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WELL-POSEDNESS OF THE SUPERCRITICAL LANE–EMDEN HEAT FLOW IN MORREY SPACES

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Abstract. For any smoothly bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 3$, and any exponent $p > 2^* = 2n/(n-2)$ we study the Lane–Emden heat flow $u_t - \Delta u = |u|^{p-2}u$ on $\Omega \times]0,T[$ and establish local and global well-posedness results for the initial value problem with suitably small initial data $u\big|_{t=0} = u_0$ in the Morrey space $L^{2,\lambda}(\Omega)$ for suitable $T \leq \infty$, where $\lambda = 4/(p-2)$. We contrast our results with results on instantaneous complete blow-up of the flow for certain large data in this space, similar to ill-posedness results of Galaktionov–Vazquez for the Lane–Emden flow on $\mathbb{R}^n$.

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1. Introduction

Let Ω be a smoothly bounded domain in $\mathbb{R}^n$, $n \geq 3$, and let $T > 0$. Given initial data u_0, we consider the Lane–Emden heat flow

$$u_t - \Delta u = |u|^{p-2}u \text{ on } \Omega \times [0,T[,\ u = 0 \text{ on } \partial\Omega \times [0,T[,\ u\big|_{t=0} = u_0 \tag{1.1}$$

for a given exponent $p > 2^* = 2n/(n-2)$, that is, in the "supercritical" regime.

As observed by Matano–Merle [14], p. 1048, the initial value problem (1.1) may be ill-posed for certain data $u_0 \in H_0^1 \cap L^p(\Omega)$; see also our results in Section 4 below. However, as we had shown in two previous papers [4], Section 6.5, [5], Remark 3.3, the Cauchy problem (1.1) is globally well-posed for suitably small data u_0 belonging to the Morrey space $H_0^{1,\mu} \cap L^{p,\mu}(\Omega)$, where $\mu = \frac{2p}{p-2} < n$. Here we go one step further and show that problem (1.1) even is well-posed for suitably small data $u_0 \in L^{2,\lambda}(\Omega) \supset L^{p,\mu}(\Omega)$, where $\lambda = \frac{2\mu}{p} = \frac{4}{p-2} = \mu - 2$, thus considerably improving on the results of Brezis–Cazenave [6] or Weissler [16] for initial data in L^q, $q \geq n(p-2)/2$. Our results are similar to results of Taylor [15] who demonstrated local and global well-posedness of the Cauchy problem for the equation

$$u_t - \Delta u = DQ(u) \text{ on } \Omega \times [0,T[,$$

for suitably small initial data $u\big|_{t=0} = u_0$ in a Morrey space, where D is a linear differential operator of first order and Q is a quadratic form in u as in the Navier–Stokes system. However, similar to the work

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of Koch–Tataru [12] on the Navier–Stokes system, in our treatment of (1.1) we are able to completely avoid the use of pseudodifferential operators in favor of simple integration by parts and Banach's fixed-point theorem.

The study of the initial value problem for (1.1) for non-smooth initial data is motivated by the question whether a solution u of (1.1) blowing up at some time $T < \infty$ can be extended as a weak solution of (1.1) on a time interval $]0, T_1[$ for some $T_1 > T$. Note that if such a continuation is possible and if the extended solution still satisfies the monotonicity formula [5], Proposition 3.1, it follows that $u(T) \in L^{2,\lambda}(\Omega)$; see Remark 3.3. Hence, the regularity assumption $u_0 \in L^{2,\lambda}(\Omega)$ is necessary from this point of view and cannot be weakened. However, our results in Section 4 show that the condition $u(T) \in L^{2,\lambda}(\Omega)$ in general is not sufficient for continuation and that a smallness condition as in our Theorems 2.1, 2.2 below is needed.

Note that the question of continuation after blow-up only is of relevance in the supercritical case when $p > 2^*$. Indeed, as shown by Baras–Cohen [3], in the subcritical case $p < 2^*$ a classical solution $u \geq 0$ to (1.1) blowing up at some time $T < \infty$ always undergoes "complete blow-up" (see Sect. 4 for a definition), and u cannot be continued as a (weak) solution to (1.1) after time T in any reasonable way. In [9] Galaktionov und Vazquez extend the Baras–Cohen result to the critical case $p = 2^*$.

In the next section we state our well-posedness results, which we prove in Section 3. In Section 4 we then contrast these results with results on instantaneous complete blow-up of the flow for certain large data $u_0 \geq 0$. These results crucially use the scaling properties of equation (1.1) and the maximum principle by comparing our solution with a family of flow solutions blowing up in finite time, with the time of blow-up arbitrarily close to zero after suitable scaling, in a way similar to the ill-posedness results of Galaktionov–Vazquez for the Lane–Emden flow on $\mathbb{R}^n$; see for instance [9], Theorem 10.4. We conclude the paper with some open problems.

Note that in dimension $n = 2$ the limit case of Sobolev's embedding is given by the Orlicz map

$$M_\alpha = \{u \in H_0^1(\Omega);\ ||\nabla u||_{L^2}^2 = \alpha\} \ni u \mapsto \mathrm{e}^{u^2} \in L^1(\Omega)$$

when $\alpha = 4\pi$. In [13], Lamm–Robert–Struwe study a variant of the corresponding Lane–Emden type flow also in a range of super-critical "energies" $\alpha > 4\pi$.

