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# ZERO-SUM AND NONZERO-SUM DIFFERENTIAL GAMES WITHOUT ISAACS CONDITION *

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**Abstract.** In this paper we study differential games without Isaacs condition. The objective is to investigate on one hand zero-sum games with asymmetric information on the pay-off, and on the other hand, for the case of symmetric information but now for a non-zero sum differential game, the existence of a Nash equilibrium pay-off. Our results extend those by Buckdahn, Cardaliaguet and Rainer [*SIAM J. Control Optim.* **43** (2004) 624–642], to the case without Isaacs condition. To overcome the absence of Isaacs condition, randomization of the non-anticipative strategies with delay of the both players are considered. They differ from those in Buckdahn, Quincampoix, Rainer and Xu [*Int. J. Game Theory* **45** (2016) 795–816]. Unlike in [*Int. J. Game Theory* **45** (2016) 795–816], our definition of NAD strategies for a game over the time interval $[t,T]$ $(0 \le t \le T)$ guarantees that a randomized strategy along a partition $\pi$ of $[0,T]$ remains a randomized NAD strategy with respect to any finer partition $\pi'$ $(\pi \subset \pi')$. This allows to study the limit behavior of upper and lower value functions defined for games in which the both players use also different partitions.

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## 1. Introduction

In our paper we study two-player differential games which dynamics is given by

$$\frac{\mathrm{d}X_s}{\mathrm{d}s} = f\left(X_s, u_s, v_s\right),\ s \in [t,T],$$

and driven by two controls $u \in \mathcal{U}_{t,T} := L^0([t,T] \to U)$ and $v \in \mathcal{V}_{t,T} := L^0([t,T] \to V)$ used by the players I and II, respectively. The control state spaces $U$ and $V$ are compact metric spaces.

Our objective is to study for such differential games two different problems, firstly that of asymmetric information, and secondly -but now in the frame of symmetric information- that of the existence of an $\varepsilon$-Nash equilibrium pay-off in the sense of [1].

*Keywords and phrases.* Zero-sum and nonzero-sum differential game, asymmetric information, Isaacs condition, Nash equilibrium payoffs, Fenchel transformation.

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The first to study differential games without Isaacs condition were Krasovskii and Subbotin [13]; in order to get a value of the game they considered relaxed controls. Interested in the use of classical controls, Buckdahn, Li and Quincampoix [4] considered differential games without Isaacs condition, in which each of the both players randomizes his control independently along a common partition of the time interval. However, as the players in differential games with Isaacs condition use non-anticipative strategies with delay (for short, NAD strategies), in order to guarantee the existence of a value (see, *e.g.*, [2,7]), and the corresponding couple of controls is rather the result of the interaction of the NAD strategies chosen by the players, it seems more natural to consider NAD strategies randomized independently by the both players along a common partition, in order to close the gap of absence of Isaacs condition. This was the basic idea in [6], in order to study differential games without Isaacs condition, but here for the more general frame of asymmetric information. Let us mention that [6] generalized Cardaliaguet's work [7] on differential games with asymmetric information to the case without Isaacs condition.

In our approach we adapt this idea of randomization of the NAD strategies along partitions $\pi = \big\{0 = t_0 < t_1 < \ldots < t_N = T\big\}$ of the time interval $[0, T]$ introduced in [6]. Letting $t \in [t_{k-1}, t_k)$, for some $0 \le k \le N$, we consider randomized NAD strategies $\alpha : \Omega \times \mathcal{V}_{t,T} \mapsto \mathcal{U}_{t,T}$ which, for instance, for Player I, are of the form

$$\alpha\left(\omega, v\right)(s) = \alpha_l\left(\left(\zeta_{1,k}, \ldots, \zeta_{1,l}\right)(\omega), v\right)(s),\ \omega \in \Omega,\ s \in [t \vee t_{l-1}, t \vee t_l),$$

$k \le l \le N$, where $\alpha_l : \mathbb{R}^{l-k+1} \times \mathcal{V}_{t,T} \mapsto \mathcal{U}_{t,T}$ is Borel measurable and non-anticipating with delay in $v$. The randomization in [6] is made through a given sequence of independent, on $[0, 1]$ uniformly distributed random variables $\zeta_{i,\ell}$, $\ell \ge 1$, $i = 1, 2$, used by the $i$th player on the $\ell$th subinterval of the partition. This has the inconvenience that randomized NAD strategies $\alpha \in \mathcal{A}^\pi(t, T)$ along a partition $\pi$ are, in general, not randomized NAD strategies along a finer partition $\pi'(\supset \pi)$. However, as we want to have in our approach $\mathcal{A}^\pi(t, T) \subset \mathcal{A}^{\pi'}(t, T)$ for the randomized NAD strategies of Player I (and, of course, also for the NAD strategies of Player II: $\mathcal{B}^\pi(t, T) \subset \mathcal{B}^{\pi'}(t, T)$), whenever the partitions $\pi$ and $\pi'$ satisfy $\pi \subset \pi'$, we prefer a randomization based on two independent Brownian motions (see Def. 2.1). Such a property also suggests to consider the spaces of NAD strategies $\mathcal{A}(t, T) := \bigcup_\pi \mathcal{A}^\pi(t, T)$, $\mathcal{B}(t, T) := \bigcup_\pi \mathcal{B}^\pi(t, T)$ and to study what happens, when one player fixes his partition $\pi$ for the randomization, while the other player prefers another partition $\pi'$: The definition of our randomized NAD strategies $\alpha \in \mathcal{A}^\pi(t, T)$, $\beta \in \mathcal{B}^{\pi'}(t, T)$ makes that both are compatible and belong to $\mathcal{A}^{\pi \cup \pi'}(t, T)$ and $\mathcal{B}^{\pi \cup \pi'}(t, T)$, respectively.

The first part of our work revisits the paper by Buckdahn, Quincampoix, Rainer and Xu [6] on differential games with asymmetric information with the absence of Isaacs condition, but now with the new kind of randomization of the NAD strategies we described above. Given a partition $\pi$ and denoting by $W^\pi$ the upper value function and by $V^\pi$ the lower value function of the differential game with randomization of the NAD strategies along the partition $\pi$ (see (2.6) and (2.8)), we show that also in this new frame we have for $|\pi| \to 0$ ($|\pi|$ denotes the maximal distance between two neighbor points of $\pi$) the uniform convergence on compacts of $W^\pi$ and of $V^\pi$ to the value function $U$ of the game, characterized as the unique dual viscosity solution (see Def. 3.1) of the Hamilton−Jacobi−Isaacs equation (3.51).

Related with these studies is the question about the behavior of the upper value function $W$ and the lower value function $V$, defined for the game with asymmetric information, when Player I disposes of all NAD strategies $\alpha \in \mathcal{A}(t, T)$ and Player II of all NAD strategies $\beta \in \mathcal{B}(t, T)$ (see (2.7) and (2.9)). It is shown in Theorem 5.3 that $W = V = U$ is the value of the game, if Isaacs condition is satisfied, and Example 5.1 shows that, if Isaacs condition doesn't hold, $W$ and $V$ don't, in general, coincide, in spite of all randomization of the NAD strategies by the players. As a byproduct a corresponding result is obtained for the upper value function $\bar{W}^\pi$ and the lower value function $\bar{V}^\pi$ of the game with asymmetric information, where Player I uses the partition $\pi$ for the randomization of his NAD strategies, while Player II has all randomized strategies from $\mathcal{B}(t, T)$ at his disposal. It is shown that under condition (5.6) which is a bit weaker than Isaacs condition-both functions $\bar{W}^\pi$ and $\bar{V}^\pi$ converge uniformly on compacts to the value function $U$, as $|\pi| \to 0$. This result allows Player I to choose a sufficiently fine partition $\pi$ for his randomization and to be sure that, as fine Player II may choose his partition for the randomization, the obtained upper and lower value functions are near to the value of the game. Our results not only generalize those by Buckdahn, Li, Quincampoix [4], to the case with asymmetric information on the pay-off, but also give the representation of the limit value (Thm. 5.3), which can be used for the numerical approach.

While the problem described above is studied for asymmetric zero-sum differential games, the existence of a Nash equilibrium pay-off is studied for non-zero symmetric differential games, but still without Isaacs condition. We consider the same dynamics of the game as above, the same type of randomization of the NAD strategies along partitions as introduced above, but we have now for every player his own pay-off functional. To each of the both cost functionals we associate a zero-sum differential game with value functions $U_1$ and $U_2$. These value functions are used to construct a Nash equilibrium pay-off $(e_1, e_2)$ and the associated $\varepsilon$-optimal randomized NAD strategies $\alpha^\varepsilon \in \mathcal{A}^\pi(t,T)$, $\beta^\varepsilon \in \mathcal{B}^\pi(t,T)$, for $|\pi| > 0$ small enough (see Def. 4.1 and Thms. 4.1 and 4.2).

Our paper is organized as follows: In Section 2 we introduce the setting of our differential games with asymmetric information and without Isaacs condition, and we give our definition of the randomized NAD strategies. Section 3 is devoted to the study of differential games with asymmetric information and without Isaacs condition. In Section 4, for non-zero sum differential games with symmetric information but still without Isaacs condition, the existence of a Nash equilibrium is investigated. Last but not least, Section 5 considers zero-sum differential games in which both players can choose different partitions for the randomization of the NAD strategies, and the convergence behavior of the associated upper and lower value functions is studied.

## 2. Preliminaries

Let $(\Omega, \mathcal{F}, P)$ be the canonical Wiener space, that is, $\Omega$ is the set of continuous functions from $[0,T]$ to $\mathbb{R}^2$, $\mathcal{F}$ is the completed $\sigma$-algebra on $\Omega$, $P$ is the Wiener measure. We define the canonical process $B_t(\omega) = (B_t^1(\omega), B_t^2(\omega)) = (\omega_1(t), \omega_2(t))$, $t \in [0,T]$, $\omega = (\omega_1, \omega_2) \in \Omega$. Then $B$ is a 2-dimensional Brownian motion on $(\Omega, \mathcal{F}, P)$ and $B^1$ is independent of $B^2$. We denote by $\{\mathcal{F}_{t,s}, s \geq t\}$ the filtration generated by the increments of the Brownian motion $B$ over time interval $[t,T)$, where $\mathcal{F}_{t,s} = \sigma\{B_r - B_t, r \in [t,s]\} \vee \mathcal{N}$, and $\mathcal{N}$ is the set of null-set of $P$.

For any given partition $\pi = \{0 = t_0 < t_1 < \ldots < t_N = T\}$ of the interval $[0,T]$, we define random variables $\zeta_{i,j}^\pi = \Phi_{0,1}(\frac{B_{t_j}^i - B_{t_{j-1}}^i}{\sqrt{t_j - t_{j-1}}})$, $i = 1,2$, $j = 1,2,\ldots,N$, where $\Phi_{0,1}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x \exp\{-\frac{y^2}{2}\} dy$, $x \in \mathbb{R}$. Obviously, $\zeta_{i,j}^\pi$, $1 \leq j \leq N$, $i = 1,2$, is a family of independent random variables with uniform distribution on $[0,1]$. Let $U$ and $V$ be compact metric spaces which are the control state spaces used by Player I and II, respectively. By $\mathcal{P}(U)$ and $\mathcal{P}(V)$ we denote the space of all probability measures over $U$ and $V$, respectively. From Skorohod's Representation Theorem we know that $\mathcal{P}(U)$ (resp., $\mathcal{P}(V)$) coincides with the set of the distributions of all $U$-valued (resp., $V$-valued) random variables.

Now we introduce the admissible controls for both players.

For any $t \in [0,T]$, the $U$-valued Lebesgue measurable functions $(u_s)_{s\in[t,T]}$ form the set of admissible controls for Player I, the $V$-valued Lebesgue measurable functions $(v_s)_{s\in[t,T]}$ those for Player II. We denote by $\mathcal{U}_{t,T}$ the set of admissible controls $(u_s)_{s\in[t,T]}$ for Player I and by $\mathcal{V}_{t,T}$ the set of admissible controls $(v_s)_{s\in[t,T]}$ for Player II. Both spaces $\mathcal{U}_{t,T}$ and $\mathcal{V}_{t,T}$ are endowed with the topology of the convergence in Lebesgue measure; by $\mathcal{B}(\mathcal{U}_{t,T})$ and $\mathcal{B}(\mathcal{V}_{t,T})$ we denote the corresponding Borel $\sigma$-fields.

For any given $t \in [0,T]$, $x \in \mathbb{R}^n$, we consider the following ordinary differential equation

$$X_s = x + \int_t^s f(X_r, u_r, v_r) dr, \ \ s \in [t,T], \tag{2.1}$$

where $u \in \mathcal{U}_{t,T}$ and $v \in \mathcal{V}_{t,T}$, and the coefficient $f : \mathbb{R}^n \times U \times V \mapsto \mathbb{R}^n$ is supposed to be bounded, continuous with respect to $(u,v)$ and Lipschitz continuous in $x$, uniformly with respect to $u$ and $v$. Therefore, equation (2.1) has a unique solution and we denote it by $X^{t,x,u,v}$. From standard estimates we obtain that there exists a constant $C > 0$ such that, for all $(t,x), (t',x') \in [0,T] \times \mathbb{R}^n$, for all $s \in [t \vee t', T]$,

$$\begin{aligned} &(1)\ |X_s^{t,x,u,v} - x| \leq C(s-t), \\ &(2)\ |X_s^{t,x,u,v} - X_s^{t',x',u,v}| \leq C(|t-t'| + |x-x'|). \end{aligned} \tag{2.2}$$

The cost functionals of the zero-sum differential game are defined by the $I \times J$ functionals $g_{ij}(X_T^{t,x,u,v})$, $i = 1, 2, \ldots, I$, $j = 1, 2, \ldots, J$, where the mappings $g_{ij} : \mathbb{R}^n \mapsto \mathbb{R}$ are Lipschitz continuous and bounded. Player I wants to minimize $g_{ij}(X_T^{t,x,u,v})$, *i.e.*, it is a cost functional for him/her, while Player II wants to maximize $g_{ij}(X_T^{t,x,u,v})$ being a payoff for him/her. The cost functionals of nonzero-sum differential games are defined in Section 4.

The rules for our zero-sum differential game with asymmetric information are as follows:

(1) At the beginning of the game, a pair $(i, j)$ is chosen randomly with the probability $(p, q) \in \Delta(I) \times \Delta(J)$, where $\Delta(I)$ is the set of probabilities $p = (p_i)_{i=1,\ldots,I}$ on $\{1, \ldots, I\}$ (*i.e.*, $p_i \geq 0$, $1 \leq i \leq I$, and $\sum_{i=1}^I p_i = 1$); $\Delta(J)$ is defined similarly. Both players know the probability $(p, q)$.
(2) The choice of $i$ is only communicated to Player I, while the choice of $j$ is only communicated to Player II. But both players observe their opponent's controls.

Generally speaking, differential games with "control against control" don't admit a dynamic programming principle and the value does, in general, not exist. Thus, we study the game of the type "nonanticipative strategy with delay against nonanticipative strategy with delay". Considering the asymmetry of the information, the players want to hide a part of their private information. For this they randomize their strategies, and the kind of randomization we choose is the key to obtain a value for our zero-sum game in a framework without Isaacs condition.

Let us consider an arbitrarily given partition $\pi = \{0 = t_0 < t_1 < \ldots < t_N = T\}$ and assume $t \in [t_{k-1}, t_k)$. We give the definition of random non-anticipative strategies with delay for a game over the time interval $[t, T]$.

**Definition 2.1.** A random non-anticipative strategy with delay (NAD, for short) along the partition $\pi$ for Player I is a mapping $\alpha : \Omega \times [t, T] \times \mathcal{V}_{t,T} \mapsto \mathcal{U}_{t,T}$ of the form

$$\alpha(\omega, v)(s) = \alpha_k(\omega, \zeta_{1,k-1}^\pi(\omega), v)(s) I_{[t,t_k)}(s) + \sum_{l=k+1}^N \alpha_l(\omega, (\zeta_{k-1}^\pi, \ldots, \zeta_{l-2}^\pi, \zeta_{1,l-1}^\pi)(\omega), v)(s) I_{[t_{l-1},t_l)}(s), \tag{2.3}$$

where $\zeta_l^\pi = (\zeta_{1,l}^\pi, \zeta_{2,l}^\pi)$, $k-1 \leq l \leq N-2$, and for $k \leq l \leq N$, the mappings $\alpha_l : \Omega \times \mathbb{R}^{2(l-k)+1} \times [t \vee t_{l-1}, t_l] \times \mathcal{V}_{t,T} \mapsto \mathcal{U}_{t,T}$ are $\mathcal{F}_{0,t_{k-2}} \otimes \mathcal{B}(\mathbb{R}^{2(l-k)+1}) \otimes \mathcal{B}([t \vee t_{l-1}, t_l]) \otimes \mathcal{B}(\mathcal{V}_{t,T})$-measurable and satisfy: For all $v, v' \in \mathcal{V}_{t,T}$, it holds that, whenever $v = v'$ a.e. on $[t, t_{l-1}]$, we have for all $\omega \in \Omega$ and all $x \in \mathbb{R}^{2(l-k)+1}$, $\alpha_l(\omega, x, v)(s) = \alpha_l(\omega, x, v')(s)$, a.e. on $[t \vee t_{l-1}, t_l]$, $k+1 \leq l \leq N$.

Similarly, a random NAD strategy along the partition $\pi$ for Player II is a mapping $\beta : \Omega \times [t, T] \times \mathcal{U}_{t,T} \mapsto \mathcal{V}_{t,T}$ of the form

$$\beta(\omega, u)(s) = \beta_k(\omega, \zeta_{2,k-1}^\pi(\omega), v)(s) I_{[t,t_k)}(s) + \sum_{l=k+1}^N \beta_l(\omega, (\zeta_{k-1}^\pi, \ldots, \zeta_{l-2}^\pi, \zeta_{2,l-1}^\pi)(\omega), v)(s) I_{[t_{l-1},t_l)}(s), \tag{2.4}$$

where $\zeta_l^\pi = (\zeta_{1,l}^\pi, \zeta_{2,l}^\pi)$, $k-1 \leq l \leq N-2$, and for $k \leq l \leq N$, the mappings $\beta_l : \Omega \times \mathbb{R}^{2(l-k)+1} \times [t \vee t_{l-1}, t_l] \times \mathcal{U}_{t,T} \mapsto \mathcal{V}_{t,T}$ are $\mathcal{F}_{0,t_{k-2}} \otimes \mathcal{B}(\mathbb{R}^{2(l-k)+1}) \otimes \mathcal{B}([t \vee t_{l-1}, t_l]) \otimes \mathcal{B}(\mathcal{U}_{t,T})$-measurable and satisfy: For all $u, u' \in \mathcal{U}_{t,T}$, it holds that, whenever $u = u'$ a.e. on $[t, t_{l-1}]$, we have for all $\omega \in \Omega$ and all $x \in \mathbb{R}^{2(l-k)+1}$, $\beta_l(\omega, x, u)(s) = \beta_l(\omega, x, u')(s)$, a.e. on $[t \vee t_{l-1}, t_l]$, $k+1 \leq l \leq N$.

The set of all such random NAD strategies for Player I along the partition $\pi$ is denoted by $\mathcal{A}^\pi(t, T)$, and similarly $\mathcal{B}^\pi(t, T)$ is that for Player II, $\mathcal{A}_0^\pi(t, T)$ and $\mathcal{B}_0^\pi(t, T)$ are the sets of pure (*i.e.* deterministic non randomized) strategies for player I and II. Then we know, for any partitions $\pi, \pi'$ of interval $[t, T]$ with $\pi \subset \pi'$, it holds $\mathcal{A}^\pi(t, T) \subset \mathcal{A}^{\pi'}(t, T)$. Moreover we define

$$\mathcal{A}(t, T) := \bigcup_\pi \mathcal{A}^\pi(t, T), \quad \mathcal{B}(t, T) := \bigcup_\pi \mathcal{B}^\pi(t, T). \tag{2.5}$$

**Definition 2.2.** We say that $\alpha \in \mathcal{A}_1^\pi(t,T)$, if $\alpha \in \mathcal{A}^\pi(t,T)$ and the $\alpha_l$'s in (2.3) don't depend on $\omega$, *i.e.*, for all $k \le l \le N$, $\alpha_l : \mathbb{R}^{2(l-k)+1} \times [t \vee t_{l-1}, t_l] \times \mathcal{V}_{t,T} \mapsto \mathcal{U}_{t,T}$, is a $\mathcal{B}(\mathbb{R}^{2(l-k)+1}) \otimes \mathcal{B}([t \vee t_{l-1}, t_l]) \otimes \mathcal{B}(\mathcal{V}_{t,T})$-measurable function satisfying: For all $v, v' \in \mathcal{V}_{t,T}$, it holds that, whenever $v = v'$ a.e. on $[t, t_{l-1}]$, we have for all $x \in \mathbb{R}^{2(l-k)+1}$, $\alpha_l(x,v)(s) = \alpha_l(x,v')(s)$, a.e. on $[t \vee t_{l-1}, t_l]$, $k+1 \le l \le N$. Similarly, we define $\beta \in \mathcal{B}_1^\pi(t,T)$.

Obviously, from the Definitions 2.1 and 2.2 we know $\mathcal{A}_0^\pi(t,T) \subset \mathcal{A}_1^\pi(t,T) \subset \mathcal{A}^\pi(t,T)$, $\mathcal{B}_0^\pi(t,T) \subset \mathcal{B}_1^\pi(t,T) \subset \mathcal{B}^\pi(t,T)$.

From the definition of an NAD strategy we get the following lemma which is crucial throughout the paper. Such a result was established the first time by Buckdahn, Cardaliaguet and Rainer ([1], Lem. 2.4).

**Lemma 2.3.** *For any $\alpha \in \mathcal{A}(t,T)$ and $\beta \in \mathcal{B}(t,T)$, there exists a unique measurable mapping $\Omega \ni \omega \mapsto (u_\omega, v_\omega) \in \mathcal{U}_{t,T} \times \mathcal{V}_{t,T}$, such that, for all $\omega \in \Omega$,*

$$\alpha(\omega, v_\omega) = u_\omega,\ \beta(\omega, u_\omega) = v_\omega,\ \textit{a.e. on } [t,T].$$

A proof of Lemma 2.3 for a similar context can be found in [6]. However, since our framework is slightly more general, for the reader's convenience we prefer to give it here.

*Proof.* For any $\alpha \in \mathcal{A}(t,T)$, from (2.5) we know there exists a partition $\pi_1$ of interval $[0,T]$, such that $\alpha \in \mathcal{A}^{\pi_1}(t,T)$. Similarly, there is a partition $\pi_2$ of interval $[0,T]$, such that $\beta \in \mathcal{B}^{\pi_2}(t,T)$. We define $\pi = \pi_1 \cup \pi_2$ which combines $\pi_1$ and $\pi_2$, and we notice that then $\alpha \in \mathcal{A}^\pi(t,T)$ and $\beta \in \mathcal{B}^\pi(t,T)$.

Indeed, if, for example, $\pi = \{0 = t_0 < t_1 < \ldots < t_N = T\}$ and $t_{l-1}, t_{l+1} \in \pi_1$, but $t_l \notin \pi_1$, then for $[t_{l-1}, t_{l+1}]$ as $j$th subinterval of the partition $\pi_1$, $\zeta_{i,j}^{\pi_1} = \Phi_{0,1}(\frac{B^i_{t_{l+1}} - B^i_{t_{l-1}}}{\sqrt{t_{l+1}-t_{l-1}}}) = \Phi_{0,1}\Big(\frac{1}{\sqrt{t_{l+1}-t_{l-1}}}\big(\sqrt{t_l - t_{l-1}}\Phi_{0,1}^{-1}(\Phi_{0,1}(\frac{B^i_{t_l} - B^i_{t_{l-1}}}{\sqrt{t_l - t_{l-1}}})) + \sqrt{t_{l+1}-t_l}\Phi_{0,1}^{-1}(\Phi_{0,1}(\frac{B^i_{t_{l+1}} - B^i_{t_l}}{\sqrt{t_{l+1}-t_l}}))\Big)$, *i.e.*, $\zeta_{i,j}^{\pi_1}$ is a measurable function of $(\zeta_{i,l}^\pi, \zeta_{i,l+1}^\pi)$, $i = 1,2$. The above situation can be extended in an obvious manner to the general case $\pi_1 \subset \pi$ and allows to show that $\mathcal{A}^{\pi_1}(t,T) \subset \mathcal{A}^\pi(t,T)$. Analogously, $\mathcal{B}^{\pi_2}(t,T) \subset \mathcal{B}^\pi(t,T)$.

