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INVESTMENT AND CONSUMPTION PROBLEM IN FINITE TIME WITH CONSUMPTION CONSTRAINT *

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Abstract. In this paper, we consider an investment-consumption problem where the consumption is subject to an upper limit. This upper limit on consumption may reflect the following fact. Investors may have to finance their consumption first by using credits then pay the balance by cashing out part of their portfolio in the stock market. Credit companies set up an upper limit for the credit, thus imposing an upper bound for consumption. We also set up our model in finite horizon, which makes the problem much harder due to the loss of stationary when $T < \infty$. We prove that the above described problem is equivalent to a free boundary problem of nonlinear parabolic equations. We aim to characterize explicitly the free boundary by applying a dual transformation technique to convert the original nonlinear parabolic equation to a linear differential equation. This trick allows us to characterize explicitly the free boundary and the optimal consumption strategy. We also prove that the regularity of the value function, which is critical for the application of Ito formula.

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1. Introduction and model formulation

We consider a standard Black−Scholes financial market with two assets: a bond and a stock. The price of the bond is driven by an ordinary differential equation

$$dP_t = rP_t \mathrm{d}t,$$

where r is the risk-free interest rate. The price of the stock is driven by a stochastic differential equation:

$$dS_t = \alpha S_t \mathrm{d}t + \sigma S_t \mathrm{d}W_t,$$

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where α is the mean return rate of the stock, σ is the volatility of the stock, and W_t is a standard one-dimensional Brownian motion on a given complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$. We denote by $\{\mathcal{F}_t = \sigma(W_u, u \leqslant t), 0 \leq t \leq T\}$ the filtration generated by the Brownian motion. The interest rate r, the mean rate of return α, and the volatility σ are assumed to be constants with $r > 0$, $\sigma > 0$, and $\mu := \alpha - r > 0$. There are no transaction fees or taxes and shorting is also allowed in the market.

Let us consider a small investor in the market. The investor's trading will not affect the market prices of the two assets. His trading strategy is self-financing meaning that there is no incoming or outgoing cash flow during the whole ivestment period. Then it is well−known that the wealth process of the investor is driven by an SDE:

$$\begin{cases} \mathrm{d}X_s = (rX_s + \pi_s\mu - c_s)\mathrm{d}s + \pi_s\sigma\mathrm{d}W_s, \quad t < s \leq T, \\ X_t = x, \end{cases} \tag{1.1}$$

where $x > 0$ is the initial endowment of the investor at time t, π_t is the amount of money invested in the stock, $c_t \geqslant 0$ is the consumption rate at time t. An admissible strategy (π_t, c_t) should be progressively measurable processes with constraints

$$E\left[\int_0^T (\pi_t^2 + c_t)\mathrm{d}t\right] < \infty, \quad \forall T > 0.$$

In this paper, we assume that bankruptcy is prohibited, that is

$$X_s \geqslant 0, \quad \forall\, 0 \leq s \leq T, \tag{1.2}$$

almost surely (a.s.). The target of the investor is to choose the best investment-consumption strategy $(\pi(\cdot), c(\cdot))$, which is subject to certain constraints specified below, to maximize the total expected (discounted) utility from consumption over an finite trading horizon

$$\text{maximize } \mathbf{E}\left[\int_t^T \mathrm{e}^{-\beta(s-t)}U(c_s)\mathrm{d}s + \mathrm{e}^{-\beta(T-t)}U(X_T)\right], \tag{1.3}$$

where $U : \mathbb{R}^+ \mapsto \mathbb{R}^+$ is the utility function of the investor, which is strictly increasing, and $\beta > 0$ is a constant discounting factor. In this paper, we consider risk-averse investor only, this is equivalent to say $U(\cdot)$ is concave.