2. Global and local well-posedness

Recall that for any $1 \leq p < \infty$, $0 < \lambda < n$ (in Adams' [1] notation) a function $f \in L^p(\Omega)$ on a domain $\Omega \subset \mathbb{R}^n$ belongs to the Morrey space $L^{p,\lambda}(\Omega)$ if

$$\|f\|_{L^{p,\lambda}(\Omega)}^p := \sup_{x_0 \in R^n,\ r>0} r^{\lambda - n} \int_{B_r(x_0)\cap\Omega} |f|^p \mathrm{d}x < \infty, \tag{2.1}$$

where $B_r(x_0)$ denotes the Euclidean ball of radius $r > 0$ centered at x_0. Moreover, we write $f \in L_0^{p,\lambda}(\Omega)$ whenever $f \in L^{p,\lambda}(\Omega)$ satisfies

$$\sup_{x_0 \in R^n,\ 0<r<r_0} r^{\lambda - n} \int_{B_r(x_0)\cap\Omega} |f|^p \mathrm{d}x \to 0 \text{ as } r_0 \downarrow 0.$$

Similarly, for any $1 \leq p < \infty$, $0 < \mu < n + 2$ a function $f \in L^p(E)$ on $E \subset \mathbb{R}^n \times \mathbb{R}$ belongs to the parabolic Morrey space $L^{p,\mu}(E)$ if

$$\|f\|_{L^{p,\mu}(E)}^p := \sup_{z_0=(x_0,t_0)\in R^{n+1}, r>0} r^{\mu-(n+2)} \int_{P_r(z_0)\cap E} |f|^p \mathrm{d}z < \infty,$$

where $P_r(x,t)$ denotes the backwards parabolic cylinder $P_r(x,t) = B_r(x) \times]t - r^2, t[$.

Note that in abuse of notation we use the symbol $L^{p,\mu}$ for both the standard and the parabolic Morrey space, where the latter is always meant on a space-time domain. For clarity, we write $\|u(t)\|_{L^{p,\mu}}$ for the standard Morrey norm of the function $u(t)$ at a fixed time t.

Given $p > 2^*$, we now fix the Morrey exponents $\mu = \frac{2p}{p-2}$ and $\lambda = \frac{4}{p-2} = \mu - 2$, which are natural for the study of problem (1.1).

Throughout the following a function u will be called a smooth solution of (1.1) on $]0,T[$ if $u \in C^1(\bar{\Omega}\times]0,T[)$ with $u_t \in L^2_{loc}(\bar{\Omega} \times [0,T[)$ solves (1.1) in the sense of distributions and achieves the initial data in the sense of traces. By standard regularity theory then u also is of class C^2 with respect to x and satisfies (1.1) classically. Schauder theory, finally, yields even higher regularity to the extent allowed by smoothness of the nonlinearity $g(v) = |v|^{p-2}v$. The function u will be called a global smooth solution of (1.1) if the above holds with $T = \infty$.

Our results on local and global well-posedness are summarized in the following theorems.

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^n$ be a smoothly bounded domain, $n \geq 3$. There exists a constant $\varepsilon_0 > 0$ such that for any function $u_0 \in L^{2,\lambda}(\Omega)$ satisfying $\|u_0\|_{L^{2,\lambda}} < \varepsilon_0$ there is a unique global smooth solution u to (1.1) on $\Omega\times]0,\infty[$.*

The smallness condition can be somewhat relaxed.

Theorem 2.2. *Let $u_0 \in L^{2,\lambda}(\Omega)$ and suppose that there exists a number $R > 0$ such that*

$$\sup_{x_0 \in R^n,\ 0<r<R} r^{\lambda-n} \int_{B_r(x_0)\cap\Omega} |u_0|^2 \mathrm{d}x \leq \varepsilon_0^2,$$

where $\varepsilon_0 > 0$ is as determined in Theorem 2.1. Then there exists a unique smooth solution u to (1.1) on an interval $]0,T_0[$, where $T_0/R^2 = C(\varepsilon_0/\|u_0\|_{L^{2,\lambda}}) > 0$.

In particular, for any $u_0 \in L^{2,\lambda}_0(\Omega)$ there exists a unique smooth solution u to (1.1) on some interval $]0,T[$, where $T = T(u_0) > 0$.

It is well-known that for smooth initial data $u_0 \in C^1(\bar{\Omega})$ there exists a smooth solution u to the Cauchy problem (1.1) on some time interval $[0,T[$, $T > 0$. By the uniqueness of the solution to (1.1) constructed in Theorem 2.1 or 2.2, the latter solution coincides with u and hence is smooth up to $t = 0$ if $u_0 \in C^1(\bar{\Omega})$.

3. Proof of Theorem 2.1

Let $n \geq 3$ and let

$$G(x,t) = (4\pi t)^{-n/2} \mathrm{e}^{-\frac{|x|^2}{4t}},\ x \in \mathbb{R}^n, t > 0,$$

be the fundamental solution to the heat equation on $\mathbb{R}^n$ with singularity at $(0,0)$. Given a domain $\Omega \subset \mathbb{R}^n$ also let $\Gamma = \Gamma(x,y,t) = \Gamma(y,x,t)$ be the corresponding fundamental solution to the heat equation on Ω with homogeneous Dirichlet boundary data $\Gamma(x,y,t) = 0$ for $x \in \partial\Omega$. Note that by the maximum principle for any $x, y \in \Omega$, any $t > 0$ there holds $0 < \Gamma(x,y,t) \leq G(x-y,t)$.

For $x \in \Omega$, $r > 0$ we let

$$\Omega_r(x) = B_r(x) \cap \Omega;$$

similarly, for $x \in \Omega$, $r, t > 0$ we define

$$Q_r(x,t) = P_r(x,t) \cap \Omega\times]0,\infty[.$$

We sometimes write $z = (x,t)$ for a generic point in space-time. The letter C will denote a generic constant, sometimes numbered for clarity.

For $f \in L^1(\Omega)$ set

$$(S_\Omega f)(x,t) := \int_\Omega \Gamma(x,y,t) f(y)\,\mathrm{d}y,\ t > 0,$$

so that $v = S_\Omega f$ solves the equation

$$v_t - \Delta v = 0 \text{ on } \Omega \times [0,\infty[\tag{3.1}$$

with boundary data $v(x,t) = 0$ for $x \in \partial\Omega$ and initial data $v\big|_{t=0} = f$ on Ω. See [7] for a thorough introduction to the concept of fundamental solutions.