Assume $\pi = \{0 = t_0 < t_1 < \ldots < t_N = T\}$, and $t \in [t_{k-1}, t_k)$, $0 \le k \le N$. For each $\omega \in \Omega$, $\alpha(\omega, v)$ (respectively, $\beta(\omega, u)$) restricted to $[t, t_k]$ depends only on $v \in \mathcal{V}_{t,T}$ (respectively, $u \in \mathcal{U}_{t,T}$) restricted to $[t, t_{k-1}]$. Since $[t, t_{k-1}]$ is empty or a singleton, from the property of delay we know $\alpha(\omega, v), \beta(\omega, u)$ restricted to $[t, t_k]$ do not depend on $v$ and $u$. Then we can define $u_\omega^1 = \alpha(\omega, v^0)$, $v_\omega^1 = \beta(\omega, u^0)$, for any $v^0 \in \mathcal{V}_{t,T}$ and $u^0 \in \mathcal{U}_{t,T}$, and the mapping $\Omega \ni \omega \mapsto (u_\omega^1, v_\omega^1) \in \mathcal{U}_{t,T} \times \mathcal{V}_{t,T}$ is measurable. Then we have

$$\alpha(\omega, v^1) = u^1,\ \beta(\omega, u^1) = v^1\ \textit{a.e.} \text{ on } [t, t_k].$$

Now we assume that for $j \ge 2$, $\alpha(\omega, v_\omega^{j-1}) = u_\omega^{j-1}$, $\beta(\omega, u_\omega^{j-1}) = v_\omega^{j-1}$, a.e. on $[t, t_{j+k-2}]$, and $\omega \mapsto (u_\omega^{j-1}, v_\omega^{j-1})$ is measurable. Then we define $u_\omega^j = \alpha(\omega, v_\omega^{j-1})$, $v_\omega^j = \beta(\omega, u_\omega^{j-1})$. Obviously, $u_\omega^j = u_\omega^{j-1}$, $v_\omega^j = v_\omega^{j-1}$, a.e. on $[t, t_{j+k-2}]$. From the property of delay, we have $\alpha(\omega, v_\omega^j) = \alpha(\omega, v_\omega^{j-1}) = u_\omega^j$, $\beta(\omega, u_\omega^j) = \beta(\omega, u_\omega^{j-1}) = v_\omega^j$, a.e. on $[t, t_{j+k-1}]$, and $\omega \mapsto (u_\omega^j, v_\omega^j)$ is measurable. Consequently, we get the existence of the measurable mapping $\Omega \ni \omega \mapsto (u_\omega, v_\omega) \in \mathcal{U}_{t,T} \times \mathcal{V}_{t,T}$ satisfying this lemma. Its uniqueness is obvious from the above construction. ☐

**Remark 2.4.** This lemma implies that, for any partition $\pi$ of $[0,T]$, for any $\alpha \in \mathcal{A}^\pi(t,T)$, $\beta \in \mathcal{B}^\pi(t,T)$, but also for any $\alpha \in \mathcal{A}(t,T)$, $\beta \in \mathcal{B}^\pi(t,T)$, and for any $\alpha \in \mathcal{A}^\pi(t,T)$, $\beta \in \mathcal{B}(t,T)$, there exists the unique mapping $\Omega \ni \omega \mapsto (u_\omega, v_\omega) \in \mathcal{U}_{t,T} \times \mathcal{V}_{t,T}$, such that for all $\omega \in \Omega$,

$$\alpha(\omega, v_\omega) = u_\omega,\ \beta(\omega, u_\omega) = v_\omega,\ \textit{a.e.} \text{ on } [t,T].$$

**Remark 2.5.** The control processes $u$ and $v$ along the partition $\pi$ satisfying Lemma 2.3 have the following form:

$$\begin{cases} u(\omega,s) = u^k(\omega, \zeta_{1,k-1}^\pi, s) \cdot I_{[t,t_k)}(s) + \sum\limits_{l=k+1}^N u^l(\omega, \zeta_{k-1}^\pi, \ldots, \zeta_{l-2}^\pi, \zeta_{1,l-1}^\pi, s) \cdot I_{[t_{l-1},t_l)}(s), \\ v(\omega,s) = v^k(\omega, \zeta_{2,k-1}^\pi, s) \cdot I_{[t,t_k)}(s) + \sum\limits_{l=k+1}^N v^l(\omega, \zeta_{k-1}^\pi, \ldots, \zeta_{l-2}^\pi, \zeta_{2,l-1}^\pi, s) \cdot I_{[t_{l-1},t_l)}(s), \end{cases}$$

where $u^l, v^l$ are $\mathcal{F}_{0,t_{k-2}} \otimes \mathcal{B}(\mathbb{R}^{2(l-k)+1}) \otimes \mathcal{B}([t \vee t_{l-1}, t_l])$-measurable functions, $k \le l \le N$. We denote by $\mathcal{U}^{\pi}_{t,T}$ and $\mathcal{V}^{\pi}_{t,T}$ the set of the processes $u$ and $v$, respectively, which have the above form. The corresponding set of control constructed with the help of strategies from $\mathcal{A}^{\pi}_1(t,T)$ and $\mathcal{B}^{\pi}_1(t,T)$ is denoted by $\mathcal{U}^{\pi,1}_{t,T}$ and $\mathcal{V}^{\pi,1}_{t,T}$. The only difference between $\mathcal{U}^{\pi}_{t,T}$ and $\mathcal{U}^{\pi,1}_{t,T}$ is that, if $u \in \mathcal{U}^{\pi,1}_{t,T}$, then $u_l$, $k \le l \le N$, is just $\mathcal{B}(\mathbb{R}^{2(l-k)+1}) \otimes \mathcal{B}([t \vee t_{l-1}, t_l])$-measurable, *i.e.*, $u_l$ is deterministic.

**Remark 2.6.** We write $\hat{\alpha} \in (\mathcal{A}^{\pi}(t,T))^I$, if $\hat{\alpha} = (\alpha_1, \dots, \alpha_I)$ and $\alpha_i \in \mathcal{A}^{\pi}(t,T)$, $i = 1, \dots, I$, and $\hat{\beta} \in (\mathcal{B}^{\pi}(t,T))^J$, if $\hat{\beta} = (\beta_1, \dots, \beta_J)$ and $\beta_j \in \mathcal{B}^{\pi}(t,T)$, $j = 1, \dots, J$. Similarly, we have $\hat{\alpha} \in (\mathcal{A}(t,T))^I$, $\hat{\beta} \in (\mathcal{B}(t,T))^J$.

For $(p,q) \in \Delta(I) \times \Delta(J)$, $(t,x) \in [0,T] \times \mathbb{R}^n$, $\pi = \{0 = t_0 < t_1 < \ldots < t_N = T\}$ and $t \in [t_{k-1}, t_k)$, we define the payoff functional

$$J(t,x,\hat{\alpha},\hat{\beta},p,q) = \sum_{i=1}^{I}\sum_{j=1}^{J} p_i q_j E\left[g_{ij}\left(X_T^{t,x,\alpha_i,\beta_j}\right)\right].$$

Let us now introduce the following upper value functions:

$$W^{\pi}(t,x,p,q) = \inf_{\hat{\alpha} \in (\mathcal{A}^{\pi}(t,T))^I} \sup_{\hat{\beta} \in (\mathcal{B}^{\pi}(t,T))^J} J(t,x,\hat{\alpha},\hat{\beta},p,q), \tag{2.6}$$

$$W(t,x,p,q) = \inf_{\hat{\alpha} \in (\mathcal{A}(t,T))^I} \sup_{\hat{\beta} \in (\mathcal{B}(t,T))^J} J(t,x,\hat{\alpha},\hat{\beta},p,q), \tag{2.7}$$

and the lower value functions:

$$V^{\pi}(t,x,p,q) = \sup_{\hat{\beta} \in (\mathcal{B}^{\pi}(t,T))^J} \inf_{\hat{\alpha} \in (\mathcal{A}^{\pi}(t,T))^I} J(t,x,\hat{\alpha},\hat{\beta},p,q), \tag{2.8}$$

$$V(t,x,p,q) = \sup_{\hat{\beta} \in (\mathcal{B}(t,T))^J} \inf_{\hat{\alpha} \in (\mathcal{A}(t,T))^I} J(t,x,\hat{\alpha},\hat{\beta},p,q), \tag{2.9}$$

respectively, which will be studied in what follows.

**Definition 2.7.** Given $\varepsilon > 0$, we say that $\hat{\alpha} \in (\mathcal{A}^{\pi}(t,T))^I$ is an $\varepsilon$-optimal randomized strategy for $W^{\pi}(t,x,p,q)$, if for all $(t,x,p,q) \in [0,T] \times \mathbb{R}^n \times \Delta(I) \times \Delta(J)$,

$$|W^{\pi}(t,x,p,q) - \sup_{\hat{\beta} \in (\mathcal{B}^{\pi}(t,T))^J} J(t,x,\hat{\alpha},\hat{\beta},p,q)| \le \varepsilon. \tag{2.10}$$

In the same sense, $\hat{\beta} \in (\mathcal{B}^{\pi}(t,T))^J$ is to be an $\varepsilon$-optimal randomized strategy for $V^{\pi}(t,x,p,q)$, if for all $(t,x,p,q) \in [0,T] \times \mathbb{R}^n \times \Delta(I) \times \Delta(J)$,

$$|V^{\pi}(t,x,p,q) - \inf_{\hat{\alpha} \in (\mathcal{A}^{\pi}(t,T))^I} J(t,x,\hat{\alpha},\hat{\beta},p,q)| \le \varepsilon. \tag{2.11}$$

Similarly, we define $\varepsilon$-optimal strategies for the upper value function $W(t,x,p,q)$ and the lower value function $V(t,x,p,q)$.

## 3. The functions $W^\pi(t,x,p,q)$ and $V^\pi(t,x,p,q)$ without Isaacs condition

In this section we mainly prove that, when the mesh of the partition $\pi$ tends to 0, the functions $W^\pi$ and $V^\pi$ converge uniformly to the same function which is the unique dual solution of some Hamilton−Jacobi−Isaacs (HJI, for short) equation. For this, we introduce the following auxiliary functions:

$$W_1^\pi(t,x,p,q) = \inf_{\hat\alpha\in(\mathcal{A}_1^\pi(t,T))^I}\ \sup_{\hat\beta\in(\mathcal{B}_1^\pi(t,T))^J} J(t,x,\hat\alpha,\hat\beta,p,q), \tag{3.1}$$

$$V_1^\pi(t,x,p,q) = \sup_{\hat\beta\in(\mathcal{B}_1^\pi(t,T))^J}\ \inf_{\hat\alpha\in(\mathcal{A}_1^\pi(t,T))^I} J(t,x,\hat\alpha,\hat\beta,p,q). \tag{3.2}$$

**Theorem 3.1.** *For any $(t,x,p,q) \in [0,T]\times\mathbb{R}^n\times\Delta(I)\times\Delta(J)$, it holds $V^\pi(t,x,p,q) = V_1^\pi(t,x,p,q)$, $W^\pi(t,x,p,q) = W_1^\pi(t,x,p,q)$.*

We only give the proof for $V^\pi(t,x,p,q) = V_1^\pi(t,x,p,q)$, the proof for $W^\pi(t,x,p,q) = W_1^\pi(t,x,p,q)$ is similar. In order to show this, we need the following auxiliary lower value function:

$$\tilde V^\pi(t,x,p,q) = \operatorname*{esssup}_{\hat\beta\in(\mathcal{B}^\pi(t,T))^J}\ \operatorname*{essinf}_{\hat\alpha\in(\mathcal{A}^\pi(t,T))^I} \sum_{i=1}^I\sum_{j=1}^J p_iq_jE\left[g_{ij}\left(X_T^{t,x,\alpha_i,\beta_j}\right)|\mathcal{F}_{0,t_{k-2}}\right], \tag{3.3}$$

$t\in[t_{k-1},t_k)$, for $\pi=\{0=t_0<t_1<\ldots<t_N=T\}$. Note this $\tilde V^\pi(t,x,p,q)$ is a prior $\mathcal{F}_{0,t_{k-2}}$-measurable. However, we have the following lemma.

**Lemma 3.2.** *For all $(t,x,p,q)\in[0,T]\times\mathbb{R}^n\times\Delta(I)\times\Delta(J)$, the function $\tilde V^\pi(t,x,p,q)$ is deterministic,* i.e., *independent of $\mathcal{F}_{0,t_{k-2}}$. As a consequence we have $\tilde V^\pi(t,x,p,q) = E[\tilde V^\pi(t,x,p,q)]$, $P$-a.s.*

*Proof.* Recall $\Omega = C([0,T];\mathbb{R}^2)$ and put

$$H=\left\{h\in\Omega : \exists \text{ Radon-Nikodym derivative } \dot h\in L^2([0,T];\mathbb{R}^2), h(s)=h(s\wedge t_{k-2}), s\in[0,T]\right\}$$

($H$ is the Cameron−Martin Space). For any $h\in H$, we define the mapping $\tau_h:\Omega\mapsto\Omega$ by setting $\tau_h(\omega):=\omega+h$, $\omega\in\Omega$. Obviously, $\tau_h$ is a bijection and $\tau_h^{-1}=\tau_{-h}$.

Recall that $\alpha\in\mathcal{A}^\pi(t,T)$ has the form

$$\alpha(\omega,v)(s)=\alpha_k(\omega,\zeta_{1,k-1}^\pi(\omega),v)(s)I_{[t,t_k)}(s)+\sum_{l=k+1}^N\alpha_l(\omega,(\zeta_{k-1}^\pi,\ldots,\zeta_{l-2}^\pi,\zeta_{1,l-1}^\pi)(\omega),v)(s)I_{[t_{l-1},t_l)}(s)$$

(see Def. 2.1). Hence, for any $h\in H$, its Girsanov transform takes the form

$$\begin{aligned}\alpha^h(\omega,v)(s)&:=\alpha(\tau_h(\omega),v)(s)\\ &=\alpha_k(\tau_h(\omega),\zeta_{1,k-1}^\pi(\omega),v)(s)I_{[t,t_k)}(s)+\sum_{l=k+1}^N\alpha_l(\tau_h(\omega),(\zeta_{k-1}^\pi,\ldots,\zeta_{l-2}^\pi,\zeta_{1,l-1}^\pi)(\omega),v)(s)I_{[t_{l-1},t_l)}(s).\end{aligned}$$

Obviously, $\alpha^h\in\mathcal{A}^\pi(t,T)$, and the mapping $\alpha\mapsto\alpha^h$ is a bijection on $\mathcal{A}^\pi(t,T)$. For any $h\in H$, $\beta\in\mathcal{B}^\pi(t,T)$, $\beta^h$ is defined similarly and also $\beta\mapsto\beta^h$ is a bijection on $\mathcal{B}^\pi(t,T)$. Moreover, we can check easily that, for all $h\in H$,

$$E\left[g_{ij}\left(X_T^{t,x,\alpha_i,\beta_j}\right)|\mathcal{F}_{0,t_{k-2}}\right]\circ\tau_h = E\left[g_{ij}\left(X_T^{t,x,\alpha_i^h,\beta_j^h}\right)|\mathcal{F}_{0,t_{k-2}}\right],\ P\text{-a.s.} \tag{3.4}$$

We now set

$$I(t,x,p,q,\hat{\beta}) := \operatorname*{essinf}_{\hat{\alpha}\in(\mathcal{A}^{\pi}(t,T))^I} \sum_{i=1}^{I}\sum_{j=1}^{J} p_i q_j E\left[g_{ij}\left(X_T^{t,x,\alpha_i,\beta_j}\right)|\mathcal{F}_{0,t_{k-2}}\right],\ \hat{\beta}\in(\mathcal{B}^{\pi}(t,T))^J.$$

Since $I(t,x,p,q,\hat{\beta}) \le \sum_{i=1}^{I}\sum_{j=1}^{J} p_i q_j E[g_{ij}(X_T^{t,x,\alpha_i,\beta_j})|\mathcal{F}_{0,t_{k-2}}]$, $P$-a.s., from (3.4) we get

$$I(t,x,p,q,\hat{\beta})\circ\tau_h \le \sum_{i=1}^{I}\sum_{j=1}^{J} p_i q_j E\left[g_{ij}\left(X_T^{t,x,\alpha_i^h,\beta_j^h}\right)|\mathcal{F}_{0,t_{k-2}}\right],\ P\text{-a.s., for all } \hat{\alpha}\in(\mathcal{A}^{\pi}(t,T))^I. \tag{3.5}$$

On the other hand, for any random variable $\xi$, such that $\xi \le \sum_{i=1}^{I}\sum_{j=1}^{J} p_i q_j E[g_{ij}(X_T^{t,x,\alpha_i^h,\beta_j^h})|\mathcal{F}_{0,t_{k-2}}]$, $P$-a.s., we have that

$$\xi\circ\tau_{-h} \le \sum_{i=1}^{I}\sum_{j=1}^{J} p_i q_j E\left[g_{ij}\left(X_T^{t,x,\alpha_i,\beta_j}\right)|\mathcal{F}_{0,t_{k-2}}\right],\ P\text{-a.s., for all } \hat{\alpha}\in(\mathcal{A}^{\pi}(t,T))^I.$$

Consequently, $\xi\circ\tau_{-h}\le I(t,x,p,q,\hat{\beta})$, $P$-a.s., which implies that $\xi \le I(t,x,p,q,\hat{\beta})\circ\tau_h$, $P$-a.s., as the law of $\tau_h$ is equivalent to $P$. Thus, we have

$$I(t,x,p,q,\hat{\beta})\circ\tau_h = \operatorname*{essinf}_{\hat{\alpha}\in(\mathcal{A}^{\pi}(t,T))^I} \sum_{i=1}^{I}\sum_{j=1}^{J} p_i q_j E\left[g_{ij}\left(X_T^{t,x,\alpha_i^h,\beta_j^h}\right)|\mathcal{F}_{0,t_{k-2}}\right],\ P\text{-a.s.} \tag{3.6}$$

Using the same method, we obtain

$$\left(\operatorname*{esssup}_{\hat{\beta}\in(\mathcal{B}^{\pi}(t,T))^J} I(t,x,p,q,\hat{\beta})\right)\circ\tau_h = \operatorname*{esssup}_{\hat{\beta}\in(\mathcal{B}^{\pi}(t,T))^J}\left(I(t,x,p,q,\hat{\beta})\circ\tau_h\right),\ P\text{-a.s.} \tag{3.7}$$

Therefore, for all $h\in H$, from (3.7) and (3.6) we get, $P$-a.s.,

$$\begin{aligned}\tilde{V}^{\pi}(t,x,p,q)\circ\tau_h &= \left(\operatorname*{esssup}_{\hat{\beta}\in(\mathcal{B}^{\pi}(t,T))^J} I(t,x,p,q,\hat{\beta})\right)\circ\tau_h\\ &= \operatorname*{esssup}_{\hat{\beta}\in(\mathcal{B}^{\pi}(t,T))^J}\ \operatorname*{essinf}_{\hat{\alpha}\in(\mathcal{A}^{\pi}(t,T))^I} \sum_{i=1}^{I}\sum_{j=1}^{J} p_i q_j E\left[g_{ij}\left(X_T^{t,x,\alpha_i^h,\beta_j^h}\right)|\mathcal{F}_{0,t_{k-2}}\right]\\ &= \operatorname*{esssup}_{\hat{\beta}\in(\mathcal{B}^{\pi}(t,T))^J}\ \operatorname*{essinf}_{\hat{\alpha}\in(\mathcal{A}^{\pi}(t,T))^I} \sum_{i=1}^{I}\sum_{j=1}^{J} p_i q_j E\left[g_{ij}\left(X_T^{t,x,\alpha_i,\beta_j}\right)|\mathcal{F}_{0,t_{k-2}}\right] = \tilde{V}^{\pi}(t,x,p,q),\end{aligned} \tag{3.8}$$

where, for the latter equality, we have used that $\{\alpha^h|\alpha\in\mathcal{A}^{\pi}(t,T)\}=\mathcal{A}^{\pi}(t,T)$ and $\{\beta^h|\beta\in\mathcal{B}^{\pi}(t,T)\}=\mathcal{B}^{\pi}(t,T)$. Then, combining this invariance of $\tilde{V}^{\pi}(t,x,p,q)$ under Girsanov transformation $\tau_h$, $h\in H$, with Lemma 4.1 in [3] we obtain the stated result. □

Now we give the proof of Theorem 3.1.

*Proof.*
**Step 1.** We prove $\tilde{V}^{\pi}(t,x,p,q)=V_1^{\pi}(t,x,p,q)$, for all $(t,x,p,q)\in[0,T]\times\mathbb{R}^n\times\Delta(I)\times\Delta(J)$.

For any $\hat{\beta}\in(\mathcal{B}_1^{\pi}(t,T))^J$ (independent of $\mathcal{F}_{0,t_{k-2}}$), as $\mathcal{B}_1^{\pi}(t,T)\subset\mathcal{B}^{\pi}(t,T)$, we have

$$\tilde{V}^{\pi}(t,x,p,q) \ge \operatorname*{essinf}_{\hat{\alpha}\in(\mathcal{A}^{\pi}(t,T))^I} \sum_{i=1}^{I}\sum_{j=1}^{J} p_i q_j E\left[g_{ij}\left(X_T^{t,x,\alpha_i,\beta_j}\right)|\mathcal{F}_{0,t_{k-2}}\right],\ P\text{-a.s.}$$

Moreover, for any $\varepsilon > 0$, there exists $\hat{\alpha} \in (\mathcal{A}^\pi(t,T))^I$ (depending on $\varepsilon$, $\hat{\beta}$), such that

$$\tilde{V}^\pi(t,x,p,q) \geq \sum_{i=1}^I \sum_{j=1}^J p_i q_j E\left[g_{ij}\left(X_T^{t,x,\alpha_i,\beta_j}\right)|\mathcal{F}_{0,t_{k-2}}\right] - \varepsilon,\ P\text{-a.s.} \tag{3.9}$$

(For the method of the construction of such $\hat{\alpha}$, see, *e.g.* [3], Lem. 4.4). Then, from Lemmas 3.2 and (3.9), we have

$$\begin{aligned}\tilde{V}^\pi(t,x,p,q) = E[\tilde{V}^\pi(t,x,p,q)] &\geq \sum_{i=1}^I \sum_{j=1}^J p_i q_j E\left[g_{ij}\left(X_T^{t,x,\alpha_i,\beta_j}\right)\right] - \varepsilon \\ &\geq \inf_{\hat{\alpha}\in(\mathcal{A}_1^\pi(t,T))^I} \sum_{i=1}^I \sum_{j=1}^J p_i q_j E\left[g_{ij}\left(X_T^{t,x,\alpha_i,\beta_j}\right)\right] - \varepsilon.\end{aligned} \tag{3.10}$$

Since (3.10) holds for all $\hat{\beta} \in (\mathcal{B}_1^\pi(t,T))^J$, we get

$$\tilde{V}^\pi(t,x,p,q) \geq \sup_{\hat{\beta}\in(\mathcal{B}_1^\pi(t,T))^J} \inf_{\hat{\alpha}\in(\mathcal{A}_1^\pi(t,T))^I} \sum_{i=1}^I \sum_{j=1}^J p_i q_j E\left[g_{ij}\left(X_T^{t,x,\alpha_i,\beta_j}\right)\right] - \varepsilon = V_1^\pi(t,x,p,q) - \varepsilon. \tag{3.11}$$

Finally, from the arbitrariness of $\varepsilon$, we obtain $\tilde{V}^\pi(t,x,p,q) \geq V_1^\pi(t,x,p,q)$.
On the other hand, for any $\varepsilon > 0$, there exists $\hat{\beta} \in (\mathcal{B}^\pi(t,T))^J$, such that, $P$-a.s.,

$$\begin{aligned}\tilde{V}^\pi(t,x,p,q) &\leq \operatorname*{essinf}_{\hat{\alpha}\in(\mathcal{A}^\pi(t,T))^I} \sum_{i=1}^I \sum_{j=1}^J p_i q_j E\left[g_{ij}\left(X_T^{t,x,\alpha_i,\beta_j}\right)|\mathcal{F}_{0,t_{k-2}}\right] + \varepsilon \\ &\leq \operatorname*{essinf}_{\hat{\alpha}\in(\mathcal{A}_1^\pi(t,T))^I} \sum_{i=1}^I \sum_{j=1}^J p_i q_j E\left[g_{ij}\left(X_T^{t,x,\alpha_i,\beta_j}\right)|\mathcal{F}_{0,t_{k-2}}\right] + \varepsilon\end{aligned} \tag{3.12}$$

Notice that, for $\hat{\alpha} \in (\mathcal{A}_1^\pi(t,T))^I$, $\hat{\beta} \in (\mathcal{B}^\pi(t,T))^J$, $E[g_{ij}(X_T^{t,x,\alpha_i,\beta_j})|\mathcal{F}_{0,t_{k-2}}](\omega) = E[g_{ij}(X_T^{t,x,\alpha_i,\beta_j^{\bar{\omega}}})]$, $P(d\omega)$-a.s., where $\bar{\omega}(s) = \omega(s \wedge t_{k-2})$, $s \in [0,T]$, and, for $\beta_j$ of form (2.4)

$$\beta_j^{\bar{\omega}}(\omega,u)(s) = \beta_k(\bar{\omega},\zeta_{2,k-1}^\pi(\omega),v)(s)I_{[t,t_k)}(s) + \sum_{l=k+1}^N \beta_l(\bar{\omega},(\zeta_{k-1}^\pi,\ldots,\zeta_{l-2}^\pi,\zeta_{2,l-1}^\pi)(\omega),v)(s)I_{[t_{l-1},t_l)}(s)$$

belongs to $\mathcal{B}_1^\pi(t,T)$, for all $\bar{\omega} = \omega(\cdot \wedge t_{k-2})$. Thus, from (3.12), taking the expectation on both sides, we have

$$\begin{aligned}\tilde{V}^\pi(t,x,p,q) &\leq \inf_{\hat{\alpha}\in(\mathcal{A}_1^\pi(t,T))^I} \sum_{i=1}^I \sum_{j=1}^J p_i q_j E\left[g_{ij}\left(X_T^{t,x,\alpha_i,\beta_j^{\bar{\omega}}}\right)\right] + \varepsilon \\ &\leq \sup_{\hat{\beta}\in(\mathcal{B}_1^\pi(t,T))^J} \inf_{\hat{\alpha}\in(\mathcal{A}_1^\pi(t,T))^I} \sum_{i=1}^I \sum_{j=1}^J p_i q_j E\left[g_{ij}\left(X_T^{t,x,\alpha_i,\beta_j}\right)\right] + \varepsilon = V_1^\pi(t,x,p,q) + \varepsilon.\end{aligned} \tag{3.13}$$

Therefore, from the arbitrariness of $\varepsilon$, we obtain $\tilde{V}^\pi(t,x,p,q) \leq V_1^\pi(t,x,p,q)$.

**Step 2.** We now prove $\tilde{V}^\pi(t,x,p,q) = V^\pi(t,x,p,q)$, for all $(t,x,p,q) \in [0,T] \times \mathbb{R}^n \times \Delta(I) \times \Delta(J)$.
For any $\varepsilon > 0$ and $\hat{\beta} \in (\mathcal{B}^\pi(t,T))^J$ there exists $\hat{\alpha} \in (\mathcal{A}^\pi(t,T))^I$, such that, $P$-a.s.,

$$\begin{aligned}\tilde{V}^\pi(t,x,p,q) &\geq \operatorname*{essinf}_{\hat{\alpha}\in(\mathcal{A}^\pi(t,T))^I} \sum_{i=1}^I \sum_{j=1}^J p_i q_j E\left[g_{ij}\left(X_T^{t,x,\alpha_i,\beta_j}\right)|\mathcal{F}_{0,t_{k-2}}\right] \\ &\geq \sum_{i=1}^I \sum_{j=1}^J p_i q_j E\left[g_{ij}\left(X_T^{t,x,\alpha_i,\beta_j}\right)|\mathcal{F}_{0,t_{k-2}}\right] - \varepsilon.\end{aligned} \tag{3.14}$$

From Lemma 3.2 and (3.14) we have

$$\tilde{V}^\pi(t,x,p,q) = E[\tilde{V}^\pi(t,x,p,q)] \geq \sum_{i=1}^{I}\sum_{j=1}^{J} p_i q_j E\left[g_{ij}\left(X_T^{t,x,\alpha_i,\beta_j}\right)\right] - \varepsilon$$

$$\geq \inf_{\hat{\alpha}\in(\mathcal{A}^\pi(t,T))^I} \sum_{i=1}^{I}\sum_{j=1}^{J} p_i q_j E\left[g_{ij}\left(X_T^{t,x,\alpha_i,\beta_j}\right)\right] - \varepsilon, \tag{3.15}$$

for all $\hat{\beta} \in (\mathcal{B}^\pi(t,T))^J$. From the arbitrariness of $\varepsilon$ and $\hat{\beta} \in (\mathcal{B}^\pi(t,T))^J$, we have $\tilde{V}^\pi(t,x,p,q) \geq V^\pi(t,x,p,q)$. On the other hand, for any $\varepsilon > 0$, there exists $\hat{\beta} \in (\mathcal{B}^\pi(t,T))^J$, such that, $P$-a.s.,

$$\tilde{V}^\pi(t,x,p,q) \leq \operatorname*{essinf}_{\hat{\alpha}\in(\mathcal{A}^\pi(t,T))^I} \sum_{i=1}^{I}\sum_{j=1}^{J} p_i q_j E\left[g_{ij}\left(X_T^{t,x,\alpha_i,\beta_j}\right)|\mathcal{F}_{0,t_{k-2}}\right] + \varepsilon$$

$$\leq \sum_{i=1}^{I}\sum_{j=1}^{J} p_i q_j E\left[g_{ij}\left(X_T^{t,x,\alpha_i,\beta_j}\right)|\mathcal{F}_{0,t_{k-2}}\right] + \varepsilon, \tag{3.16}$$

for all $\hat{\alpha} \in (\mathcal{A}^\pi(t,T))^I$. Thus, from Lemma 3.2 we get

$$\tilde{V}^\pi(t,x,p,q) = E[\tilde{V}^\pi(t,x,p,q)] \leq \sum_{i=1}^{I}\sum_{j=1}^{J} p_i q_j E\left[g_{ij}\left(X_T^{t,x,\alpha_i,\beta_j}\right)\right] + \varepsilon,$$

and taking into account the arbitrariness of $\hat{\alpha} \in (\mathcal{A}^\pi(t,T))^I$, this yields

$$\tilde{V}^\pi(t,x,p,q) \leq \inf_{\hat{\alpha}\in(\mathcal{A}^\pi(t,T))^I} \sum_{i=1}^{I}\sum_{j=1}^{J} p_i q_j E\left[g_{ij}\left(X_T^{t,x,\alpha_i,\beta_j}\right)\right] + \varepsilon$$

$$\leq \sup_{\hat{\beta}\in(\mathcal{B}^\pi(t,T))^J}\inf_{\hat{\alpha}\in(\mathcal{A}^\pi(t,T))^I} \sum_{i=1}^{I}\sum_{j=1}^{J} p_i q_j E\left[g_{ij}\left(X_T^{t,x,\alpha_i,\beta_j}\right)\right] + \varepsilon = V^\pi(t,x,p,q) + \varepsilon. \tag{3.17}$$

Thus, we obtain $\tilde{V}^\pi(t,x,p,q) \leq V^\pi(t,x,p,q)$. Finally, from Steps 1 and 2, we have $V^\pi(t,x,p,q) = \tilde{V}^\pi(t,x,p,q) = V_1^\pi(t,x,p,q)$. ☐

We now prove that, when the mesh of the partition $\pi$ tends to 0, the functions $W_1^\pi$ and $V_1^\pi$ converge uniformly to the same function which is the unique dual solution of some HJI equation.

