\dot{u}_n,\dot{u}_n)\dot{u}_n\in\mathrm{L}^2(S^1).$$

Thus $v_n\in\mathrm{W}^{1,2}_{u_n}(S^1,TM)$, and

$$\begin{aligned}\langle\dot{u}_n,v_n\rangle&=1-\sigma_n^2\left(2\left\langle D_t^2\dot{u}_n,\dot{u}_n\right\rangle+3\kappa^2(u_n)\right) &(7.2)\\ &=1-\sigma_n^2\kappa^2(u_n) &(7.3)\end{aligned}$$

as $|\dot{u}_n|=1$, so $0=2\langle D_t\dot{u}_n,\dot{u}_n\rangle$, and we have

$$0=\left\langle D_t^2\dot{u}_n,\dot{u}_n\right\rangle+\langle D_t\dot{u}_n,D_t\dot{u}_n\rangle=\left\langle D_t^2\dot{u}_n,\dot{u}_n\right\rangle+\kappa^2(u_n).$$

Remark that $v_n\in\mathrm{W}^{1,2}([0,L_n])$ implies that u_n is in $C^{\nu-1}([0,L_n],M)$. Indeed,

$$2\sigma_n^2D_t^2\dot{u}_n=\dot{u}_n-v_n-3\sigma_n^2\kappa^2(u_n)\dot{u}_n\in\mathrm{L}^1([0,L_n])$$

so $D_t^2\dot{u}_n\in L^1([0,L_n])$, and $u\in\mathrm{W}^{3,1}([0,L_n])$. An immediate bootstrap argument implies that $u_n\in\mathrm{W}^{\nu,1}(S^1,M)\subset C^{\nu-1}(S^1,M)$ and $v_n\in\mathrm{W}^{\nu-1,1}(S^1,M)\subset C^{\nu-2}(S^1,M)$. Furthermore, $D_tv_n\underset{n\to\infty}{\longrightarrow}0$ in $\mathrm{L}^2(S^1)$. Indeed (recall $L_n=\mathfrak{L}(u_n)$),

$$\int_0^{L_n}\sigma_n^2|R(D_t\dot{u}_n,\dot{u}_n)\dot{u}_n|^2\mathrm{d}\mathscr{L}^1\leq\|R\|_{\mathrm{L}^\infty(M)}\int_0^{L_n}\sigma_n^2|D_t\dot{u}_n|^2\mathrm{d}\mathscr{L}^1=\|R\|_{\mathrm{L}^\infty(M)}\int_{S^1}\kappa^2(u_n)|\dot{u}_n|\mathrm{d}\mathscr{L}^1,$$

We deduce that

$$\|D_tv_n\|_{\mathrm{L}^2(S^1)}\leq 2\,\|R\|_{\mathrm{L}^\infty(M)}\,\sigma_n\left(\sigma_n^2\int_{S^1}\kappa^2(u_n)|\dot{u}_n|\mathrm{d}\mathscr{L}^1\right)^{\frac{1}{2}}\underset{n\to\infty}{\longrightarrow}0$$

In particular, there exist $\overline{v} \in M$ such that

$$\|v_n - \overline{v}\|_{\mathrm{L}^\infty(S^1)} \underset{n\to\infty}{\longrightarrow} 0.$$

If we set $\overline{v_n} = \dfrac{1}{2\pi}\displaystyle\int_{S^1} v_n \mathrm{d}\mathscr{L}^1$, this is equivalent to

$$\|v_n - \overline{v_n}\|_{\mathrm{L}^\infty(S^1)} \underset{n\to\infty}{\longrightarrow} 0,$$

which in turn implies that

$$\begin{aligned}\int_0^{L_n} v_n \cdot \overline{v_n}\mathrm{d}\mathscr{L}^1 &= \int_0^{L_n} |\overline{v_n}|^2 \mathrm{d}\mathscr{L}^1 + \int_0^{L_n} (v_n - \overline{v_n}) \cdot \overline{v_n}\mathrm{d}\mathscr{L}^1 \\ &= L_n|\overline{v_n}|^2 + \int_0^{L_n} (v_n - \overline{v_n}) \cdot \overline{v_n}\mathrm{d}\mathscr{L}^1\end{aligned}$$

and

$$\varepsilon_n = \int_0^{L_n} (v_n - \overline{v_n}) \cdot \overline{v_n}\mathrm{d}\mathscr{L}^1 \leq L_n|\overline{v_n}|\,\|v_n - \overline{v_n}\|_{\mathrm{L}^\infty(S^1)} \underset{n\to\infty}{\longrightarrow} 0.$$

On the other hand,

$$\int_0^{L_n} v_n \cdot \overline{v_n}\mathrm{d}\mathscr{L}^1 = \int_0^{L_n} \dot{u}_n \cdot \overline{v_n}\mathrm{d}\mathscr{L}^1 - 3\sigma_n^2 \int_0^{L_n} \kappa^2(u_n)\dot{u_n} \cdot \overline{v_n}\mathrm{d}\mathscr{L}^1 \leq L_n|v_n| + 3\sigma_n^2 \int_0^{L_n} \kappa^2(u_n)\mathrm{d}\mathscr{L}^1|v_n|$$

so

$$L_n|\overline{v_n}|^2 + \varepsilon_n \leq L_n|\overline{v_n}| + 3\sigma_n^2 \int_{S^1} \kappa^2(u_n)|\dot{u}_n|\mathrm{d}\mathscr{L}^1.$$

Now, $L_n \geq \beta(0)$ pour tout $n \in \mathbb{N}$, so

$$|\overline{v_n}| \leq 1 + \frac{3}{\beta(0)}\sigma_n^2 \int_{S^1} \kappa^2(u_n)|\dot{u}_n|\mathrm{d}\mathscr{L}^1 \underset{n\to\infty}{\longrightarrow} 1.$$

And we get

$$|\overline{v}| = \lim_{n\to\infty} \frac{1}{L_n}\int_0^{L_n} |v_n|\mathrm{d}\mathscr{L}^1 \leq 1. \tag{7.4}$$

Step 2. Weak convergence

The sequence $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $\mathrm{W}^{1,\infty}(S^1, M)$, as (M, g) is compact, and $\{L_n\}_{n\in\mathbb{N}}$ is bounded. Therefore Arzelà–Ascoli and Banach–Alaoglu theorems imply that we can extract a subsequence from $\{u_n\}_{n\in\mathbb{N}}$ (which is still denoted $\{u_n\}_{n\in\mathbb{N}}$), such that $\{u_n\}_{n\in\mathbb{N}}$ in L^∞ and weakly-* to a function $u \in \mathrm{W}^{1,\infty}([0, L], M)$. In particular $\{\dot{u}_n\}_{n\in\mathbb{N}}$ converges almost everywhere to $\dot{u}$, for all interval I such that for n large enough, $I \subset [0, L_n]$, we have

$$\int_I \langle \dot{u}_n, \overline{v}_n\rangle\,\mathrm{d}\mathscr{L}^1 \underset{n\to\infty}{\longrightarrow} \int_I \langle \dot{u}, \overline{v}\rangle\,\mathrm{d}\mathscr{L}^1$$

and according to (7.2),

$$\frac{1}{\mathscr{L}^1(I)}\int_I \langle \dot{u}_n, \overline{v}_n\rangle = 1 - \frac{1}{\mathscr{L}^1(I)}\int_I \sigma_n^2\kappa^2(u_n)\mathrm{d}\mathscr{L}^1 \underset{n\to\infty}{\longrightarrow} 1.$$

Furthermore, as $|\dot{u}_n| = 1$, and $\{\dot{u}_n\}_{n\in\mathbb{N}}$ converges almost everywhere to $\dot{u}$ so $|\dot{u}| \leq 1$. According to (7.4), $|\overline{v}| \leq 1$, so thanks of Cauchy−Schwarz inequality, we have $|\dot{u}| = 1$, and $|\overline{v}| = 1$. We deduce that

$$L = \int_{S^1} |\dot{u}| \mathrm{d}\mathscr{L}^1 = \lim_{n\to\infty} L_n \geq \beta(0),$$

As $E_{\sigma_n}(u_n) = \beta(\sigma_n)$, and $\beta(\sigma_n) \xrightarrow[n\to\infty]{} \beta(0)$, we get $L = \beta(0)$. Indeed,

$$\beta(0) = \beta(\sigma_n) + o(1) = L_n + \int_{S^1} \sigma_n^2 \kappa^2(u_n)|\dot{u}_n| \mathrm{d}\mathscr{L}^1 + o(1),$$

so $L_n \xrightarrow[n\to\infty]{} \beta(0)$.

Step 3. Limiting equation.

We wish now to pass to the limit in the Euler−Lagrange equation. We need the following technical lemma, stated separately for the sake of clarity.

Lemma 7.2. *Let $v \in \mathrm{W}^{2,2}_u(S^1, TM)$, and for all $n \in \mathbb{N}$, $v_n = P(u_n)v \in \mathrm{W}^{2,2}_{u_n}(S^1, TM)$, where $P(u_n)$ is the orthogonal projection on $T_{u_n}M$. We have*

(1)
$$v_n \xrightarrow[n\to\infty]{\mathrm{L}^\infty} v,$$

(2)
$$\{D_t v_n\}_{n\in\mathbb{N}} \text{ is bounded in } \mathrm{L}^\infty \text{ and } D_t v_n \xrightarrow[n\to\infty]{\mathrm{L}^2} D_t v,$$

(3)
$$\int_0^{L_n} \sigma_n^2 |D_t^2 v_n| \mathrm{d}\mathscr{L}^1 \xrightarrow[n\to\infty]{} 0.$$

Proof of Lemma 7.2. If $\mathfrak{n}$ is a normal vector field to M, the orthogonal projection $P(u_n) : \mathbb{R}^q \to T_{u_n}M$ is given by

$$P(u_n)v = v - \mathfrak{n}(u_n)\langle \mathfrak{n}(u_n), v\rangle = v - \mathfrak{n}(u_n)\langle \mathfrak{n}(u_n) - \mathfrak{n}(u), v\rangle$$

and $u_n \xrightarrow[n\to\infty]{} u$ in L^∞, so $P(u_n)v \xrightarrow[n\to\infty]{} v$ in L^∞.

$$\begin{aligned} D_t(P(u_n)v - v) = & -D\mathfrak{n}(u_n)[\dot{u}_n]\langle \mathfrak{n}(u_n) - \mathfrak{n}(u), v\rangle - \mathfrak{n}(u_n)\langle D\mathfrak{n}(u_n)[\dot{u}_n] - D\mathfrak{n}(u)[\dot{u}], v\rangle \\ & - \mathfrak{n}(u_n)\langle \mathfrak{n}(u_n) - \mathfrak{n}(u), D_t v\rangle . \end{aligned}$$