Similar to [4], Proposition 4.3, by adapting the methods of Adams [1] we can show that S_Ω is well-behaved on Morrey spaces. Recall that $\mu = \frac{2p}{p-2}$ with $2 < \mu < n$, and $\lambda = \mu - 2 = \frac{4}{p-2} > 0$.

Lemma 3.1.

(i) *For any* $p > 2^* = \frac{2n}{n-2}$ *the map*

$$S_\Omega\colon L^{2,\lambda}(\Omega) \ni f \mapsto (v, \nabla v) \in L^{p,\mu} \times L^{2,\mu}(\Omega \times [0,\infty[)$$

is well-defined and bounded. Moreover, we have the bounds

$$\|v(t)\|^2_{L^\infty} \le Ct^{-\lambda/2}\|f\|^2_{L^{2,\lambda}},\ \|v(t)\|^2_{L^{2,\lambda}} \le C\|f\|^2_{L^{2,\lambda}},\ t > 0. \tag{3.2}$$

(ii) *Let* $f \in L^{2,\lambda}(\Omega)$ *and suppose that for a given* $\varepsilon_0 > 0$ *there exists a number* $R > 0$ *such that*

$$\sup_{x_0\in\Omega,\ 0<r<R}\left(r^{\lambda-n}\int_{\Omega_r(x_0)}|f|^2\mathrm{d}x\right)^{1/2} \le \varepsilon_0.$$

Then with a constant $C > 0$ *for* $v = S_\Omega f$ *there holds the estimate*

$$\sup_{x_0\in\Omega,\ 0<r^2\le t_0\le T_0}\left(r^{\mu-n-2}\int_{Q_r(x_0,t_0)}|v|^p\mathrm{d}z\right)^{1/p} \le C\varepsilon_0,$$

where $T_0/R^2 = C(\varepsilon_0/\|f\|_{L^{2,\lambda}(\Omega)}) > 0$.

Proof.

(i) Let $f \in L^{2,\lambda}(\mathbb{R}^n)$ and set $v = S_\Omega f$ as above. Recall the definition of the fractional maximal functions

$$M_\alpha f(x) := \sup_{r>0} M_{\alpha,r}f(x),\ M_{\alpha,r}f(x) := r^{\alpha-n}\int_{\Omega_r(x)}|f(y)|\,\mathrm{d}y,\ \alpha > 0.$$

Note that Hölder's inequality gives the uniform bound

$$\left(M_{\lambda/2}f\right)^2 \le M_\lambda(|f|^2) \le \|f\|^2_{L^{2,\lambda}}. \tag{3.3}$$

Following the scheme outlined by Adams [1], proof of Proposition 3.1, we first derive pointwise estimates for v and bounds on parabolic cylinders $P_r(x_0,t_0)$ with radius r satisfying $0 < 2r^2 < t_0$. Using the well known estimate

$$G(x-y,t) \le C(|x-y| + \sqrt{t})^{-n}$$

for the heat kernel and recalling that $\Gamma(x,y,t) \le G(x-y,t)$, for any $t > 0$ we can bound

$$\begin{aligned}
|v(x,t)| &\le C\int_\Omega (|x-y|+\sqrt{t})^{-n}|f(y)|\,\mathrm{d}y\\
&\le C\int_{\Omega_{\sqrt{t}}(x)}(|x-y|+\sqrt{t})^{-n}|f(y)|\,\mathrm{d}y\\
&\quad + C\sum_{k=1}^\infty\int_{\Omega_{2^k\sqrt{t}}(x)\setminus\Omega_{2^{k-1}\sqrt{t}}(x)}(|x-y|+\sqrt{t})^{-n}|f(y)|\,\mathrm{d}y\\
&\le C\sum_{k=0}^\infty(2^k\sqrt{t})^{-n}(2^k\sqrt{t})^{n-\lambda/2}M_{\lambda/2,2^k\sqrt{t}}f(x) \le Ct^{-\lambda/4}M_{\lambda/2}f(x).
\end{aligned}$$

Hence by (3.3) with a uniform constant $C > 0$ for any $t > 0$ there holds

$$\|v(t)\|^2_{L^\infty} \le Ct^{-\lambda/2}\|M_{\lambda/2}f\|^2_{L^\infty} \le Ct^{-\lambda/2}\|f\|^2_{L^{2,\lambda}},$$

as claimed in (3.2). Moreover, for any $x_0 \in \mathbb{R}^n$, any $t_0 > 0$ and any $0 < r < \sqrt{t_0/2}$ we obtain the bounds

$$\|v(t_0)\|^2_{L^2(\Omega_r(x_0))} \le Cr^n t_0^{-\lambda/2}\|f\|^2_{L^{2,\lambda}} \le Cr^{n-\lambda}\|f\|^2_{L^{2,\lambda}}, \tag{3.4}$$

and similarly

$$\|v\|^p_{L^p(Q_r(x_0,t_0))} \le Cr^{n+2}t_0^{-p\lambda/4}\|f\|^p_{L^{2,\lambda}} \le Cr^{n+2-\mu}\|f\|^p_{L^{2,\lambda}}, \tag{3.5}$$

where we also used that $\mu = 2p\lambda/4$.

In order to derive (3.5) also for radii $r \ge \sqrt{t_0/2}$ we need to argue slightly differently. We may assume that $x_0 = 0$. Moreover, after enlarging t_0, if necessary, we may assume that $t_0 = 2r^2$. Let $\psi = \psi_0 = \psi_0(x)$ be a smooth cut-off function satisfying $\chi_{B_r(0)} \le \psi \le \chi_{B_{2r}(0)}$ and with $|\nabla\psi|^2 \le 4r^{-2}$. Set $r =: r_0$ and let $r_i = 2^i r_0$, $\psi_i(x) = \psi(2^{-i}x)$, $i \in \mathbb{N}$. For ease of notation in the following estimates we drop the index i.