**Lemma 3.3.** *The functions $W_1^\pi$ and $V_1^\pi$ are Lipschitz continuous with respect to $(t,x,p,q)$, uniformly with respect to $\pi$.*

*Proof.* We just give the proof for $V_1^\pi$, the proof of $W_1^\pi$ is similar. Since the cost functionals $g_{ij}$ are bounded, from the definition of $V_1^\pi$, we obviously have that $V_1^\pi$ is Lipschitz with respect to $p$ and $q$. For any $t \in [0,T]$, $(u,v) \in \mathcal{U}_{t,T} \times \mathcal{V}_{t,T}$, from (2.2) it follows that the functional $g_{ij}(X_T^{t,x,u,v})$ is uniformly Lipschitz continuous with respect to $x$. Thus, for all $(\hat{\alpha},\hat{\beta}) \in (\mathcal{A}_1^\pi(t,T))^I \times (\mathcal{B}_1^\pi(t,T))^J$, we have that $J(t,x,\hat{\alpha},\hat{\beta},p,q)$ is Lipschitz continuous with respect to $x$. Moreover, the Lipschitz constant only depends on the Lipschitz constants of $g_{ij}$ and the bound of $f$. Thus we have $V_1^\pi$ is Lipschitz with respect to $x$.

Now we only need to show $V_1^\pi$ is Lipschitz with respect to $t$. Let $x \in \mathbb{R}^n$, $(p,q) \in \Delta(I) \times \Delta(J)$, and $t < t' < T$ be arbitrarily fixed. Let $\varepsilon > 0$ and $\hat{\beta} = (\beta_j)_{j=1,2,\ldots,J} \in (\mathcal{B}_1^\pi(t,T))^J$ be an $\varepsilon$-optimal strategy for $V_1^\pi(t,x,p,q)$. We define a strategy $\beta_j' \in \mathcal{B}_1^\pi(t',T)$ associated with $\beta_j$. For this end, we put for all $u \in \mathcal{U}_{t',T}$,

$$\tilde{\beta}_j(\omega,u) = \beta_j(\omega,\tilde{u}), \text{ where } \tilde{u}(s) = \begin{cases} \bar{u}, & s \in [t,t'), \\ u(s), & s \in [t',T], \end{cases}$$

and $\bar{u} \in U$ is an arbitrarily given constant control.

If $t' < t_k$, then $\tilde{\beta}_j \in \mathcal{B}_1^\pi(t',T)$ and we define $\beta'_j = \tilde{\beta}_j$. Otherwise, we let $l \geq k+1$ be such that $t_{l-1} \leq t' < t_l$, and we consider $2(l-k)+1$ random variables $\eta^i_{k-1}, \dots, \eta^i_{l-2}, \eta^1_{l-1}$, $i = 1,2$, defined on $([0,1], \mathcal{B}([0,1]), dx)$ with $\eta^1_{l-1}(x) = x,\ x \in [0,1]$, which are mutually independent, independent of $\zeta^\pi_{i,j},\ (i,j) \neq (2, l-1)$, and uniformly distributed on $[0,1]$. Then also the composed random variables $\eta^1_{k-1} \circ \zeta^\pi_{2,l-1}, \eta^2_{k-1} \circ \zeta^\pi_{2,l-1}, \dots, \eta^1_{l-1} \circ \zeta^\pi_{2,l-1}$ are mutually independent, independent of all $\zeta^\pi_{i,j},\ (i,j) \neq (2,l-1)$, and uniformly distributed over $[0,1]$. For any $u \in \mathcal{U}_{t',T},\ s \in [t',T]$, we define

$$\beta'_j(\omega,u)(s) = \sum_{m=l}^{N} \tilde{\beta}_{j,m}\big((\eta^1_{k-1} \circ \zeta^\pi_{2,l-1}, \eta^2_{k-1} \circ \zeta^\pi_{2,l-1}, \dots,$$

$$\eta^1_{l-1} \circ \zeta^\pi_{2,l-1}, \zeta^\pi_{1,l-1}, \zeta^\pi_l, \dots, \zeta^\pi_{m-2}, \zeta^\pi_{2,m-1})(\omega), u\big)(s) \times I_{[t' \vee t_{m-1}, t_m)}(s).$$

where $\tilde{\beta}_{j,m}(\omega,u)(s) = \tilde{\beta}_j(\omega,u)(s) I_{[t \vee t_{m-1}, t_m)}(s)$. Obviously, $\beta'_j \in \mathcal{B}_1^\pi(t',T)$. Notice that for all $u \in \mathcal{U}_{t',T},\ \beta'_j(u)$ and $\tilde{\beta}_j(u)$ obey the same law knowing $\zeta^\pi_{l-1}, \dots, \zeta^\pi_{N-1}$. Therefore, $E[g_{ij}(X_T^{t',x,u,\beta'_j(u)})] = E[g_{ij}(X_T^{t',x,u,\tilde{\beta}_j(u)})]$. Consequently, for all $\hat{\alpha} \in (\mathcal{A}_1^\pi(t',T))^I$,

$$J(t',x,\hat{\alpha},(\beta'_j),p,q) = J(t',x,\hat{\alpha},(\tilde{\beta}_j),p,q). \tag{3.18}$$

Now for any $\alpha \in \mathcal{A}_1^\pi(t',T)$, we define a strategy $\alpha' \in \mathcal{A}_1^\pi(t,T)$ associated with $\alpha$ as follows: For all $v \in \mathcal{V}_{t,T}$, we put

$$\alpha'(\omega,v)(s) = \begin{cases} \bar{u}(s), & s \in [t,t'), \\ \alpha(\omega, v|_{[t',T]})(s), & s \in [t',T]. \end{cases}$$

Through the above construction and from Lemma 2.3 we see that the couples of admissible controls related to the couples of strategies $(\alpha', \beta_j)$ and $(\alpha, \tilde{\beta}_j)$ coincide on the interval $[t',T]$. Hence, using standard estimates and Gronwall's inequality, we have

$$E\left[\left|X_s^{t,x,\alpha',\beta_j} - X_s^{t',x,\alpha,\tilde{\beta}_j}\right|\right] \leq M|t'-t|,\ s \in [t',T], \tag{3.19}$$

where the constant $M$ only depends on the bound of $f$ as well as the Lipschitz constant of $f$. Thus, for any $\hat{\alpha} \in (\mathcal{A}_1^\pi(t',T))^I$, from (3.18), (3.19) and (2.11) we obtain

$$\begin{aligned} &J(t',x,\hat{\alpha},(\beta'_j),p,q) = J(t',x,\hat{\alpha},(\tilde{\beta}_j),p,q) \geq J(t,x,\hat{\alpha}',\hat{\beta},p,q) - C|t'-t| \\ &\geq \inf_{\hat{\alpha}'' \in (\mathcal{A}_1^\pi(t,T))^I} J(t,x,\hat{\alpha}'',\hat{\beta},p,q) - C|t'-t| \geq V_1^\pi(t,x,p,q) - \varepsilon - C|t'-t|. \end{aligned}$$

Therefore,

$$V_1^\pi(t',x,p,q) \geq V_1^\pi(t,x,p,q) - \varepsilon - C|t'-t|. \tag{3.20}$$

Similarly, if we assume that $\hat{\beta} \in (\mathcal{B}_1^\pi(t',T))^J$ is $\varepsilon$-optimal for $V_1^\pi(t',x,p,q)$, we get

$$V_1^\pi(t,x,p,q) \geq V_1^\pi(t',x,p,q) - \varepsilon - C|t'-t|. \tag{3.21}$$

Finally, from the arbitrariness of $\varepsilon > 0$ we obtain $V_1^\pi$ is Lipschitz continuous in $t$. □

**Lemma 3.4.** *For any $(t,x) \in [0,T] \times \mathbb{R}^n$, the functions $W_1^\pi(t,x,p,q)$ and $V_1^\pi(t,x,p,q)$ are convex in $p \in \Delta(I)$ and concave in $q \in \Delta(J)$.*

*Proof.* We just give the proof for $V_1^\pi$, the proof of $W_1^\pi$ is similar.

As it is obvious that

$$V_1^\pi(t,x,p,q) = \sup_{(\beta_j) \in (\mathcal{B}_1^\pi(t,T))^J} \sum_{i=1}^{I} p_i \inf_{\alpha \in \mathcal{A}_1^\pi(t,T)} \sum_{j=1}^{J} q_j E\left[g_{ij}\left(X_T^{t,x,\alpha,\beta_j}\right)\right], \tag{3.22}$$

the convexity of $p \mapsto V_1^\pi(t,x,p,q)$ is immediate.

Now we prove that $V_1^\pi(t,x,p,q)$ is concave in $q$. Let $(t,x,p)\in[0,T]\times\mathbb{R}^n\times\Delta(I)$, $q^0,q^1\in\Delta(J),\lambda\in(0,1)$, and let $\hat\beta^0=(\beta_j^0)_{j=1,\dots,J}\in(\mathcal{B}_1^\pi(t,T))^J$ and $\hat\beta^1=(\beta_j^1)_{j=1,\dots,J}\in(\mathcal{B}_1^\pi(t,T))^J$ be $\varepsilon$-optimal for $V_1^\pi(t,x,p,q^0)$ and $V_1^\pi(t,x,p,q^1)$, respectively. For $q^0=(q_1^0,\dots,q_J^0)$ and $q^1=(q_1^1,\dots,q_J^1)$, we define $q_j^\lambda=(1-\lambda)q_j^0+\lambda q_j^1$, and obviously $q^\lambda=(q_1^\lambda,\dots,q_J^\lambda)\in\Delta(J)$. Without loss of generality we assume $q_j^\lambda>0$, $j=1,\dots,J$, and we put $c_j=\frac{(1-\lambda)q_j^0}{q_j^\lambda}$, $j=1,\dots,J$. For $\omega\in\Omega$, $u\in\mathcal{U}_{t,T}$, $s\in[t,T)$, we introduce the strategy $\hat\beta^\lambda=(\beta_j^\lambda)_{j=1,\dots,J}$ by putting

$$\begin{aligned}\beta_j^\lambda(y_1,\dots,y_{2(N-k)+1},u)(s)&=\beta_j^0(y_1,y_2,\dots,y_{2(N-k)},\frac{1}{c_j}y_{2(N-k)+1},u)(s)\cdot I_{[0,c_j]}(y_{2(N-k)+1})\\&\quad+\beta_j^1(y_1,y_2,\dots,y_{2(N-k)},\frac{1}{1-c_j}(y_{2(N-k)+1}-c_j),u)(s)\cdot I_{[c_j,1]}(y_{2(N-k)+1}),\end{aligned}$$

where $\beta_j^i((\zeta_{k-1}^\pi,\dots,\zeta_{N-2}^\pi,\zeta_{2,N-1}^\pi)(\omega),u)(s)=\sum_{l=k}^N\beta_{lj}^i((\zeta_{k-1}^\pi,\dots,\zeta_{l-2}^\pi,\zeta_{2,l-1}^\pi)(\omega),u)I_{[t\vee t_{l-1},t_l]}(s)$, $i=0,1$, respectively. Then $(\beta_j^\lambda)\in(\mathcal{B}_1^\pi(t,T))^J$, and we have

$$\begin{aligned}&\inf_{\alpha\in(\mathcal{A}_1^\pi(t,T))^I}J(t,x,\hat\alpha,\hat\beta^\lambda,p,q^\lambda)=\sum_{i=1}^Ip_i\inf_{\alpha\in\mathcal{A}_1^\pi(t,T)}\sum_{j=1}^Jq_j^\lambda E\left[g_{ij}\left(X_T^{t,x,\alpha,\beta_j^\lambda}\right)\right]\\&=\sum_{i=1}^Ip_i\inf_{\alpha\in\mathcal{A}_1^\pi(t,T)}\sum_{j=1}^Jq_j^\lambda\left(\int_{[0,c_j]}E\left[g_{ij}\left(X_T^{t,x,\alpha,\beta_j^0((\zeta_{2,k-1}^\pi,\zeta_{1,k-1}^\pi,\zeta_k^\pi,\dots,\zeta_{N-2}^\pi,\frac{1}{c_j}y_{2(N-k)+1})(\omega),\cdot)}\right)\right]\mathrm{d}y_{2(N-k)+1}\right.\\&\quad\left.+\int_{[c_j,1]}E\left[g_{ij}\left(X_T^{t,x,\alpha,\beta_j^1((\zeta_{2,k-1}^\pi,\zeta_{1,k-1}^\pi,\zeta_k^\pi,\dots,\zeta_{N-2}^\pi,\frac{1}{1-c_j}(y_{2(N-k)+1}-c_j))(\omega),\cdot)}\right)\right]\mathrm{d}y_{2(N-k)+1}\right)\\&=\sum_{i=1}^Ip_i\inf_{\alpha\in\mathcal{A}_1^\pi(t,T)}\sum_{j=1}^Jq_j^\lambda\left[\frac{(1-\lambda)q_j^0}{q_j^\lambda}E\left[g_{ij}\left(X_T^{t,x,\alpha,\beta_j^0}\right)\right]+\frac{\lambda q_j^1}{q_j^\lambda}E\left[g_{ij}\left(X_T^{t,x,\alpha,\beta_j^1}\right)\right]\right]\\&\geq(1-\lambda)\sum_{i=1}^Ip_i\inf_{\alpha\in\mathcal{A}_1^\pi(t,T)}\sum_{j=1}^Jq_j^0E\left[g_{ij}\left(X_T^{t,x,\alpha,\beta_j^0}\right)\right]+\lambda\sum_{i=1}^Ip_i\inf_{\alpha\in\mathcal{A}_1^\pi(t,T)}\sum_{j=1}^Jq_j^1E\left[g_{ij}\left(X_T^{t,x,\alpha,\beta_j^1}\right)\right]\\&\geq(1-\lambda)V_1^\pi(t,x,p,q^0)+\lambda V_1^\pi(t,x,p,q^1)-\varepsilon,\end{aligned}$$

since $\hat\beta^0$ and $\hat\beta^1$ are $\varepsilon$-optimal strategies for $V_1^\pi(t,x,p,q^0)$ and $V_1^\pi(t,x,p,q^1)$, respectively. Thus,

$$V_1^\pi(t,x,p,q^\lambda)\geq\inf_{\alpha\in(\mathcal{A}_1^\pi(t,T))^I}J\left(t,x,\hat\alpha,\hat\beta^\lambda,p,q^\lambda\right)\geq(1-\lambda)V_1^\pi(t,x,p,q^0)+\lambda V_1^\pi(t,x,p,q^1)-\varepsilon.\tag{3.23}$$

Thanks to the arbitrariness of $\varepsilon$, we obtain the stated result. □

Now we introduce the Fenchel transforms (we refer to [7]). Let $\psi:[0,T]\times\mathbb{R}^n\times\Delta(I)\times\Delta(J)\to\mathbb{R}$ be convex in $p$ and concave in $q$ on $\Delta(I)$ and $\Delta(J)$, respectively. Then we define its convex conjugate (with respect to variable $p$) $\psi^*$ by

$$\psi^*(t,x,\bar p,q)=\sup_{p\in\Delta(I)}\{\bar p\cdot p-\psi(t,x,p,q)\},\ (t,x,\bar p,q)\in[0,T]\times\mathbb{R}^n\times\mathbb{R}^I\times\Delta(J),\tag{3.24}$$

and its concave conjugate (with respect to variable $q$) $\psi^\#$ by

$$\psi^\#(t,x,p,\bar q)=\inf_{q\in\Delta(J)}\{\bar q\cdot q-\psi(t,x,p,q)\},\ (t,x,p,\bar q)\in[0,T]\times\mathbb{R}^n\times\Delta(I)\times\mathbb{R}^J.\tag{3.25}$$

Using these notations we denote by $V_1^{\pi *}$(respectively, $W_1^{\pi\#}$) the convex (respectively, concave) conjugate of $V_1^\pi$ (respectively, $W_1^\pi$) with respect to $p$ (respectively, $q$).

**Lemma 3.5.** *For all $(t,x,\bar p,q)\in[0,T]\times\mathbb{R}^n\times\mathbb{R}^I\times\Delta(J)$, we have*

$$V_1^{\pi *}(t,x,\bar p,q)=\inf_{(\beta_j)\in(\mathcal{B}_1^\pi(t,T))^J}\ \sup_{\alpha\in\mathcal{A}_1^\pi(t,T)}\ \max_{i\in\{1,\dots,I\}}\left\{\bar p_i-\sum_{j=1}^J q_jE\left[g_{ij}\left(X_T^{t,x,\alpha,\beta_j}\right)\right]\right\}. \tag{3.26}$$

*Proof.* We define

$$F(t,x,\bar p,q)=\inf_{(\beta_j)\in(\mathcal{B}_1^\pi(t,T))^J}\ \sup_{\alpha\in\mathcal{A}_1^\pi(t,T)}\ \max_{i\in\{1,\dots,I\}}\left\{\bar p_i-\sum_{j=1}^J q_jE\left[g_{ij}\left(X_T^{t,x,\alpha,\beta_j}\right)\right]\right\}. \tag{3.27}$$

We claim that $F(t,x,\bar p,q)$ is convex with respect to $\bar p$.

Indeed, for any $\varepsilon>0$ and $(t,x,q)\in[0,T]\times\mathbb{R}^n\times\Delta(J)$, $\bar p^0,\bar p^1\in\mathbb{R}^I$, $\lambda\in(0,1)$, let $(\beta_j^0)\in(\mathcal{B}_1^\pi(t,T))^J$ and $(\beta_j^1)\in(\mathcal{B}_1^\pi(t,T))^J$ be the $\varepsilon$-optimal strategies for $F(t,x,\bar p^0,q)$ and $F(t,x,\bar p^1,q)$, respectively. We assume that $\bar p^\lambda=(1-\lambda)\bar p^0+\lambda\bar p^1$, and for $\omega\in\Omega$, $u\in\mathcal{U}_{t,T}$, $s\in[t,T)$, we define the strategy $\hat\beta^\lambda=(\beta_j^\lambda)_{j=1,\dots,J}$ and

$$\begin{aligned}&\beta_j^\lambda(y_1,\dots,y_{2(N-k)+1},u)(s)\\&=\beta_j^1(y_1,y_2,\dots,y_{2(N-k)},\frac{1}{\lambda}y_{2(N-k)+1},u)(s)\cdot I_{[0,\lambda]}(y_{2(N-k)+1})\\&\quad+\beta_j^0(y_1,y_2,\dots,y_{2(N-k)},\frac{1}{1-\lambda}(y_{2(N-k)+1}-\lambda),u)(s)\cdot I_{[\lambda,1]}(y_{2(N-k)+1}),\end{aligned}$$

where $\beta_j^i((\zeta_{k-1}^\pi,\dots,\zeta_{N-2}^\pi,\zeta_{2,N-1}^\pi)(\omega),u)(s)=\sum_{l=k}^N\beta_{lj}^i((\zeta_{k-1}^\pi,\dots,\zeta_{l-2}^\pi,\zeta_{2,l-1}^\pi)(\omega),u)I_{[t\vee t_{l-1},t_l]}(s)$, $i=0,1$, respectively. Then we have $(\beta_j^\lambda)\in(\mathcal{B}_1^\pi(t,T))^J$. For all $\alpha\in\mathcal{A}_1^\pi(t,T)$, we have

$$\begin{aligned}&\max_{i\in\{1,\dots,I\}}\left\{\bar p_i^\lambda-\sum_{j=1}^J q_jE\left[g_{ij}\left(X_T^{t,x,\alpha,\beta_j^\lambda}\right)\right]\right\}\\
&=\max_{i\in\{1,\dots,I\}}\left\{(1-\lambda)\left(\bar p_i^0-\sum_{j=1}^J q_jE\left[g_{ij}\left(X_T^{t,x,\alpha,\beta_j^0}\right)\right]\right)+\lambda\left(\bar p_i^1-\sum_{j=1}^J q_jE\left[g_{ij}\left(X_T^{t,x,\alpha,\beta_j^1}\right)\right]\right)\right\}\\
&\le(1-\lambda)\max_{i\in\{1,\dots,I\}}\left\{\bar p_i^0-\sum_{j=1}^J q_jE\left[g_{ij}\left(X_T^{t,x,\alpha,\beta_j^0}\right)\right]\right\}+\lambda\max_{i\in\{1,\dots,I\}}\left\{\bar p_i^1-\sum_{j=1}^J q_jE\left[g_{ij}\left(X_T^{t,x,\alpha,\beta_j^1}\right)\right]\right\}\\
&\le(1-\lambda)\sup_{\alpha\in\mathcal{A}_1^\pi(t,T)}\max_{i\in\{1,\dots,I\}}\left\{\bar p_i^0-\sum_{j=1}^J q_jE\left[g_{ij}\left(X_T^{t,x,\alpha,\beta_j^0}\right)\right]\right\}\\
&\quad+\lambda\sup_{\alpha\in\mathcal{A}_1^\pi(t,T)}\max_{i\in\{1,\dots,I\}}\left\{\bar p_i^1-\sum_{j=1}^J q_jE\left[g_{ij}\left(X_T^{t,x,\alpha,\beta_j^1}\right)\right]\right\}\\
&\le(1-\lambda)F(t,x,\bar p^0,q)+\lambda F(t,x,\bar p^1,q)+\varepsilon.\end{aligned}$$

Therefore,

$$\begin{aligned}F(t,x,\bar p^\lambda,q)&\le\sup_{\alpha\in\mathcal{A}_1^\pi(t,T)}\max_{i\in\{1,\dots,I\}}\left\{\bar p_i^\lambda-\sum_{j=1}^J q_jE\left[g_{ij}\left(X_T^{t,x,\alpha,\beta_j^\lambda}\right)\right]\right\}\\&\le(1-\lambda)F(t,x,\bar p^0,q)+\lambda F(t,x,\bar p^1,q)+\varepsilon,\end{aligned}$$

from the arbitrariness of $\varepsilon$, we know $F(t,x,\bar p,q)$ is convex with respect to $\bar p$.

We also see that $F(t,x,\bar{p},q) = V_1^{\pi *}(t,x,\bar{p},q)$.

Indeed, from (3.24) and (3.27) we have

$$
\begin{aligned}
F^*(t,x,p,q) &= \sup_{\bar{p}\in\mathbb{R}^I}\left\{p\cdot\bar{p} - \inf_{(\beta_j)}\max_{i\in\{1,\ldots,I\}}\left\{\bar{p}_i - \inf_{\alpha}\sum_{j=1}^{J} q_j E\left[g_{ij}\left(X_T^{t,x,\alpha,\beta_j}\right)\right]\right\}\right\}\\
&= \sup_{(\beta_j)}\sup_{\bar{p}\in\mathbb{R}^I}\min_{i\in\{1,\ldots,I\}}\left\{p\cdot\bar{p} - \bar{p}_i + \inf_{\alpha}\sum_{j=1}^{J} q_j E\left[g_{ij}\left(X_T^{t,x,\alpha,\beta_j}\right)\right]\right\}\\
&= \sup_{(\beta_j)}\sup_{\bar{p}\in\mathbb{R}^I}\min_{i\in\{1,\ldots,I\}}\{p\cdot\bar{p} - \bar{p}_i + h_i\},
\end{aligned} \tag{3.28}
$$

where we have put $h_i := \inf_{\alpha}\sum_{j=1}^{J} q_j E[g_{ij}(X_T^{t,x,\alpha,\beta_j})]$, $1 \le i \le I$.

On the other hand,

$$
\begin{aligned}
\sup_{\bar{p}\in\mathbb{R}^I}\min_{i\in\{1,\ldots,I\}}\{p\cdot\bar{p} - \bar{p}_i + h_i\} &= \sup_{\bar{p}\in\mathbb{R}^I}\left\{p\cdot\bar{p} + \min_{i\in\{1,\ldots,I\}}\{h_i - \bar{p}_i\}\right\} = \sup_{\bar{p}\in\mathbb{R}^I}\left\{p\cdot\bar{p} + \inf_{\bar{\bar{p}}\in\Delta(I)}(h-\bar{p})\bar{\bar{p}}\right\}\\
&= \sup_{\bar{p}\in\mathbb{R}^I}\inf_{\bar{\bar{p}}\in\Delta(I)}\{(h-\bar{p})\bar{\bar{p}} + p\cdot\bar{p}\} = \inf_{\bar{\bar{p}}\in\Delta(I)}\sup_{\bar{p}\in\mathbb{R}^I}\{(p-\bar{\bar{p}})\bar{p} + h\cdot\bar{\bar{p}}\}\\
&= h\cdot p.
\end{aligned} \tag{3.29}
$$

From (3.28), (3.29) and (3.22), we get

$$
F^*(t,x,p,q) = \sup_{(\beta_j)}\sum_{i=1}^{I} p_i \inf_{\alpha}\sum_{j=1}^{J} q_j E\left[g_{ij}\left(X_T^{t,x,\alpha,\beta_j}\right)\right] = V_1^{\pi}(t,x,p,q). \tag{3.30}
$$

Finally, since $F$ is convex in $\bar{p}$, we have $V_1^{\pi *} = (F^*)^* = F$. $\square$

Using the definition of $V_1^{\pi *}$ and $W_1^{\pi \#}$, from Lemma 3.3 we have the following statement.