If no constraint on the consumption rate and investment strategy exists, problem (1.3) becomes the classical Merton's investment-consumption problem [4, 5]. However, in practice, constraint on the consumption rate always exists; for example, the consumption rate cannot be too low because an investor has basic needs, which are the minimal amount of resources necessary required for long-term physical well-being; this is the so-called the subsistence consumption requirement [1, 2, 8–10]. Those five references are all in infinite time horizon case with constant lower bound for consumption.

Another practical example is when the manager of a fund requests a fixed salary and a proportion of the managed wealth as a bonus. Of course, most of the wealth still belongs to the owner, and consequently, the manager cannot take excessive amounts from the total wealth. These scenarios motivated us to consider an upper constraint on the consumption rate. In [11] it is assumed that the consumption rate is upper bounded by a time-invariant linear function of the wealth, $kX_t + \ell$, in infinite time horizon.

In this paper, specifically, we assume that the consumption rate is bounded from above by a positive constant ℓ:

$$0 \leqslant c_s \leqslant \ell. \tag{1.4}$$

We justify our assumption based on the following reasoning. Investors may not be able to finance their consumption through the investment account directly. But instead consumption has to be financed by a debt card or credit card issued by some credit companies which set up an credit limit or withdraw limit at each time t. Thus consumption at each time is subject to an upper bound ℓ set up by the credit companies, to make things simple, we assume the investor pay off his credit balance instantaneously (within the grace period) by cashing out part of their portfolio. We also set up our model in finite horizon. It should be noted that finite horizon set up makes the investment-consumption problem nonstationary, which tends to be much harder to handle due to the explicit dependence of the value function on time t.

Denote the value function by

$$V(x,t) := \sup_{(\pi(\cdot),c(\cdot))} \mathbf{E}\left[\int_t^T \mathrm{e}^{-\beta(s-t)}U(c_s)\mathrm{d}s + \mathrm{e}^{-\beta(T-t)}U(X_T)\right]. \tag{1.5}$$

where the investment-consumption strategy $(\pi(\cdot), c(\cdot))$ is subject to the constraints (1.2), and (1.4).

In this paper, we focus on the constant relative risk aversion (CRRA) type utility function

$$U(c) = \frac{c^p}{p}, \quad c \geqslant 0, \quad 0 < p < 1. \tag{1.6}$$

In next section we derive an explicit solution of value function for the unconstraint case. In Section 3, we introduce a dual transformation of the value function and reduce the nonlinear parabolic HJB equation to a linear one. This allows us to obtain an explicit formula for the free boundary in Section 4. Section 5 proves the strict concavity of the value function. Section 6 looks at the monotonicity of the free boundary. Section 7 proves the regularity property of the value function, it will be proved that the value function is a classical solution of the HJB equation, which is critical for the validity of Ito formula for proving verification theorem. Section 8 presents some financial interpretations and concludes the paper.

2. Unconstraint case

We first solve a benchmark case without consumption constraint. As we shall see the solution in this case will be useful in presenting the solution in the constrained case.

If there are no constraints on the consumption, problem (1.5) becomes the classical Merton's investment-consumption problem ([4, 5]). This situation is equivalent to the case of $\ell = +\infty$, the HJB equation of problem (1.5) is

$$V_t^\infty = \beta V^\infty + \frac{\mu^2}{2\sigma^2}\frac{(V_x^\infty)^2}{V_{xx}^\infty} - rxV_x^\infty + \frac{p-1}{p}(V_x^\infty)^{\frac{p}{p-1}}, \quad x>0,\ 0<t<T \tag{2.1}$$

$$V^\infty(0,t) = 0, \tag{2.2}$$

$$V^\infty(x,T) = \frac{1}{p}\,x^p, \quad x>0, \tag{2.3}$$

Let us assume the value function takes the following functional form:

$$V^\infty(x,t) = \frac{1}{p}\,x^p f(t),$$

which satisfies (2.2). Substituting it into (2.1) and (2.3), we obtain an ODE for $f(t)$:

$$f'(t) = [\beta - p(\theta + r)]f(t) + (p-1)[f(t)]^{\frac{p}{p-1}}, \quad 0<t<T \tag{2.4}$$

$$f(T) = 1, \tag{2.5}$$

where

$$\theta := \frac{\mu^2}{2\sigma^2(1-p)} > 0. \tag{2.6}$$

(2.4) is the well known Bernoulli equation, the solution of (2.4) and (2.5) is given by

$$f(t) = \left[\frac{1+(\lambda-1)\mathrm{e}^{\lambda(t-T)}}{\lambda}\right]^{1-p},$$

where

$$\lambda := \frac{\beta - p(\theta + r)}{1-p}\cdot \tag{2.7}$$

It follows that

$$\pi^*(x,t) = -\frac{\mu}{\sigma^2}\frac{V_x^\infty(x,t)}{V_{xx}^\infty(x,t)} = \frac{\mu x}{\sigma^2(1-p)}, \tag{2.8}$$

$$c^*(x,t) = [V_x^\infty(x,t)]^{\frac{1}{p-1}} = \frac{\lambda x}{1+(\lambda-1)\mathrm{e}^{\lambda(t-T)}}\cdot \tag{2.9}$$

Theorem 2.1: *If $\lambda > 0$ and there is no constraint on the consumption rate,* i.e., *$\ell = +\infty$, then the optimal investment-consumption strategy for problem* (1.5) *is given by*

$$(\pi_t^*, c_t^*) = \left(\frac{\mu}{\sigma^2(1-p)}X_t,\ \frac{\lambda}{1+(\lambda-1)\mathrm{e}^{\lambda(t-T)}}X_t\right),$$

and the optimal value

$$V^\infty(x,t) = \frac{1}{p}x^p\left[\frac{1+(\lambda-1)\mathrm{e}^{\lambda(t-T)}}{\lambda}\right]^{1-p}\cdot \tag{2.10}$$

The optimal value in (2.10) will be shown to serve as an upper bound for the optimal value in the constrained case.

3. HJB equation and dual transformation

Now we turn our attention to the finite horizon constrained case. We start with proving some basic properties of the value function. The following result says the value function in the unconstrained case serves as an upper bound for the constrained case.

Proposition 3.1. *If $\lambda > 0$, then the value function $V(x,t)$ of problem* (1.5) *satisfies*

$$V(x,t) \leqslant \frac{1}{p}x^p\left[\frac{1+(\lambda-1)\mathrm{e}^{\lambda(t-T)}}{\lambda}\right]^{1-p}, \quad x > 0. \tag{3.1}$$

Moreover, $V(\cdot,t)$ is continuous, increasing, and concave on $x \in [0,+\infty)$ with $V(0,t) = 0$.

Proof. Both the set of admissible controls and the optimal value of problem (1.5) are increasing in ℓ and consequently, an upper bound of the optimal value is given by the scenario $\ell = +\infty$. So the inequality (3.1) follows from (2.10).

If the initial endowment of problem (1.5) is 0 at time t, then the unique admissible investment-consumption strategy is $(\pi(\cdot), c(\cdot)) \equiv (0,0)$, so $V(0,t) = 0$ and consequently, $V(\cdot,t)$ is continuous at 0 from (3.1). By the definition of $V(x,t)$ and (1.6), it is not hard to prove its the concavity and monotonicity. The continuity of $V(\cdot,t)$ on $(0,+\infty)$ follows from its finiteness and concavity since the concave function is continuous in the interior of the definition domain. □