Therefore, $D_t(P(u_n)v)$ is bounded in L^∞, and converges into L^2 to $D_t v$. Finally,

$$D_t^2(P(u_n)v) = D^2P(u_n)[\dot{u}_n, \dot{u}_n]v + DP(u_n)[D_t\dot{u}_n]v + 2P(u_n)[\dot{u}_n]D_t v + P(u_n)D_t^2 v$$

and P is $C^{\nu-1}$ (and $\nu - 1 \geq 2$), so there exist a constant C independent of n such that

$$\int_0^{L_n} \sigma_n^2 |D_t^2(P(u_n)v)|^2 \mathrm{d}\mathscr{L}^1 \leq C\sigma_n^2 \|v\|_{\mathrm{W}^{2,2}(S^1)} + C\sigma_n \|v\|_{\mathrm{L}^2(S^1)} \left(\int_0^{L_n} \sigma_n^2\kappa^2(u_n)\mathrm{d}\mathscr{L}^1\right)^{\frac{1}{2}} \xrightarrow[n\to\infty]{} 0.$$

which completes the proof of the lemma. □

For all $\sigma > 0$, define $F_\sigma = DE_\sigma - DE_0$. We have

$$\begin{aligned}\sigma_n^2 F_{\sigma_n}(u_n)\cdot v_n &= \sigma_n^2\int_0^{L_n} 2\left\langle D_t^2 v_n, D_t\dot{u}_n\right\rangle + 3\kappa^2(u_n)\left\langle \dot{u}_n, D_t v_n\right\rangle + 2\left\langle R(D_t\dot{u}_n,\dot{u}_n)\dot{u}_n, v_n\right\rangle \mathrm{d}\mathscr{L}^1\\ &\leq 2\left(\int_0^{L_n}\sigma_n^2|D_t^2 v_n|\mathrm{d}\mathscr{L}^1\right)^{\frac{1}{2}}\left(\int_0^{L_n}\sigma_n^2|D_t\dot{u}_n|\mathrm{d}\mathscr{L}^1\right)^{\frac{1}{2}} + 3\left\|D_t v_n\right\|_{\mathrm{L}^\infty(S^1)}\int_0^{L_n}\sigma_n^2\kappa^2(u_n)\mathrm{d}\mathscr{L}^1\\ &\quad +2\left\|R\right\|_{\mathrm{L}^\infty(M)}\left\|v_n\right\|_{\mathrm{L}^2(S^1)}\sigma_n\left(\int_0^{L_n}\sigma_n^2\kappa^2(u_n)\mathrm{d}\mathscr{L}^1\right)^{\frac{1}{2}} \xrightarrow[n\to\infty]{} 0\end{aligned}$$

whereas

$$\int_0^{L_n}\left\langle D_t v_n, \dot{u}_n\right\rangle \mathrm{d}\mathscr{L}^1 \xrightarrow[n\to\infty]{} \int_0^{L}\left\langle D_t v, \dot{u}\right\rangle \mathrm{d}\mathscr{L}^1.$$

As for all $n\in\mathbb{N}$, u_n is a critical point of E_{σ_n}, we have

$$DE_{\sigma_n}(u_n)\cdot v_n = \int_0^{L_n}\left\langle D_t v_n, \dot{u}_n\right\rangle \mathrm{d}\mathscr{L}^1 + \sigma_n^2 F_{\sigma_n}(u_n)\cdot v_n = 0$$

so we deduce

$$\int_0^{L}\left\langle D_t v, \dot{u}\right\rangle \mathrm{d}\mathscr{L}^1 = 0$$

for all $v\in \mathrm{W}^{2,2}_u(S^1, TM)$, *i.e.* u is a distributional solution of

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}u + \mathbb{I}(\dot{u},\dot{u}) = 0 \tag{7.5}$$

where $\mathbb{I}$ is the second fundamental form of the immersion $u : S^1 \to (M^m, h)$. This implies that $\frac{\mathrm{d}^2}{\mathrm{d}t^2}u \in L^\infty([0,L],M)$, so $u\in \mathrm{W}^{2,\infty}([0,L])$, and by a immediate bootstrap argument, we get that actually $u\in C^\nu([0,L])$, $|\dot{u}| = 1$,

$$D_t u = \frac{\mathrm{d}^2}{\mathrm{d}t^2}u + \mathbb{I}(\dot{u},\dot{u}) = 0 \tag{7.6}$$

We conclude that u is a non-trivial closed geodesic of length $\beta(0) > 0$. □

8. Admissible family construction

Theorem 8.1. *Let (M^m, h) a compact Riemannian manifold of class C^ν ($\nu \geq 3$). We assume if M^m is simply connected that the first non-trivial class of higher homotopy group is homotopic to an immersion. Then there exists an admissible set $\mathscr{A}$ in $\mathrm{W}^{2,2}(S^1, M)$, in the sense of Definition 6.1.*

Proof. Since M^m is a compact manifold, $H_m(M^m,\mathbb{Z})\simeq\mathbb{Z}$ hence there exists $k\leq m$ such that $\pi_k(M)\neq\{1\}$. According to Hurewicz theorem, if $\pi_1(M) = 1$ (otherwise, we can minimize directly on a non-trivial homotopy class), there exists and integer k such that $H_i(M) = \{1\}$ for all $i<k$, $H_k(M)\neq\{1\}$, then $\pi_i(M) = \{1\}$ for all $i<k$, and $\pi_k(M)\simeq H_k(M)$. As $H_m(M^m,\mathbb{Z})\simeq\mathbb{Z}$, there exist $k\leq m$ such that $\pi_k(M)\neq\{1\}$. Let $f : S^k\to M$ a continuous homotopically non-trivial. We may assume that f is of class C^ν, because according to Whitney theorem, every continuous map between manifolds is homotopic to a regular map (see [24, 55]). We further

assume by hypothesis that f is an immersion. On S^k let us consider the following canonical sweep-out (see Fig. 1)

$$\{x_3 = 1 - 2t_3, \dots, x_{k+1} = 1 - 2t_{k+1}\} \tag{8.1}$$

where $t_3, \dots, t_{k+1} \in [0,1]$. This gives a map $g : S^k \to S^k$ of degree 1. Write, for $t \in [0,1]^{k-1}$, $u_t : S^1 \to S^k$ the circle defined by (8.1). Then for all but finitely $t \in [0,1]^{k-1}$, u_t is an immersed curve. We define

$$\mathscr{A} = \left\{ \{\varphi \circ f \circ u_t\}_{t\in[0,1]^{k-1}}, \varphi \in \mathrm{Homeo}_0(\mathrm{W}^{2,2}_\iota(S^1, M)) \right\}$$

where $\mathrm{Homeo}_0(\mathrm{W}^{2,2}_\iota(S^1, M))$ is the set of locally Lipschitz homeomorphisms of $\mathrm{W}^{2,2}_\iota(S^1, M)$ isotopic to the identity map. The manifold (M^m, g) being compact, its injectivity radius $\mathrm{inj}(M)$ is positive. Let us show that for all $\varphi \in \mathrm{Homeo}_0(\mathrm{W}^{2,2}_\iota(S^1, M))$,

$$\sup_{t\in[0,1]^{k-1}} \mathfrak{L}(\varphi \circ f \circ u_t) \geq \mathrm{inj}(M). \tag{8.2}$$

We argue by contradiction. If if (8.2) is not satisfied, for all $t \in [0,1]^{k-1}$, the curve $\varphi \circ f \circ u_t : S^1 \to M$ is null-homotopic. This implies that $\varphi \circ f \circ g$ is null-homotopic. Now φ is isotopic to the identity map, so $\varphi \circ f \circ g$ is homotopic to $f \circ g$. But as $g : S^k \to S^k$ is a degree one map, and $f : S^k \to M$ homotopically non-trivial, $f \circ g : S^k \to M$ cannot be null-homotopic. We deduce that

$$\beta(0) = \inf_{A\in\mathscr{A}} \sup_{u\in\mathscr{A}} \mathfrak{L}(u) \geq \mathrm{inj}(M) > 0,$$

which concludes the proof of the theorem. □

Remark 8.2. We remark that in case of $m = 2$, if M^2 is not simply connected, the existence of closed geodesics is trivial, and otherwise, M^2 is diffeomorphic to the two-sphere S^2, so and using the same argument furnishes an admissible family. Furthermore, if the first non-trivial map $f : S^k \to M^m$ is such that $2k \leq m$, as immersions are generic [24], the hypothesis is empty. We also remark that one could replace this construction by a more elaborated one using regular homotopy: if there exists $k \leq m$ such that $\pi_1(\mathrm{Imm}(S^k, M^m)) \neq \{0\}$, then we can also produce an admissible family by an immediate adaptation of this argument. We do not know if this condition always holds for a simply connected compact manifold M^m.

9. Lower semi-continuity of the index

Motivating by the construction by a min-max viscosity method of minimal surfaces of given index, we aim at proving here that the index of the constructed curves in lower semi-continuous.

Definition 9.1. Let $\sigma \geq 0$, and u a critical point of E_σ. The index of u, noted $\mathrm{Ind}(u) \in \mathbb{N} \cup \{\infty\}$, is equal to the dimension of the larger subspace of $\mathrm{W}^{2,2}_u(S^1, TM)$, on which the second derivative $D^2E_\sigma(u)$ (defined by (4.1)) is negative semi-definite.

The proof of the index lower semi-continuity will be an easy consequence of the following lemma.

Lemma 9.2. *Let $\{\sigma_n\}_{n\in\mathbb{N}}$ a sequence of positive real numbers converging to 0. If $\{u_n\}_{n\in\mathbb{N}}$ is a sequence of critical points associated to $\{E_{\sigma_n}\}_{n\in\mathbb{N}}$, such that $\{u_n\}_{n\in\mathbb{N}}$ (resp. $\{\dot{u}_n\}_{n\in\mathbb{N}}$) converge in L^∞ (resp. almost everywhere) to a closed non-trivial geodesic $u \in \mathrm{W}^{2,2}_\iota(S^1, M)$ (resp. to $\dot{u}$) of length $L > 0$. If $\{v_n\}_{n\in\mathbb{N}}$ is a sequence verifying $v_n \in \mathrm{W}^{2,2}_{u_n}(S^1, TM)$, $v_n \xrightarrow[n\to\infty]{\mathrm{L}^\infty} v \in \mathrm{W}^{2,2}_u(S^1, TM)$, $D_t v_n \xrightarrow[n\to\infty]{\mathrm{L}^2} D_t v$, $\{v_n\}_{n\in\mathbb{N}}$ is bounded in L^∞, and*

$$\int_{S^1} \sigma_n^2 \kappa^2(u_n)|\dot{u}_n| d\mathscr{L}^1 \xrightarrow[n\to\infty]{} 0, \quad \int_{S^1} \sigma_n^2 |D_t^2 v_n|^2 |\dot{u}_n| d\mathscr{L}^1 \xrightarrow[n\to\infty]{} 0.$$

Then

$$D^2E_{\sigma_n}(u_n)[v_n,v_n] \xrightarrow[n\to\infty]{} D^2E_0(u)[v,v] = \int_0^L \left(|D_tv|^2 - \langle D_tv,\dot{u}\rangle^2 - \langle R(\dot{u},v)v,\dot{u}\rangle\right)\mathrm{d}\mathscr{L}^1.$$

Proof. The hypothesis implies that $\{v_n\}_{n\in\mathbb{N}}$ is bounded in $L^\infty(S^1,M)$, and in $\mathrm{W}^{1,2}(S^1,M)$. Furthermore, Theorem 7.1 shows that $\{\dot{u_n}\}_{n\in\mathbb{N}}$ converges to $L^p(S^1,M)$ (if $L_n=\mathfrak{L}(u)$, we have $L_n \xrightarrow[n\to\infty]{} L$) for all $1\leq p<\infty$ according to Lebesgue's dominated convergence theorem.