Upon multipying (3.1) with $v\psi^2$ we find the equation

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(|v|^2\psi^2) - \operatorname{div}(v\nabla v\psi^2) + |\nabla v|^2\psi^2 \\ = -2v\nabla v\psi\nabla\psi \le \frac{1}{2}|\nabla v|^2\psi^2 + 2|v|^2|\nabla\psi|^2.\end{aligned}$$

Integrating over $\Omega\times]0,t_1[$ and using the bound $|\nabla\psi|^2 \le 4r^{-2}$, for any $0 < t_1 < t_0$ we obtain

$$\begin{aligned}\int_{\Omega_{2r}(0)}|v(t_1)|^2\psi^2\mathrm{d}x + \int_{\Omega_{2r}(0)\times]0,t_1[}|\nabla v|^2\psi^2\mathrm{d}x\mathrm{d}t \\ \le \int_{\Omega_{2r}(0)}|f|^2\psi^2\mathrm{d}x + 16r^{-2}\int_{\Omega_{2r}(0)\times]0,t_1[}|v|^2\mathrm{d}x\mathrm{d}t.\end{aligned} \tag{3.6}$$

For $r = r_i$, $i \in \mathbb{N}_0$, set

$$\Psi(r) := \sup_{x_0\in\Omega, 0<t<t_0} r^{\lambda-n}\int_{\Omega_r(x_0)}|v(t)|^2\mathrm{d}x.$$

Recalling that $\lambda = \mu - 2$, then from the previous inequality (3.6) with the uniform constants $C_1 = 2^{n-\lambda}$, $C_2 = 32C_1$ we obtain

$$\begin{aligned}\Psi(r_i) &\le r_i^{\lambda-n}\Big(\int_{\Omega_{2r_i}(0)}|f|^2\mathrm{d}x + 16t_0r_i^{-2}\sup_{0<t<t_0}\int_{\Omega_{2r_i}(0)}|v(t)|^2\mathrm{d}x\Big) \\ &\le C_1\|f\|^2_{L^{2,\lambda}} + C_2 2^{-2i}\Psi(r_{i+1}).\end{aligned}$$

By iteration, for any $k_0 \in \mathbb{N}$ there results

$$\begin{aligned}\Psi(r_0) &\le C_1\|f\|^2_{L^{2,\lambda}} + C_2\Psi(r_1) \le C_1(1+C_2)\|f\|^2_{L^{2,\lambda}} + C_2^2 2^{-2}\Psi(r_2) \le \dots \\ &\le C_1\sum_{k=0}^{k_0} C_2^k 2^{(1-k)k}\|f\|^2_{L^{2,\lambda}} + C_2^{k_0+1}2^{-k_0(k_0+1)}\Psi(r_{k_0+1}).\end{aligned}$$

Passing to the limit $k_0 \to \infty$, we obtain that $\Psi(r_1) \le C\|f\|^2_{L^{2,\lambda}}$. Inserting this information into (3.6), where we again set $r = r_0$, then we find

$$\Psi(r) + \sup_{x_0\in\Omega} r^{\mu-2-n}\int_{\Omega_r(x_0)\times]0,t_0[}|\nabla v|^2\mathrm{d}x\mathrm{d}t \le C\|f\|^2_{L^{2,\lambda}}. \tag{3.7}$$

In particular, together with (3.4) we have now shown the bound

$$\|v(t)\|^2_{L^{2,\lambda}} \le C\|f\|^2_{L^{2,\lambda}} \text{ for all } t > 0,$$

and thus have verified (3.2) completely.

To complete the proof of (3.5) for $r = r_0 = \sqrt{t_0/2}$, let $\psi = \psi_0$ as above and let $\tau(t) = \min\{t, t_0 - t\}$. Multiplying (3.1) with the function $v|v|^{p-2}\psi^2\tau$ then we obtain

$$\begin{aligned}&\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}(|v|^p\psi^2\tau) - \frac{1}{p}\frac{\mathrm{d}\tau}{\mathrm{d}t}|v|^p\psi^2 - \mathrm{div}(|v|^{p-2}v\nabla v\psi^2\tau) + (p-1)|\nabla v|^2|v|^{p-2}\psi^2\tau\\ &\quad = -2|v|^{p-2}v\nabla v\psi\nabla\psi\tau \ge -|\nabla v|^2|v|^{p-2}\psi^2\tau - |v|^p|\nabla\psi|^2\tau.\end{aligned}$$

Integrating over $\Omega\times]0, t_0[$ and using that $\frac{\mathrm{d}\tau}{\mathrm{d}t} = 1$ for $0 < t < t_0/2$, $\frac{\mathrm{d}\tau}{\mathrm{d}t} = -1$ for $t_0/2 < t < t_0$, as well as the fact that the region $\Omega_{2r}(0)\times]t_0/2, t_0[$ may be covered by a collection of at most $L = L(n)$ cylinders $Q_r(x_l, t_0)$, $1 \le l \le L$, we find

$$\begin{aligned}\int_{Q_r(x_0,t_0/2)} |v|^p\mathrm{d}z &\le L \sup_{1\le l\le L}\int_{Q_r(x_l,t_0)} |v|^p\mathrm{d}z + Cr^{-2}\int_{\Omega_{2r}(0)\times]0,t_0[} |v|^p\tau\,\mathrm{d}x\mathrm{d}t\\ &\quad + C\int_{\Omega_{2r}(0)\times]0,t_0[} |\nabla v|^2|v|^{p-2}\tau\,\mathrm{d}x\mathrm{d}t.\end{aligned}$$