**Lemma 3.6.** *For all partition $\pi$ of the interval $[0,T]$, the convex conjugate function $V_1^{\pi *}(t,x,\bar{p},q)$ is Lipschitz with respect to $(t,x,\bar{p},q)$, and the concave conjugate function $W_1^{\pi \#}(t,x,p,\bar{q})$ is Lipschitz with respect to $(t,x,p,\bar{q})$. The Lipschitz constants are independent of $\pi$.*

Generally speaking, the game with asymmetric information does not have the dynamic programming principle, but it satisfies a sub-dynamic programming principle.

**Lemma 3.7.** *For all $(t,x,\bar{p},q) \in [t_{k-1},t_k) \times \mathbb{R}^n \times \mathbb{R}^I \times \Delta(J)$, and for all $l$ $(k \le l \le N)$, we have*

$$
V_1^{\pi *}(t,x,\bar{p},q) \le \inf_{\beta\in\mathcal{B}_1^{\pi}(t,t_l)}\sup_{\alpha\in\mathcal{A}_1^{\pi}(t,t_l)} E\left[V_1^{\pi *}(t_l, X_{t_l}^{t,x,\alpha,\beta}, \bar{p}, q)\right]. \tag{3.31}
$$

*Proof.* We define

$$
G(t,t_l,x,\bar{p},q) = \inf_{\beta\in\mathcal{B}_1^{\pi}(t,t_l)}\sup_{\alpha\in\mathcal{A}_1^{\pi}(t,t_l)} E\left[V_1^{\pi *}(t_l, X_{t_l}^{t,x,\alpha,\beta}, \bar{p}, q)\right]. \tag{3.32}
$$

For any given $\varepsilon > 0$, let $\beta^0 \in \mathcal{B}_1^{\pi}(t,t_l)$ be an $\varepsilon$-optimal strategy for $G(t,t_l,x,\bar{p},q)$, *i.e.*,

$$
|G(t,t_l,x,\bar{p},q) - \sup_{\alpha\in\mathcal{A}_1^{\pi}(t,t_l)} E[V_1^{\pi *}(t_l, X_{t_l}^{t,x,\alpha,\beta^0}, \bar{p}, q)]| \le \varepsilon. \tag{3.33}
$$

For all $y \in \mathbb{R}^n$, there exists an $\varepsilon$-optimal strategy $\hat{\beta}^y = (\beta^y_j)_{j=1,\dots,J} \in (\mathcal{B}^\pi_1(t_l,T))^J$ for $V^{\pi*}_1(t_l,y,\bar{p},q)$ for Player II, *i.e.*,

$$|V^{\pi*}_1(t_l,y,\bar{p},q) - \sup_{\alpha\in\mathcal{A}^\pi_1(t_l,T)} \max_{i\in\{1,\dots,I\}} \left\{ \bar{p}_i - \sum_{j=1}^J q_j E\left[ g_{ij}\left( X_T^{t_l,y,\alpha,\beta^y_j} \right)\right] \right\}| \le \varepsilon. \tag{3.34}$$

Because $V^{\pi*}_1(t_l,y,\bar{p},q)$ is Lipschitz continuous with respect to $y$ (with some Lipschitz constant $C$), $\hat{\beta}^y$ is a $(2\varepsilon)$-optimal strategies for $V^{\pi*}_1(t_l,z,\bar{p},q)$, if $z\in B_r(y)$, where $B_r(y)$ is the ball with diameter $r := \frac{\varepsilon}{C}$.

Since the coefficient $f$ is bounded, there exists some $R>0$ large enough such that all values of $X^{t,x,\alpha,\beta}_{t_l}$ belong to the ball $B_R(0)$. Let $O_n$, $n=1,\dots,n_0$, be a finite Borel partition of $B_R(0)$ with $\operatorname{diam}(O_n)\le r$. Fixing arbitrarily $x_n\in O_n$, we put $\beta^n_j = \beta^{x_n}_j$, $n=1,\dots,n_0$. Then the strategy $\beta^n_j$ is $(2\varepsilon)$-optimal for $V^{\pi*}_1(t_l,z,\bar{p},q)$, for all $z\in O_n$. For $\omega\in\Omega$ and $u\in\mathcal{U}_{t,T}$, we define

$$\beta_j(\omega,u)(s) = \begin{cases} \beta^0(\omega,u)(s), & s\in[t,t_l),\\ \sum\limits_{n=1}^{n_0} \beta^n_j(\omega,u|_{[t_l,T)})\cdot I_{\{X^{t,x,u,\beta^0(u)}_{t_l}\in O_n\}}, & s\in[t_l,T]. \end{cases}$$

Obviously, $\beta_j\in\mathcal{B}^\pi_1(t,T)$. The strategies $\alpha\in\mathcal{A}^\pi_1(t,T)$ have the following form (see, Def. 2.2):

$$\begin{aligned}\alpha(\omega,v)(s) = &\sum_{m=k}^{l} \alpha_m((\zeta^\pi_{1,k-1},\zeta^\pi_{2,k-1},\dots,\zeta^\pi_{1,m-1})(\omega),v)(s) I_{[t\vee t_{m-1},t_m)}(s)\\ &+ \sum_{m=l+1}^{N} \alpha_m((\zeta^\pi_{1,k-1},\zeta^\pi_{2,k-1},\dots,\zeta^\pi_{1,l-1},\zeta^\pi_{2,l-1},\zeta^\pi_{1,l},\zeta^\pi_{2,l},\zeta^\pi_{l+1},\dots,\zeta^\pi_{m-2},\zeta^\pi_{1,m-1})(\omega),v)(s)I_{[t_{m-1},t_m)}(s).\end{aligned}$$

For $s\in[t_l,T]$, we define $\tilde{\alpha}(\omega,Q,v)(s) = \sum_{m=l+1}^N \alpha_m(Q,(\zeta^\pi_{1,l},\zeta^\pi_{2,l},\dots,\zeta^\pi_{1,m-1})(\omega),v)(s)I_{[t_{m-1},t_m)}(s)$, where $Q$ is a $2(l-k)+2$-dimensional constant vector. Obviously, $\tilde{\alpha}(Q)\in\mathcal{A}^\pi_1(t_l,T)$. Then, for any $\alpha\in\mathcal{A}^\pi_1(t,T)$, as $X^{t,x,\alpha,\beta^0}_{t_l}$ and $Q_0 = (\zeta^\pi_{1,k-1},\zeta^\pi_{2,k-1},\dots,\zeta^\pi_{1,l-1},\zeta^\pi_{2,l-1})$ are $\mathcal{F}_{t_{k-2},t_{l-1}}$-measurable, and $\beta^n_j$ as well as $\tilde{\alpha}$ are $\mathcal{F}_{t_{l-1},T}$-measurable, we have

$$\begin{aligned} E\left[g_{ij}\left(X_T^{t,x,\alpha,\beta_j}\right)\right] &= E\left[\sum_{n=1}^{n_0} g_{ij}\left(X_T^{t_l,X^{t,x,\alpha,\beta^0}_{t_l},\tilde{\alpha}(Q_0),\beta^n_j}\right)\cdot I_{\left\{X^{t,x,\alpha,\beta^0}_{t_l}\in O_n\right\}}\right]\\ &= E\left[\sum_{n=1}^{n_0} E\left[g_{ij}\left(X_T^{t_l,y,\tilde{\alpha}(Q),\beta^n_j}\right)\right]_{y=X^{t,x,\alpha,\beta^0}_{t_l},Q=Q_0}\cdot I_{\left\{X^{t,x,\alpha,\beta^0}_{t_l}\in O_n\right\}}\right]. \end{aligned} \tag{3.35}$$

From (3.35) and (3.33), we have

$$\begin{aligned} &\max_{i\in\{1,\dots,I\}}\left\{\bar{p}_i - \sum_{j=1}^J q_j E\left[g_{ij}\left(X_T^{t,x,\alpha,\beta_j}\right)\right]\right\}\\ &= \max_{i\in\{1,\dots,I\}}\left\{\bar{p}_i - \sum_{j=1}^J q_j E\left[\sum_{n=1}^{n_0} E\left[g_{ij}\left(X_T^{t_l,y,\tilde{\alpha}(Q),\beta^n_j}\right)\right]_{y=X^{t,x,\alpha,\beta^0}_{t_l},Q=Q_0}\cdot I_{\left\{X^{t,x,\alpha,\beta^0}_{t_l}\in O_n\right\}}\right]\right\}\\ &\le E\left[\sum_{n=1}^{n_0}\max_{i\in\{1,\dots,I\}}\left\{\bar{p}_i - \sum_{j=1}^J q_j E\left[g_{ij}\left(X_T^{t_l,y,\tilde{\alpha}(Q),\beta^n_j}\right)\right]\right\}_{y=X^{t,x,\alpha,\beta^0}_{t_l},Q=Q_0}\cdot I_{\left\{X^{t,x,\alpha,\beta^0}_{t_l}\in O_n\right\}}\right] \end{aligned}$$

$$\leq E\left[\sum_{n=1}^{n_0}\sup_{\alpha'\in\mathcal{A}_1^{\pi}(t_l,T)}\max_{i\in\{1,\dots,I\}}\left\{\bar{p}_i-\sum_{j=1}^{J}q_jE\left[g_{ij}\left(X_T^{t_l,y,\alpha',\beta_j^n}\right)\right]\right\}_{y=X_{t_l}^{t,x,\alpha,\beta^0}}\cdot I_{\{X_{t_l}^{t,x,\alpha,\beta^0}\in O_n\}}\right]$$
$$\leq E\left[\sum_{n=1}^{n_0}V_1^{\pi*}(t_l,X_{t_l}^{t,x,\alpha,\beta^0},\bar{p},q)\cdot I_{\{X_{t_l}^{t,x,\alpha,\beta^0}\in O_n\}}\right]+2\varepsilon$$
$$\leq G(t,t_l,x,\bar{p},q)+3\varepsilon,$$

which means that $V_1^{\pi*}(t,x,\bar{p},q)\leq G(t,t_l,x,\bar{p},q)$. □

We consider a sequence partitions $(\pi_n)_{n\geq 1}$ of the interval $[0,T]$ satisfying that, when $n\to\infty$, the mesh of the partition $\pi_n$ tends to zero. From Lemma 3.6 and Arzelà−Ascoli Theorem applied to $V_1^{\pi_n*}(t,x,\bar{p},q)$ and $W_1^{\pi_n\#}(t,x,p,\bar{q})$, we have the following result.

**Lemma 3.8.** *There exists a subsequence of $(\pi_n)_{n\geq1}$, still denoted by $(\pi_n)_{n\geq1}$, and two functions $\tilde{V}:[0,T]\times\mathbb{R}^n\times\mathbb{R}^I\times\Delta(J)\mapsto\mathbb{R}$ and $\tilde{W}:[0,T]\times\mathbb{R}^n\times\Delta(I)\times\mathbb{R}^J\mapsto\mathbb{R}$ such that $(V_1^{\pi_n*},W_1^{\pi_n\#})\to(\tilde{V},\tilde{W})$ uniformly on compacts in $[0,T]\times\mathbb{R}^n\times\Delta(I)\times\Delta(J)\times\mathbb{R}^I\times\mathbb{R}^J$.*

**Remark 3.9.** Notice that from Lemma 3.6, the limit functions $\tilde{V}$ and $\tilde{W}$ are Lipschitz continuous with respect to all their variables.

Now we prove that the limit functions $\tilde{V}$ and $\tilde{W}$ are a viscosity subsolution and a viscosity supersolution of some HJI equation, respectively. For more details on viscosity solutions, the reader is referred to [9].

**Lemma 3.10.** *For all $(\bar{p},q)\in\mathbb{R}^I\times\Delta(J)$, the limit function $\tilde{V}(t,x,\bar{p},q)$ is a viscosity subsolution of the following HJI equation*

$$\begin{cases}\dfrac{\partial\tilde{V}}{\partial t}(t,x)+H^*(x,D\tilde{V}(t,x))=0, & (t,x)\in[0,T]\times\mathbb{R}^n,\\ \tilde{V}(T,x)=\max\limits_{i\in\{1,\dots,I\}}\left\{\bar{p}_i-\sum\limits_{j=1}^{J}q_jg_{ij}(x)\right\}, x\in\mathbb{R}^n,\end{cases}\tag{3.36}$$

*where*

$$\begin{aligned}H^*(x,\xi)=-H(x,-\xi)&=\inf_{\nu\in\mathcal{P}(V)}\sup_{\mu\in\mathcal{P}(U)}\left(\int_{U\times V}f(x,u,v)\mu(\mathrm{d}u)\nu(\mathrm{d}v)\cdot\xi\right)\\&=\sup_{\mu\in\mathcal{P}(U)}\inf_{\nu\in\mathcal{P}(V)}\left(\int_{U\times V}f(x,u,v)\mu(\mathrm{d}u)\nu(\mathrm{d}v)\cdot\xi\right),\end{aligned}$$

*and, for shortness, $\tilde{V}(t,x):=\tilde{V}(t,x,\bar{p},q)$.*

*Proof.* For any fixed $(t,x)\in[0,T]\times\mathbb{R}^n$, since the coefficient $f$ is bounded, there is some $M>0$ such that, $\bar{B}_M(x)\supset\{X_r^{s,y,\alpha,\beta},\ (s,y)\in[0,T]\times\bar{B}_1(x),(\alpha,\beta)\in\mathcal{A}_1^{\pi}(s,T)\times\mathcal{B}_1^{\pi}(s,T),r\in[s,T]\}$, where $\bar{B}_M(x)$ is the closed ball with the center $x$ and the radius $M$. From Lemma 3.8 we know $V_1^{\pi_n*}$ converges to $\tilde{V}$ uniformly on $[0,T]\times\bar{B}_M(x)$. Let $\varphi\in C_b^1([0,T]\times\mathbb{R}^n)$ (the set of bounded continuous functions with bounded, continuous first order partial derivative) be a test function such that

$$(\tilde{V}-\varphi)(t,x)>(\tilde{V}-\varphi)(s,y),\ \text{for all }(s,y)\in[0,T]\times\bar{B}_M(x)\setminus\{(t,x)\}.\tag{3.37}$$

Let $(s_n,x_n)\in[0,T]\times\bar{B}_M(x)$ be the maximum point of $V_1^{\pi_n*}-\varphi$ over $[0,T]\times\bar{B}_M(x)$. Then there exists a subsequence of $(s_n,x_n)$ still denoted by $(s_n,x_n)$, such that $(s_n,x_n)$ converges to $(t,x)$.

Indeed, since $[0,T]\times\bar{B}_M(x)$ is a compact set, there exists a subsequence $(s_n,x_n)$ and $(\bar{s},\bar{x})\in[0,T]\times\bar{B}_M(x)$ such that $(s_n,x_n)\to(\bar{s},\bar{x})$. Due to $(V_1^{\pi_n*}-\varphi)(s_n,x_n)\geq(V_1^{\pi_n*}-\varphi)(t,x)$, for $n\geq1$, we have

$$(\tilde{V}-\varphi)(\bar{s},\bar{x})\geq(\tilde{V}-\varphi)(t,x), \tag{3.38}$$

and from (3.37) and (3.38) we conclude $(\bar{s},\bar{x})=(t,x)$.

For the partition $\pi_n=\{0=t_0^n<\ldots<t_{N_n}^n=T\}$, we assume $t_{k_{n-1}}^n\leq s_n<t_{k_n}^n$, and for simplicity we write $t_{k-1}^n\leq s_n<t_k^n$. Since $x_n\to x$, there is a positive integer $N$ such that for all $n\geq N$, we have $|x_n-x|\leq1$. On the other hand, from Lemma 3.7 we get

$$\begin{aligned}\varphi(s_n,x_n)=V_1^{\pi_n*}(s_n,x_n)&\leq\inf_{\beta\in\mathcal{B}_1^{\pi_n}(s_n,t_k^n)}\sup_{\alpha\in\mathcal{A}_1^{\pi_n}(s_n,t_k^n)}E\left[V_1^{\pi_n*}\left(t_k^n,X_{t_k^n}^{s_n,x_n,\alpha,\beta}\right)\right]\\&\leq\inf_{\beta\in\mathcal{B}_1^{\pi_n}(s_n,t_k^n)}\sup_{\alpha\in\mathcal{A}_1^{\pi_n}(s_n,t_k^n)}E\left[\varphi\left(t_k^n,X_{t_k^n}^{s_n,x_n,\alpha,\beta}\right)\right].\end{aligned}\tag{3.39}$$

Thus,

$$\begin{aligned}0&\leq\inf_{\beta\in\mathcal{B}_1^{\pi_n}(s_n,t_k^n)}\sup_{\alpha\in\mathcal{A}_1^{\pi_n}(s_n,t_k^n)}E\left[\varphi\left(t_k^n,X_{t_k^n}^{s_n,x_n,\alpha,\beta}\right)-\varphi(s_n,x_n)\right]\\&=\inf_{\beta\in\mathcal{B}_1^{\pi_n}(s_n,t_k^n)}\sup_{\alpha\in\mathcal{A}_1^{\pi_n}(s_n,t_k^n)}E\left[\int_{s_n}^{t_k^n}\left(\frac{\partial\varphi}{\partial r}(r,X_r^{s_n,x_n,\alpha,\beta}\right)+f\left(X_r^{s_n,x_n,\alpha,\beta},\alpha_r,\beta_r\right)\cdot D\varphi\left(r,X_r^{s_n,x_n,\alpha,\beta}\right))\,\mathrm{d}r\right].\end{aligned}\tag{3.40}$$

For $(u,v)\in\mathcal{U}_{t,T}\times\mathcal{V}_{t,T}$, we introduce the following continuity modulus,

$$m(\delta):=\sup_{\substack{|r-s|+|y-\bar{x}|\leq\delta,\\ u\in U,\,v\in V,\,\bar{x},\,y\in\bar{B}_M(x)}}\left|\left(\frac{\partial\varphi}{\partial r}(r,y)+f(y,u,v)\cdot D\varphi(r,y)\right)-\left(\frac{\partial\varphi}{\partial r}(s,\bar{x})+f(\bar{x},u,v)\cdot D\varphi(s,\bar{x})\right)\right|.\tag{3.41}$$

Obviously, $m(\delta)$ is nondecreasing in $\delta$ and $m(\delta)\to0$, as $\delta\downarrow0$. From (2.2), considering that $|X_r^{s_n,x_n,\alpha,\beta}-x_n|\leq C|r-s_n|\leq C|t_k^n-s_n|,\ r\in[s_n,t_k^n]$, and from (3.41) we obtain that

$$\begin{aligned}&\left|\left(\frac{\partial\varphi}{\partial r}(r,X_r^{s_n,x_n,\alpha,\beta})+f(X_r^{s_n,x_n,\alpha,\beta},\alpha_r,\beta_r)\cdot D\varphi\left(r,X_r^{s_n,x_n,\alpha,\beta}\right)\right)\right.\\&\left.-\left(\frac{\partial\varphi}{\partial r}(s_n,x_n)+f(x_n,\alpha_r,\beta_r)\cdot D\varphi(s_n,x_n)\right)\right|\leq m(C|t_k^n-s_n|),\ r\in[s_n,t_k^n].\end{aligned}\tag{3.42}$$

It follows from (3.40) and (3.42) that

$$\begin{aligned}-(t_k^n-s_n)\left(\frac{\partial\varphi}{\partial r}(s_n,x_n)+m(C|t_k^n-s_n|)\right)&\leq\inf_{\beta\in\mathcal{B}_1^{\pi_n}(s_n,t_k^n)}\sup_{\alpha\in\mathcal{A}_1^{\pi_n}(s_n,t_k^n)}E\left[\int_{s_n}^{t_k^n}f(x_n,\alpha_r,\beta_r)\cdot D\varphi(s_n,x_n)\mathrm{d}r\right]\\&\leq\sup_{\alpha\in\mathcal{A}_1^{\pi_n}(s_n,t_k^n)}E\left[\int_{s_n}^{t_k^n}f(x_n,\alpha_r,\tilde{\beta}_r)\cdot D\varphi(s_n,x_n)\mathrm{d}r\right],\end{aligned}\tag{3.43}$$

where we take $\tilde{\beta}_r=\tilde{v}(\zeta_{2,k-1}^{\pi_n})$, $r\in[s_n,t_k^n]$, and $\tilde{v}$ is a $V$-valued measurable function. Define $\rho_n=(t_k^n-s_n)^2$. From (3.43) we see that there exists a $\rho_n$-optimal strategy $\alpha^n$ (depending on $\tilde{\beta}$) such that

$$\begin{aligned}&-(t_k^n-s_n)\left(\frac{\partial\varphi}{\partial r}(s_n,x_n)+m(C|t_k^n-s_n|)+(t_k^n-s_n)\right)\leq E\left[\int_{s_n}^{t_k^n}f(x_n,\alpha_r^n,\tilde{\beta}_r)\cdot D\varphi(s_n,x_n)\mathrm{d}r\right]\\&=\int_{s_n}^{t_k^n}E\left[f\left(x_n,\alpha_r^n(\zeta_{1,k-1}^{\pi_n},\tilde{v}),\tilde{v}(\zeta_{2,k-1}^{\pi_n})\right)\cdot D\varphi(s_n,x_n)\right]\mathrm{d}r.\end{aligned}\tag{3.44}$$

Notice that on the interval $[s_n, t_k^n]$, $\alpha^n$ does not depend on the control $\tilde{v}$ due to the delay property. Then, thanks to the independence between $\zeta_{1,k-1}^{\pi_n}$ and $\zeta_{2,k-1}^{\pi_n}$, from (3.44) we get

$$\begin{aligned}&-(t_k^n - s_n)\left(\frac{\partial\varphi}{\partial r}(s_n, x_n) + m(C|t_k^n - s_n|) + (t_k^n - s_n)\right)\\&\leq \int_{s_n}^{t_k^n} \sup_{\mu\in\mathcal{P}(U)} \int_U E\left[f\left(x_n, u, \tilde{v}\left(\zeta_{2,k-1}^{\pi_n}\right)\right) \cdot D\varphi(s_n, x_n)\right]\mu(\mathrm{d}u)\mathrm{d}r.\end{aligned} \tag{3.45}$$

From the arbitrariness of $\tilde{v}$, it follows that

$$\begin{aligned}&-(t_k^n - s_n)\left(\frac{\partial\varphi}{\partial r}(s_n, x_n) + m(C|t_k^n - s_n|) + (t_k^n - s_n)\right)\\&\leq (t_k^n - s_n) \inf_{\nu\in\mathcal{P}(V)} \sup_{\mu\in\mathcal{P}(U)} \int_{U\times V} f(x_n, u, v) \cdot D\varphi(s_n, x_n)\mu(\mathrm{d}u)\nu(\mathrm{d}v),\end{aligned} \tag{3.46}$$

which means that

$$-\left(\frac{\partial\varphi}{\partial r}(s_n, x_n) + m(C|t_k^n - s_n|) + (t_k^n - s_n)\right)s \leq \inf_{\nu\in\mathcal{P}(V)} \sup_{\mu\in\mathcal{P}(U)} \int_{U\times V} f(x_n, u, v) \cdot D\varphi(s_n, x_n)\mu(\mathrm{d}u)\nu(\mathrm{d}v). \tag{3.47}$$

Recalling that $(s_n, x_n) \to (t, x)$ and $0 \leq (t_k^n - s_n) \leq (t_k^n - t_{k-1}^n) \leq |\pi_n|$, taking the limit we get

$$\frac{\partial\varphi}{\partial t}(t, x) + \inf_{\nu\in\mathcal{P}(V)} \sup_{\mu\in\mathcal{P}(U)} \int_{U\times V} f(x, u, v) \cdot D\varphi(t, x)\mu(\mathrm{d}u)\nu(\mathrm{d}v) \geq 0. \tag{3.48}$$

□

Now we want to prove $\tilde{W}$ is a viscosity supersolution of the HJI equation (3.36). Notice that

$$-W_1^\pi(t, x, p, q) = \sup_{(\alpha_i)\in(\mathcal{A}_1^\pi(t,T))^I} \inf_{(\beta_j)\in(\mathcal{B}_1^\pi(t,T))^J} \sum_{i=1}^I \sum_{j=1}^J p_i q_j E\left[-g_{ij}\left(X_T^{t,x,\alpha_i,\beta_j}\right)\right]. \tag{3.49}$$

Then $-W_1^\pi(t, x, p, q)$ has the same form as $V_1^\pi$, the only change concerns the role of players. Thus, the convex conjugate of $-W_1^\pi(t, x, p, q)$ with respect to $q$, *i.e.*, $-(W_1^{\pi\#}(t, x, p, -\bar{q}))$ satisfies a sub-dynamic programming principle. Then similar to Lemma 3.7 and Theorem 3.10 we have the following result.

**Lemma 3.11.** *For any $(t, x, p, \bar{q}) \in [0, T] \times \mathbb{R}^n \times \Delta(I) \times \mathbb{R}^J$, and for all $l$ $(k \leq l \leq n)$, we have*

$$W_1^{\pi\#}(t, x, p, \bar{q}) \geq \sup_{\alpha\in\mathcal{A}_1^\pi(t,t_l)} \inf_{\beta\in\mathcal{B}_1^\pi(t,t_l)} E\left[W_1^{\pi\#}\left(t_l, X_{t_l}^{t,x,\alpha,\beta}, p, \bar{q}\right)\right], \tag{3.50}$$

*and $\tilde{W}$ (Recall Lem. 3.8) is a supersolution of the HJI equation (3.36).*

We now give the definition of dual solutions for the following HJI equation

$$\begin{cases}\dfrac{\partial V}{\partial t}(t, x) + H(x, DV(t, x)) = 0, \ (t, x) \in [0, T] \times \mathbb{R}^n,\\ V(T, x) = \sum_{i,j} p_i q_j g_{ij}(x), \qquad x \in \mathbb{R}^n,\end{cases} \tag{3.51}$$

where $H(x, \xi) = \inf_{\mu\in\mathcal{P}(U)} \sup_{\nu\in\mathcal{P}(V)} \left(\int_{U\times V} f(x, u, v)\mu(\mathrm{d}u)\nu(\mathrm{d}v) \cdot \xi\right)$.

**Definition 3.12.** A function $w : [0,T] \times \mathbb{R}^n \times \Delta(I) \times \Delta(J) \mapsto \mathbb{R}$ is called a dual viscosity subsolution of the equation (3.51) if, firstly, $w$ is Lipschitz continuous with all its variables, convex with respect to $p$ and concave with respect to $q$, and secondly, for any $(p, \bar{q}) \in \Delta(I) \times \mathbb{R}^J$, $w^{\#}(t,x,p,\bar{q})$ is a viscosity supersolution of the dual HJI equation

$$\frac{\partial V}{\partial t}(t,x) + H^*(x, DV(t,x)) = 0,\ (t,x) \in [0,T] \times \mathbb{R}^n, \tag{3.52}$$

where $H^*(x,\xi) = -H(x,-\xi)$.