All estimates are elementary, using only Cauchy−Schwarz inequality, and otherwise, R being a $C^{\nu-2}$ ($\nu-2\geq 1$) tensor on the compact C^ν manifold (M,g) implies that

$$\max\{\|R\|_{\mathrm{L}^\infty(M)},\|\nabla R\|_{\mathrm{L}^\infty(M)}\}<\infty$$

As for all $n\in\mathbb{N}$, $|\dot{u}_n|=1$,

$$\begin{aligned}&\sup_{n\in\mathbb{N}}\|\nabla_{\dot{u}_n}R\|_{\mathrm{L}^\infty(M)}\leq\|\nabla R\|_{\mathrm{L}^\infty(M)}<\infty\\ &\sup_{n\in\mathbb{N}}\|\nabla_{v_n}R\|_{\mathrm{L}^\infty(M)}\leq\|\nabla R\|_{\mathrm{L}^\infty(M)}\sup_{n\in\mathbb{N}}\|v_n\|_{\mathrm{L}^\infty(S^1)}<\infty\end{aligned}$$

We have

$$\sigma_n^2\int_0^{L_n}|D_t^2v_n|^2+|R(v_n,\dot{u}_n)\dot{u}_n|^2\mathrm{d}\mathscr{L}^1\leq\int_0^{L_n}\sigma_n^2|D_t^2v_n|^2\mathrm{d}\mathscr{L}^1+\sigma_n^2\|R\|_{\mathrm{L}^\infty(M)}\|v_n\|^2_{\mathrm{L}^2([0,L_n])}\xrightarrow[n\to\infty]{}0$$

We write

$$K_n=\left(\int_{S^1}\sigma_n^2\kappa^2(u_n)|\dot{u}_n|\mathrm{d}\mathscr{L}^1\right)^{\frac{1}{2}}=\left(\int_0^{L_n}\sigma_n^2\kappa^2(u_n)\mathrm{d}\mathscr{L}^1\right)^{\frac{1}{2}}\xrightarrow[n\to\infty]{}0.$$

We estimate the other terms as following.

$$\begin{aligned}&\sigma_n^2\left|\int_0^{L_n}\left(4\langle D_tv_n,\dot{u}_n\rangle^2+2\langle D_tv_n,\dot{u}_n\rangle-|D_tv_n|^2+\langle R(\dot{u}_n,v_n)v_n,\dot{u}_n\rangle\right)\kappa^2(u_n)\mathrm{d}\mathscr{L}^1\right|\\ &\leq\left(4\|D_tv_n\|^2_{\mathrm{L}^\infty([0,L_n])}+2\|D_tv_n\|_{\mathrm{L}^\infty([0,L_n])}+\|R\|_{\mathrm{L}^\infty(M)}\|v_n\|_{\mathrm{L}^\infty([0,L_n])}\right)K_n^2\end{aligned}$$

$$\sigma_n^2\left|\int_0^{L_n}\left(\langle D_t^2v_n,\dot{u}_n\rangle+\langle D_tv_n,D_t\dot{u}_n\rangle\right)^2\mathrm{d}\mathscr{L}^1\right|\leq 2\int_0^{L_n}\sigma_n^2\left\|D_t^2v_n\right\|^2_{\mathrm{L}^2([0,L_n])}+2\|D_tv_n\|^2_{\mathrm{L}^\infty([0,L_n])}K_n^2$$

$$\sigma_n^2\left|\int_0^{L_n}\langle\nabla_{\dot{u}_n}R(v_n,\dot{u}_n)v_n,D_t\dot{u}_n\rangle\,\mathrm{d}\mathscr{L}^1\right|\leq\sigma_n\|\nabla R\|_{\mathrm{L}^\infty(M)}\|v_n\|_{\mathrm{L}^\infty([0,L_n])}\|v_n\|_{\mathrm{L}^2([0,L_n])}K_n$$

$$\sigma_n^2\left|\int_0^{L_n}\langle\nabla_{v_n}R(v_n,\dot{u}_n)\dot{u}_n,D_t\dot{u}_n\rangle\,\mathrm{d}\mathscr{L}^1\right|\leq\sigma_n\|R\|_{\mathrm{L}^\infty(M)}\|v_n\|_{\mathrm{L}^\infty([0,L_n])}\|v_n\|_{\mathrm{L}^2([0,L_n])}K_n$$

$$\sigma_n^2\left|\int_0^{L_n}\langle R(D_tv_n,\dot{u}_n)v_n,D_t\dot{u}_n\rangle\,\mathrm{d}\mathscr{L}^1\right|\leq\sigma_n\|\nabla R\|_{\mathrm{L}^\infty(M)}\|v_n\|_{\mathrm{L}^\infty([0,L_n])}\|D_tv_n\|_{\mathrm{L}^2([0,L_n])}K_n$$

$$\sigma_n^2\left|\int_0^{L_n}\langle R(D_t\dot{u}_n,v_n)v_n,D_t\dot{u}_n\rangle\,\mathrm{d}\mathscr{L}^1\right|\leq\|R\|_{\mathrm{L}^\infty(M)}\|v_n\|^2_{\mathrm{L}^\infty([0,L_n])}K_n^2$$

$$\sigma_n^2\left|\int_0^{L_n}\langle R(v_n,\dot{u}_n)D_tv_n,D_t\dot{u}_n\rangle\,\mathrm{d}\mathscr{L}^1\right|\leq\sigma_n\|R\|_{\mathrm{L}^\infty(M)}\|v_n\|_{\mathrm{L}^\infty([0,L_n])}\|D_tv_n\|_{\mathrm{L}^2([0,L_n])}K_n$$

$$\sigma_n^2\left|\int_0^{L_n}\langle R(\dot{u}_n,D_tv_n)\dot{u}_n,D_t\dot{u}_n\rangle\,\mathrm{d}\mathscr{L}^1\right|\leq\sigma_n\|R\|_{\mathrm{L}^\infty(M)}\|D_tv_n\|_{\mathrm{L}^2([0,L_n])}K_n$$

$$\sigma_n^2\left|\int_0^{L_n}\langle R(v_n,\dot{u}_n)D_tv_n,D_t\dot{u}_n\rangle\,\mathrm{d}\mathscr{L}^1\right|\leq\sigma_n\|R\|_{\mathrm{L}^\infty(M)}\|v_n\|_{\mathrm{L}^\infty([0,L_n])}\|D_tv_n\|_{\mathrm{L}^2([0,L_n])}K_n$$

$$\sigma_n^2\left|\int_0^{L_n}\langle D_tv_n,\dot{u}_n\rangle\langle R(v_n,\dot{u}_n)\dot{u}_n,D_t\dot{u}_n\rangle\,\mathrm{d}\mathscr{L}^1\right|\leq\sigma_n\|R\|_{\mathrm{L}^\infty(M)}\|v_n\|_{\mathrm{L}^\infty([0,L_n])}\|D_tv_n\|_{\mathrm{L}^2([0,L_n])}K_n$$

$$\begin{aligned}&\sigma_n^2\left|\int_0^{L_n}\langle D_tv_n,\dot{u}_n\rangle\left(\langle D_t^2v_n,D_t\dot{u}_n\rangle-2\langle D_tv_n,\dot{u}_n\rangle\kappa^2(u_n)+\langle R(v_n,\dot{u}_n)\dot{u}_n,D_t\dot{u}_n\rangle\right)\mathrm{d}\mathscr{L}^1\right|\\&\leq\sigma_n^2\|D_tv_n\|_{\mathrm{L}^\infty([0,L_n])}\left\|D_t^2v_n\right\|^2_{\mathrm{L}^2(S^1)}K_n+2\|D_tv_n\|^2_{\mathrm{L}^\infty([0,L_n])}K_n^2\\&\quad+\sigma_n\|R\|_{\mathrm{L}^\infty(M)}\|v_n\|_{\mathrm{L}^\infty([0,L_n])}\|D_tv_n\|_{\mathrm{L}^2([0,L_n])}K_n\end{aligned}$$

$$\begin{aligned}&\sigma_n^2\left|\int_0^{L_n}\left(|D_tv_n|^2-\langle D_tv_n,\dot{u}_n\rangle^2-\langle R(\dot{u}_n,v_n)v_n,\dot{u}_n\rangle\right)\kappa^2(u_n)\mathrm{d}\mathscr{L}^1\right|\\&\leq\left(\|D_tv_n\|^2_{\mathrm{L}^\infty([0,L_n])}+\|R\|_{\mathrm{L}^\infty(M)}\|v_n\|^2_{\mathrm{L}^\infty([0,L_n])}\right)K_n^2.\end{aligned}$$

Finally, the unit vector sequence $\{\dot{u}_n\}_{n\in\mathbb{N}}$ converge almost everywhere to $\dot{u}$ (which is also a unit vector), so we can apply Lebesgue's dominated convergence theorem to get

$$\begin{aligned}&\int_{S^1}\left(|\nabla_{\frac{\dot{u}_n}{|\dot{u}_n|}}v_n|^2-\left\langle\nabla_{\frac{\dot{u}_n}{|\dot{u}_n|}}v_n,\frac{\dot{u}_n}{|\dot{u}_n|}\right\rangle^2-\left\langle R\left(\frac{\dot{u}_n}{|\dot{u}_n|},v_n\right)v_n,\frac{\dot{u}_n}{|\dot{u}_n|}\right\rangle\right)|\dot{u_n}|\mathrm{d}\mathscr{L}^1\\&\underset{n\to\infty}{\longrightarrow}\int_{S^1}\left(|\nabla_{\frac{\dot{u}}{|\dot{u}|}}v|^2-\left\langle\nabla_{\frac{\dot{u}}{|\dot{u}|}}v,\dot{u}\right\rangle^2-\left\langle R\left(\frac{\dot{u}}{|\dot{u}|},v\right)v,\frac{\dot{u}}{|\dot{u}|}\right\rangle\right)|\dot{u}|\mathrm{d}\mathscr{L}^1,\end{aligned}$$

which completes the proof of the lemma. □

Theorem 9.3. *Under the hypothesis of 7.1, if $\{u_n\}_{n\in\mathbb{N}}$ is the sequence of critical points associated to $\{E_{\sigma_n}\}$, to a non-trivial closed geodesic $u\in\mathrm{W}^{2,2}_\iota(S^1,M)$, we have*

$$\mathrm{Ind}(u)\leq\liminf_{n\to\infty}\mathrm{Ind}(u_n).$$

Proof. If $v \in \mathrm{W}^{2,2}_u(S^1, TM)$, and $P(u_n)$ is the orthogonal projection $\mathbb{R}^q \to T_{u_n}M$, if $v_n = P(u_n)$, la suite $\{v_n\}_{n\in\mathbb{N}}$ thanks of Lemma 7.2, $\{v_n\}_{n\in\mathbb{N}}$ satisfies the hypothesis of Lemma 9.2. If $v^1, \ldots, v^I \in \mathrm{W}^{1,2}(S^1, M)$ is a free orthonormal family in $\mathrm{L}^2(S^1, M)$ such that D^2E_0 is negative semi-definite on Span $\left\{v^1, \ldots, v^I\right\}$, then, if $v^j_n = P(u_n)v^j$, the family $\left\{v^1_n, \ldots, v^I_n\right\}$ is free in $\mathrm{W}^{2,2}_{u_n}(S^1, M)$, for n large enough. As $D^2E_0(u)[v^j, v^j] < 0$, we have

$$D^2E_{\sigma_n}(u_n)[v_n, v_n] \xrightarrow[n\to\infty]{} D^2E_0(u)[v^j, v^j] < 0$$

so for n large enough, $D^2E_{\sigma_n}(u_n)[v_n, v_n] < 0$, and $\left\{v^1_n, \ldots, v^I_n\right\}$ is free, thus

$$I \leq \liminf_{n\to\infty} \mathrm{Ind}(u_n).$$

This implies that

$$\mathrm{Ind}(u) \leq \liminf_{n\to\infty} \mathrm{Ind}(u_n).$$

which concludes the proof of the theorem. □

10. Counter-examples

10.1. Counter-examples in dimension 1

Let (M^2, h) a compact C^3 Riemannian surface of constant Gauss curvature $K_M \in \mathbb{R}$ (which is just equal to the sectional curvature in our convention). Let $\sigma > 0$, and u_σ a critical point of E_σ. We know that u is C^2 and satisfies (10.1)

$$D_t\dot{u} = \sigma^2\left\{D_t\left(2D_t^2\dot{u} + 3\kappa^2\dot{u}\right) + 2R(D_t\dot{u}, \dot{u})\dot{u}\right\} \tag{10.1}$$

Let ν a normal vector to the curve u, and k the signed curvature, defined as

$$D_t\dot{u} = k\nu$$

As $\langle D_t\dot{u}, \dot{u}\rangle = 0$, k is well-defined. Moreover, Frénet equations in dimension 2 imply that

$$D_t\nu = -k\dot{u},$$

so

$$\begin{aligned}
&D_t^2\dot{u} = \dot{k}\nu - k^2\dot{u},\\
&D_t^3\dot{u} = \ddot{k}\nu + \dot{k}(-k\dot{u}) - 2\dot{k}k\dot{u} - k^2(k\nu) = \ddot{k}\nu - 3\dot{k}k\dot{u} - k^3\nu.
\end{aligned}$$

As

$$D_t(k^2\dot{u}) = 2\dot{k}k\dot{u} + k^3\nu$$

the equation (10.1) is equivalent to

$$k\nu = \sigma^2(2\ddot{k} + k^3)\nu + 2\sigma^2 kR(\nu, \dot{u})\dot{u}$$

Taking the scalar product with ν, we get

$$k_\sigma(t) = \sigma^2(2\ddot{k}_\sigma(t) + k_\sigma^3(t) + 2K_M k_\sigma(t)) \tag{10.2}$$

We can explicitly solve this equation thanks of the Jacobi elliptic functions (see [13, 33]).