But by (3.2) we have $|v|^{p-2}\tau \le |v|^{p-2}t \le C\|f\|^{p-2}_{L^{2,\lambda}}$, and from (3.7) we obtain

$$\begin{aligned}&r^{-2}\int_{\Omega_{2r}(0)\times]0,t_0[} |v|^p\tau\,\mathrm{d}x\mathrm{d}t + \int_{\Omega_{2r}(0)\times]0,t_0[} |\nabla v|^2|v|^{p-2}\tau\,\mathrm{d}x\mathrm{d}t\\ &\quad \le C\|f\|^{p-2}_{L^{2,\lambda}}\Big(r^{n-\lambda}\Psi(2r) + \int_{\Omega_{2r}(0)\times]0,t_0[} |\nabla v|^2\mathrm{d}x\mathrm{d}t\Big) \le Cr^{n-\lambda}\|f\|^p_{L^{2,\lambda}}.\end{aligned}$$

Recalling that for each cylinder $Q_r(x_l, t_0)$, $1 \le l \le L$, there holds (3.5), we then obtain

$$\int_{Q_r(x_0,t_0/2)} |v|^p\mathrm{d}z \le L \sup_{1\le l\le L}\int_{Q_r(x_l,t_0)} |v|^p\mathrm{d}z + Cr^{n-\lambda}\|f\|^p_{L^{2,\lambda}} \le Cr^{n-\lambda}\|f\|^p_{L^{2,\lambda}},$$

and (3.5) follows since $\lambda = \mu - 2$.

Finally, for $t_0 \le r^2$ and any $x_0 \in \Omega$ equation (3.6) yields the gradient bound

$$\begin{aligned}\int_{Q_r(0,t_0)} |\nabla v|^2\mathrm{d}z &\le \int_{\Omega_{2r}(0)} |f|^2\psi^2\mathrm{d}x + 16r^{-2}\int_{\Omega_{2r}(0)\times]0,t_0[} |v|^2\mathrm{d}x\mathrm{d}t\\ &\le Cr^{n-\lambda}\big(\|f\|^2_{L^{2,\lambda}} + \Psi(2r)\big) \le Cr^{n-\lambda}\|f\|^2_{L^{2,\lambda}}.\end{aligned}$$

In view of (3.2) the same bound also holds for $t_0 > r^2$ as can be seen by shifting time by $t_0 - r^2$ and replacing f with the function $\tilde{f}(x) = v(x, t_0 - r^2) \in L^{2,\lambda}(\Omega)$. With $\lambda = \mu - 2$ we obtain the bound $\|\nabla v\|_{L^{2,\mu}} \le C\|f\|_{L^{2,\lambda}}$, as desired.

(ii) Set $L_0 := \|f\|_{L^{2,\lambda}}$. As before, for any $x \in \Omega$ we have the bound

$$|v(x,t)| \le C\sum_{k=0}^{\infty}(2^k\sqrt{t})^{-\lambda/2}M_{\lambda/2,2^k\sqrt{t}}f(x).$$

By assumption for $r = 2^k\sqrt{t} \le R$ we can estimate

$$M_{\lambda/2,r}(|f|)(x) \le \big(M_{\lambda,r}(|f|^2)(x)\big)^{1/2} \le \varepsilon_0,$$

whereas for any $r > 0$ we have

$$M_{\lambda/2,r}(|f|)(x) \le \big(M_{\lambda,r}(|f|^2)(x)\big)^{1/2} \le \|f\|_{L^{2,\lambda}} = L_0.$$

Let $k_0 \in \mathbb{N}$ such that $2^{-k_0\lambda/2}L_0 \le \varepsilon_0$. Then for $0 < t < T := 2^{-2k_0}R^2$ we find the uniform estimate

$$|v(x,t)| \le Ct^{-\lambda/4}\Big(\sum_{k=0}^{k_0} 2^{-k\lambda/2}\varepsilon_0 + \sum_{k=k_0+1}^{\infty} 2^{-k\lambda/2}L_0\Big) \le Ct^{-\lambda/4}\varepsilon_0.$$

Proceeding as in part (i) of the proof, for any $0 < t < T$, any $x_0 \in \Omega$, and any $0 < r < \sqrt{t/2}$ we then obtain the bound

$$\|v(t)\|^2_{L^2(\Omega_r(x_0))} \le Cr^n t^{-\lambda/2}\varepsilon_0^2 \le Cr^{n-\lambda}\varepsilon_0^2;$$

similarly, we find

$$\|v\|^p_{L^p(Q_r(x_0,t_0))} \le Cr^{n+2}t^{-p\lambda/4}\varepsilon_0^p \le Cr^{n+2-\mu}\varepsilon_0^p \tag{3.8}$$

whenever $0 < 2r^2 < t_0 < T$. In order to derive the latter bound also for radii $r > 0$ with $t_0/2 \le r^2 \le t_0 \le T$ as in i) we may assume that $x_0 = 0$ and fix some numbers $0 < t_0 < T$, $r_0 \ge \sqrt{t_0/2}$. Setting

$$\Psi(r) := \sup_{0<t<t_0} r^{\lambda-n}\int_{B_r(0)} |v(t)|^2 dx,\ r > 0,$$

for $r = r_i = 2^i r_0$, $i \in \mathbb{N}_0$, from (3.6) we obtain the bound

$$\begin{aligned}\Psi(r_i) &\le r_i^{\lambda-n}\int_{B_{2r_i}(0)} |f|^2 dx + 16C_1 t_0 r_i^{-2}\Psi(2r_i)\\ &\le C_1 M_{\lambda,r_{i+1}}(|f|^2)(0) + C_2 2^{-2i}\Psi(r_{i+1})\end{aligned}$$

for any $i \in \mathbb{N}$, with constants $C_1 = 2^{n-\lambda}$, $C_2 = 32C_1$ as before.
Suppose that $r_{i_0} \le R$ for some $i_0 \in \mathbb{N}$. Bounding $M_{\lambda,r_i}(|f|^2)(x) \le \varepsilon_0^2$ for $i \le i_0$ and $M_{\lambda,r_i}(|f|^2)(x) \le L_0^2$ else, by iteration we then obtain

$$\begin{aligned}\Psi(r_0) &\le C_1\varepsilon_0^2 + C_2\Psi(r_1) \le C_1(1 + C_2)\varepsilon_0^2 + C_2^2 2^{-2}\Psi(r_2) \le \dots\\ &\le C_1\sum_{i=0}^{i_0-1} C_2^i 2^{(1-i)i}\varepsilon_0^2 + C_1\sum_{i=i_0}^{k} C_2^i 2^{(1-i)i}L_0^2 + C_2^{k+1}2^{-k(k+1)}\Psi(r_{k+1}).\end{aligned}$$