A function $w : [0,T] \times \mathbb{R}^n \times \Delta(I) \times \Delta(J) \mapsto \mathbb{R}$ is called a dual viscosity supersolution of the equation (3.51) if, firstly, $w$ is Lipschitz continuous with all its variables, convex with respect to $p$ and concave with respect to $q$, and secondly, for any $(\bar{p}, q) \in \mathbb{R}^I \times \Delta(J)$, $w^*(t,x,\bar{p},q)$ is a viscosity subsolution of the dual HJI equation (3.52).

The function $w$ is called the dual viscosity solution of the equation (3.51) if $w$ is a dual viscosity subsolution and a dual viscosity supersolution of the equation (3.51).

**Lemma 3.13.** *Let $w_1, w_2 : [0,T] \times \mathbb{R}^n \times \Delta(I) \times \Delta(J) \mapsto \mathbb{R}$ be a dual viscosity subsolution and a dual viscosity supersolution of the HJI equation (3.51), respectively. If, for all $(x,p,q) \in \mathbb{R}^n \times \Delta(I) \times \Delta(J)$, $w_1(T,x,p,q) \leq w_2(T,x,p,q)$, then we have $w_1 \leq w_2$ on $[0,T] \times \mathbb{R}^n \times \Delta(I) \times \Delta(J)$.*

For the proof of Lemma 3.13 the reader is referred to Theorem 5.1 in [7].

**Theorem 3.14.** *The functions $(V_1^{\pi_n})$ and $(W_1^{\pi_n})$ converge uniformly on compacts to the same Lipschitz function $U$ when the mesh of the partition $\pi_n$ tends to $0$. Moreover, the function $U$ is the unique dual viscosity solution of the HJI equation (3.51).*

To prove this statement we first consider the following proposition, then we get Theorem 3.14 directly.

**Proposition 3.15.** *For any sequence of partitions $\pi_n$ with $|\pi_n| \to 0$, there exists a subsequence still denoted by $(\pi_n)_{n\geq 1}$, such that $(V_1^{\pi_n})$ and $(W_1^{\pi_n})$ converge uniformly on compacts to the same function $U$, and the function $U$ is the unique dual viscosity solution of the HJI equation (3.51).*

**Remark 3.16.** If Proposition 3.15 holds, then for all subsequence $(\pi_n)$ with $|\pi_n| \to 0$, there exists a sub-subsequence $(\pi_{n_l})$ such that $(V_1^{\pi_{n_l}}, W_1^{\pi_{n_l}})$ converges uniformly to the function $(U,U)$, and the limit $U$ is the unique dual solution of the HJI equation (3.51). Therefore, the limits of all converging sub-subsequences are the same, then Theorem 3.14 holds.

Now we prove (of Prop. 3.15).

*Proof.* From Lemma 3.3, using the Arzelà−Ascoli Theorem we know there exist two bounded Lipschitz functions $V_1$ and $W_1 : [0,T] \times \mathbb{R}^n \times \Delta(I) \times \Delta(J) \mapsto \mathbb{R}$ such that $(V_1^{\pi_n}, W_1^{\pi_n}) \to (V_1, W_1)$ uniformly on compacts in $[0,T] \times \mathbb{R}^n \times \Delta(I) \times \Delta(J)$, and $V_1, W_1$ are convex in $p$, concave in $q$.

From Lemma 3.8, we have $\tilde{W} = \lim_{n\to\infty} W_1^{\pi_n\#}$, $\tilde{V} = \lim_{n\to\infty} V_1^{\pi_n *}$, and due to Lemmas 3.10 and 3.11 $\tilde{V}^*$ and $\tilde{W}^{\#}$ is a dual viscosity supersolution and a dual viscosity subsolution of HJI equation (3.51), respectively, with the terminal value $\tilde{V}^*(T,x,p,q) = \tilde{W}^{\#}(T,x,p,q) = \sum_{ij} p_i q_j g_{ij}(x)$. Then from Lemma 3.13, it follows

$$\tilde{V}^* \geq \tilde{W}^{\#},\ \text{on } [0,T] \times \mathbb{R}^n \times \Delta(I) \times \Delta(J). \tag{3.53}$$

Let $\rho > 0$. Since, for any $M > 0$, $V_1(t,x,p,q) = \lim_{n\to\infty} V_1^{\pi_n}(t,x,p,q)$, uniformly in $(t,x,p,q) \in [0,T] \times \bar{B}_M(0) \times \Delta(I) \times \Delta(J)$, there exists a positive integer $N_{\rho,M}$ such that, for all $(t,x,p,q) \in [0,T] \times \bar{B}_M(0) \times \Delta(I) \times \Delta(J)$, it holds $|V_1^{\pi_n}(t,x,p,q) - V_1(t,x,p,q)| \leq \rho$. Thus, from the definition of convex conjugate we have

$$\begin{aligned} |V_1^{\pi_n *}(t,x,\bar{p},q) - V_1^*(t,x,\bar{p},q)| &= \left| \sup_{p\in\Delta(I)} \{\bar{p}\cdot p - V_1^{\pi_n}(t,x,p,q)\} - \sup_{p\in\Delta(I)} \{\bar{p}\cdot p - V_1(t,x,p,q)\} \right| \\ &\leq \sup_{p\in\Delta(I)} |V_1^{\pi_n}(t,x,p,q) - V_1(t,x,p,q)| \leq \rho. \end{aligned}$$

Hence, $V_1^*(t,x,\bar{p},q) = \lim_{n\to\infty} V_1^{\pi_n *}(t,x,\bar{p},q)$ uniformly on compacts, and $\tilde{V} = \lim_{n\to\infty} V_1^{\pi_n *} = V_1^*$. Since $V_1$ is convex in $p$, we have $V_1 = (V_1^*)^* = \tilde{V}^*$. Similarly, we obtain $W_1 = \tilde{W}^{\#}$. Thus, from (3.53) we have

$$W_1 \leq V_1, \text{ on } [0,T]\times\mathbb{R}^n\times\Delta(I)\times\Delta(J). \tag{3.54}$$

On the other hand, as $W_1^{\pi_n} \geq V_1^{\pi_n}$, for all $n \geq 1$, it follows that

$$W_1 \geq V_1, \text{ on } [0,T]\times\mathbb{R}^n\times\Delta(I)\times\Delta(J). \tag{3.55}$$

Finally, (3.54) and (3.55) imply that $U := V_1 = W_1 (= \tilde{V}^* = \tilde{W}^{\#})$ on $[0,T]\times\mathbb{R}^n\times\Delta(I)\times\Delta(J)$, and as $U = \tilde{V}^*$ is a dual viscosity supersolution and $U = \tilde{W}^{\#}$ a dual viscosity subsolution, $U$ is a dual viscosity solution of HJI equation (3.51). Its uniqueness follows from Lemma 3.13. □

From Theorems 3.1 and 3.14 we obtain immediately the following result.

**Theorem 3.17.** *The functions $(V^{\pi_n})$ and $(W^{\pi_n})$ converge uniformly on compacts to the same Lipschitz continuous function $U$, when the mesh of the partition $\pi_n$ tends to 0. Moreover, the function $U$ is the unique dual viscosity solution of the HJI equation (3.51).*

**Remark 3.18.** Note that usually differential games with asymmetric information don't admit the dynamic programming principle (See Lem. 3.7). To solve this problem, we adopt the dual approach introduced by Cardaliaguet [7] to prove the existence of the value for our differential games. Another way to deal with the absence of the dynamic programming principle is to study the associated partial differential equation (PDE, for short) with obstacle. Cardaliaguet [8] proved that a mapping $w$ is a dual viscosity solution of PDE (3.51) if and only if $w$ is a viscosity solution of the related PDE with obstacle.

## 4. Nash equilibrium payoffs for nonzero-sum differential games with symmetric information and without Isaacs condition

In this section, in the same framework as before, we consider the existence of Nash equilibrium payoffs for nonzero-sum differential games, but with symmetric information (i.e., $I = J = 1$) and without Isaacs condition. Due to Theorem 3.1 we only need to consider strategies $\alpha \in \mathcal{A}_1^\pi(t,T)$ and $\beta \in \mathcal{B}_1^\pi(t,T)$ for our nonzero-sum games. Let $g_1 : \mathbb{R}^n \mapsto \mathbb{R}$ and $g_2 : \mathbb{R}^n \mapsto \mathbb{R}$ be two bounded Lipschitz continuous functions. For $(t,x) \in [0,T]\times\mathbb{R}^n$, $(u,v) \in \mathcal{U}_{t,T}^{\pi,1} \times \mathcal{V}_{t,T}^{\pi,1}$ (For the definition of $\mathcal{U}_{t,T}^{\pi,1}$ and $\mathcal{V}_{t,T}^{\pi,1}$ we refer to Rem. 2.5), we define

$$J_1(t,x,u,v) = E\left[g_1(X_T^{t,x,u,v})\right] \text{ and } J_2(t,x,u,v) = E\left[g_2(X_T^{t,x,u,v})\right], \tag{4.1}$$

where $X^{t,x,u,v}$ is the solution of equation (2.1). From Remark 2.4 we know that, for all $(\alpha,\beta) \in \mathcal{A}_1^\pi(t,T) \times \mathcal{B}_1^\pi(t,T)$, there exists $(u,v) \in \mathcal{U}_{t,T}^{\pi,1} \times \mathcal{V}_{t,T}^{\pi,1}$, such that $\alpha(v) = u, \beta(u) = v$. This allows to define $J_m(t,x,\alpha,\beta) = J_m(t,x,u,v),\ m = 1,2$.

Here, in our nonzero-sum differential game Player I wants to maximize $J_1(t,x,\alpha,\beta)$, while Player II aims to maximize $J_2(t,x,\alpha,\beta)$. A Nash equilibrium point is a couple of strategies $(\bar{\alpha},\bar{\beta})$ such that for any other couples of strategies $(\alpha,\beta)$, it holds

$$J_1(t,x,\bar{\alpha},\bar{\beta}) \geq J_1(t,x,\alpha,\bar{\beta}), \text{ and } J_2(t,x,\bar{\alpha},\bar{\beta}) \geq J_2(t,x,\bar{\alpha},\beta), \tag{4.2}$$

and the pair $(J_1(t,x,\bar{\alpha},\bar{\beta}), J_2(t,x,\bar{\alpha},\bar{\beta}))$ is called a Nash equilibrium payoff. However, working without Isaacs condition, we cannot expect to find $(\bar{\alpha},\bar{\beta}) \in \mathcal{A}^\pi(t,T)\times\mathcal{B}^\pi(t,T)$ for a suitable partition $\pi$, such that (4.2) holds. For this reason, in our paper, we only investigate the existence of a Nash equilibrium payoff which can be approximated by pairs of payoff functionals $(J_1(t,x,\bar{\alpha}^\epsilon,\bar{\beta}^\epsilon), J_2(t,x,\bar{\alpha}^\epsilon,\bar{\beta}^\epsilon))$, when $\epsilon$ tends to 0. Let us be more precise and give the definition of a Nash equilibrium payoff for our nonzero-sum differential game.

**Definition 4.1.** A couple $(e_1, e_2) \in \mathbb{R}^2$ is called a Nash equilibrium payoff (NEP, for short) at the position $(t, x)$, if for any $\epsilon > 0$, there exists $\delta_\epsilon$ small enough satisfying that, for any partition $\pi$ of the interval $[0, T]$ with $|\pi| \le \delta_\epsilon$, there exist $(\alpha^\epsilon, \beta^\epsilon) \in \mathcal{A}_1^\pi(t, T) \times \mathcal{B}_1^\pi(t, T)$ such that for all $(\alpha, \beta) \in \mathcal{A}_1^\pi(t, T) \times \mathcal{B}_1^\pi(t, T)$

$$J_1(t, x, \alpha^\epsilon, \beta^\epsilon) \ge J_1(t, x, \alpha, \beta^\epsilon) - \epsilon \text{ and } J_2(t, x, \alpha^\epsilon, \beta^\epsilon) \ge J_2(t, x, \alpha^\epsilon, \beta) - \epsilon, \tag{4.3}$$

and

$$|J_m(t, x, \alpha^\epsilon, \beta^\epsilon) - e_m| \le \epsilon, \; m = 1, 2. \tag{4.4}$$

The following lemma gives an equivalent condition of assumption (4.3) which will be frequently used in this section.

**Lemma 4.2.** *Let $\epsilon > 0$ and $(\alpha^\epsilon, \beta^\epsilon) \in \mathcal{A}_1^\pi(t, T) \times \mathcal{B}_1^\pi(t, T)$. Assumption (4.3) holds if and only if, for any $(u, v) \in \mathcal{U}_{t,T}^{\pi,1} \times \mathcal{V}_{t,T}^{\pi,1}$,*

$$J_1(t, x, \alpha^\epsilon, \beta^\epsilon) \ge J_1(t, x, u, \beta^\epsilon(u)) - \epsilon \text{ and } J_2(t, x, \alpha^\epsilon, \beta^\epsilon) \ge J_2(t, x, \alpha^\epsilon(v), v) - \epsilon. \tag{4.5}$$

*Proof.* We assume (4.3) holds. For any fixed $u \in \mathcal{U}_{t,T}^{\pi,1}$, we put $\alpha(v) \equiv u$, for all $v \in \mathcal{V}_{t,T}^{\pi,1}$. Obviously, $\alpha \in \mathcal{A}_1^\pi(t, T)$. Then, from condition (4.3) we have $J_1(t, x, \alpha^\epsilon, \beta^\epsilon) \ge J_1(t, x, u, \beta^\epsilon(u)) - \epsilon$. Similarly, for any $v \in \mathcal{V}_{t,T}^{\pi,1}$, we obtain $J_2(t, x, \alpha^\epsilon, \beta^\epsilon) \ge J_2(t, x, \alpha^\epsilon(v), v) - \epsilon$. Thus, condition (4.5) holds true.

Conversely, suppose now that (4.5) holds. For all $\alpha \in \mathcal{A}_1^\pi(t, T)$, there exists $(u, v) \in \mathcal{U}_{t,T}^{\pi,1} \times \mathcal{V}_{t,T}^{\pi,1}$ such that, $\alpha(v) = u, \beta^\epsilon(u) = v$. Then,

$$J_1(t, x, \alpha, \beta^\epsilon) - \epsilon = J_1(t, x, u, \beta^\epsilon(u)) - \epsilon \le J_1(t, x, \alpha^\epsilon, \beta^\epsilon),$$

and the symmetric argument applied to $J_2$ yields (4.3). □

Due to Theorem 3.14, the upper value function $W_1^\pi$ and the lower value function $V_1^\pi$ associated with a terminal cost function $g$ converge to the same function as the mesh $|\pi|$ of partition $\pi$ tends to 0. Thus, we can introduce the following functions $U_1(t, x)$ and $U_2(t, x)$, associated with $g_1$ and $g_2$, respectively.

$$U_1(t, x) = \lim_{|\pi| \to 0} \inf_{\beta \in \mathcal{B}_1^\pi(t,T)} \sup_{\alpha \in \mathcal{A}_1^\pi(t,T)} J_1(t, x, \alpha, \beta) = \lim_{|\pi| \to 0} \sup_{\alpha \in \mathcal{A}_1^\pi(t,T)} \inf_{\beta \in \mathcal{B}_1^\pi(t,T)} J_1(t, x, \alpha, \beta), \tag{4.6}$$

and

$$U_2(t, x) = \lim_{|\pi| \to 0} \sup_{\beta \in \mathcal{B}_1^\pi(t,T)} \inf_{\alpha \in \mathcal{A}_1^\pi(t,T)} J_2(t, x, \alpha, \beta) = \lim_{|\pi| \to 0} \inf_{\alpha \in \mathcal{A}_1^\pi(t,T)} \sup_{\beta \in \mathcal{B}_1^\pi(t,T)} J_2(t, x, \alpha, \beta). \tag{4.7}$$

Now we state the following both main results for our nonzero-sum differential game.

**Theorem 4.3** (Characterization)**.** *A couple $(e_1, e_2) \in \mathbb{R}^2$ is an NEP at the position $(t, x)$ if and only if, for all $\epsilon > 0$, there exists $\delta_\epsilon > 0$ satisfying that for any partition $\pi = \{0 = t_0 < t_1 < \ldots < t_N = T\}$ with $|\pi| < \delta_\epsilon$ and $t = t_{k-1}$, there exists $(u^\epsilon, v^\epsilon) \in \mathcal{U}_{t,T}^{\pi,1} \times \mathcal{V}_{t,T}^{\pi,1}$ such that, for $i = k, \ldots, N$ and $m = 1, 2$, respectively,*

$$P\left\{E\left[g_m(X_T^{t,x,u^\epsilon,v^\epsilon})|\mathcal{F}_{t_{k-2},t_{i-2}}\right] \ge U_m\left(t_{i-1}, X_{t_{i-1}}^{t,x,u^\epsilon,v^\epsilon}\right) - \epsilon\right\} \ge 1 - \epsilon, \tag{4.8}$$

*and*

$$\left|E\left[g_m\left(X_T^{t,x,u^\epsilon,v^\epsilon}\right)\right] - e_m\right| \le \epsilon. \tag{4.9}$$

**Theorem 4.4.** *For any initial position $(t, x) \in [0, T] \times \mathbb{R}^n$, there exists an NEP at the position $(t, x)$.*

The rest of this section is devoted to the proof of the above theorems. We first prove Theorem 4.3 and then, with the help of Theorem 4.3 we show the existence result (Thm. 4.4). First of all, let us give the following lemma which will be used in the proofs of the Theorems 4.3 and 4.4.

**Lemma 4.5.** *a) Let* $(t,x)\in[0,T]\times\mathbb{R}^n$ *and* $\epsilon>0$*. Then for any partition* $\pi=\{0=t_0<t_1<\ldots<t_N=T\}$ *with* $|\pi|<\delta_\epsilon$ *(*$\delta_\epsilon>0$ *small enough)* $t=t_{k-1}$*, and for any given* $u'\in\mathcal{U}^{\pi,1}_{t,T}$*, there exist strategies* $\alpha^i\in\mathcal{A}^\pi_1(t,T)$*,* $i=k,\ldots,N$*, such that for all* $v\in\mathcal{V}^{\pi,1}_{t,T}$*,*

$$\alpha^i(v)\equiv u',\ P\text{-a.s., on } [t,t_{i-1}], E\left[g_2\left(X_T^{t,x,\alpha^i(v),v}\right)|\mathcal{F}_{t_{k-2},t_{i-2}}\right]\le U_2(t_{i-1},X_{t_{i-1}}^{t,x,\alpha^i(v),v})+\epsilon,\ P\text{-a.s.} \tag{4.10}$$

*b) Let* $(t,x)\in[0,T]\times\mathbb{R}^n$ *and* $\epsilon>0$*. Then for any partition* $\pi=\{0=t_0<t_1<\ldots<t_N=T\}$ *with* $|\pi|<\delta_\epsilon$ *(*$\delta_\epsilon>0$ *small enough)* $t=t_{k-1}$*, and for any given* $u'\in\mathcal{U}^{\pi,1}_{t,T}$*, there exist strategies* $\alpha^i\in\mathcal{A}^\pi_1(t,T)$*,* $i=k,\ldots,N$*, such that for all* $v\in\mathcal{V}^{\pi,1}_{t,T}$*,*

$$\alpha^i(v)\equiv u',\ P\text{-a.s., on } [t,t_{i-1}], E\left[g_1(X_T^{t,x,\alpha^i(v),v})|\mathcal{F}_{t_{k-2},t_{i-2}}\right]\ge U_1\left(t_{i-1},X_{t_{i-1}}^{t,x,\alpha^i(v),v}\right)-\epsilon,\ P\text{-a.s.} \tag{4.11}$$

*Proof.* We just give the proof for $a$), the proof of $b$) is analogous.

For any $\epsilon>0$, $y\in\mathbb{R}^n$, and $i$ $(1\le i\le N)$, it follows from the definition of the value function $U_2$ that there exists a strategy $\alpha^i_y\in\mathcal{A}^\pi_1(t_{i-1},T)$ such that, for $|\pi|<\delta_\epsilon$, $\delta_\epsilon>0$ small enough,

$$\begin{aligned}
U_2(t_{i-1},y)&=\lim_{|\pi|\to0}\inf_{\alpha\in\mathcal{A}^\pi_1(t_{i-1},T)}\sup_{\beta\in\mathcal{B}^\pi_1(t_{i-1},T)}E\left[g_2\left(X_T^{t_{i-1},y,\alpha,\beta}\right)\right]\\
&\ge\inf_{\alpha\in\mathcal{A}^\pi_1(t_{i-1},T)}\sup_{\beta\in\mathcal{B}^\pi_1(t_{i-1},T)}E\left[g_2\left(X_T^{t_{i-1},y,\alpha,\beta}\right)\right]-\frac{\epsilon}{4}\\
&\ge\inf_{\alpha\in\mathcal{A}^\pi_1(t_{i-1},T)}\sup_{v\in\mathcal{V}^{\pi,1}_{t_{i-1},T}}E\left[g_2\left(X_T^{t_{i-1},y,\alpha(v),v}\right)\right]-\frac{\epsilon}{4}\ge\sup_{v\in\mathcal{V}^{\pi,1}_{t_{i-1},T}}E\left[g_2\left(X_T^{t_{i-1},y,\alpha^i_y(v),v}\right)\right]-\frac{\epsilon}{2}.
\end{aligned}\tag{4.12}$$

Since the coefficient $f$ is bounded, there exists a constant $R>0$ such that $|X^{t,x,u,v}|\le R$, for all $(u,v)\in\mathcal{U}^{\pi,1}_{t,T}\times\mathcal{V}^{\pi,1}_{t,T}$. Let us choose a finite Borel partition $(O_l)_{l=1,2,\ldots,n}$ of the closed ball $\bar{B}_R(0)$, with $\mathrm{diam}(O_l)\le\epsilon/(4C)$, $1\le l\le n$, and let $y_l$ denote an arbitrarily chosen element in $O_l$. Then it follows from the fact that $U_2(t_{i-1},z)$ and $\sup_{v\in\mathcal{V}^{\pi,1}_{t_{i-1},T}}E[g_2(X_T^{t_{i-1},z,\alpha^i_{y_l}(v),v})]$ are Lipschitz continuous with respect to $z$, with some Lipschitz constant $C$:

$$\sup_{v\in\mathcal{V}^{\pi,1}_{t_{i-1},T}}E\left[g_2\left(X_T^{t_{i-1},z,\alpha^i_{y_l}(v),v}\right)\right]\le U_2(t_{i-1},z)+\epsilon,\ z\in O_l. \tag{4.13}$$

Recall that all $v\in\mathcal{V}^{\pi,1}_{t,T}$ is of the following form (We refer to Rem. 2.5)

$$v(\omega,s)=v^k(s,\zeta^\pi_{2,k-1})I_{[t,t_k)}(s)+\sum_{l=k+1}^N v^l(s,\zeta^\pi_{k-1},\ldots,\zeta^\pi_{l-2},\zeta^\pi_{2,l-1})I_{[t_{l-1},t_l)}(s), \tag{4.14}$$

where the $v^l$'s are Borel functions over suitable spaces and with values in V. For $k\le i\le N$ and $s\in[t_{i-1},T]$, we put

$$v''(\omega,Q,s)=\sum_{l=i}^N v^l(s,Q,\zeta^\pi_{2,i-1},\zeta^\pi_{1,i-1},\zeta^\pi_{2,i},\ldots,\zeta^\pi_{2,l-1})I_{[t_{l-1},t_l)}(s), \tag{4.15}$$

where $Q\in\mathbb{R}^{2(i-k)}$. Obviously, $v''\in\mathcal{V}^{\pi,1}_{t_{i-1},T}$, and we define the following strategy $\alpha^i$ $(k+1\le i\le N)$,

$$\alpha^i(v):=\begin{cases}u', & \text{on } [t,t_{i-1}],\\ \alpha^i_{y_l}(v''(Q_0,s)), & \text{on } (t_{i-1},T]\times\{X^{t,x,u',v}_{t_{i-1}}\in O_l\},\end{cases}\quad v\in\mathcal{V}^{\pi,1}_{t,T}, \tag{4.16}$$

where $Q_0 = (\zeta^\pi_{2,k-1}, \zeta^\pi_{1,k-1}, \dots, \zeta^\pi_{2,i-2}, \zeta^\pi_{1,i-2})$. Then, $\alpha^i \in \mathcal{A}^\pi_1(t,T)$. Notice that $Q_0$ and $X^{t,x,u',v}_{t_{i-1}}$ are all $\mathcal{F}_{t_{k-2},t_{i-2}}$-measurable, $v''(Q,s)$ and $\alpha^i_{y_l}(v''(Q,s))$ are all $\mathcal{F}_{t_{i-2},T}$-measurable. Hence, for all $v \in \mathcal{V}^{\pi,1}_{t,T}$ (of the form (4.14)) from (4.13) we have, $P$-a.s.,

$$E\left[g_2\left(X^{t,x,\alpha^i(v),v}_T\right)|\mathcal{F}_{t_{k-2},t_{i-2}}\right] = \sum_{l=1}^n E\left[g_2\left(X_T^{t_{i-1},z,\alpha^i_{y_l}(v''(Q,s)),v''(Q,s)}\right)\right]_{Q=Q_0,z=X^{t,x,u',v}_{t_{i-1}}} \cdot I_{\left\{X^{t,x,u',v}_{t_{i-1}} \in O_l\right\}}$$
$$\le \sum_{l=1}^n U_2(t_{i-1}, X^{t,x,u',v}_{t_{i-1}}) \cdot I_{\left\{X^{t,x,u',v}_{t_{i-1}} \in O_l\right\}} + \epsilon = U_2\left(t_{i-1}, X^{t,x,\alpha^i(v),v}_{t_{i-1}}\right) + \epsilon. \tag{4.17}$$

□

Now with the help of Lemma 4.5, we can prove Theorem 4.3.

*Proof. Sufficient condition.*

Let us assume that $(e_1, e_2)$ satisfies condition (4.8) and (4.9) of Theorem 4.3, namely, for all $\epsilon > 0$, there exists $\delta_\epsilon > 0$ small enough satisfying that for any partition $\pi = \{0 = t_0 < t_1 < \ldots < t_N = T\}$ with $|\pi| < \delta_\epsilon$ and for $t = t_{k-1}$, there exists $(u^\epsilon, v^\epsilon) \in \mathcal{U}^{\pi,1}_{t,T} \times \mathcal{V}^{\pi,1}_{t,T}$ such that for $i = k, \dots, N$ and $m = 1, 2$,

$$P\left\{E\left[g_m\left(X^{t,x,u^\epsilon,v^\epsilon}_T\right)|\mathcal{F}_{t_{k-2},t_{i-2}}\right] \ge U_m\left(t_{i-1}, X^{t,x,u^\epsilon,v^\epsilon}_{t_{i-1}}\right) - \epsilon\right\} \ge 1 - \epsilon, \tag{4.18}$$

and

$$\left|E\left[g_m\left(X^{t,x,u^\epsilon,v^\epsilon}_T\right)\right] - e_m\right| \le \epsilon/2. \tag{4.19}$$

We prove that $(e_1, e_2)$ is an NEP for the initial position $(t, x)$. For this, we construct $(\alpha^\epsilon, \beta^\epsilon) \in \mathcal{A}^\pi_1(t,T) \times \mathcal{B}^\pi_1(t,T)$ satisfying (4.3) and (4.4).