Let $0 \leq p < 1$, we define $f_p : \mathbb{R} \to \mathbb{R}$

$$f_p(t) = \int_0^t \frac{\mathrm{d}\theta}{\sqrt{1 - p^2 \sin^2 \theta}} \tag{10.3}$$

and Jacobi elliptic functions sn, cn and dn as

$$\begin{aligned}
\mathrm{sn}(t,p) &= \sin f_p^{-1}(t)\\
\mathrm{cn}(t,p) &= \cos f_p^{-1}(t)\\
\mathrm{dn}(t,p) &= \sqrt{1 - p^2 \mathrm{sn}^2(t)}
\end{aligned}$$

and the function $\mathscr{K} : [0,1[\to \mathbb{R}_+$ by

$$\mathscr{K}(p) = f_p\left(\frac{\pi}{2}\right) = \int_0^{\frac{\pi}{2}} \frac{\mathrm{d}\theta}{\sqrt{1 - p^2 \sin^2 \theta}}.$$

The functions sn, cn, are $4\mathscr{K}$-periodic, and dn is $2\mathscr{K}$-periodic. If we write $\mathrm{sn}_p = \mathrm{sn}(\cdot, p)$, $\mathrm{cn}_p = \mathrm{cn}(\cdot, p)$, $\mathrm{dn}_p = \mathrm{dn}(\cdot, p)$, we have

$$\begin{aligned}
\dot{\mathrm{sn}}_p &= \mathrm{cn}_p \mathrm{dn}_p\\
\dot{\mathrm{cn}}_p &= -\mathrm{cn}_p \mathrm{dn}_p\\
\dot{\mathrm{dn}}_p &= -p^2 \mathrm{sn}_p \mathrm{cn}_p
\end{aligned}$$

and if $\mathrm{dn} = \mathrm{dn}(\cdot, p)$ $(0 \leq p < 1)$,

$$\ddot{\mathrm{dn}} + 2\mathrm{dn}^3 - (2 - p^2)\mathrm{dn} = 0.$$

The function $t \mapsto u(t) = a\,\mathrm{dn}(bt, p)$, u is a solution of the differential equation

$$\ddot{u} + 2\left(\frac{b}{a}\right)^2 u^3 - b^2(2 - p^2)u = 0$$

Fix $0 \leq p < 1$, we have

$$k_\sigma(t) = \pm \left(\frac{1}{\sigma^2} \frac{2(1 - 2\sigma^2 K_M)}{(2 - p^2)}\right)^{\frac{1}{2}} \mathrm{dn}\left(\left(\frac{1 - 2\sigma^2 K_M}{2(2 - p^2)}\right)^{\frac{1}{2}} \frac{t}{\sigma}, p\right) \tag{10.4}$$

and

$$\sigma^2 k_\sigma^2(t) = 2\frac{1 - 2\sigma^2 K_M}{2 - p^2}\left(1 - p^2 \mathrm{sn}^2\left(\left(\frac{1 - 2\sigma^2 K_M}{2(2 - p^2)}\right)^{\frac{1}{2}} \frac{t}{\sigma}, p\right)\right)$$

If $C(\sigma)$ is the constant

$$C(\sigma) = \left(\frac{1 - 2\sigma^2 K_M}{2(2 - p(\sigma)^2)}\right)^{\frac{1}{2}}$$

Then, k_σ^2 is a $2\sigma C(\sigma)^{-1}\mathscr{K}(p(\sigma))$-periodic function and $L(\sigma)$-periodic ($L(\sigma) = \mathfrak{L}(u)$). Thus there exists $m(\sigma) \in \mathbb{N}$ such that $L(\sigma) = 2\sigma m(\sigma) C(\sigma)^{-1} \mathscr{K}(p(\sigma))$. In particular (see [33], p. 19),

$$\begin{aligned}
\int_0^{L(\sigma)} \sigma^2 k_\sigma^2(t)\mathrm{d}t &= 8\sigma\, m(\sigma) C(\sigma) \int_0^{\mathscr{K}(\sigma)} \mathrm{dn}_{p(\sigma)}^2(t)\mathrm{d}t\\
&= 4\sigma\, m(\sigma)\sqrt{\frac{2 - 4\sigma^2 K_M}{2 - p(\sigma)^2}} \int_0^{\frac{\pi}{2}} \sqrt{1 - p(\sigma)^2 \sin^2(t)}\mathrm{d}t
\end{aligned}$$

If $\{\sigma_n\}_{n\in\mathbb{N}}$ is a sequence of positive numbers converging to 0 such that

$$L = \lim_{n\to\infty} L(\sigma_n) > 0,$$

if $p(\sigma_n) \xrightarrow[n\to\infty]{} p \in [0,1[$, we have $\mathscr{K}(p(\sigma_n)) \xrightarrow[n\to\infty]{} \mathscr{K}(p) \in \left[\frac{\pi}{2}, \infty\right)$ and

$$\begin{aligned}\int_0^{L(\sigma_n)} \sigma_n^2 k_{\sigma_n}^2(t)\mathrm{d}t &= \frac{2L(\sigma_n)C(\sigma_n)^2}{\mathscr{K}(p(\sigma_n))}\int_0^{\frac{\pi}{2}}\sqrt{1-p(\sigma_n)^2\sin^2(t)}\mathrm{d}t \\ &\xrightarrow[n\to\infty]{} \frac{4L}{(2-p^2)\mathscr{K}(p)}\int_0^{\frac{\pi}{2}}\sqrt{1-p^2\sin^2(t)}\,\mathrm{d}t > 0,\end{aligned} \tag{10.5}$$

which give a family of counter-examples, as we will see in next section.

10.1.1. Explicit counter-example on S^2

The goal of this section is to prove the following result.

Proposition 10.1. *On S^2 equipped with its standard metric, let $\mathscr{A}$ the admissible set of curves given by the canonical sweep-out on S^2. There exists a sequence $\{\sigma_n\}_{n\in\mathbb{N}}$ of positive real numbers converging to 0 and a sequence of critical points $\{u_n\}_{n\in\mathbb{N}}$ of $\{E_{\sigma_n}\}_{n\in\mathbb{N}}$, and a curve $u \in \mathrm{W}^{1,2}(S^1, M)$, such that*

$$E_{\sigma_n}(u_n) \xrightarrow[n\to\infty]{} \beta(0) = \pi, \quad \mathfrak{L}(u_n) \xrightarrow[n\to\infty]{} \frac{\pi}{2}$$

and

$$u_n \xrightarrow[n\to\infty]{\mathrm{L}^\infty} u \quad \text{strongly}, \qquad u_n \xrightarrow[n\to\infty]{\mathrm{W}^{1,2}} u \quad \text{weakly}, \quad \text{and} \quad \dot{u}_n \underset{n\to\infty}{\not\longrightarrow} \dot{u} \quad \text{a.e.}$$

Furthermore, there exists a negligible subset $N \subset S^1$ such that $\{\dot{u}_n(t)\}_{n\in\mathbb{N}}$ has no limit point for all $t \in S^1 \setminus N$, and for all open interval $I \subset S^1$,

$$\mathfrak{L}(u|I) < \liminf_{n\to\infty} \mathfrak{L}(u_n|I).$$

Proof. The shortest closed geodesics on S^2 equipped with the standard metric are of length π (the great circles). We choose $p = 0$ in and define

$$u_\sigma(t) = \frac{\sigma}{(1-2\sigma^2)^{\frac{1}{2}}}\left(\cos\left((1-2\sigma^2)^{\frac{1}{2}}\frac{t}{\sigma}\right), \sin\left((1-2\sigma^2)^{\frac{1}{2}}\frac{t}{\sigma}\right), \frac{(1-2\sigma^2)^{\frac{1}{2}}}{\sigma}\sqrt{1-\frac{\sigma^2}{1-2\sigma^2}}\right)$$

then $|\dot{u}_\sigma| = 1$, and on S^2,

$$D_t\dot{u}_\sigma(t) = \ddot{u}_\sigma(t) = -\frac{(1-2\sigma^2)^{\frac{1}{2}}}{\sigma}u_\sigma(t)$$

so

$$k(u_\sigma(t)) = -\frac{(1-2\sigma^2)^{\frac{1}{2}}}{\sigma}$$

and u_σ is a critical point of E_σ for all $\sigma > 0$. And for all $\{\sigma_n\}_{n\in\mathbb{N}}$ converging to 0,

$$u_{\sigma_n} \xrightarrow[n\to\infty]{\mathrm{L}^\infty} (0,0,1)$$

and as $C(\sigma)=\frac{1}{2}(1-2\sigma^2)^{\frac{1}{2}}$, $\mathscr{K}(0)=\frac{\pi}{2}$, we have

$$E_\sigma(u_\sigma)=2L(\sigma)(1-\sigma^2)=4\pi\sigma m(\sigma)\frac{1-\sigma^2}{(1-2\sigma^2)^{\frac{1}{2}}},$$

where $m(\sigma)$ is an arbitrary integer. So if we choose

$$\sigma_n=\frac{1}{4n},\quad m(\sigma_n)=n$$

then writing $u_n=u_{\sigma_n}$,

$$E_{\sigma_n}(u_n)\xrightarrow[n\to\infty]{}\pi=\beta(0),\quad L(u_n)\xrightarrow[n\to\infty]{}\frac{\pi}{2}=\frac{\beta(0)}{2}$$

while

$$u_n\xrightarrow[n\to\infty]{\mathrm{L}^\infty}u\equiv(0,0,1)$$

and according to Riemann−Lebesgue lemma, $\{\dot{u}_n\}_{n\in\mathbb{N}}$ converges weakly in L^2 to $\dot{u}=0$, and if we consider $\{\dot{u}_n\}_{n\in\mathbb{N}}$ as a sequence of functions on $\mathbb{R}$ (extended by periodicity), for all $t\in\mathbb{R}/\mathbb{Q}$, $\{\dot{u}_n(t)\}_{n\in\mathbb{N}}$ has no limit point (as for all $\alpha\in\mathbb{R}/\mathbb{Q}$, $\{\cos(n\alpha)\}_{n\in\mathbb{N}}$ and $\{\sin(n\alpha)\}_{n\in\mathbb{N}}$ are dense in $[-1,1]$). So finally, we have

$$u_n\xrightarrow[n\to\infty]{}u\quad\text{weakly in } \mathrm{W}^{1,2}(S^1,S^2)$$

and for all open interval $I\subset S^1$,

$$0=\mathfrak{L}(u(I))<\liminf_{n\to\infty}\mathfrak{L}(u_n(I))=\frac{\pi}{2}\mathscr{L}^1(I).$$