Thus, if i_0 is such that $C_2 2^{1-i_0} \le (\varepsilon_0/L_0)^2 \le 1/2$, that is, if

$$\sqrt{2t_0} \le 2r_0 = 2^{1-i_0}r_{i_0} \le 2^{1-i_0}R \le C_2^{-1}(\varepsilon_0/L_0)^2 R,$$

upon passing to the limit $k \to \infty$ we obtain $\Psi(r_0) \le C\varepsilon_0^2$ and the analogue of (3.7) with ε_0 in place of $\|f\|_{L^{2,\lambda}}$.

Recalling the definition $T = 2^{-2k_0}R^2$ with $k_0 \in \mathbb{N}$ satisfying $2^{-k_0\lambda/2}L_0 \le \varepsilon_0$, we see that these bounds hold true for

$$0 < t_0/2 \le r_0^2 \le t_0 \le T_0 := R^2 \cdot \min\{(\varepsilon_0/L_0)^{4/\lambda}, C_2^{-2}(\varepsilon_0/L_0)^4\}.$$

Using (3.8), the remainder of the proof of (3.5) in part i) now may be copied unchanged to yield the claim. $\square$

The assertions of Theorems 2.1 and 2.2 now are a consequence of the following result.

Lemma 3.2.

(i) *For any* $p > 2^*$ *there exists a constant* $\varepsilon_0 > 0$ *such that for any* $u_0 \in L^{2,\lambda}(\Omega)$ *with* $\|u_0\|_{L^{2,\lambda}} \le \varepsilon_0$ *there exists a unique solution* $u \in L^{p,\mu}(\Omega\times]0,\infty[)$ *to the Cauchy problem* (1.1) *such that*

$$\|u\|_{L^{p,\mu}} \le C\|u_0\|_{L^{2,\lambda}}. \tag{3.9}$$

(ii) *Let* $u_0 \in L^{2,\lambda}(\Omega)$ *and suppose that there exists a number* $R > 0$ *such that*

$$\sup_{x_0\in\Omega,\ 0<r<R} r^{\lambda-n}\int_{\Omega_r(x_0)} |u_0|^2 \mathrm{d}x \le \varepsilon_0^2,$$

where $\varepsilon_0 > 0$ *is as determined in (i). Then there exists a unique smooth solution* u *to* (1.1) *on an interval* $]0,T_0[$, *where* $T_0/R^2 = C(\varepsilon_0^{-1}\|u_0\|_{L^{2,\lambda}(\Omega)}) > 0$.

Proof. For $u_0 \in L^{2,\lambda}(\mathbb{R}^n)$ set $w_0 = S_\Omega u_0$. For suitable $a > 0$ let

$$X := \{v \in L^{p,\mu}(\Omega\times]0,T_0[);\ \|v\|_{L^{p,\mu}} \le a\},$$

where $T_0 > 0$ in the case of the assumptions in (i) may be chosen arbitrarily large and otherwise is as in assertion (ii) of Lemma 3.1.

Then X is a closed subset of the Banach space $L^{p,\mu} = L^{p,\mu}(\Omega\times]0,T_0[)$. Moreover, for any $v \in X$ we have $|v|^{p-2}v \in L^{p/(p-1),\mu}$. By Lemma 4.1 in [4] there exists a unique solution $w = S(v|v|^{p-2}) \in L^{p,\mu}$ of the Cauchy problem

$$w_t - \Delta w = |v|^{p-2}v \text{ on } \Omega\times]0,T_0[,\ w\big|_{t=0} = 0,$$

with

$$\|w\|_{L^{p,\mu}} \le C\|v\|_{L^{p,\mu}}^{p-1} \le Ca^{p-1}.$$

For sufficiently small $\varepsilon_0, a > 0$ then the map

$$\Phi\colon X \ni v \mapsto w_0 + w \in X,$$

and for $v_{1,2} \in X$ with corresponding $w_i = S(v_i|v_i|^{p-2})$, $i = 1,2$, we can estimate

$$\|\Phi(v_1) - \Phi(v_2)\|_{L^{p,\mu}} = \|w_1 - w_2\|_{L^{p,\mu}} \le C\|v_1|v_1|^{p-2} - v_2|v_2|^{p-2}\|_{L^{p/(p-1),\mu}}.$$

The latter can be bounded

$$\|v_1|v_1|^{p-2} - v_2|v_2|^{p-2}\|_{L^{p/(p-1),\mu}} \le C\big(\|v_1\|_{L^{p,\mu}}^{p-2} + \|v_2\|_{L^{p,\mu}}^{p-2}\big)\|v_1 - v_2\|_{L^{p,\mu}}.$$

Thus for sufficiently small $a > 0$ we find

$$\|\Phi(v_1) - \Phi(v_2)\|_{L^{p,\mu}} \le Ca^{p-2}\|v_1 - v_2\|_{L^{p,\mu}} \le \frac{1}{2}\|v_1 - v_2\|_{L^{p,\mu}}.$$

By Banach's theorem the map Φ has a unique fixed point $u \in X$, and u solves the initial value problem (1.1) in the sense of distributions. Finally, for sufficiently small $a, \varepsilon_0 > 0$ we can invoke Proposition 4.1 in [4] to show that u, in fact, is a smooth global solution of (1.1). □