Since $g_m, m = 1, 2$, is bounded, we can assume without loss of generality that $g_m \ge 0$, which has as consequence that $W_m \ge 0$. Let $\epsilon > 0$, $\epsilon_0 = \frac{\epsilon}{8+4NC}$ and $(\bar{u}, \bar{v}) = (u^{\epsilon_0}, v^{\epsilon_0})$, and observe that (4.18) and (4.19) also hold for $\epsilon = \epsilon_0$. From Lemma 4.5 *a*), putting $u' := \bar{u}$, we see that there exist strategies $\alpha_i \in \mathcal{A}^\pi_1(t,T)$, $i = k, \dots, N$, such that, for all $v \in \mathcal{V}^{\pi,1}_{t,T}$,

$$\alpha_i(v) \equiv \bar{u}, P\text{-a.s., on } [t, t_{i-1}], E\left[g_2\left(X^{t,x,\alpha_i(v),v}_T\right)|\mathcal{F}_{t_{k-2},t_{i-2}}\right] \le U_2(t_{i-1}, X^{t,x,\alpha_i(v),v}_{t_{i-1}}) + \frac{\epsilon}{8}, \ P\text{-a.s.} \tag{4.20}$$

Given any $v \in \mathcal{V}^{\pi,1}_{t,T}$, we introduce the stopping times $S^v = \inf\{s | v_s \ne \bar{v}_s, \ t \le s \le T\} \wedge T$, $\tau^v = \inf\{t_{i-1} | t_{i-1} > S^v, k+1 \le i \le N\} \wedge T$, and we define $\alpha^\epsilon$ as follows:

$$\alpha^\epsilon(v) = \begin{cases} \bar{u}, & \text{on } [[t, \tau^v]], \\ \alpha_i(v), & \text{on } (t_{i-1}, T] \times \{\tau^v = t_{i-1}\}, \end{cases} \ v \in \mathcal{V}^{\pi,1}_{t,T}. \tag{4.21}$$

Obviously, $\alpha^\epsilon \in \mathcal{A}^\pi_1(t,T)$. Observe that, as $\tau^{\bar{v}} = T$, we have, in particular $\alpha(\bar{v}) = \bar{u}$. Furthermore, for any $v \in \mathcal{V}^{\pi,1}_{t,T}$,

$$X^{t,x,\alpha^\epsilon(v),v} = \begin{cases} X^{t,x,\bar{u},v}, & \text{on } [[t, \tau^v]], \ P\text{-a.s.}, \\ \sum_{i=k+1}^N X^{t,x,\alpha_i(v),v} \cdot I_{\{\tau^v = t_{i-1}\}}, & \text{on } [[\tau^v, T]], \ P\text{-a.s.} \end{cases} \tag{4.22}$$

Then, since $\{\tau^v = t_{i-1}\} \in \mathcal{F}_{t_{k-2},t_{i-2}}$, from (4.20) we get

$$E\left[g_2\left(X^{t,x,\alpha^\epsilon(v),v}_T\right)|\mathcal{F}_{t_{k-2},\tau^v}\right] \le U_2(\tau^v, X^{t,x,\alpha^\epsilon(v),v}_{\tau^v}) + \frac{\epsilon}{8}, \ P\text{-a.s.} \tag{4.23}$$

Taking the expectation on both sides, we obtain

$$J_2(t, x, \alpha^\epsilon(v), v) \le E\left[U_2\left(\tau^v, X^{t,x,\alpha^\epsilon(v),v}_{\tau^v}\right)\right] + \frac{\epsilon}{8}. \tag{4.24}$$

On the other hand, as $X^{t,x,\alpha^\epsilon(v),v}_{S^v} = X^{t,x,\bar{u},\bar{v}}_{S^v}$ and the dynamics coefficient $f$ is bounded, we have, for $\rho := |\pi| > 0$,

$$E\left[\sup_{0\le r\le\rho}\left|X^{t,x,\alpha^\epsilon(v),v}_{(S^v+r)\wedge T} - X^{t,x,\bar{u},\bar{v}}_{(S^v+r)\wedge T}\right|\right] \le C\rho.$$

Moreover, since $U_2(s,x)$ is Lipschitz in $x$, uniformly with respect to $s$ and $S^v \le \tau^v \le S^v + \rho$, we obtain

$$E\left[\left|U_2\left(\tau^v, X^{t,x,\alpha^\epsilon(v),v}_{\tau^v}\right) - U_2\left(\tau^v, X^{t,x,\bar{u},\bar{v}}_{\tau^v}\right)\right|\right] \le C\rho \le \frac{\epsilon}{8}, \tag{4.25}$$

for $\rho = |\pi| \le \epsilon/(8C)$. Then, combining (4.24) and (4.25), we see that

$$J_2(t,x,\alpha^\epsilon(v),v) \le E[U_2(\tau^v, X^{t,x,\bar{u},\bar{v}}_{\tau^v})] + \frac{\epsilon}{4}\cdot \tag{4.26}$$

Putting

$$\Omega_i := \left\{E\left[g_2(X^{t,x,\bar{u},\bar{v}}_T)|\mathcal{F}_{t_{k-2},t_{i-2}}\right] \ge U_2(t_{i-1}, X^{t,x,\bar{u},\bar{v}}_{t_{i-1}}) - \epsilon_0\right\}, \tag{4.27}$$

we have from (4.18) that $P(\Omega_i) \ge 1-\epsilon_0$. From (4.26), (4.27) and (4.19), we deduce

$$\begin{aligned}
&J_2(t,x,\alpha^\epsilon(v),v)\\
&\le \sum_{i=k+1}^{N} E[U_2(t_{i-1},X_{t_{i-1}})\cdot I_{\{\tau^v=t_{i-1}\}}\cdot I_{\Omega_i}] + \sum_{i=k+1}^{N} E[U_2\left(t_{i-1},X_{t_{i-1}}\right)\cdot I_{\{\tau^v=t_{i-1}\}}\cdot I_{\Omega_i^c}] + \frac{\epsilon}{4}\\
&\le \sum_{i=k+1}^{N} E\left[\left(E\left[g_2(X_T)|\mathcal{F}_{t_{k-2},t_{i-2}}\right]+\epsilon_0\right)\cdot I_{\{\tau^v=t_{i-1}\}}\cdot 1_{\Omega_i}\right] + \sum_{i=k+1}^{N} CP(\Omega_i^c\cap\{\tau^v=t_{i-1}\}) + \frac{\epsilon}{4}\\
&\le E[g_2(X_T)] + \epsilon_0 + \sum_{i=k+1}^{N} CP(\Omega_i^c) + \frac{\epsilon}{4} \le e_2 + (2+NC)\epsilon_0 + \frac{\epsilon}{4} = e_2 + \frac{\epsilon}{2},
\end{aligned} \tag{4.28}$$

where $X_\cdot := X^{t,x,\bar{u},\bar{v}}_\cdot$ (Recall also that $g_2 \ge 0$). Finally, combining (4.28) and (4.21), we obtain that, for all $v \in \mathcal{V}^{\pi,1}_{t,T}$,

$$J_2(t,x,\alpha^\epsilon(v),v) \le e_2 + \frac{\epsilon}{2},\ \text{and}\ \alpha^\epsilon(\bar{v}) = \bar{u}. \tag{4.29}$$

Similarly, we can construct $\beta^\epsilon \in \mathcal{B}^\pi_1(t,T)$ such that, for all $u \in \mathcal{U}^{\pi,1}_{t,T}$,

$$J_1(t,x,u,\beta^\epsilon(u)) \le e_1 + \frac{\epsilon}{2}, \text{and}\ \beta^\epsilon(\bar{u}) = \bar{v}. \tag{4.30}$$

From (4.29), (4.30) and (4.19) we obtain, for $m = 1, 2$,

$$|J_m(t,x,\alpha^\epsilon,\beta^\epsilon) - e_m| = |J_m(t,x,\bar{u},\bar{v}) - e_m| \le \frac{\epsilon}{2}, \tag{4.31}$$

which is just (4.4). Moreover, from (4.31), for $m = 1, 2$,

$$e_m \le J_m(t,x,\alpha^\epsilon,\beta^\epsilon) + \frac{\epsilon}{2}\cdot \tag{4.32}$$

Consequently, (4.29), (4.30) and (4.32) yield

$$\begin{aligned}
J_2(t,x,\alpha^\epsilon(v),v) &\le e_2 + \frac{\epsilon}{2} \le J_2(t,x,\alpha^\epsilon,\beta^\epsilon) + \epsilon,\\
J_1(t,x,u,\beta^\epsilon(u)) &\le e_1 + \frac{\epsilon}{2} \le J_1(t,x,\alpha^\epsilon,\beta^\epsilon) + \epsilon.
\end{aligned}$$

Finally, Lemma 4.2 gives (4.3) and allows to conclude.

*Necessary condition.*

We assume there exists an NEP $(e_1, e_2) \in \mathbb{R}^2$ at the position $(t, x)$, *i.e.*, for any $\epsilon > 0$, there exists $\delta_\epsilon > 0$ small enough, satisfying that, for any partition $\pi = \{0 = t_0 < \ldots < t_N = T\}$ with $|\pi| < \delta_\epsilon$ and with $t = t_{k-1}$, there exists $(\alpha^\epsilon, \beta^\epsilon) \in \mathcal{A}_1^\pi(t,T) \times \mathcal{B}_1^\pi(t,T)$ such that, for all $(u, v) \in \mathcal{U}_{t,T}^{\pi,1} \times \mathcal{V}_{t,T}^{\pi,1}$, the following inequalities hold:

$$J_1(t,x,\alpha^\epsilon,\beta^\epsilon) \geq J_1(t,x,u,\beta^\epsilon(u)) - \frac{\epsilon^2}{2} \text{ and } J_2(t,x,\alpha^\epsilon,\beta^\epsilon) \geq J_2(t,x,\alpha^\epsilon(v),v) - \frac{\epsilon^2}{2}, \tag{4.33}$$

and

$$|J_m(t,x,\alpha^\epsilon,\beta^\epsilon) - e_m| \leq \frac{\epsilon^2}{2},\ m = 1,2. \tag{4.34}$$

Due to Remark 2.4 there exists $(u^\epsilon, v^\epsilon) \in \mathcal{U}_{t,T}^{\pi,1} \times \mathcal{V}_{t,T}^{\pi,1}$ such that $\alpha^\epsilon(v^\epsilon) = u^\epsilon$ and $\beta^\epsilon(u^\epsilon) = v^\epsilon$. Then, obviously, (4.9) holds. Let us suppose that (4.8) does not hold. This means that there exists $\epsilon > 0$ arbitrarily small such that, for all $\delta > 0$, there is a partition $\pi = \{0 = t_0 < \ldots < t = t_{k-1} < \ldots < t_N = T\}$ such that for all $(u, v) \in \mathcal{U}_{t,T}^{\pi,1} \times \mathcal{V}_{t,T}^{\pi,1}$ and, thus, in particular, for $(u^\epsilon, v^\epsilon)$, there is some $j \in \{k, \ldots, N\}$, for which, without loss of generality, for $m = 1$,

$$P\left\{E\left[g_1\left(X_T^{t,x,u^\epsilon,v^\epsilon}\right)|\mathcal{F}_{t_{k-2},t_{j-2}}\right] < U_1(t_{j-1}, X_{t_{j-1}}^{t,x,u^\epsilon,v^\epsilon}) - \epsilon\right\} > \epsilon. \tag{4.35}$$

Let us put

$$A = \left\{E\left[g_1\left(X_T^{t,x,u^\epsilon,v^\epsilon}\right)|\mathcal{F}_{t_{k-2},t_{j-2}}\right] < U_1\left(t_{j-1}, X_{t_{j-1}}^{t,x,u^\epsilon,v^\epsilon}\right) - \epsilon\right\} \in \mathcal{F}_{t_{k-2},t_{j-2}}. \tag{4.36}$$

Applying Lemma 4.5 *b*) to $u' := u^\epsilon \in \mathcal{U}_{t,T}^{\pi,1}$, we see that there exists a strategy $\alpha \in \mathcal{A}_1^\pi(t,T)$ such that, for all $v \in \mathcal{V}_{t,T}^{\pi,1}$, $\alpha(v) = u^\epsilon$, on $[t, t_{j-1}]$, $P$-a.s., and

$$E\left[g_1\left(X_T^{t,x,\alpha(v),v}\right)|\mathcal{F}_{t_{k-2},t_{j-2}}\right] \geq U_1\left(t_{j-1}, X_{t_{j-1}}^{t,x,\alpha(v),v}\right) - \frac{\epsilon}{2},\ P\text{-}a.s. \tag{4.37}$$

For $(\alpha, \beta^\epsilon) \in \mathcal{A}_1^\pi(t,T) \times \mathcal{B}_1^\pi(t,T)$, there exists a pair $(u, v) \in \mathcal{U}_{t,T}^{\pi,1} \times \mathcal{V}_{t,T}^{\pi,1}$ such that $\alpha(v) = u$ and $\beta^\epsilon(u) = v$. Notice that $u \equiv u^\epsilon, v \equiv v^\epsilon$, on $[t, t_{j-1}]$. We put

$$\bar{u} = \begin{cases} u^\epsilon, \text{ on } ([t, t_{j-1}] \times \Omega) \cup \left((t_{j-1}, T] \times A^c\right), \\ u, \text{ on } (t_{j-1}, T] \times A. \end{cases}$$

Obviously, $\bar{u} \in \mathcal{U}_{t,T}^\pi$, and $\beta^\epsilon(\bar{u}) \equiv v^\epsilon$, on $[t, t_{j-1})$, while for $s \in [t_{j-1}, T]$, $\beta^\epsilon(\bar{u})_s = v_s I_A + v_s^\epsilon I_{A^c}$. This implies that $X^{t,x,\bar{u},\beta^\epsilon(\bar{u})} \equiv X^{t,x,u^\epsilon,v^\epsilon}$, on $[t, t_{j-1}]$, and $X^{t,x,\bar{u},\beta^\epsilon(\bar{u})} = X^{t,x,\alpha(v),v} I_A + X^{t,x,u^\epsilon,v^\epsilon} I_{A^c}$. Consequently,

$$\begin{aligned} J_1(t,x,\bar{u},\beta^\epsilon(\bar{u})) &= E\left[g_1\left(X_T^{t,x,u^\epsilon,v^\epsilon}\right) \cdot I_{A^c}\right] + E\left[g_1(X_T^{t,x,\alpha(v),v}) \cdot I_A\right] \\ &= E\left[g_1(X_T^{t,x,u^\epsilon,v^\epsilon}) \cdot I_{A^c}\right] + E\left[E\left[g_1(X_T^{t,x,\alpha(v),v})|\mathcal{F}_{t_{k-2},t_{j-2}}\right] \cdot I_A\right] \\ &\geq E[g_1(X_T^{t,x,u^\epsilon,v^\epsilon}) \cdot I_{A^c}] + E\left[U_1\left(t_{j-1}, X_{t_{j-1}}^{t,x,\alpha(v),v}\right) \cdot I_A\right] - \frac{\epsilon}{2}P(A) \quad \text{(from (4.37))} \\ &\geq E\left[g_1\left(X_T^{t,x,u^\epsilon,v^\epsilon}\right)\right] + \frac{\epsilon}{2}P(A) \quad \text{(from (4.36))} \\ &> J_1(t,x,\alpha^\epsilon,\beta^\epsilon) + \frac{\epsilon^2}{2}, \quad \text{(from (4.35) and (4.36))} \end{aligned} \tag{4.38}$$

which is in contradiction with (4.33). Therefore, (4.8) holds true. □

To prove Theorem 4.4 we only need to prove that, for any $\epsilon > 0$, there exists $\delta_\epsilon > 0$ small enough satisfying that, for any partition $\pi = \{0 = t_0 < t_1 < \ldots < t_N = T\}$ with $|\pi| < \delta_\epsilon$ and $t = t_{k-1}$, there is a pair $(u^\epsilon, v^\epsilon)$ satisfying (4.8) and (4.9) in Theorem 4.3. For this we show the following stronger result.

**Proposition 4.6.** *For any $\epsilon > 0$, there exists $\delta_\epsilon > 0$ small enough satisfying that, for any partition $\pi = \{0 = t_0 < t_1 < \ldots < t_N = T\}$ with $|\pi| < \delta_\epsilon$ and $t = t_{k-1}$, there exists a pair $(u^\epsilon, v^\epsilon) \in \mathcal{U}_{t,T}^{\pi,1} \times \mathcal{V}_{t,T}^{\pi,1}$ such that, for all $k \le i \le l \le N$, and $m = 1, 2$,*

$$E[U_m(t_l, X_{t_l})|\mathcal{F}_{t_{k-2},t_{i-2}}] \ge U_m(t_{i-1}, X_{t_{i-1}}) - \epsilon,\ P\text{-a.s.}, \tag{4.39}$$

*where $X_\cdot = X_\cdot^{t,x,u^\epsilon,v^\epsilon}$.*

**Remark 4.7.** Proposition 4.6 implies Theorem 4.3. Indeed, setting $l = N$, we have $U_m(T, x) = g_m(x)$ and $U_m(T, X_T^{t,x,u^\epsilon,v^\epsilon}) = g_m(X_T^{t,x,u^\epsilon,v^\epsilon})$. Consequently, the pair $(u^\epsilon, v^\epsilon)$ satisfies (4.8) in Theorem 4.3. In order to get (4.9), it suffices to observe that the sequence $\{(E[g_m(X_T^{t,x,u^\epsilon,v^\epsilon})])_{m=1,2},\ \epsilon > 0\}$ is bounded and, consequently, converges along a subsequence, as $\epsilon \downarrow 0$, to a limit $(e_1, e_2) \in \mathbb{R}^2$. The relations (4.8) and (4.9) are obvious now.

We first give the following lemma.

**Lemma 4.8.** *For any $\epsilon > 0$, there exists $\delta_\epsilon > 0$ small enough satisfying that, for any partition $\pi = \{0 = t_0 < t_1 < \ldots < t_N = T\}$ with $|\pi| < \delta_\epsilon$ and $t = t_{k-1}$, there exists a pair $(u^\epsilon, v^\epsilon) \in \mathcal{U}_{t,T}^{\pi,1} \times \mathcal{V}_{t,T}^{\pi,1}$ such that, for $m = 1, 2$,*

$$E\left[U_m\left(t_k, X_{t_k}^{t,x,u^\epsilon,v^\epsilon}\right)\right] \ge U_m(t, x) - \epsilon. \tag{4.40}$$

*Proof.* From the definition of $U_1(t, x)$ and $U_2(t, x)$ (See (4.6) and (4.7)), it follows that there is some $\delta_\epsilon > 0$ such that, for $|\pi| < \delta_\epsilon$,

$$U_1(t,x) = \lim_{|\pi|\to 0} \sup_{\alpha\in\mathcal{A}_1^\pi(t,T)} \inf_{\beta\in\mathcal{B}_1^\pi(t,T)} J_1(t,x,\alpha,\beta) \le \sup_{\alpha\in\mathcal{A}_1^\pi(t,T)} \inf_{\beta\in\mathcal{B}_1^\pi(t,T)} J_1(t,x,\alpha,\beta) + \frac{\epsilon}{4},$$

$$U_2(t,x) = \lim_{|\pi|\to 0} \sup_{\beta\in\mathcal{B}_1^\pi(t,T)} \inf_{\alpha\in\mathcal{A}_1^\pi(t,T)} J_2(t,x,\alpha,\beta) \le \sup_{\beta\in\mathcal{B}_1^\pi(t,T)} \inf_{\alpha\in\mathcal{A}_1^\pi(t,T)} J_2(t,x,\alpha,\beta) + \frac{\epsilon}{4}\cdot$$

Then we choose $\alpha^\epsilon \in \mathcal{A}_1^\pi(t, T)$ and $\beta^\epsilon \in \mathcal{B}_1^\pi(t, T)$ such that

$$\begin{aligned} U_1(t,x) &\le \inf_{\beta\in\mathcal{B}_1^\pi(t,T)} J_1(t,x,\alpha^\epsilon,\beta) + \frac{\epsilon}{2} \le \inf_{v\in\mathcal{V}_{t,T}^{\pi,1}} J_1(t,x,\alpha^\epsilon(v),v) + \frac{\epsilon}{2}, \\ U_2(t,x) &\le \inf_{\alpha\in\mathcal{A}_1^\pi(t,T)} J_2(t,x,\alpha,\beta^\epsilon) + \frac{\epsilon}{2} \le \inf_{u\in\mathcal{U}_{t,T}^{\pi,1}} J_2(t,x,u,\beta^\epsilon(u)) + \frac{\epsilon}{2}\cdot \end{aligned} \tag{4.41}$$

Let $(u^\epsilon, v^\epsilon) \in \mathcal{U}_{t,T}^{\pi,1} \times \mathcal{V}_{t,T}^{\pi,1}$ be such that, $\alpha^\epsilon(v^\epsilon) = u^\epsilon$ and $\beta^\epsilon(u^\epsilon) = v^\epsilon$ (see, Rem. 2.4). We prove that $(u^\epsilon, v^\epsilon)$ satisfies (4.40). For this, we suppose (4.40) does not hold, *i.e.*, for $m = 2$ (or, similarly, for $m = 1$) we have

$$E\left[U_2(t_k, X_{t_k}^{t,x,u^\epsilon,v^\epsilon})\right] < U_2(t,x) - \epsilon. \tag{4.42}$$

From Lemma 4.5 *a*), we see that, for $u' := u^\epsilon$, there exists an NAD strategy $\alpha \in \mathcal{A}_1^\pi(t, T)$ such that for all $v \in \mathcal{V}_{t,T}^{\pi,1}$, $\alpha(v) = u^\epsilon$, $P$-a.s., on $[t, t_k]$, and

$$E\left[g_2(X_T^{t,x,\alpha(v),v})|\mathcal{F}_{t_{k-2},t_{k-1}}\right] \le U_2\left(t_k, X_{t_k}^{t,x,\alpha(v),v}\right) + \frac{\epsilon}{2},\ P\text{-a.s.} \tag{4.43}$$

Let now $(\bar{u}, \bar{v}) \in \mathcal{U}_{t,T}^{\pi,1} \times \mathcal{V}_{t,T}^{\pi,1}$ be such that, $\alpha(\bar{v}) = \bar{u}$ and $\beta^\epsilon(\bar{u}) = \bar{v}$. Since $\bar{u} \equiv u^\epsilon$ and $\bar{v} \equiv v^\epsilon$, on $[t, t_k]$, we have $X_{t_k}^{t,x,\bar{u},\bar{v}} = X_{t_k}^{t,x,\alpha(\bar{v}),\bar{v}} = X_{t_k}^{t,x,u^\epsilon,v^\epsilon}$, $P$-a.s., and from (4.43) and (4.42) it follows that

$$\begin{aligned} J_2(t,x,\bar{u},\beta^\epsilon(\bar{u})) = J_2(t,x,\alpha(\bar{v}),\bar{v}) &= E\left[E\left[g_2(X_T^{t,x,\alpha(\bar{v}),\bar{v}})|\mathcal{F}_{t_{k-2},t_{k-1}}\right]\right] \\ &\le E\left[U_2\left(t_k, X_{t_k}^{t,x,\alpha(\bar{v}),\bar{v}}\right)\right] + \frac{\epsilon}{2} < U_2(t,x) - \frac{\epsilon}{2}, \end{aligned} \tag{4.44}$$

which contradicts (4.41). Hence, (4.40) holds true. □

Finally, we give the proof of Proposition 4.6.

*Proof.* First, we show that when $l = i$, Proposition 4.6 holds true.

Similarly to Lemma 4.8, we see that for any $\epsilon > 0$, there exists $\delta_\epsilon > 0$ small enough satisfying that, for any partition $\pi = \{0 = t_0 < t_1 < \ldots < t_N = T\}$ with $|\pi| < \delta_\epsilon$ and $t = t_{k-1}$, and for all $y \in \mathbb{R}^n$, there exists $(u_j^{\epsilon,y}, v_j^{\epsilon,y}) \in \mathcal{U}_{t_j,T}^{\pi,1} \times \mathcal{V}_{t_j,T}^{\pi,1}$, $j = k-1, \ldots, N-1$, such that, for $m = 1, 2$,

$$E\left[U_m\left(t_{j+1}, X_{t_{j+1}}^{t_j,y,u_j^{\epsilon,y},v_j^{\epsilon,y}}\right)\right] \geq U_m(t_j, y) - \epsilon. \tag{4.45}$$

We now construct $(u^\epsilon, v^\epsilon) \in \mathcal{U}_{t,T}^{\pi,1} \times \mathcal{V}_{t,T}^{\pi,1}$ by induction over $i$, in order to have, for all $i = k, \ldots, N$,

$$E\left[U_m(t_i, X_{t_i}^{t,x,u^\epsilon,v^\epsilon})|\mathcal{F}_{t_{k-2},t_{i-2}}\right] \geq U_m\left(t_{i-1}, X_{t_{i-1}}^{t,x,u^\epsilon,v^\epsilon}\right) - \epsilon, \ P\text{-a.s.} \tag{4.46}$$

For $i = k$, it follows from (4.45) that there is $(u_{k-1}^{\epsilon,x}, v_{k-1}^{\epsilon,x}) \in \mathcal{U}_{t_{k-1},T}^{\pi,1} \times \mathcal{V}_{t_{k-1},T}^{\pi,1}$ satisfying (4.46). We define $u^\epsilon := u_{k-1}^{\epsilon,x}|_{[t_{k-1},t_k)}$ and $v^\epsilon := v_{k-1}^{\epsilon,x}|_{[t_{k-1},t_k)}$.