which concludes the proof of the proposition. □

10.1.2. Surfaces of Constant Gauss Curvature

Proposition 10.2. *Let (M^2,h) a compact Riemannian surface of constant Gauss curvature, and $\beta(0)$ is the length of the shortest closed geodesic in (M^2,g). For all $1\leq 2\varepsilon<2$, there exists a sequence of positive numbers $\{\sigma_n\}_{n\in\mathbb{N}}$ converging to 0, and a sequence of critical points $\{u_n\}_{n\in\mathbb{N}}$ associated to $\{E_{\sigma_n}\}_{n\in\mathbb{N}}$ such that*

$$E_{\sigma_n}(u_n)\xrightarrow[n\to\infty]{}\beta(0),\quad\mathfrak{L}(u_n)\xrightarrow[n\to\infty]{}\varepsilon\beta(0)$$

and

$$u_n\xrightarrow[n\to\infty]{\mathrm{L}^\infty}u\quad\textit{strongly},\quad u_n\xrightarrow[n\to\infty]{\mathrm{W}^{1,2}}u\quad\textit{weakly},\quad\textit{and}\quad\dot{u}_n\not\xrightarrow[n\to\infty]{}\dot{u}\quad\textit{a.e.}$$

and

$$\mathfrak{L}(u)<\liminf_{n\to\infty}\mathfrak{L}(u_n).$$

Proof. We fix $0\leq p<1$, and recall that $K_M\in\mathbb{R}$ is the Gauss curvature. We consider a sequence $\{u_n\}_{n\in\mathbb{N}}$ of critical points of $\{E_{\sigma_n}\}_{n\in\mathbb{N}}$ given by (10.4) (*i.e.* the solutions of $D_t\dot{u}=k\nu$, where we choose $\{\sigma_n\}_{n\in\mathbb{N}}$ and $\{m(\sigma_n)\}_{n\in\mathbb{N}}$ such that

$$E_{\sigma_n}(u_n)\xrightarrow[n\to\infty]{}\beta(0)$$

The sequence $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $\mathrm{W}^{1,2}(S^1,M)$, so we can extract a subsequence (still denotes $\{u_n\}_{n\in\mathbb{N}}$) strongly converging in $\mathrm{L}^\infty(S^1,M)$, and weakly converging in $\mathrm{W}^{1,2}(S^1,M)$ to a function $u\in\mathrm{W}^{1,2}(S^1,M)$.

For all interval $I\subset[0,L]$ such that $I\subset[0,L_n]$ for n large enough, we have

$$\mathfrak{L}(u|_I)^2=\left(\int_I|\dot{u}|\mathrm{d}\mathscr{L}^1\right)^2\leq|I|\int_I|\dot{u}|^2\mathrm{d}\mathscr{L}^1\leq|I|\liminf_{n\to\infty}\int_I|\dot{u}_n|^2\mathrm{d}\mathscr{L}^1$$

and $\{u_n\}_{n\in\mathbb{N}}$ is in arc-length parametrization, so

$$\mathfrak{L}(u_n)^2=|I|\int_I|\dot{u}_n|^2\mathrm{d}\mathscr{L}^1$$

and

$$\mathfrak{L}(u|_I)\leq\liminf_{n\to\infty}\mathfrak{L}(u_n|_I). \tag{10.6}$$

Furthermore, we have

$$\mathfrak{L}(u)<\liminf_{n\to\infty}\mathfrak{L}(u_n). \tag{10.7}$$

We prove this assertion by contradiction. Assume that we have the equality in 10.7, then $\{\dot{u}_n\}_{n\in\mathbb{N}}$ converges almost everywhere to $\dot{u}$, and in particular,

$$L_n=\mathfrak{L}(u_n)\xrightarrow[n\to\infty]{}\mathfrak{L}(u)=L,$$

and $u\in\mathrm{W}^{2,2}_\iota(S^1,M)$, as we can pass to the limit in the arc-length expression $|\dot{u}_n|=1$. Thanks of the proof of theorem (7.1), for all $v\in\mathrm{W}^{2,2}_u$, if $v_n=P(u_n)v$ we have

$$\int_0^{L_n}\langle\dot{u}_n,D_tv_n\rangle=-3\int_0^{L_n}\sigma_n^2\kappa^2(u_n)\langle\dot{u}_n,D_tv_n\rangle+o(1) \tag{10.8}$$

and $\{D_tv_n\}_{n\in\mathbb{N}}$ is bounded in L^∞, so $\{\langle\dot{u}_n,D_tv_n\rangle\}_{n\in\mathbb{N}}$ is bounded in L^∞ and is a sequence of continuous functions, while $\left\{\sigma_n^2\kappa^2(u_n)\right\}_{n\in\mathbb{N}}$ converges weakly in L^2 to 1. Indeed,

$$\sigma_n^2\kappa^2(u_n)=\frac{2(1-2\sigma_n^2K_M)}{2-p^2}\left(1-\frac{p^2}{2}\right)+\frac{p^2(1-2\sigma_n^2K_M)}{2-p^2}\mathrm{cn}_p\left(\left(\frac{2(1-2\sigma_n^2K_M)}{2-p^2}\right)^{\frac{1}{2}}\frac{t}{\sigma}\right)$$

and the last term converges weakly in L^2 to 0 according to Riemann−Lebesgue theorem. So we can pass to the limit in (10.8) to find that

$$\int_0^L\langle\dot{u},D_tv\rangle=-3\int_0^L\langle\dot{u},D_tv\rangle\,\mathrm{d}\mathscr{L}^1$$

so u is a closed geodesic. As we have chosen $\{\sigma_n\}_{n\in\mathbb{N}}$, and $\{m(\sigma_n\})$ such that

$$E_{\sigma_n}(u_n)\xrightarrow[n\to\infty]{}\beta(0),$$

then

$$L=\varepsilon(p)\beta(0)=\left(1+\frac{4}{(2-p^2)\mathscr{K}(p)}\int_0^{\frac{\pi}{2}}\sqrt{1-p^2\sin^2(t)}\mathrm{d}t\right)^{-1}\beta(0)<\beta(0)$$

so u is a non-trivial closed geodesic of length strictly inferior that the length of the shortest closed geodesic, which yields the desired contradiction. Finally as $0\leq p<1$, and

$$\varepsilon(p)\xrightarrow[p\to0]{}\frac{1}{2},\quad\varepsilon(p)\xrightarrow[p\to1]{}1,$$

this completes the proof of the proposition. □

10.1.3. General Surfaces

In the case of a general surface, we get

$$k_\sigma(t) = \sigma^2\left(2\ddot{k}_\sigma(t) + k_\sigma^3(t) + 2K\left(\dot{u}(t),\nu(t)\right)k_\sigma(t)\right)\right) \tag{10.9}$$

where $K(\dot{u}(t),\nu(t))$ is the sectional curvature of the 2-plan $\dot{u}(t)\wedge\nu(t)$. Let K_M^+ (resp. K_M^-) the maximum (resp. minimum) of sectional curvature of (M,g). If s is the sign function

$$\begin{aligned}
&\ddot{k}_\sigma(t) + \frac{1}{2}k_\sigma^3(t) + \left(K_M^{\mathrm{s}(k_\sigma(t))} - \frac{1}{2\sigma^2}\right)k_\sigma(t) \geq 0,\\
&\ddot{k}_\sigma(t) + \frac{1}{2}k_\sigma^3(t) + \left(K_M^{-\mathrm{s}(k_\sigma(t))} - \frac{1}{2\sigma^2}\right)k_\sigma(t) \leq 0.
\end{aligned}$$

If we write $C(\sigma,K) = \left(\dfrac{1-2\sigma^2K}{2(2-p(\sigma)^2)}\right)$, an elementary application of comparison principle implies that there exists a solution k_σ of (10.9) such that

$$\frac{2}{\sigma}C(\sigma,K_M^+)\mathrm{dn}\left(C(\sigma,K_M^+)\frac{t}{\sigma},p(\sigma)\right) \leq k_\sigma(t) \leq \frac{2}{\sigma}C(\sigma,K_M^-)\mathrm{dn}\left(C(\sigma,K_M^-)\frac{t}{\sigma},p(\sigma)\right). \tag{10.10}$$

Furthermore, thanks of a result of Joel Langer and David A. Singer [33], we can extend this procedure to get counter-examples in any dimension $m \geq 2$. Finally inequality (10.10) permits to extend the result of the general result of the previous subsection.

Proposition 10.3. *If (M^2,h) is a Riemannian surface, for all $\beta > 0$, for all $1 < 2\varepsilon < 2$, there exists a sequence of positive numbers $\{\sigma_n\}_{n\in\mathbb{N}}$ converging to 0, and a sequence of critical points $\{u_n\}_{n\in\mathbb{N}}$ associated to $\{E_{\sigma_n}\}_{n\in\mathbb{N}}$ such that*

$$E_{\sigma_n}(u_n) \xrightarrow[n\to\infty]{} \beta, \quad \mathfrak{L}(u_n) \xrightarrow[n\to\infty]{} \varepsilon\beta$$

and

$$u_n \xrightarrow[n\to\infty]{\mathrm{L}^\infty} u \quad \textit{strongly}, \quad \textit{and} \quad u_n \xrightarrow[n\to\infty]{\mathrm{W}^{1,2}} u \quad \textit{weakly}, \quad \dot{u}_n \not\xrightarrow[n\to\infty]{} \dot{u} \quad \textit{a.e.}$$

and

$$\mathfrak{L}(u) < \liminf_{n\to\infty}\mathfrak{L}(u_n).$$

10.2. Counter-examples in dimension 2

Thanks of an article of Pinkall (see [41]), if u is a critical point of E_σ, then thanks of the Hopf fibration, we can create an Hopf torus which is a critical point of the Willmore energy. We will take slightly different conventions than the article of Pinkall. Let $p: S^3 \to S^2$ the map defined by

$$p(w,z) = (|w|^2 - |z|^2, 2w\overline{z})$$

for all $(w,z)\in S^3$, where

$$S^3 = \mathrm{C}^2 \cap \left\{(w,z) : |w|^2 + |z|^2 = 1\right\}.$$

We recall that p is surjective, and we see that it is invariant by the action of S^1 by rotation. It will be convenient for computations to use quaternions for writing the Hopf fibration. Le $q \mapsto \tilde{q}$ is the real vector space automorphism such that q leaves 1, j and k unchanged, and which sends i to $-i$. It is easy to verify that the Hopf fibration is given by

$$p(q) = \widetilde{q}q$$

if we identify $q = (w, z) \in S^3$.

Let γ a curve $\gamma : [0, L] \to S^2$ a closed curve of length $L > 0$. Let $\varGamma$ a lift of γ by the fibration p, *i.e.* a curve $\varGamma : [0, L] \to S^3$ such that $p \circ \varGamma = \gamma$. We now parametrize $\varGamma$ by arc-length, and we defined the Hopf torus of γ as

$$\varGamma(t, \theta) = \mathrm{e}^{i\theta}\varGamma(t)$$

and assume that $t \mapsto \dot{\varGamma}(t)$ is orthogonal to $\partial_\theta \varGamma$. As $\varGamma \neq 0$, $\dot{\varGamma}$ is proportional to $\varGamma$ in $\mathbb{H}$: there exists a smooth function $\lambda : [0, L] \to \mathbb{H}$ such that

$$\dot{\varGamma}(t) = \lambda(t)\varGamma(t), \quad \forall t \in [0, L].$$

Moreover, $\left\langle \dot{\varGamma}, \varGamma \right\rangle = 0$ implies that $\mathrm{Re}\,\lambda = 0$, and as $\partial_t \varGamma$ is proportional to $\partial_\theta \varGamma$, λ is orthogonal to $\mathrm{e}^{i\theta}$ for all $\theta \in S^1$, so $\lambda \in \mathrm{Span}(j, k)$. To produce the counter-example, we now proceed with the derivation of the mean curvature of the Hopf torus $\varGamma$.