Remark 3.3. As already pointed out in the introduction, the assumption $u_0 \in L^{2,\lambda}(\Omega)$ is natural in the context of weak continuations of the flow (1.1). Indeed, suppose that a solution u of (1.1) blowing up at some time $T < \infty$ can be extended as a weak solution of (1.1) on a time interval $]0,T_1[$ for some $T_1 > T$ and assume that the extended solution still satisfies the monotonicity formula [5], Proposition 3.1. In the notation of [5], for any $x_1 = 0 \in \Omega$ and any $0 < T < t_1 < T_1$ choose (x_1, t_1) as center of scaling and integrate the scaled energy function H^φ given by (2.13) in [5]. Using that $rF_2^\varphi(r) \to 0$ as $r \downarrow 0$, for any $0 < R \le R_1 \le \sqrt{t_1}$ similar to (4.7) in [5] we then obtain the inequality

$$F_2^\varphi(R) + \frac{1}{R}\int_0^R \big(D^\varphi(r) + F_p^\varphi(r)\big)\,\mathrm{d}r \le CH^\varphi(R_1) + C\int_0^{R_1} |B_-^\varphi(r)|\frac{\mathrm{d}r}{r} + C_0\delta(\rho, R_1),$$

where the integral involving $B_-^\varphi(r)$ on the right can be bounded uniformly in (x_1, t_1) by means of [5], Lemmas 4.1 and 4.3. Choosing $R = \sqrt{t_1 - T}$, for sufficiently small $t_1 > T$ we have $\varphi \equiv 1$ on $B_R(0)$ and thus we are able to bound

$$R^{\lambda-n}\int_{\Omega_R(x_1)} |u(T)|^2\,\mathrm{d}x \le CF_2^\varphi(R) \le C$$

with constants $C > 0$ independent of x_1 and $R > 0$; that is, $u(T) \in L^{2,\lambda}(\Omega)$.

4. Ill-posedness for "large" data

4.1. Minimal solutions for non-negative initial data

In order to obtain a notion of solution of (1.1) on $\Omega\times]0,\infty[$ for arbitrary nonnegative initial data $u_0 \ge 0$, following Baras–Cohen [3] for $n \in \mathbb{N}$ we solve the initial value problem

$$u_{n,t} - \Delta u_n = f_n(u_n) = \min\{u_n^{p-1}, n^{p-1}\} \text{ on } \Omega\times]0,\infty[,\ u = 0 \text{ on } \partial\Omega\times]0,\infty[, \tag{4.1}$$

with initial data

$$u_n(x,0) = u_{0n}(x) := \min\{u_0(x), n\} \ge 0. \tag{4.2}$$

As the right-hand side $f_n(u_n)$ in (4.1) is uniformly bounded, for any $n \in \mathbb{N}$ there exists a unique global solution of (4.1), (4.2). By the maximum principle, positivity of the initial data is preserved and u_n is monotonically increasing in n. Hence, the pointwise limit $u(x,t) := \lim_{n\to\infty} u_n(x,t) \le \infty$ exists. Inspired by Baras and Cohen [3] we call this limit the *minimal solution* of problem (1.1) for the given data u_0. Moreover, similar to their Proposition 2.1 we have $u \le v$ for any v which is an *integral solution* v of (1.1) in the sense that

$$v(t) = S_t u_0 + \int_0^t S_{t-s} v^{p-1}(s)\mathrm{d}s, \tag{4.3}$$

where for brevity we now write $(S_t)_{t\ge 0}$ for the heat semigroup on Ω, defined by

$$S_t w(x) = \int_\Omega \Gamma(x,y,t) w(y)\mathrm{d}y,$$

with $\Gamma > 0$ denoting the fundamental solution of the heat equation on Ω.

Indeed, by Duhamel's principle the u_n satisfy the integral equation

$$u_n(t) = S_t u_{0n} + \int_0^t S_{t-s} f_n(u_n(s))\mathrm{d}s. \tag{4.4}$$

Recalling that the sequence u_n is monotonically increasing in n, from Beppo–Levi's theorem on monotone convergence we find that u satisfies (4.3). On the other hand, for each n and any integral solution v of (1.1) clearly there holds $u_n \le v$.

With these prerequisites we now show that there are initial data $u_0 \in L^{p,\mu}(\Omega)$ with even $\nabla u_0 \in L^{2,\mu}$ such that the minimal solution u to (1.1) satisfies $u \equiv \infty$ on $\Omega\times]0,\infty[$, that is, undergoes *complete instantaneous blow-up*. The following arguments are modelled on corresponding results on complete instantaneous blow-up by Galaktionov and Vazquez [9] in the case when $\Omega = \mathbb{R}^n$.

4.2. Complete instantaneous blow-up

It is well-known that on a bounded domain Ω equation (1.1) may be interpreted as the negative gradient flow of the energy

$$E(u) = E_\Omega(u) = \int_\Omega \left(\frac{1}{2}|\nabla u|^2 - \frac{1}{p}|u|^p\right) \mathrm{d}x.$$

As observed by Ball ([2], Thm. 3.2), sharpening an earlier result of Kaplan [11], for data u_0 with $E(u_0) < 0$ the solution to (1.1) blows up in finite time. Indeed, Ball [2], Theorem 3.2, observes that testing equation (1.1) with u leads to the differential inequality

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2_{L^2} &= -\int_{\Omega\times\{t\}} \left(|\nabla u|^2 - |u|^p\right)\mathrm{d}x = -2E(u(t)) + \frac{p-2}{p}\|u(t)\|^p_{L^p} \\ &\geq -2E(u_0) + c_0\|u(t)\|^p_{L^2} \geq c_0\|u(t)\|^p_{L^2}\end{aligned}$$

for some constant $c_0 > 0$. Hence we find

$$\|u(t)\|_{L^2} \geq \left(\|u_0\|^{(2-p)/2}_{L^2} - c_0(p-2)t\right)^{-2/(p-2)},$$

and $u(t)$ must blow up at the latest at time $T = c_0^{-1}(p-2)^{-1}\|u_0\|^{(2-p)/2}_{L^2}$.