For $i = k+1$, (4.45) yields that, for any $y \in \mathbb{R}^n$, there is $(u_k^{\epsilon,y}, v_k^{\epsilon,y}) \in \mathcal{U}_{t_k,T}^{\pi,1} \times \mathcal{V}_{t_k,T}^{\pi,1}$ such that,

$$E\left[U_m\left(t_{k+1}, X_{t_{k+1}}^{t_k,y,u_k^{\epsilon,y},v_k^{\epsilon,y}}\right)\right] \geq U_m(t_k, y) - \frac{\epsilon}{2}, \ m = 1, 2. \tag{4.47}$$

Since the dynamics coefficient $f$ is bounded, there is some $R > 0$ such that $|X_{t_k}^{t,x,u^\epsilon,v^\epsilon}| < R$. Then there exists a finite Borel partition $(O_l)_{l=1}^n$ of $\bar{B}_R(0)$ with $\mathrm{diam}(O_l) \leq \delta_\epsilon$, for $\delta_\epsilon > 0$ small enough, such that, for all $z \in O_l$,

$$E\left[U_m\left(t_{k+1}, X_{t_{k+1}}^{t_k,z,u_k^{\epsilon,y_l},v_k^{\epsilon,y_l}}\right)\right] \geq U_m(t_k, z) - \epsilon. \tag{4.48}$$

Indeed, use (4.47) for $y = y_l$ and the uniform Lipschitz property of $U_m(t_i, \cdot)$ and $z \mapsto X_{t_{k+1}}^{t_k,z,u_k^{\epsilon,y_l},v_k^{\epsilon,y_l}}$. Now we extend $u^\epsilon$ and $v^\epsilon$ from $[t_{k-1}, t_k)$ to $[t_k, t_{k+1})$ by putting, on $[t_k, t_{k+1})$, $u^\epsilon := \sum_{l=1}^n u_k^{\epsilon,y_l} I_{\{X_{t_k}^{t,x,u^\epsilon,v^\epsilon} \in O_l\}}$, $v^\epsilon := \sum_{l=1}^n v_k^{\epsilon,y_l} I_{\{X_{t_k}^{t,x,u^\epsilon,v^\epsilon} \in O_l\}}$. Then from (4.48) we obtain

$$\begin{aligned} E\left[U_m(t_{k+1}, X_{t_{k+1}}^{t,x,u^\epsilon,v^\epsilon})|\mathcal{F}_{t_{k-2},t_{k-1}}\right] &= \sum_{l=1}^n E[U_m(t_{k+1}, X_{t_{k+1}}^{t_k,z,u_k^{\epsilon,y_l},v_k^{\epsilon,y_l}})]_{z=X_{t_k}^{t,x,u^\epsilon,v^\epsilon}} I_{\left\{X_{t_k}^{t,x,u^\epsilon,v^\epsilon} \in O_l\right\}} \\ \geq \sum_{l=1}^n [U_m(t_k, z) - \epsilon]_{z=X_{t_k}^{t,x,u^\epsilon,v^\epsilon}} I_{\left\{X_{t_k}^{t,x,u^\epsilon,v^\epsilon} \in O_l\right\}} &= U_m\left(t_k, X_{t_k}^{t,x,u^\epsilon,v^\epsilon}\right) - \epsilon, \ P\text{-a.s.} \end{aligned} \tag{4.49}$$

Repeating the above step, we construct $(u^\epsilon, v^\epsilon) \in \mathcal{U}_{t,T}^{\pi,1} \times \mathcal{V}_{t,T}^{\pi,1}$ satisfying (4.46).

Next for $l > i$, from (4.46) with using $\epsilon := \frac{\epsilon}{N}$ we get

$$\begin{aligned} &E\left[U_m(t_l, X_{t_l}^{t,x,u^\epsilon,v^\epsilon})|\mathcal{F}_{t_{k-2},t_{i-2}}\right] = E\left[E\left[U_m(t_l, X_{t_l}^{t,x,u^\epsilon,v^\epsilon})|\mathcal{F}_{t_{k-2},t_{l-2}}\right]|\mathcal{F}_{t_{k-2},t_{i-2}}\right] \\ &\geq E\left[U_m(t_{l-1}, X_{t_{l-1}}^{t,x,u^\epsilon,v^\epsilon})|\mathcal{F}_{t_{k-2},t_{i-2}}\right] - \frac{\epsilon}{N} \ldots \geq U_m(t_{i-1}, X_{t_{i-1}}^{t,x,u^\epsilon,v^\epsilon}) - \epsilon, \ P\text{-a.s.} \end{aligned} \tag{4.50}$$

□

## 5. Characterization for the functions $W(t, x, p, q)$ and $V(t, x, p, q)$

In this section we devote again to zero-sum games with asymmetric information, we characterize the functions $W(t, x, p, q)$ and $V(t, x, p, q)$ defined by (2.7) and (2.9). Under the Isaacs condition we prove that $W(t, x, p, q) = V(t, x, p, q) = U(t, x, p, q)$, the value function (see, Thm. 3.17). This characterization guarantees that combined with the Definitions (5.2)–(5.5) and the proof that these functions converge all to $W(= V)$ is the key for numerical approaches. It allows, *e.g.*, that Player I makes his computations along a partition $\pi$ and guarantees him that, whatever the partition $\pi'$ is Player II chooses for his strategies (Player II, knowing that Player I chooses $\pi$, may prefer to take a finer partition, in order to get after shorter time intervals, the information about the randomization chosen by Player II), the value function $\bar{W}^\pi(t,x)$ along $\pi$ (see (5.2)) always converges to the value of the game (see the Thms. 5.12 and 5.13).

Let us slightly change the definition of the strategies along a partition $\pi = \{0 = t_0 < t_1 < \ldots < t_N = T\}$. We suppose that the both Players I and II do not randomize their strategies over the first small time interval $[t, t_k)$. With this change the strategies $\alpha \in \mathcal{A}^\pi(t,T)$ take the form (see, Def. 2.1)

$$\alpha(\omega, v)(s) = \alpha_k(v)(s) I_{[t,t_k)}(s) + \sum_{l=k+1}^{N} \alpha_l((\zeta^\pi_{1,k}, \zeta^\pi_{2,k}, \zeta^\pi_{1,k+1}, \ldots, \zeta^\pi_{1,l-1})(\omega), v) I_{[t_{l-1},t_l)}(s),\ \ s \in [t,T], \tag{5.1}$$

where $\alpha_k : [t,t_k) \times \mathcal{V}_{t,T} \mapsto \mathcal{U}_{t,T}$, $\alpha_l : \mathbb{R}^{2(l-k)-1} \times [t_{l-1}, t_l) \times \mathcal{V}_{t,T} \mapsto \mathcal{U}_{t,T}$, $k+1 \le l \le N$, are Borel measurable functions satisfying: For all $v, v' \in \mathcal{V}_{t,T}$, it holds that, whenever $v = v'$ a.e. on $[t, t_{l-1}]$, we have for all $x \in \mathbb{R}^{2(l-k)-1}$, $\alpha_l(x,v)(s) = \alpha_l(x,v')(s)$, a.e. on $[t_{l-1}, t_l]$, $k+1 \le l \le N$.

Obviously, the strategy $\alpha$ such defined is a special case of Definition 2.1. We continue to write $\mathcal{A}^\pi(t,T)$ for the set of the strategies $\alpha$ of the above form. In the same manner we redefine $\mathcal{B}^\pi(t,T)$. Obviously, for $\pi' \subset \pi$, we have $\mathcal{A}^{\pi'}(t,T) \subset \mathcal{A}^\pi(t,T)$. $\mathcal{A}(t,T)$ and $\mathcal{B}(t,T)$ are again defined as the union of $\mathcal{A}^\pi(t,T)$ and $\mathcal{B}^\pi(t,T)$ over all partitions $\pi$, respectively. The other definitions are the same with that defined in Section 2.

For our approach we introduce the following upper and lower value functions, respectively:

$$\bar{W}^\pi(t,x,p,q) = \inf_{\hat{\alpha} \in (\mathcal{A}^\pi(t,T))^I} \sup_{\hat{\beta} \in (\mathcal{B}(t,T))^J} J(t,x,\hat{\alpha},\hat{\beta},p,q), \tag{5.2}$$

$$\bar{V}^\pi(t,x,p,q) = \sup_{\hat{\beta} \in (\mathcal{B}(t,T))^J} \inf_{\hat{\alpha} \in (\mathcal{A}^\pi(t,T))^I} J(t,x,\hat{\alpha},\hat{\beta},p,q), \tag{5.3}$$

$$\bar{\bar{W}}^\pi(t,x,p,q) = \inf_{\hat{\alpha} \in (\mathcal{A}(t,T))^I} \sup_{\hat{\beta} \in (\mathcal{B}^\pi(t,T))^J} J(t,x,\hat{\alpha},\hat{\beta},p,q), \tag{5.4}$$

$$\bar{\bar{V}}^\pi(t,x,p,q) = \sup_{\hat{\beta} \in (\mathcal{B}^\pi(t,T))^J} \inf_{\hat{\alpha} \in (\mathcal{A}(t,T))^I} J(t,x,\hat{\alpha},\hat{\beta},p,q). \tag{5.5}$$

We begin our approach with proving that $(\bar{W}^\pi(t,x,p,q), \bar{V}^\pi(t,x,p,q))$ as well as $(\bar{\bar{W}}^\pi(t,x,p,q), \bar{\bar{V}}^\pi(t,x,p,q))$ converge uniformly on compacts to the same couple $(U(t,x,p,q), U(t,x,p,q))$, as $|\pi| \to 0$. For this we have to assume the condition

$$\inf_{u \in U} \sup_{\nu \in \mathcal{P}(V)} f(x,u,\nu) \cdot \xi = \sup_{\nu \in \mathcal{P}(V)} \inf_{u \in U} f(x,u,\nu) \cdot \xi, \tag{5.6}$$

$$\sup_{v \in V} \inf_{\mu \in \mathcal{P}(U)} f(x,\mu,v) \cdot \xi = \inf_{\mu \in \mathcal{P}(U)} \sup_{v \in V} f(x,\mu,v) \cdot \xi, \tag{5.7}$$

respectively, where $f(x,\mu,v) := \int_U f(x,u,v)\mu(du)$, $f(x,u,\nu) := \int_V f(x,u,v)\nu(\mathrm{d}v)$, and the function $U(t,x,p,q)$ is the unique solution of the HJI equation (3.51). In a second step we will show that, under the conditions (5.6) and (5.7), the functions $W(t,x,p,q) = U(t,x,p,q) = V(t,x,p,q)$ all coincide.

**Remark 5.1.** The assumptions (5.6) and (5.7) hold both, if and only if the (classical) Isaacs condition holds:

$$\inf_{u \in U} \sup_{v \in V} f(x,u,v) \cdot \xi = \sup_{v \in V} \inf_{u \in U} f(x,u,v) \cdot \xi. \tag{5.8}$$

Indeed, as

$$\inf_{\mu\in\mathcal{P}(U)} f(x,\mu,v)\cdot\xi = \inf_{\mu\in\mathcal{P}(U)} \int_U f(x,u,v)\cdot\xi \mathrm{d}\mu(u) \geq \inf_{u\in U} f(x,u,v)\cdot\xi \geq \inf_{\mu\in\mathcal{P}(U)} f(x,\mu,v)\cdot\xi,$$

we have $\inf_{\mu\in\mathcal{P}(U)} f(x,\mu,v)\cdot\xi = \inf_{u\in U} f(x,u,v)\cdot\xi$, and analogously, $\sup_{\nu\in\mathcal{P}(V)} f(x,u,\nu)\cdot\xi = \sup_{v\in V} f(x,u,v)\cdot\xi$.

If (5.6) and (5.7) hold, we have

$$\inf_{u\in U}\sup_{v\in V} f(x,u,v)\cdot\xi = \inf_{u\in U}\sup_{\nu\in\mathcal{P}(V)} f(x,u,\nu)\cdot\xi = \sup_{\nu\in\mathcal{P}(V)}\inf_{u\in U} f(x,u,\nu)\cdot\xi = \sup_{\nu\in\mathcal{P}(V)}\inf_{\mu\in\mathcal{P}(U)} f(x,\mu,\nu)\cdot\xi,$$

$$\sup_{v\in V}\inf_{u\in U} f(x,u,v)\cdot\xi = \sup_{v\in V}\inf_{\mu\in\mathcal{P}(U)} f(x,\mu,v)\cdot\xi = \inf_{\mu\in\mathcal{P}(U)}\sup_{v\in V} f(x,\mu,v)\cdot\xi = \inf_{\mu\in\mathcal{P}(U)}\sup_{\nu\in\mathcal{P}(V)} f(x,\mu,\nu)\cdot\xi.$$

As $\sup_{\nu\in\mathcal{P}(V)}\inf_{\mu\in\mathcal{P}(U)} f(x,\mu,\nu)\cdot\xi = \inf_{\mu\in\mathcal{P}(U)}\sup_{\nu\in\mathcal{P}(V)} f(x,\mu,\nu)\cdot\xi$, we get the classical Isaacs condition (5.8). On the other hand, if (5.8) holds, then

$$\inf_{u\in U}\sup_{\nu\in\mathcal{P}(V)} f(x,u,\nu)\cdot\xi = \inf_{u\in U}\sup_{v\in V} f(x,u,v)\cdot\xi = \sup_{v\in V}\inf_{u\in U} f(x,u,v)\cdot\xi = \sup_{\nu\in\mathcal{P}(V)}\inf_{u\in U} f(x,u,\nu)\cdot\xi,$$

*i.e.*, (5.6) is satisfied. Similarly, we get (5.7).

A consequence of the redefinition of $\mathcal{A}^\pi(t,T)$ and $\mathcal{B}^\pi(t,T)$ is that now Isaacs condition is needed for the following result.

**Proposition 5.2.** *Under the condition (5.6) and (5.7) the functions $(V^{\pi_n})$ and $(W^{\pi_n})$ converge uniformly on compacts to a same Lipschitz continuous function $U$, when the mesh of the partition $\pi_n$ tends to 0. Moreover, the function $U$ is the unique dual viscosity solution of the HJI equation (3.51).*

The proof follows that of Theorem 3.14, but the redefinition of $\mathcal{A}^\pi(t,T)$ and $\mathcal{B}^\pi(t,T)$ makes that the argument used for (3.45)–(3.46) does not apply any more, now Isaacs condition has to be used.

Let us now investigate the convergence of $(\bar{W}^\pi(t,x,p,q), \bar{V}^\pi(t,x,p,q))$; that of $(\bar{\bar{W}}^\pi(t,x,p,q), \bar{\bar{V}}^\pi(t,x,p,q))$ can be studied similarly. Notice that, for all $\beta\in\mathcal{B}(t,T)$, there exists a partition $\bar{\pi}$ such that $\beta\in\mathcal{B}^{\bar{\pi}}(t,T)$. This allows to apply arguments used in Section 3, to prove the following assertions.

**Lemma 5.3.** *The functions $\bar{W}^\pi$ and $\bar{V}^\pi$ are Lipschitz in $(t,x,p,q)$, uniformly with respect to $\pi$.*

**Lemma 5.4.** *For all $(t,x)\in[0,T]\times\mathbb{R}^n$, the functions $\bar{W}^\pi(t,x,p,q)$ and $\bar{V}^\pi(t,x,p,q)$ are convex in $p\in\Delta(I)$ and concave in $q\in\Delta(J)$.*

**Lemma 5.5.** *For all $(t,x,\bar{p},q)\in[0,T]\times\mathbb{R}^n\times\mathbb{R}^I\times\Delta(J)$, we have*

$$\bar{V}^{\pi*}(t,x,\bar{p},q) = \inf_{(\beta_j)\in(\mathcal{B}(t,T))^J}\ \sup_{\alpha\in\mathcal{A}_0^\pi(t,T)}\ \max_{i\in\{1,\dots,I\}}\left\{\bar{p}_i - \sum_{j=1}^J q_j E\left[g_{ij}\left(X_T^{t,x,\alpha,\beta_j}\right)\right]\right\}. \tag{5.9}$$

*Proof.* Let us denote by $\bar{V}_1^{\pi*}(t,x,\bar{p},q)$ the right hand side of (5.9). Similarly to the proof of Lemma 3.5 we have

$$\bar{V}^{\pi*}(t,x,\bar{p},q) = \inf_{(\beta_j)\in(\mathcal{B}(t,T))^J}\ \sup_{\alpha\in\mathcal{A}^\pi(t,T)}\ \max_{i\in\{1,\dots,I\}}\left\{\bar{p}_i - \sum_{j=1}^J q_j E\left[g_{ij}\left(X_T^{t,x,\alpha,\beta_j}\right)\right]\right\} \tag{5.10}$$

(Note that, unlike in (5.9), we have here $\mathcal{A}^\pi(t,T)$ instead of $\mathcal{A}_0^\pi(t,T)$). Since $\mathcal{A}_0^\pi(t,T)\subset\mathcal{A}^\pi(t,T)$, we have $\bar{V}_1^{\pi*}(t,x,\bar{p},q) \leq \bar{V}^{\pi*}(t,x,\bar{p},q)$. We have still to show that $\bar{V}_1^{\pi*}(t,x,\bar{p},q) \geq \bar{V}^{\pi*}(t,x,\bar{p},q)$. Recall that $\alpha\in$

$\mathcal{A}^\pi(t,T)$ is of the form (5.1) and notice that for $y=(y_1,y_2,\dots,y_{2(N-k)-1})\in\mathbb{R}^{2(N-k)-1}$, it holds $\alpha(y,\cdot)\in\mathcal{A}_0^\pi(t,T)$. Thus, for any $(\beta_j)\in(\mathcal{B}(t,T))^J$,

$$\begin{aligned}
&\sup_{\alpha\in\mathcal{A}^\pi(t,T)}\max_{i\in\{1,\dots,I\}}\left\{\bar p_i-\sum_{j=1}^J q_jE\left[g_{ij}\left(X_T^{t,x,\alpha,\beta_j}\right)\right]\right\}\\
&\le \sup_{\alpha\in\mathcal{A}^\pi(t,T)}\int_{[0,1]^{2(N-k)-1}}\max_{i\in\{1,\dots,I\}}\left\{\bar p_i-\sum_{j=1}^J q_jE\left[g_{ij}\left(X_T^{t,x,\alpha((y_1,y_2,\dots,y_{2(N-k)-1}),\cdot),\beta_j}\right)\right]\right\}\mathrm{d}y_1\dots\mathrm{d}y_{2(N-k)-1}\\
&\le \sup_{\alpha\in\mathcal{A}^\pi(t,T)}\sup_{y\in[0,1]^{2(N-k)-1}}\max_{i\in\{1,\dots,I\}}\left\{\bar p_i-\sum_{j=1}^J q_jE\left[g_{ij}\left(X_T^{t,x,\alpha((y_1,y_2,\dots,y_{2(N-k)-1}),\cdot),\beta_j}\right)\right]\right\}\\
&\le \sup_{\alpha\in\mathcal{A}_0^\pi(t,T)}\max_{i\in\{1,\dots,I\}}\left\{\bar p_i-\sum_{j=1}^J q_jE\left[g_{ij}\left(X_T^{t,x,\alpha,\beta_j}\right)\right]\right\}.
\end{aligned}\tag{5.11}$$

Finally, taking infimum over $(\beta_j)\in(\mathcal{B}(t,T))^J$ on both sides we get the stated result. ☐

Following the scheme of Section 3, we can state now the following lemmas.

**Lemma 5.6.** *For all partition $\pi$ of the interval $[0,T]$, the convex conjugate function $\bar V^{\pi *}(t,x,\bar p,q)$ is Lipschitz in $(t,x,\bar p,q)$, the concave conjugate function $\bar W^{\pi\#}(t,x,p,\bar q)$ is Lipschitz in $(t,x,p,\bar q)$.*

**Lemma 5.7.** *For all $(t,x,\bar p,q)\in[t_{k-1},t_k)\times\mathbb{R}^n\times\mathbb{R}^I\times\Delta(J)$, and all $l$ $(k\le l\le N)$, we have the following sub-dynamic programming principle:*

$$\begin{aligned}
\bar V^{\pi *}(t,x,\bar p,q)&\le \inf_{\beta\in\mathcal{B}(t,t_l)}\sup_{\alpha\in\mathcal{A}_0^\pi(t,t_l)}E\left[\bar V^{\pi *}(t_l,X_{t_l}^{t,x,\alpha,\beta},\bar p,q)\right]\\
&\le \inf_{\beta\in\mathcal{B}(t,t_l)}\sup_{\alpha\in\mathcal{A}^\pi(t,t_l)}E\left[\bar V^{\pi *}(t_l,X_{t_l}^{t,x,\alpha,\beta},\bar p,q)\right].
\end{aligned}\tag{5.12}$$

*Proof.* The proof of the first inequality, using Lemma 5.5, is analogous to the proof of Lemma 3.7. Finally, the second inequality is obvious, since $\mathcal{A}_0^\pi(t,t_l)\subset\mathcal{A}^\pi(t,t_l)$. ☐

**Lemma 5.8.** *For any sequence of partitions $\pi_n$, $n\ge1$, with $|\pi_n|\to0$ $(n\to\infty)$, there exists a subsequence, still denoted by $(\pi_n)_{n\ge1}$, and two functions $\tilde V:[0,T]\times\mathbb{R}^n\times\mathbb{R}^I\times\Delta(J)\mapsto\mathbb{R}$ and $\tilde W:[0,T]\times\mathbb{R}^n\times\Delta(I)\times\mathbb{R}^J\mapsto\mathbb{R}$ such that $(\bar V^{\pi_n *},\bar W^{\pi_n\#})\to(\tilde V,\tilde W)$ uniformly on compacts on $[0,T]\times\mathbb{R}^n\times\Delta(I)\times\Delta(J)\times\mathbb{R}^I\times\mathbb{R}^J$. Furthermore, the functions $\tilde V$ and $\tilde W$ are Lipschitz in all their variables.*

**Lemma 5.9.** *The limit function $\tilde V(t,x,\bar p,q)$ is a viscosity subsolution of the HJI equation (3.36).*

On the other hand we observe that, since

$$-\bar W^\pi(t,x,p,q)=\sup_{(\alpha_i)\in(\mathcal{A}^\pi(t,T))^I}\inf_{(\beta_j)\in(\mathcal{B}(t,T))^J}\sum_{i=1}^I\sum_{j=1}^J p_iq_jE\left[-g_{ij}\left(X_T^{t,x,\alpha_i,\beta_j}\right)\right],\tag{5.13}$$

also the convex conjugate $-(\bar W^{\pi\#}(t,x,p,-\bar q))$ of $(-\bar W^\pi)$ with respect to $q$, satisfies a sub-dynamic programming principle. Consequently, we can formulate the following result.

**Corollary 5.10.** *For all $(t,x,p,\bar q)\in[t_{k-1},t_k)\times\mathbb{R}^n\times\Delta(I)\times\mathbb{R}^J$, and all $l$ $(k\le l\le N)$, we have the following super-dynamic programming principle:*

$$\bar W^{\pi\#}(t,x,p,\bar q)\ge\sup_{\alpha\in\mathcal{A}^\pi(t,t_l)}\inf_{\beta\in\mathcal{B}(t,t_l)}E[\bar W^{\pi\#}(t_l,X_{t_l}^{t,x,\alpha,\beta},p,\bar q)].\tag{5.14}$$

**Proposition 5.11.** *Under the condition* (5.6), *the limit function* $\tilde{W}(t,x,p,\bar{q})$ *is a viscosity supersolution of the HJI equation* (3.36).

*Proof.* We fix arbitrarily $(p,\bar{q}) \in \Delta(I)\times\mathbb{R}^J$, and for simplicity, we write $\tilde{W}(t,x)$ for $\tilde{W}(t,x,p,\bar{q})$. As the coefficient $f$ of the dynamics is bounded, for any fixed $(t,x)\in[0,T]\times\mathbb{R}^n$, there is some $M>0$ such that, $\bar{B}_M(x)\supset\{X_r^{s,y,\alpha,\beta},\ (s,y)\in[0,T]\times\bar{B}_1(x),(\alpha,\beta)\in\mathcal{A}^\pi(s,T)\times\mathcal{B}(s,T), r\in[s,T]\}$, where $\bar{B}_M(x)$ denotes the closed ball with the center $x$ and the radius $M$. By Lemma 5.8 we know that $\bar{W}^{\pi_n\#}$ converges uniformly to $\tilde{W}$ on $[0,T]\times\bar{B}_{M+1}(x)$. Let $\varphi\in C_b^1([0,T]\times\mathbb{R}^n)$ be a test function such that

$$(-\tilde{W}-(-\varphi))(t,x) > (-\tilde{W}-(-\varphi))(s,y),\ \text{for all } (s,y)\in[0,T]\times\bar{B}_M(x)\setminus\{(t,x)\}, \tag{5.15}$$

and let $(s_n,x_n)\in[0,T]\times\bar{B}_M(x)$ be the maximum point of $-\bar{W}^{\pi_n\#}-(-\varphi)$ over $[0,T]\times\bar{B}_M(x)$. Then it is standard that there exists a subsequence of $(s_n,x_n)$ still denoted by $(s_n,x_n)$, such that $(s_n,x_n)$ converges to $(t,x)$ (See also the proof of Lem. 3.10).