We have

$$\dot{\gamma} = 2\widetilde{\varGamma}\lambda\varGamma.$$

so $|\dot{\gamma}| = 2$. We should be now careful that $\gamma : [0, \frac{L}{2}] \to S^2$, and $\varGamma : [0, \frac{L}{2}] \times S^1 \to S^3$. We can easily check that $\mathfrak{n} : [0, \frac{L}{2}] \to S^3$ is a unit normal vector field to the surface $\varGamma$, if

$$\mathfrak{n}(t, \theta) = i\mathrm{e}^{i\theta}\lambda(t)\varGamma(t).$$

If we define the function κ by the formula

$$\lambda' = 2i\kappa\lambda,$$

then

$$\begin{cases} \partial_t \mathfrak{n}(t, \theta) = -2\kappa(t)\partial_t \varGamma(t, \theta) - \partial_\theta \varGamma(t, \theta) \\ \partial_\theta \mathfrak{n}(t, \theta) = -\partial_t \varGamma(t, \theta) \end{cases}$$

and κ is also the curvature of the curve γ. Indeed,

$$\ddot{\gamma} = 2\widetilde{\varGamma}\lambda'\varGamma - 4\gamma$$

so

$$\nabla_{\frac{\dot{\gamma}}{|\dot{\gamma}|}} \frac{\dot{\gamma}}{|\dot{\gamma}|} = \frac{1}{4} 4\kappa\widetilde{\varGamma}i\lambda\varGamma = \kappa\nu$$

if ν is the normal of γ. If we had taken in the beginning a curve parametrized by arc-length, and if we write γ_0 the new curve in the arc-length parametrization of $\varGamma$, then

$$\gamma_0(t) = \gamma(2t),$$

and if we now write the curvature with the original curve, we get

$$\begin{cases} \partial_t \mathfrak{n}(t, \theta) = -2\kappa(t)\partial_t \varGamma(t, \theta) - \partial_\theta \varGamma(t, \theta) \\ \partial_\theta \mathfrak{n}(t, \theta) = -\partial_t \varGamma(t, \theta). \end{cases} \tag{10.11}$$

The mean curvature is defined as

$$H(t,\theta)=\frac{1}{2}\mathrm{Tr}\,\mathrm{d}\mathfrak{n}(t,\theta)$$

and the Gaussian curvature by

$$K(t,\theta)=\det \mathrm{d}\mathfrak{n}(t,\theta).$$

With the new convention about κ, we have

$$H(t,\theta)=\kappa(2t),\quad K(t,\theta)=-1.$$

We now define the Willmore σ-energy, by

$$W_\sigma(\vec{\Phi})=\int_\Sigma (1+\sigma^2|\vec{H}|^2)\mathrm{dvol}_g$$

is H is a the average of the principal curvature of an immersion $\vec{\Phi}$ from a Riemannian surface Σ in S^3. Then $\vec{\Phi}$ is a critical point of W_σ if and only if

$$2H=\sigma^2(\Delta_g H+2H(H^2-2K)) \tag{10.12}$$

if Δ_g is the Laplace operator for the metric g induced by $\vec{\Phi}$ on Σ by the metric of S^3, and K is the Gauss curvature. As $|\partial_t\Gamma|=|\partial_\theta\Gamma|=1$, and $\partial_t\Gamma$ is orthogonal to $\partial_\theta\Gamma$, we have

$$\Delta_g H(t,\theta)=\frac{\mathrm{d}^2}{\mathrm{d}t^2}\kappa(2t)=4\ddot{\kappa}(2t)$$

so (10.12) is equivalent to

$$2\kappa(2t)=\sigma^2(4\ddot{\kappa}(2t)+2\kappa(2t)^3+4\kappa(2t))$$

which is equivalent to

$$\kappa=\sigma^2(2\ddot{\kappa}+\kappa^3+2\kappa).$$

This last expression is nothing else than (10.1), so Γ is a critical point of W_σ if and only if γ is a critical point of E_σ. And

$$W_\sigma(\Gamma)=\int_{[0,\frac{L}{2}]}(1+\sigma^2\kappa(2t))\mathrm{d}t\,\mathrm{d}\theta=\pi E_\sigma(\gamma).$$

Furthermore, the second fundamental form $|\mathbb{I}_\Gamma|^2$ of Γ is equal to $4H^2-2K=4\kappa^2+2$, so

$$\begin{aligned}A_\sigma(\Gamma)&=\int_\Gamma(1+\sigma^2|\mathbb{I}_\Gamma|^2)\mathrm{dvol}_g=\int_{[0,\frac{L}{2}]\times S^1}(1+2\sigma^2+4\sigma^2\kappa^2(2t))\mathrm{d}t\mathrm{d}\theta\\&=(1+2\sigma^2)\pi E_{\sigma'}(\gamma),\quad \sigma'=\frac{2\sigma}{\sqrt{1+2\sigma^2}}.\end{aligned}$$

and as $|\mathbb{I}|^2$ depends only of H, Γ is a critical point of A_σ if and only if it is a critical point of $W_{\sigma'}$, every 1-dimension elliptic Jacobi function constructed in the preceding section can be lifted to a critical point of A_σ.

Proposition 10.4. *For all $\beta > 0$, there exists a sequence $\{\sigma_n\}_{n\in\mathbb{N}}$ of positive real numbers converging to 0, a sequence of flat torii $\left\{T_n^2\right\}_{n\in\mathbb{N}}$ converging to a torus T^2, and a sequence $\{\vec{\Phi}_n : T_n^2 \to S^3\}_{n\in\mathbb{N}}$ of conformal immersions which are critical points associated to $\{A_{\sigma_n}\}_{n\in\mathbb{N}}$, such that*

$$\lim_{n\to\infty} A_{\sigma_n}(\vec{\Phi}_n) = \beta, \quad \lim_{n\to\infty} \mathscr{H}^2(\vec{\Phi}_n(T_n^2)) = \frac{\beta}{2},$$

and $\{\vec{\Phi}_n\}_{n\in\mathbb{N}}$ weakly converges to a limiting map $\vec{\Phi} \in \mathrm{W}^{1,2}(T^2, S^3)$, but $\{\vec{\Phi}_n\}_{n\in\mathbb{N}}$ nowhere strongly converges; for all open subset $U \subset T^2$

$$\mathscr{H}^2(\vec{\Phi}(T^2 \cap U)) < \liminf_{n\to\infty} \mathscr{H}^2(\vec{\Phi}_n(T_n^2 \cap U)).$$

Proof. The proof is now an easy consequence of (10.7), as we can lift the degenerate family of critical $\{u_n\}_{n\in\mathbb{N}}$ constructed in the preceding subsection in a family of immersions $\{\vec{\Phi}_n\}_{n\in\mathbb{N}}$ which are critical points of $\{A_{\sigma_n}\}$[**], where for all $n \in \mathbb{N}$,

$$T_n = [0, a_n] \times S^1 = \mathbb{R}^2 / \left(a_n + 2\pi i\right) \mathbb{Z}^2 \quad \left(a_n = \frac{L_n}{2}\right)\cdot$$

As $L_n \xrightarrow[n\to\infty]{} L$, and the lifted curves are conformal immersions $\{\vec{\Phi}_n\}_{n\in\mathbb{N}}$ such that

$$|\partial_t \vec{\Phi}_n| = |\partial_\theta \vec{\Phi}_n| = 1,$$

this sequence is bounded in $\mathrm{W}^{1,2}(S^1, M)$, so converges weakly to an element $\vec{\Phi} \in \mathrm{W}^{1,2}(S^1, M)$, and thanks of Proposition 10.1, for all open subset $U \subset T^2$

$$\mathscr{H}^2(\vec{\Phi}(T^2 \cap U)) < \liminf_{n\to\infty} \mathscr{H}^2(\vec{\Phi}_n(T_n^2 \cap U)),$$

which concludes the proof. □

Appendix A. Completeness of the space of immersions

Let $(M^m, h) \subset \mathbb{R}^q$ a C^ν $(\nu \geq 3)$ embedded Riemannian submanifold of $\mathbb{R}^q$. We recall the definitions

$$\mathrm{W}^{2,2}(S^1, M) = \mathrm{W}^{2,2}(S^1, \mathbb{R}^q) \cap \left\{\gamma :\ \gamma(t) \in M \text{ for } \mathscr{L}^1 \text{ almost all } t \in S^1\right\}$$
$$\mathrm{W}^{2,2}_\iota(S^1, M) = \mathrm{W}^{2,2}(S^1, M) \cap \left\{\gamma :\ \dot{\gamma}(t) \neq 0 \text{ for } \mathscr{L}^1 \text{almost all}\, t \in S^1\right\}$$

and for all $\gamma \in \mathrm{W}^{2,2}_\iota(S^1, M)$, we define

$$\mathrm{W}^{2,2}_\gamma(S^1, TM) = \mathrm{W}^{2,2}(S^1, \mathbb{R}^q) \cap \left\{v :\ v(t) \in T_{\gamma(t)}M \text{ for } \mathscr{L}^1 \text{almost all } t \in S^1\right\}.$$

Then we define for all $\gamma \in \mathrm{W}^{2,2}_\iota(S^1, M)$ the following norm for $v \in \mathrm{W}^{2,2}_\gamma(S^1, TM)$

$$\|v\|_\gamma = \left(\int_{S^1} \left(|v|^2 + |\nabla v|_g^2 + |\nabla^2 v|_g^2\right) d\mathrm{vol}_g\right)^{\frac{1}{2}}$$

where $g = \gamma^* h$ (locally $g = |\dot{\gamma}| d\mathscr{L}^1$), and $\nabla = \gamma^* \nabla^h$ is the pull-back of the Levi−Civita connection ∇^h of (M^m, h). We first make a the following simple remark that this norm controls the L^∞ norm of $|\nabla v|_g$. Indeed,

[**]Changing the σ_n of the 1-dimensional counter-example into σ'_n.

we have by Cauchy−Schwarz inequality, taking arc-length parametrisation (where $L = \mathfrak{L}(u)$)

$$\begin{aligned}\int_{S^1} |\nabla^2 v|^2 \mathrm{dvol}_g &= \int_0^L |\nabla^2_{\partial_t \gamma} v|^2 \mathrm{d}t \\ &\geq \frac{1}{L}\left(\int_0^L |\nabla_{\partial_t\gamma}(\nabla_{\partial_t\gamma} v)|\mathrm{d}t\right) \\ &\geq \frac{1}{L} \| |\nabla_{\gamma_t} v| \|^2_{\mathrm{L}^\infty([0,L])} \\ &= \frac{1}{L}\, \| |\nabla v|_g \|^2_{\mathrm{L}^\infty(S^1)}\end{aligned}$$

so

$$\| |\nabla v|_g \|_{\mathrm{L}^\infty(S^1)} \leq \sqrt{L}\left(\int_{S^1} |\nabla^2 v|_g^2 \mathrm{dvol}_g\right)^{\frac{1}{2}} \leq \mathfrak{L}(\gamma)\, \|v\|_\gamma \,. \tag{A.1}$$

and in particular, the factor $|\nabla v|_g^2$ is irrelevant, and only added for convenience in the proof of the following theorem.