In order to obtain data $u_0 \in L^{p,\mu}$ leading to instantaneous complete blow-up, we combine this observation with the following well-known scaling property of equation (1.1): whenever u is a solution of (1.1) on Ω, then for any $R > 0$, any $x_0 \in \mathbb{R}^n$ the function

$$u_{R,x_0}(x,t) = R^{-\alpha}u(R^{-1}(x-x_0), R^{-2}t) \tag{4.5}$$

with $\alpha = \frac{2}{p-2}$ is a solution of (1.1) on the scaled domain

$$\Omega_{R,x_0} := \{x \in \mathbb{R}^n;\ R^{-1}(x-x_0) \in \Omega\}.$$

Clearly we may assume that $0 \in \Omega$.

Theorem 4.1. *Let $0 \leq w_0 \in C_c^\infty(B_1(0))$ with $E_{B_1(0)}(w_0) < 0$. Set*

$$M = M_{w_0} = \sup_{|y|\leq 1}\left(|y|^\alpha w_0(y)\right),$$

where $\alpha = \frac{2}{p-2}$ as above. Then for every initial data $0 \leq u_0 \in C^0(\Omega \setminus \{0\})$ satisfying

$$\liminf_{x\to 0}\left(u_0(x) - M|x|^{-\alpha}\right) > 0$$

the minimal solution u to (1.1) blows up completely instantaneously.

Proof. By Ball's above result, the solution w to (1.1) on $B_1(0)\times]0,T[$ with initial data $w(0) = w_0$ blows up after some finite time T at a point y_0.

Fix $R_0 > 0$ with $B_{R_0}(0) \subset \Omega$ and such that

$$u_0(x) > M|x|^{-\alpha} \text{ for } |x| \leq R_0.$$

For $R < R_0$ and $x_0 \in \Omega$ with $|x_0| \leq R_0 - R$ consider the rescaled solutions

$$w_{R,x_0}(x,t) := R^{-\alpha}w(R^{-1}(x-x_0), R^{-2}t)$$

on $B_R(x_0) \times [0, R^2T[$ that blow up at time R^2T.

Since by assumption we have

$$w_{R,0}(x,0) = R^{-\alpha} w_0(R^{-1}x) \le M|x|^{-\alpha} < u_0(x) \text{ on } B_R(0),$$

by continuity of u_0 away from $x = 0$ and continuity of w_0 there is a number $\delta = \delta(R) > 0$ such that

$$w_{R,x_0}(x,0) < u_0(x) \text{ on } B_R(x_0)$$

for all x_0 with $|x_0| < \delta$. Since in addition $u \ge 0 = w_{R,x_0}$ on $\partial B_R(x_0) \times [0, R^2T[$, by the maximum principle for any $\varepsilon > 0$, any $n \ge \|w_{R,x_0}\|_{L^\infty(B_R(x_0)\times[0,R^2T-\varepsilon])}$ there holds

$$u(x,t) \ge u_n(x,t) \ge w_{R,x_0}(x,t) \text{ on } B_R(x_0) \times [0, R^2T - \varepsilon],$$

where u_n solves (4.1) for each $n \in \mathbb{N}$. Passing to the limit $\varepsilon \to 0$, we then find

$$\begin{aligned} u(x_0 + Ry_0, R^2T) &= \left(S_{R^2T} u_0 + \int_0^{R^2T} S_{R^2T-s} f(u(s))\mathrm{d}s \right)(x_0 + Ry_0) \\ &= \lim_{n\to\infty} \left(S_{R^2T} u_{0n} + \int_0^{R^2T} S_{R^2T-s} f_n(u_n(s))\mathrm{d}s \right)(x_0 + Ry_0) \\ &\ge \lim_{t \uparrow R^2T} w_{R,x_0}(x_0 + Ry_0, t) = \infty \end{aligned}$$

for all $x_0 \in B_\delta(0)$.

Since $R > 0$ may be chosen arbitrarily small, we conclude that for any sufficiently small $t > 0$ there holds $\mathcal{L}^n(\{x \in \Omega; u(x,t) = \infty\}) > 0$. But then positivity of Γ and Duhamel's principle (4.3) yield

$$u(x,t) = \left(S_t u_0 + \int_0^t S_{t-s} u^{p-1}(s)\mathrm{d}s \right)(x) = \infty,$$

for any $t > 0$ and any $x \in \Omega$. □

5. Open problems

An obvious question to be investigated is whether the pathological situation that leads to instantaneous complete blow-up of the flow (1.1) can arise under "natural" hypotheses. In particular, is it possible that a smooth solution u of (1.1) on $[0,T[$ blowing up at time $T > 0$ with bounded energy $|E(u(t))| \le C < \infty$ for $0 < t < T$ has a "trace" $u(T)$ giving rise to instantaneous complete blow-up? Of course, it would also be of interest to quantify the smallness conditions in Theorems 2.1 and 2.2.

Conversely, one might try to determine the smallest number $M > 0$ so that the conclusion of Theorem 4.1 holds true. Can one show that at least for exponents p strictly less than the Joseph–Lundgren [10] exponent

$$p_{JL} = 2 + \frac{4}{n - 4 - 2\sqrt{n-1}} \text{ if } n \ge 11, \ p_{JL} = \infty \text{ if } n \le 10,$$

we have $M = \alpha(n - 2 - \alpha) =: c_*$, where c_* appears as coefficient in the singular solution $u_*(x) := c_*|x|^{-\alpha}$ of the time-independent equation (1.1) on $\mathbb{R}^n$? The significance of the exponent p_{JL} is illustrated for instance in Lemma 9.3 of [9].

Hopefully, we will be able to answer some of these questions in the future.

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