For the partition $\pi_n$ we assume $t^n_{k_{n-1}}\le s_n<t^n_{k_n}$, and for simplicity we write $t^n_{k-1}\le s_n<t^n_k$. Since $x_n\to x$, there is a positive integer $N$ such that, for all $n\ge N$, we have $|x_n-x|\le 1$. From Lemma 5.10 we get

$$\begin{aligned}-\varphi(s_n,x_n)=-\bar{W}^{\pi_n\#}(s_n,x_n)&\le\inf_{\alpha\in\mathcal{A}^{\pi_n}(s_n,t^n_k)}\sup_{\beta\in\mathcal{B}(s_n,t^n_k)}E\left[-\bar{W}^{\pi_n\#}\left(t^n_k,X^{s_n,x_n,\alpha,\beta}_{t^n_k}\right)\right]\\&\le\inf_{\alpha\in\mathcal{A}^{\pi_n}(s_n,t^n_k)}\sup_{\beta\in\mathcal{B}(s_n,t^n_k)}E\left[-\varphi\left(t^n_k,X^{s_n,x_n,\alpha,\beta}_{t^n_k}\right)\right].\end{aligned}\tag{5.16}$$

Thus,

$$\begin{aligned}0&\le\inf_{\alpha\in\mathcal{A}^{\pi_n}(s_n,t^n_k)}\sup_{\beta\in\mathcal{B}(s_n,t^n_k)}E\left[-\varphi\left(t^n_k,X^{s_n,x_n,\alpha,\beta}_{t^n_k}\right)-(-\varphi(s_n,x_n))\right]\\&=\inf_{\alpha\in\mathcal{A}^{\pi_n}(s_n,t^n_k)}\sup_{\beta\in\mathcal{B}(s_n,t^n_k)}E\left[-\int_{s_n}^{t^n_k}\left(\frac{\partial\varphi}{\partial r}(r,X^{s_n,x_n,\alpha,\beta}_r\right)+f(X^{s_n,x_n,\alpha,\beta}_r,\alpha_r,\beta_r)\cdot D\varphi\left(r,X^{s_n,x_n,\alpha,\beta}_r\right))\mathrm{d}r\right].\end{aligned}\tag{5.17}$$

For $\delta>0$ let us introduce the following continuity modulus,

$$m(\delta):=\sup_{\substack{|r-s|+|y-\bar{x}|\le\delta,\\ u\in U,v\in V,\bar{x},y\in\bar{B}_M(x)}}\left|\left(\frac{\partial\varphi}{\partial r}(r,y)+f(y,u,v)\cdot D\varphi(r,y)\right)-\left(\frac{\partial\varphi}{\partial r}(s,\bar{x})+f(\bar{x},u,v)\cdot D\varphi(s,\bar{x})\right)\right|.\tag{5.18}$$

Obviously, $m(\delta)$ is nondecreasing in $\delta$ and $m(\delta)\to 0$, as $\delta\downarrow 0$. From (2.2) we have $|X_r^{s_n,x_n,\alpha,\beta}-x_n|\le C|r-s_n|\le C|t^n_k-s_n|,\ r\in[s_n,t^n_k]$. Hence,

$$\begin{aligned}&\Bigg|\left(\frac{\partial\varphi}{\partial r}(r,X^{s_n,x_n,\alpha,\beta}_r)+f(X^{s_n,x_n,\alpha,\beta}_r,\alpha_r,\beta_r)\cdot D\varphi(r,X^{s_n,x_n,\alpha,\beta}_r)\right)\\&\qquad-\left(\frac{\partial\varphi}{\partial r}(s_n,x_n)+f(x_n,\alpha_r,\beta_r)\cdot D\varphi(s_n,x_n)\right)\Bigg|\le m(C|t^n_k-s_n|),\ r\in[s_n,t^n_k].\end{aligned}\tag{5.19}$$

Thus, it follows from (5.17) and (5.19) that

$$\begin{aligned}-(t^n_k-s_n)\left(-\frac{\partial\varphi}{\partial r}(s_n,x_n)+m(C|t^n_k-s_n|)\right)&\le\inf_{\alpha\in\mathcal{A}^{\pi_n}(s_n,t^n_k)}\sup_{\beta\in\mathcal{B}(s_n,t^n_k)}E\left[\int_{s_n}^{t^n_k}(-f)(x_n,\alpha_r,\beta_r)\cdot D\varphi(s_n,x_n)\mathrm{d}r\right]\\&\le\sup_{\beta\in\mathcal{B}(s_n,t^n_k)}E\left[\int_{s_n}^{t^n_k}(-f)(x_n,\tilde{\alpha}_r,\beta_r)\cdot D\varphi(s_n,x_n)\mathrm{d}r\right],\end{aligned}\tag{5.20}$$

where we have taken $\tilde{\alpha}_r = \tilde{u}_k$, $r \in [t, t_k^n]$, $\tilde{u}_k \in U$. Let $\rho_n = (t_k^n - s_n)^2$. From (5.20) we see that there exists a $\rho_n$-optimal strategy $\beta^n \in \mathcal{B}(s_n, t_k^n)$ (depending on $\tilde{\alpha}_r$) such that

$$-(t_k^n - s_n)\left(-\frac{\partial\varphi}{\partial r}(s_n, x_n) + m(C|t_k^n - s_n|) + (t_k^n - s_n)\right) \le E\left[\int_{s_n}^{t_k^n} (-f\,(x_n, \tilde{\alpha}_r, \beta_r^n) \cdot D\varphi(s_n, x_n)\mathrm{d}r\right]. \quad (5.21)$$

Since $\beta^n \in \mathcal{B}(s_n, t_k^n)$ there is some partition $\pi^0$ such that $\beta^n \in \mathcal{B}^{\pi^0}(s_n, t_k^n)$, and without loss of generality we can assume that $\pi^0 \supset \pi_n$. Supposing that $\{s_n = \theta_0 < \theta_1 < \ldots < \theta_m = t_k^n\} \subset \pi^0$, we obtain

$$E\left[\int_{s_n}^{t_k^n} (-f)(x_n, \tilde{\alpha}_r, \beta_r^\rho) \cdot D\varphi(s_n, x_n)\mathrm{d}r\right] = \sum_{l=1}^{m} \int_{\theta_{l-1}}^{\theta_l} E\left[(-f)(x_n, \tilde{\alpha}_r, \beta_r^\rho) \cdot D\varphi(s_n, x_n)\right]\mathrm{d}r, \quad (5.22)$$

where $\beta^\rho$ depends on $(\zeta_{2,1}^{\pi^0}, \ldots, \zeta_{2,l-1}^{\pi^0}, \tilde{u}_k)$ on $[\theta_{l-1}, \theta_l]$. Then for $r \in [\theta_{l-1}, \theta_l]$, we have,

$$\begin{aligned} E\left[(-f)(x_n, \tilde{\alpha}_r, \beta_r^\rho) \cdot D\varphi(s_n, x_n)\right] &= E\left[(-f)\left(x_n, \tilde{u}_k, \beta_r^\rho(\zeta_{2,1}^{\pi^0}, \ldots, \zeta_{2,l-1}^{\pi^0}, \tilde{u}_k)\right) \cdot D\varphi(s_n, x_n)\right] \\ &= \int_V (-f)(x_n, \tilde{u}_k, v) \cdot D\varphi(s_n, x_n) P_{\beta_r^\rho(\zeta_{2,1}^{\pi^0}, \ldots, \zeta_{2,l-1}^{\pi^0}, \tilde{u}_k)}(\mathrm{d}v) \le \sup_{\nu \in \mathcal{P}(V)} \int_V (-f)(x_n, \tilde{u}_k, v) \cdot D\varphi(s_n, x_n)\nu(\mathrm{d}v). \end{aligned} \quad (5.23)$$

Putting $I(\tilde{u}_k) := \sup_{\nu \in \mathcal{P}(V)} \int_V (-f)(x_n, \tilde{u}_k, v) \cdot D\varphi(s_n, x_n)\nu(\mathrm{d}v)$, from the arbitrariness of $\tilde{u}_k$ and the continuity of $I$ in $\tilde{u}_k$, allow to choose $\tilde{u}_k := u^*$ such that $I(u^*) = \min_{\tilde{u}_k \in U} I(\tilde{u}_k)$. Then, from (5.23), we have for all $u \in U$,

$$\int_{\theta_{l-1}}^{\theta_l} I(\tilde{u}_k)\mathrm{d}r = \int_{\theta_{l-1}}^{\theta_l} I(u^*)\mathrm{d}r \le \int_{\theta_{l-1}}^{\theta_l} \sup_{\nu \in \mathcal{P}(V)} \int_V (-f)(x_n, u, v) \cdot D\varphi(s_n, x_n) d\nu(v)\mathrm{d}r. \quad (5.24)$$

Hence, (5.24) and condition (5.6) yield

$$\begin{aligned} \int_{\theta_{l-1}}^{\theta_l} I(\tilde{u}_k)\mathrm{d}r \le &(\theta_l - \theta_{l-1}) \inf_{u \in U} \sup_{\nu \in \mathcal{P}(V)} \int_V (-f)(x_n, u, v) \cdot D\varphi(s_n, x_n) d\nu(v) \\ = &(\theta_l - \theta_{l-1}) \sup_{\nu \in \mathcal{P}(V)} \inf_{u \in U} \int_V (-f)(x_n, u, v) \cdot D\varphi(s_n, x_n) d\nu(v) \\ = &(\theta_l - \theta_{l-1}) \sup_{\nu \in \mathcal{P}(V)} \inf_{\mu \in \mathcal{P}(U)} \int_{U \times V} (-f)(x_n, u, v) \cdot D\varphi(s_n, x_n)\mathrm{d}\mu(u) d\nu(v). \end{aligned} \quad (5.25)$$

Consequently, combining (5.21)–(5.23) and (5.25), we have

$$\begin{aligned} &-(t_k^n - s_n)\left(-\frac{\partial\varphi}{\partial r}(s_n, x_n) + m(C|t_k^n - s_n|) + (t_k^n - s_n)\right) \le \\ &\qquad (t_k^n - s_n) \sup_{\nu \in \mathcal{P}(V)} \inf_{\mu \in \mathcal{P}(U)} \int_{U \times V} (-f)(x_n, u, v) \cdot D\varphi(s_n, x_n)\mu(\mathrm{d}u)\nu(\mathrm{d}v), \end{aligned} \quad (5.26)$$

*i.e.*,

$$\frac{\partial\varphi}{\partial r}(s_n, x_n) - m(C|t_k^n - s_n|) - (t_k^n - s_n)) \le \sup_{\nu \in \mathcal{P}(V)} \inf_{\mu \in \mathcal{P}(U)} \int_{U \times V} (-f)(x_n, u, v) \cdot D\varphi(s_n, x_n)\mu(\mathrm{d}u)\nu(\mathrm{d}v). \quad (5.27)$$

Finally, recalling that $(s_n, x_n) \to (t, x)$ and $0 \le (t_k^n - s_n) \le (t_k^n - t_{k-1}^n) \le |\pi_n|$, and taking the limit as $n \to \infty$, we get

$$\frac{\partial\varphi}{\partial t}(t, x) + \inf_{\nu \in \mathcal{P}(V)} \sup_{\mu \in \mathcal{P}(U)} \int_{U \times V} f(x, u, v) \cdot D\varphi(t, x)\mu(\mathrm{d}u)\nu(\mathrm{d}v) \le 0. \quad (5.28)$$

Therefore, $\tilde{W}(t, x, p, \bar{q})$ is a viscosity supersolution of the HJI equation (3.36). □

Similar to Section 3 (Thm. 3.14), we have the following result.

**Theorem 5.12.** *Suppose condition (5.6) holds. Then, for all sequences of partitions $(\pi_n)$ with $|\pi_n| \to 0$, the sequences $(\bar{V}^{\pi_n})$ and $(\bar{W}^{\pi_n})$ converge uniformly on compacts to the same Lipschitz continuous function $U$. Moreover, the function $U$ is the unique dual solution of the HJI equation (3.51).*

Similarly to $(\bar{W}^{\pi_n}, \bar{V}^{\pi_n})$, we obtain the following result for $(\bar{\bar{V}}^{\pi_n}, \bar{\bar{W}}^{\pi_n})$ (Recall its Def. (5.4) and (5.5)).

**Theorem 5.13.** *Suppose condition (5.7) holds. Then, for all sequences of partitions $(\pi_n)$ with $|\pi_n| \to 0$, the sequences $(\bar{\bar{V}}^{\pi_n})$ and $(\bar{\bar{W}}^{\pi_n})$ converge uniformly on compacts to the unique dual solution $U$ of the HJI equation (3.51).*

Finally, combining the Theorems 5.12 and 5.13 with Proposition 5.2, we get the main result of this section.

**Theorem 5.14.** *Under Isaacs condition, the functions $W(t,x,p,q)$ and $V(t,x,p,q)$ coincide, for all $(t,x,p,q) \in [0,T] \times \mathbb{R}^n \times \Delta(I) \times \Delta(J)$.*

*Proof.* We have shown that the value functions $W^\pi(t,x,p,q), V^\pi(t,x,p,q)$, $\bar{W}^\pi(t,x,p,q), \bar{V}^\pi(t,x,p,q)$, $\bar{\bar{W}}^\pi(t,x,p,q)$ and $\bar{\bar{V}}^\pi(t,x,p,q)$ converge uniformly on compacts to the function $U(t,x,p,q)$, as $|\pi| \to 0$. For this we have used the assumptions (5.6) and (5.7) which, both together, are equivalent to Isaacs condition.

Then, due to the definition of $W(t,x,p,q)$, for any $\varepsilon > 0$, there exists $\hat{\alpha}^\varepsilon \in (\mathcal{A}(t,T))^I$ such that

$$\varepsilon + W(t,x,p,q) \geq \sup_{\hat{\beta} \in (\mathcal{B}(t,T))^J} J(t,x,\hat{\alpha}^\varepsilon,\hat{\beta},p,q). \tag{5.29}$$

For $\hat{\alpha}^\varepsilon \in (\mathcal{A}(t,T))^I$, there exists a partition $\pi^\varepsilon$ such that $\hat{\alpha}^\varepsilon \in (\mathcal{A}^{\pi^\varepsilon}(t,T))^I$. Thus for all $\pi \supset \pi^\varepsilon$ (Recall that $\pi^\epsilon \subset \pi$ implies $\mathcal{A}^{\pi^\varepsilon}(t,T) \subset \mathcal{A}^\pi(t,T)$), it holds that

$$\varepsilon + W(t,x,p,q) \geq \sup_{\hat{\beta} \in (\mathcal{B}^\pi(t,T))^J} J(t,x,\hat{\alpha}^\varepsilon,\hat{\beta},p,q) \geq W^\pi(t,x,p,q). \tag{5.30}$$

From the arbitrariness of $\varepsilon > 0$, we have $W(t,x,p,q) \geq W^\pi(t,x,p,q)$, and letting $|\pi| \to 0$, we obtain

$$W(t,x,p,q) \geq U(t,x,p,q). \tag{5.31}$$

With a symmetric argument we prove

$$U(t,x,p,q) \geq V(t,x,p,q). \tag{5.32}$$

On the other hand, since $W(t,x,p,q) \leq \inf_{\hat{\alpha} \in (\mathcal{A}^\pi(t,T))^I} \sup_{\hat{\beta} \in (\mathcal{B}(t,T))^J} J(t,x,\hat{\alpha},\hat{\beta},p,q) = \bar{W}^\pi(t,x,p,q)$, taking the limit $|\pi| \to 0$, yields

$$W(t,x,p,q) \leq U(t,x,p,q). \tag{5.33}$$

Analogously, as $V(t,x,p,q) \geq \bar{\bar{V}}^\pi(t,x,p,q)$, passing to the limit $|\pi| \to 0$ gives

$$V(t,x,p,q) \geq U(t,x,p,q). \tag{5.34}$$

Finally, combining the above results (5.31)−(5.33) and (5.34), we get $W(t,x,p,q) = U(t,x,p,q) = V(t,x,p,q)$. □

**Remark 5.15.** Theorems 5.12-5.14 have without doubt their own importance also for numerical approaches. They guarantee that the value function $W(= U)$ can be numerically computed, for instance, by Player 1: He can choose due to the definition of $\bar{W}^\pi(t,x,p,q)$ a sufficiently fine partition $\pi$ and an $\varepsilon$-optimal randomized strategy, well knowing that this $\varepsilon$-optimality holds for all possible partition chosen by Player 2. Without such a result, if one used only the definitions (2.1) and (2.2) for the value functions, the choice of a partition $\pi$ and a randomized strategy $\hat{\alpha}$ by Player 1 would, due to (2.1) morally oblige Player 2 to use the same partition, but, of course, Player 2 is not interested in; he is more interested in an optimal choice of the partition given that chosen by Player 1.

Let us conclude by giving an example which shows that the conditions (5.6) and (5.7) (or, equivalently, Isaacs condition) are necessary for Theorem 5.14, even for the games with symmetric information.

**Example 5.16.** We assume $U = V = [-1, 1]$, $I = J = 1$, $g(x) = x$, $f(x, u, v) = |u - v|^2$. For any given $(t, x) \in [0, T] \times \mathbb{R}$, the dynamic is of the form

$$X_s^{t,x,u,v} = x + \int_t^s |u_s - v_s|^2 \mathrm{d}s,\ s \in [t, T],\ x \in \mathbb{R},\ (u, v) \in \mathcal{U}_{t,T} \times \mathcal{V}_{t,T},$$

and the pay-off is given by

$$J(t, x, \alpha, \beta) = E\left[g\left(X_T^{t,x,\alpha,\beta}\right)\right] = E\left[X_T^{t,x,\alpha,\beta}\right].$$

Let $(u, v) \in \mathcal{U}_{t,T} \times \mathcal{V}_{t,T}$ be such that $\alpha(v) = u$ and $\beta(u) = v$. Then,

$$J(t, x, \alpha, \beta) = x + E\left[\int_t^T |u_r - v_r|^2 \mathrm{d}r\right]. \tag{5.35}$$

For $x \in \mathbb{R}$, $(u, v) \in U \times V$ and $p \in \mathbb{R}$, the Hamiltonian function is of the form $H(x, u, v, p) = |u - v|^2 p$. Let us put

$$\tilde{H}^+(x, p) \triangleq \inf_{u \in U} \sup_{v \in V} H(x, u, v, p) = \inf_{u \in U} (1 + |u|)^2 p^+ = p^+;$$

$$\tilde{H}^-(x, p) \triangleq \sup_{v \in V} \inf_{u \in U} H(x, u, v, p) = \sup_{v \in V} \big( - (1 + |v|)^2 p^- \big) = -p^-.$$

Obviously, for any $p \neq 0$, $\tilde{H}^-(x, p) = -p^- < p^+ = \tilde{H}^+(x, p)$, *i.e.*, Isaacs condition does not hold. For the measure-valued controls $\mu \in \mathcal{P}(U), \nu \in \mathcal{P}(V)$ and $(x, p) \in \mathbb{R}^2$, the Hamiltonian function takes the form $H(x, \mu, \nu, p) = \int_U \int_V |u - v|^2 p \nu(\mathrm{d}v)\mu(\mathrm{d}u)$, and we put again

$$H^+(x, p) = \inf_{\mu \in \mathcal{P}(U)} \sup_{\nu \in \mathcal{P}(V)} H(x, \mu, \nu, p); \quad H^-(x, p) = \sup_{\nu \in \mathcal{P}(V)} \inf_{\mu \in \mathcal{P}(U)} H(x, \mu, \nu, p).$$

Then obviously $H(x, p) := H^+(x, p) = H^-(x, p)$. Now we compute $H^-(x, p)$, first for $p \geq 0$, since

$$\begin{aligned} H(x, \mu, \nu, p) = \int_V \int_U |u - v|^2 p \mu(\mathrm{d}u)\nu(\mathrm{d}v) &= \int_V \int_U \big(|v - \int_V v\nu(\mathrm{d}v)|^2 + |u - \int_V v\nu(\mathrm{d}v)|^2\big) p \mu(\mathrm{d}u)\nu(\mathrm{d}v) \\ &\geq \int_V |v - \int_V v\nu(\mathrm{d}v)|^2 p \nu(\mathrm{d}v) = H(x, \delta_{\int_V v\nu(\mathrm{d}v)}, \nu, p), \end{aligned} \tag{5.36}$$

we have $\inf_{\mu \in \mathcal{P}(U)} H(x, \mu, \nu, p) = H(x, \delta_{\int_V v\nu(\mathrm{d}v)}, \nu, p) = \big(\int_V |v - \int_V v\nu(\mathrm{d}v)|^2 \nu(\mathrm{d}v)\big)p$. Therefore, $H^-(x, p) = \sup_{\nu \in \mathcal{P}(V)} (\int_V |v - \int_V v\nu(\mathrm{d}v)|^2 \nu(\mathrm{d}v))p = \sup_{\nu \in \mathcal{P}(V)} \Big(\int_V v^2 \nu(\mathrm{d}v) - (\int_V v\nu(\mathrm{d}v))^2\Big)p \leq p$, and for $\nu = \frac{1}{2}(\delta_1 + \delta_{-1})$ it attains the maximum value $p$.

For the case $p < 0$, $H^-(x, p) = -\Big(\inf_{\nu \in \mathcal{P}(V)} \sup_{\mu \in \mathcal{P}(U)} \int_U \int_V |u - v|^2 \mu(\mathrm{d}u)\nu(\mathrm{d}v)(-p)\Big) = -(-p)^+ = p$. Thus, we get $H(x, p) = H^-(x, p) = p$.

The corresponding HJI equation is as follows:

$$\begin{cases} \partial_t V + \partial_x V = \partial_t V + H(x, \partial_x V) = 0, \\ V(T, x) = g(x) = x. \end{cases} \tag{5.37}$$

Notice that $V(t, x) = x + T - t$, $t \in [0, T], x \in \mathbb{R}$ is the solution of this equation.

If both players use the same partition $\pi$, as $|\pi| \to 0$, we have due to Theorem 3.14 that $W^\pi(t, x)$ and $V^\pi(t, x)$ converge to the value function $V(t, x)$. Let us now compute

$$\bar{W}^\pi(t, x) = \inf_{\alpha \in \mathcal{A}^\pi(t,T)} \sup_{\beta \in \mathcal{B}(t,T)} E\left[g\left(X_T^{t,x,\alpha,\beta}\right)\right].$$

For Player I we consider the partition $\pi = \{0 = t_0 < t_1 < \ldots < t_N = T\}$, $t \in [t_{k-1}, t_k)$, and we suppose without loss of generality that $t = t_{k-1}$ and $\alpha_r = \alpha_k(r)$, $r \in [t_{k-1}, t_k]$ ($\alpha_k$ is independent of $\zeta^\pi_{1,k-1}$, but, notice that even if $\alpha_k$ depends on $\zeta^\pi_{1,k-1}$, from the first definition of $\mathcal{A}^\pi(t,T)$, the argument still works).

For Player II, we consider the partition $\pi_n = \{0 = (t_0 =)S_0 < S_1 < \ldots < S_{2^n}(= t_1) < S_{2^n+1} < \ldots < S_{2^n+2^n}(= t_2) < \ldots < S_{(l-1)2^n+j} < \ldots < S_{(N-1)2^n+2^n}(= t_N) = T\}$, where $S_{(l-1)2^n+j} = t_{l-1} + j(t_l - t_{l-1})2^{-n}$, $0 \le j \le 2^n$, $1 \le l \le N$. On the subinterval $\Delta^{l,n}_m := [S_{(l-1)2^n+m-1}, S_{(l-1)2^n+m}]$, Player II uses the strategy $\beta_r = \beta_{S_{(l-1)2^n+m}}(\zeta^{\pi_n}_{2,(l-1)2^n+m-1}, \alpha|_{[t,S_{(l-1)2^n+m-1}]})_r$, $1 \le m \le 2^n$, $k \le l \le N$. Obviously, $\beta_r(u) \triangleq \varphi(r, u_{r-|\pi_n|})$ is such a strategy, where $\varphi : [0,T] \times [-1,1] \to [-1,1]$ is a Borel function. Choosing $\beta_r(u) \triangleq \begin{cases} -\operatorname{sgn}(u_{r-|\pi_n|}), & r \in [t+|\pi_n|, T], \\ 0, & r \in [t, t+|\pi_n|], \end{cases}$ we have from equation (5.35)

$$\begin{aligned} J(t,x,\alpha,\beta) &= x + \int_t^{t+|\pi_n|} E[|\alpha_r|^2]\mathrm{d}r + E\left[\int_{t+|\pi_n|}^T |\alpha_r + \operatorname{sgn}(\alpha_{r-\pi_n})|^2 \mathrm{d}r\right] \\ &= x + E\left[\int_t^T |\alpha_r + \operatorname{sgn}(\alpha_r)|^2\mathrm{d}r\right] + R_n = x + E\left[\int_t^T (1+|\alpha_r|)^2\mathrm{d}r\right] + R_n, \end{aligned} \tag{5.38}$$

where $R_n = \int_t^{t+|\pi_n|} E[|\alpha_r|^2]\mathrm{d}r - \int_t^{t+|\pi_n|} E[|\alpha_r + \operatorname{sgn}(\alpha_{r-|\pi_n|})|^2]\mathrm{d}r + E[\int_t^T (|\alpha_r + \operatorname{sgn}(\alpha_{r-|\pi_n|})|^2 - |\alpha_r + \operatorname{sgn}(\alpha_r)|^2)\mathrm{d}r] \le |\pi_n| + 4|\pi_n| + 4E[\int_t^T |\operatorname{sgn}(\alpha_r) - \operatorname{sgn}(\alpha_{r-|\pi_n|})|\mathrm{d}r]$. Using the well-known result that $\lim_{\varepsilon\to 0}\int_0^T |u_s - u_{s-\varepsilon}|^2 \mathrm{d}s = 0$, for all $u \in L^2([0,T])$, we see that $R_n \to 0$, as $n \to \infty$. Thus, with our special choice of $\beta \in \mathcal{B}^{\pi_n}(t,T)$ we have

$$J(t,x,\alpha,\beta) = x + E[\int_t^T (1+|\alpha_r|)^2\mathrm{d}r] + R_n, \ \ R_n \to 0, \text{as } n \to \infty. \tag{5.39}$$

On the other hand, for all $\beta \in \mathcal{B}(t,T) \triangleq \cup_{\pi' \supset \pi}\mathcal{B}^{\pi'}(t,T)$,

$$J(t,x,\alpha,\beta) = x + E\left[\int_t^T |\alpha_r - \beta_r|^2 \mathrm{d}r\right] \le x + E\left[\int_t^T (1+|\alpha_r|)^2\mathrm{d}r\right]. \tag{5.40}$$

From (5.39) and (5.40), we see that $\sup_{\beta\in\mathcal{B}(t,T)} J(t,x,\alpha,\beta) = x + E\left[\int_t^T (1+|\alpha_r|)^2 \mathrm{d}r\right]$. Consequently,

$$\bar{W}^\pi(t,x) = \inf_{\alpha\in\mathcal{A}^\pi(t,T)} \sup_{\beta\in\mathcal{B}(t,T)} E[g(X_T^{t,x,\alpha,\beta})] = x + (T-t) = V(t,x). \tag{5.41}$$

We assume $\pi$ and $\pi_n$ are as before. Changing the roles between $\alpha$ and $\beta$ and considering now $\alpha \in \mathcal{A}^{\pi_n}(t,T)$ and $\beta \in \mathcal{B}^\pi(t,T)$, we choose $\alpha_r = \beta_{r-|\pi_n|} I_{[t+|\pi_n|,T]}$. Then, $J(t,x,\alpha,\beta) = x + E[\int_t^{t+|\pi_n|} |\beta_r|^2\mathrm{d}r] + E[\int_{t+|\pi_n|}^T |\beta_r - \beta_{r-\pi_n}|^2 dr]$, and taking the limit as $n \to \infty$, we see that

$$\inf_{\alpha\in\mathcal{A}(t,T)} E\left[g\left(X_T^{t,x,\alpha,\beta}\right)\right] = x.$$

Consequently,

$$\bar{\bar{V}}^\pi(t,x) = \sup_{\beta\in\mathcal{B}^\pi(t,T)} \inf_{\alpha\in\mathcal{A}(t,T)} E\left[g\left(X_T^{t,x,\alpha,\beta}\right)\right] = x. \tag{5.42}$$

Let now $\varepsilon > 0$. Then, from the definition of $\mathcal{A}(t,T)$ it follows that there is a partition $\pi$ and $\alpha \in \mathcal{A}^\pi(t,T)$, s.t.,

$$\bar{W}^\pi(t,x) + \varepsilon \ge W(t,x) + \varepsilon \ge \sup_{\beta\in\mathcal{B}(t,T)} E\left[g\left(X_T^{t,x,\alpha,\beta}\right)\right] \ge \bar{W}^\pi(t,x) = x + (T-t). \tag{5.43}$$

Hence, $W(t,x) = \inf_{\alpha\in\mathcal{A}(t,T)} \sup_{\beta\in\mathcal{B}(t,T)} E[g(X_T^{t,x,\alpha,\beta})] = x + (T-t)$. Asymmetric argument shows

$$V(t,x) = \sup_{\beta\in\mathcal{B}(t,T)} \inf_{\alpha\in\mathcal{A}(t,T)} E\left[g\left(X_T^{t,x,\alpha,\beta}\right)\right] = x. \tag{5.44}$$

This prove that the upper value function $W(t,x)$ is not necessarily equal to the lower value function $V(t,x)$ if we do not consider the conditions (5.6) and (5.7), or equivalently, Isaacs condition.

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