Then, by an immediate adaptation of the arguments in the book of Klingenberg for the case $\mathrm{W}^{1,2}(S^1, M)$ ([27], 1.2), $\mathrm{W}^{2,2}(S^1, M)$ is a complete C^2 Hilbert submanifold of the Hilbert space $\mathrm{W}^{2,2}(S^1, \mathbb{R}^q)$. As $\mathrm{W}^{2,2}_\iota(S^1, M)$ is an open subset of $\mathrm{W}^{2,2}(S^1, M)$, it is also a C^2 Hilbert manifold and for all $\gamma \in \mathrm{W}^{2,2}_\iota(S^1, M)$ the tangent space $T_\gamma \mathrm{W}^{2,2}_\iota(S^1, M)$ is simply

$$T_\gamma \mathrm{W}^{2,2}_\iota(S^1, M) = \mathrm{W}^{2,2}_\gamma(S^1, TM).$$

Therefore, equipped with the family of norms $\left\{ \|\cdot\|_\gamma \right\}_{\gamma \in \mathrm{W}^{2,2}_\iota(S^1, M)}$, the space of immersions $\mathrm{W}^{2,2}_\iota(S^1, M)$ is a C^2 Finsler manifold taking local trivialisation induced by the preceding Hilbert manifold structure. Then if $\gamma_0, \gamma_1 \in \mathrm{W}^{2,2}_\iota(S^1, M)$, we define

$$d(\gamma_0, \gamma_1) = \inf\left\{\int_0^1 \|\partial_s \gamma(s,\cdot)\|_{\gamma(s,\cdot)}\, ds : \gamma \in C^1([0,1], \mathrm{W}^{2,2}_\iota(S^1, M)), \gamma(0) = \gamma_0, \gamma(1) = \gamma_1\right\}$$

A classical result of Palais [39] asserts that d is a distance on $\mathrm{W}^{2,2}_\iota(S^1, M)$. However, this construction does not address the problem of completeness of $\mathrm{W}^{2,2}_\iota(S^1, M)$ equipped with this distance d, and this issue is treated in the following theorem.

Theorem A.1. *The Finsler manifold* $(\mathrm{W}^{2,2}_\iota(S^1, M), d)$ *is a complete metric space.*

Proof. We first need a simple form of Grönwall's lemma

Lemma A.2. *Let* $f \in C^1([0,1])$, $g \in \mathrm{L}^1([0,1])$, *such that for all* $s \in (0,1)$

$$f'(s) \leq g(s)(1 + f(s)).$$

Then for all $s \in [0,1]$, *we have*

$$f(s) \leq -1 + (1 + f(0))\mathrm{e}^{\int_0^s g(\tau)\mathrm{d}\tau}.$$

Proof. We simply differentiate

$$F(s) = (1 + f(s))\, \mathrm{e}^{-\int_0^s g(\tau)\mathrm{d}\tau}$$

to get $F'(s) \leq 0$. Therefore F is decreasing and $F(s) \leq F(0)$ for all $s \in [0,1]$, which implies the afore mentioned conclusion. ☐

We now come back to the proof of the theorem.

Step 1. Uniform control of the $\mathrm{W}^{2,2}$ Finsler norm.

Let $\gamma \in C^1([0,1], \mathrm{W}_\iota^{2,2}(S^1, M))$ a path such that

$$\mathscr{E}(\gamma) = \int_0^1 \|\partial_s \gamma(s,\cdot)\|_{\gamma(s,\cdot)}\,\mathrm{d}s < \infty$$

We want to make sure that under this hypothesis, we have $\gamma(1) \in \mathrm{W}_\iota^{2,2}(S^1, M)$ *i.e.* that no degeneracy can occur. We first check that γ_1 has controlled $\mathrm{W}^{2,2}$ norm. This will actually result of an uniform control as the $\mathrm{W}^{2,2}$ norm. We recall the notations

$$\gamma_t = \partial_t \gamma, \quad \gamma_s = \partial_s \gamma, \quad \text{and } \overline{\gamma}_t = \frac{\gamma_t}{|\gamma_t|}.$$

Then, if $u \in \mathrm{W}_\iota^{2,2}(S^1, M)$, we make the decomposition

$$\|u\|_{\mathrm{W}^{2,2}(S^1)}^2 = \int_{S^1} \left(|u|^2 + |\nabla u|_g^2 + |\nabla^2 u|_g^2\right)\mathrm{dvol}_g = \|u\|_{\mathrm{L}^2(S^1)}^2 + \|u\|_{\dot{\mathrm{W}}^{1,2}(S^1)}^2 + \|u\|_{\dot{\mathrm{W}}^{2,2}(S^1)}^2$$

First by definition we have $|\nabla \gamma|_g = 1$, so

$$\|\gamma\|_{\dot{\mathrm{W}}^{1,2}(S^1)} = \int_{S^1} |\nabla \gamma|_g^2 \mathrm{dvol}_g = \int_{S^1} \mathrm{dvol}_g = \int_{S^1} |\gamma_t|\mathrm{d}t$$

and by Cauchy−Schwarz inequality,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\|\gamma\|_{\dot{\mathrm{W}}^{1,2}(S^1)}^2 &= \int_{S^1} \left\langle \nabla_{\overline{\gamma}_t}\gamma_s, \overline{\gamma}_t\right\rangle |\gamma_t|\mathrm{d}t \\ &\leq \left(\int_{S^1} |\nabla \gamma_s|_g^2 \mathrm{dvol}_g\right)^{\frac{1}{2}} \left(\int_{S^1} |\gamma_t|\mathrm{d}t\right)^{\frac{1}{2}} \\ &\leq \|\partial_s \gamma(s,\cdot)\|_{\gamma(s,\cdot)} \|\gamma\|_{\dot{\mathrm{W}}^{1,2}(S^1)}\end{aligned}$$

so

$$\sup_{s\in[0,1]} \|\gamma(s,\cdot)\|_{\dot{\mathrm{W}}^{1,2}(S^1)} \leq \|\gamma(0,\cdot)\|_{\dot{\mathrm{W}}^{1,2}(S^1)} + \frac{1}{2}\int_0^1 \|\partial_s \gamma(s,\cdot)\|_{\gamma(s,\cdot)}\,\mathrm{d}s = \Gamma_1 \tag{A.2}$$

Then we have by (A.2) and (A.1)

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\|\gamma\|_{\mathrm{L}^2(S^1)}^2 = \frac{\mathrm{d}}{\mathrm{d}s}\int_{S^1}|\gamma|^2\mathrm{dvol}_g &= \int_{S^1}\left(2\left\langle \gamma_s, \gamma\right\rangle + |\gamma|^2\left\langle \nabla_{\overline{\gamma}_t}\gamma_s, \overline{\gamma_t}\right\rangle\right)|\gamma_t|\mathrm{d}t \\ &\leq 2\left(\int_{S^1}|\gamma|^2\mathrm{dvol}_g\right)^{\frac{1}{2}}\left(\int_{S^1}|\gamma_s|^2\mathrm{dvol}_g\right)^{\frac{1}{2}} + \|\,|\nabla\gamma_s|_g\,\|_{\mathrm{L}^\infty(\mathrm{S}^1)}\int_{S^1}|\gamma|^2\mathrm{dvol}_g \\ &\leq 2\left(\int_{S^1}|\gamma|^2\mathrm{dvol}_g\right)^{\frac{1}{2}}\left(\int_{S^1}|\gamma_s|^2\mathrm{dvol}_g\right)^{\frac{1}{2}} + \Gamma_1\left(\int_0^1|\nabla^2\gamma_s|^2\mathrm{dvol}_g\right)^{\frac{1}{2}}\int_{S^1}|\gamma|^2\mathrm{dvol}_g.\end{aligned}$$

so by Young's inequality

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\|\gamma(s,\cdot)\|_{\mathrm{L}^2(S^1)}^2 &\leq (2+\Gamma_1)\|\partial_s\gamma(s,\cdot)\|_{\gamma(s,\cdot)}\left(\|\gamma(s,\cdot)\|_{\mathrm{L}^2(S^1)} + \|\gamma(s,\cdot)\|_{\mathrm{L}^2(S^1)}^2\right) \\ &\leq 2(2+\Gamma_1)\|\partial_s\gamma(s,\cdot)\|_{\gamma(s,\cdot)}\left(1 + \|\gamma(s,\cdot)\|_{\mathrm{L}^2(S^1)}^2\right)\end{aligned}$$

while by Grönwall's lemma, we have

$$\sup_{s\in[0,1]}\|\gamma(s,\cdot)\|^2_{\mathrm{L}^2(S^1)} \leq -1+\left(1+\|\gamma(0,\cdot)\|^2_{\mathrm{L}^2(S^1)}\right)\mathrm{e}^{2(2+\Gamma_1)\int_0^1\|\partial_s\gamma(s,\cdot)\|_{\gamma(s,\cdot)}\mathrm{d}s}$$
$$\leq \left(1+\|\gamma(0,\cdot)\|^2_{\mathrm{L}^2(S^1)}\right)\mathrm{e}^{2(2+\Gamma_1)\mathscr{E}(\gamma)}. \tag{A.3}$$

Finally we deduce by (A.1) that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}s}\|\gamma\|^2_{\dot{\mathrm{W}}^{2,2}(S^1)} = \frac{\mathrm{d}}{\mathrm{d}s}\int_{S^1}|\nabla_{\gamma_t}\gamma_t|^2|\gamma_t|^{-3}\mathrm{d}t\\
&= \int_{S^1} 2\left\langle \nabla_{\gamma_s}\nabla_{\gamma_t}\gamma_t,\nabla_{\gamma_t}\gamma_t\right\rangle|\gamma_t|^{-3}\mathrm{d}t - 3\int_{S^1}|\nabla_{\gamma_t}\gamma_t|^2\left\langle\nabla_{\gamma_t}\gamma_s,\gamma_t\right\rangle|\gamma_t|^{-5}\mathrm{d}t\\
&= 2\int_{S^1}\left\langle\nabla^2_{\gamma_t}\gamma_s+R(\gamma_s,\gamma_t)\gamma_t,\nabla_{\gamma_t}\gamma_t\right\rangle|\gamma_t|^{-3}\mathrm{d}t - 3\int_{S^1}|\nabla^2\gamma|_g^2\left\langle\nabla_{\overline{\gamma}_t}\gamma_s,\overline{\gamma}_t\right\rangle|\gamma_t|\mathrm{d}t\\
&= 2\int_{S^1}\left(\frac{1}{|\gamma_t|^4}\left\langle\nabla^2_{\gamma_t}\gamma_s,\nabla_{\gamma_t}\gamma_t\right\rangle+\frac{1}{|\gamma_t|^2}\left\langle R(\gamma_s,\overline{\gamma}_t)\overline{\gamma}_t,\nabla_{\gamma_t}\gamma_t\right\rangle\right)|\gamma_t|\,\mathrm{d}t - 3\int_{S^1}|\nabla^2\gamma|_g^2\left\langle\nabla_{\overline{\gamma}_t}\gamma_s,\overline{\gamma}_t\right\rangle|\gamma_t|\mathrm{d}t\\
&\leq 2\left(\int_{S^1}|\nabla^2\gamma_s|_g^2\mathrm{dvol}_g\right)^{\frac12}\left(\int_{S^1}|\nabla^2\gamma|^2\mathrm{dvol}_g\right)^{\frac12}+2\|R\|_{\mathrm{L}^\infty(M)}\left(\int_{S^1}|\gamma_s|_g^2\mathrm{dvol}_g\right)^{\frac12}\left(\int_{S^1}|\nabla^2\gamma|_g^2\mathrm{dvol}_g\right)^{\frac12}\\
&+3\||\nabla\gamma_s|_g\|_{L^\infty(S^1)}\int_{S^1}|\nabla^2\gamma|_g^2\mathrm{dvol}_g\\
&\leq 2\left(\int_{S^1}|\nabla^2\gamma_s|_g^2\mathrm{dvol}_g\right)^{\frac12}\left(\int_{S^1}|\nabla^2\gamma|^2\mathrm{dvol}_g\right)^{\frac12}+2\|R\|_{\mathrm{L}^\infty(M)}\left(\int_{S^1}|\gamma_s|_g^2\mathrm{dvol}_g\right)^{\frac12}\left(\int_{S^1}|\nabla^2\gamma|_g^2\mathrm{dvol}_g\right)^{\frac12}\\
&+3\Gamma_1\left(\int_0^1|\nabla^2\gamma_s|^2\mathrm{dvol}_g\right)^{\frac12}\int_{S^1}|\nabla^2\gamma|_g^2\mathrm{dvol}_g
\end{aligned}$$

so for all $s\in[0,1]$, we have by Young's inequality

$$\frac{\mathrm{d}}{\mathrm{d}s}\|\gamma(s,\cdot)\|^2_{\dot{W}^{2,2}(S^1)} \leq \|\partial_s\gamma(s,\cdot)\|_{\gamma(s,\cdot)}\left(2(1+\|R\|_{\mathrm{L}^\infty(M)})\|\gamma(s,\cdot)\|_{\dot{W}^{2,2}(S^1)}+3\Gamma_1\|\gamma(s,\cdot)\|^2_{\dot{W}^{2,2}(S^1)}\right)$$
$$\leq\left(1+3\Gamma_1+\|R\|_{\mathrm{L}^\infty(S^1)}\right)\|\partial_s\gamma(s,\cdot)\|_{\gamma(s,\cdot)}\left(1+\|\gamma(s,\cdot)\|^2_{\dot{W}^{2,2}(S^1)}\right)$$

Therefore, we obtain by Grönwall's lemma for $\Gamma_2 = 1++3\Gamma_1+\|R\|_{\mathrm{L}^\infty(S^1)}$

$$\sup_{s\in[0,1]}\|\gamma(s,\cdot)\|^2_{\dot{\mathrm{W}}^{2,2}(S^1)}\leq\left(1+\|\gamma(0,\cdot)\|^2_{\dot{\mathrm{W}}^{2,2}(S^1)}\right)\mathrm{e}^{\Gamma_2\mathscr{E}(\gamma)}<\infty \tag{A.4}$$

So finally, we obtain by (A.3), (A.2) and (A.4) for some constant $0<C_0<\infty$ (depending on $\mathscr{E}(\gamma)$ and the curvature of M)

$$\sup_{s\in[0,1]}\|\gamma(s,\cdot)\|^2_{\mathrm{W}^{2,2}(S^1)}\leq C_0\left(1+\|\gamma(0,\cdot)\|^2_{\mathrm{W}^{2,2}(S^1)}\right)\mathrm{e}^{C_0\mathscr{E}(\gamma)}<\infty. \tag{A.5}$$

Step 2. We now want to show that γ_1 is still an immersion, *i.e.*

$$\sup_{s\in[0,1]}\|\log|\dot{\gamma}(s)|\|_{\mathrm{L}^\infty(S^1)}<\infty.$$

We aim at proving the following finite energy inequality

$$\int_0^1\int_{S^1}\left|\frac{\partial^2}{\partial s\partial t}\log|\partial_t\gamma(s,t)|\right|\mathrm{d}t\mathrm{d}s<\infty.$$

Abbreviating $\gamma_s = \partial_s\gamma$, $\gamma_t = \partial_t\gamma$, we compute by compatibility of the Levi−Civita connection, and its absence of torsion

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{\partial}{\partial s}\log|\gamma_t|\right) &= \frac{\partial}{\partial t}\left(\frac{\langle\nabla_{\gamma_s}\gamma_t,\gamma_t\rangle}{|\gamma_t|^2}\right) = \frac{\partial}{\partial t}\left(\left\langle\nabla_{\frac{\gamma_t}{|\gamma_t|}}\gamma_s,\frac{\gamma_t}{|\gamma_t|}\right\rangle\right)\\ &= \left\langle\nabla^2_{\frac{\gamma_t}{|\gamma_t|}}\gamma_s,\frac{\gamma_t}{|\gamma_t|}\right\rangle|\gamma_t| + \left\langle\nabla_{\frac{\gamma_t}{|\gamma_t|}}\gamma_s,\nabla_{\frac{\gamma_t}{|\gamma_t|}}\frac{\gamma_t}{|\gamma_t|}\right\rangle|\gamma_t|\end{aligned}$$

therefore by Cauchy−Schwarz inequality

$$\begin{aligned}&\int_0^1\int_{S^1}\left|\frac{\partial^2}{\partial s\partial t}\log|\partial_t\gamma(s,t)|\right|\mathrm{d}t\mathrm{d}s \leq \int_0^1\int_{S^1}\left(\left|\nabla^2_{\frac{\gamma_t}{|\gamma_t|}}\gamma_s\right||\gamma_t| + |\nabla_{\frac{\gamma_t}{|\gamma_t|}}\gamma_s|\kappa(\gamma)|\gamma_t|\right)\mathrm{d}t\mathrm{d}s\\ &\leq \int_0^1\left(\int_{S^1}|\nabla^2\gamma_s|_g^2|\gamma_t|\mathrm{d}t\right)^{\frac{1}{2}}\left(\int_{S^1}|\gamma_t|\mathrm{d}t\right)^{\frac{1}{2}}\mathrm{d}s + \int_0^1\left(\int_{S^1}|\nabla\gamma_s|_g^2|\gamma_t|\mathrm{d}t\right)^{\frac{1}{2}}\left(\int_{S^1}\kappa^2(\gamma)|\gamma_t|\mathrm{d}t\right)^{\frac{1}{2}}\mathrm{d}s\end{aligned}$$

Then by (A.2), we have for all $s \in [0,1]$

$$C_1^2 = \sup_{s\in[0,1]}\int_{S^1}|\partial_t\gamma(s,t)|\mathrm{d}t = \sup_{s\in[0,1]}\|\gamma(s,\cdot)\|^2_{\dot{W}^{1,2}(S^1)} \leq \|\gamma(0,\cdot)\|^2_{\dot{W}^{1,2}(S^1)} + \frac{1}{2}\int_0^1\|\partial_s\gamma(s,\cdot)\|_{\gamma(s,\cdot)}\mathrm{d}s$$

On the other hand,

$$\begin{aligned}\kappa^2(\gamma) &= |\nabla_{\overline{\gamma}_t}\overline{\gamma_t}|^2 = \frac{1}{|\gamma_t|^2}\left|\frac{1}{|\gamma_t|}\nabla_{\gamma_t}\gamma_t - \frac{1}{|\gamma_t|^3}\langle\nabla_{\gamma_t}\gamma_t,\gamma_t\rangle\gamma_t\right|^2\\ &= \frac{1}{|\gamma_t|^2}\left(\frac{1}{|\gamma_t|^2}|\nabla_{\gamma_t}\gamma_t|^2 - \frac{2}{|\gamma_t|^4}\langle\nabla_{\gamma_t}\gamma_t,\gamma_t\rangle^2 + \frac{1}{|\gamma_t|^6}\langle\nabla_{\gamma_t}\gamma_t,\gamma_t\rangle^2|\gamma_t|^2\right)\\ &= \frac{1}{|\gamma_t|^4}|\nabla_{\gamma_t}\gamma_t|^2 - \frac{\langle\nabla_{\gamma_t}\gamma_t,\gamma_t\rangle^2}{|\gamma_t|^6}\\ &= |\nabla^2\gamma|_g^2 - \frac{\langle\nabla_{\gamma_t}\gamma_t,\gamma_t\rangle^2}{|\gamma_t|^6}\\ &\leq |\nabla^2\gamma|_g^2\end{aligned}$$

therefore by (A.4)

$$C_2^2 = \sup_{s\in[0,1]}\int_{S^1}\kappa^2(\gamma(s,\cdot))\mathrm{dvol}_g \leq \sup_{s\in[0,1]}\left\|\nabla^2\gamma(s,\cdot)\right\|^2_{\dot{\mathrm{W}}^{2,2}(S^1)} \leq \left(1 + \|\gamma(0,\cdot)\|^2_{\dot{\mathrm{W}}^{2,2}(S^1)}\right)\mathrm{e}^{\Gamma_2\mathscr{E}(\gamma)} < \infty$$

and

$$\int_0^1\int_{S^1}\left|\frac{\partial^2}{\partial s\partial t}\log|\partial_t\gamma(s,t)|\right|\mathrm{d}t\mathrm{d}s \leq \max\{C_1,C_2\}\,\mathscr{E}(\gamma) < \infty$$

and by Sobolev injection, we get the result. Indeed, for $\mathscr{L}^1$ almost all $s \in [0,1]$ we have $t \to \partial_s\log|\gamma_t(s,t)| \in \mathrm{W}^{1,1}(S^1)$, so by Sobolev embedding $\mathrm{W}^{1,1}(S^1) \subset \mathrm{L}^\infty(S^1)$, we have

$$\int_0^1\|\partial_s\log|\dot{\gamma}(s)|\,\|_{\mathrm{L}^\infty(S^1)}\mathrm{d}s < \infty.$$

Now we define

$$\begin{aligned}f : [0,1] &\to \mathbb{R}_+\\ s &\mapsto \|\log|\dot{\gamma}(s)|\|_{\mathrm{L}^\infty(S^1)}\end{aligned}$$

and for all $m \in \mathbb{N}$, and $0 = s_0 < \ldots < s_n = 1$, we have by the triangle inequality

$$\begin{aligned}\sum_{i=1}^{n} |f(t_i) - f(t_{i-1})| &= \sum_{i=1}^{m} \left| \|\log|\dot{\gamma}(s_i)|\|_{\mathrm{L}^\infty(S^1)} - \|\log|\dot{\gamma}(s_{i-1})|\|_{\mathrm{L}^\infty(S^1)} \right| \\ &\leq \sum_{i=1}^{n} \|\log|\dot{\gamma}(s_i)| - \log|\dot{\gamma}(s_{i-1})|\,\|_{\mathrm{L}^\infty(S^1)} \\ &= \sum_{i=1}^{n} \left\| \int_{s_{i-1}}^{s_i} \partial_s \log|\dot{\gamma}(s)| \mathrm{d}s \right\|_{\mathrm{L}^\infty(S^1)} \\ &\leq \sum_{i=1}^{n} \int_{s_{i-1}}^{s_i} \|\partial_s \log|\dot{\gamma}(s)|\|_{\mathrm{L}^\infty(S^1)} \mathrm{d}s \\ &= \int_0^1 \|\partial_s \log|\dot{\gamma}(s)|\|_{\mathrm{L}^\infty(S^1)} \mathrm{d}s\end{aligned}$$

and following [19] 2.5.16, if $g : \mathbb{R} \to \mathbb{R}$, and $-\infty < a < b < \infty$, we define the total variation of g between a and b as

$$\mathrm{V}_a^b\, g = \sup \sum_{i=1}^{n} |g(s_i) - g(s_{i-1})|$$

corresponding to all finite sequences $a = s_0 < \ldots < s_n = b$. Then by 2.9.19(2) of [19] if $V_a^b\, g < \infty$, the derivative g' exists $\mathscr{L}^1$ almost everywhere on (a, b) and

$$\int_a^b |g'| \mathrm{d}\mathscr{L}^1 \leq V_a^b\, g < \infty.$$

Therefore, we deduce that f is a function of bounded variation on $[0, 1]$ and

$$\int_0^1 \left| \partial_s \|\log|\dot{\gamma}(s)|\|_{\mathrm{L}^\infty(S^1)} \right| \mathrm{d}s \leq \int_0^1 \|\partial_s \log|\dot{\gamma}(s)|\,\|_{\mathrm{L}^\infty(S^1)} < \infty$$

so $f \in W^{1,1}([0, 1])$ so by Sobolev embedding, we conclude that

$$\sup_{s\in[0,1]} \|\log|\dot{\gamma}(s)|\,\|_{\mathrm{L}^\infty(S^1)} \leq \int_0^1 \|\partial_s \log|\dot{\gamma}(s)|\,\|_{\mathrm{L}^\infty(S^1)} \leq \max\{C_1, C_2\}\, \mathscr{E}(\gamma) < \infty. \tag{A.6}$$

Therefore, γ_1 is an immersion. Therefore, this inequality (A.6) together with (A.5) shows the completeness of the metric space $(\mathrm{W}_\iota^{2,2}(S^1, M), d)$. □

Acknowledgements. We would like to thank the referee for numerous useful comments and remarks which allowed us to improve the quality of this manuscript and to clarify some obscure points.

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