Applying Dynamic Programming Principle [7], the value function $V(x,t)$ of problem (1.5) satisfies HJB equation

$$\begin{aligned} V_t &= \beta V - \sup_{\pi}\left(\tfrac{1}{2}\sigma^2\pi^2 V_{xx} + \pi\mu V_x\right) - \sup_{0\leqslant c\leqslant \ell}\left(U(c) - cV_x\right) - rxV_x \\ &= \beta V + \frac{\mu^2}{2\sigma^2}\frac{V_x^2}{V_{xx}} + (c(x,t) - rx)V_x - \frac{c^p(x,t)}{p}, \quad x>0,\ 0<t<T \end{aligned} \tag{3.2}$$

$$V(0,t) = 0, \tag{3.3}$$

$$V(x,T) = \frac{1}{p}\,x^p \tag{3.4}$$

where

$$c(x,t) := \min\left\{(V_x(x,t))^{\frac{1}{p-1}}, \ell\right\}, \quad x>0,\ 0<t<T. \tag{3.5}$$

Note that equation (3.2) is a fully nonlinear parabolic equation. (3.5) shows that

$$V_x(x,t) = \ell^{p-1} \tag{3.6}$$

implicitly determines a free boundary $x = g(t)$. On the left side of the free boundary $x = g(t)$ in the (x,t) plane, consumption $c(x,t) = (V_x(x,t))^{\frac{1}{p-1}}$, and on the right side, $c(x,t) = \ell$. Due to the complexity of the equation (3.2), it is hard to find the properties of the free boundary $x = g(t)$.

Dual transformation can be used to simplify some type ([6,7]), like (3.2), into a linear equation. Now we try to use it to the equation (3.2). Fortunately, we will get the explicit formula of the free boundary $x = g(t)$ in next section.

Define dual transformation of $V(x,t)$ by

$$v(y,t) := \max_{x>0}(V(x,t) - xy), \quad y>0. \tag{3.7}$$

Then $v(\cdot,t)$ is a finite decreasing convex function on $(0,+\infty)$. Suppose $V_x(\cdot,t)$ is strictly decreasing, which is equivalent to strict concavity of $V(\cdot,t)$ (We will confirm this fact in Sect. 5). Making coordinate transformation

$$y = V_x(x,t), \tag{3.8}$$

its inverse function is denoted by

$$x = I(y,t). \tag{3.9}$$

From (3.7),

$$v(y,t) = [V(x,t) - xV_x(x,t)]\Big|_{x=I(y,t)} = V(I(y,t),t) - yI(y,t). \tag{3.10}$$

Differentiating with respect to y and t,

$$v_y(y,t) = V_x(I(y,t),t)I_y(y,t) - yI_y(y,t) - I(y,t) = -I(y,t), \tag{3.11}$$

$$v_{yy}(y,t) = -I'(y,t) = -\frac{1}{V_{xx}(I(y,t),t)}, \tag{3.12}$$

$$v_t(y,t) = V_t(I(y,t),t) + V_x(I(y,t),t)I_t(y,t) - yI_t(y,t) = V_t(I(y,t),t), \tag{3.13}$$

Insert (3.11) into (3.10),

$$V(I(y,t),t) = v(y,t) - yv_y(y,t). \tag{3.14}$$

With the transformation (3.8), and applying (3.9)–(3.14), HJB equation (3.2) becomes

$$v_t = \beta(v - yv_y) - \frac{\mu^2}{2\sigma^2} y^2 v_{yy} + yd(y,t) + ryv_y - \frac{1}{p} d^p(y,t), \ \ y > 0, \ 0 < t < T, \tag{3.15}$$

where

$$d(y,t) := c(x,t) = \min\left\{ y^{\frac{1}{p-1}}, \ \ell \right\}. \tag{3.16}$$

Now we derive the terminal condition for $v(y,T)$. Note that $V_x(x,T) = x^{p-1}$, *i.e.*, $[V_x(x,T)]^{\frac{1}{p-1}} = x$, it follows that $y^{\frac{1}{p-1}} = I(y,T) = -v_y(y,T)$ and by (3.14),

$$v(y,T) = V(I(y,T),T) + yv_y(y,T) = \frac{1}{p} y^{\frac{p}{p-1}} - y^{\frac{p}{p-1}}. \tag{3.17}$$

Combining (3.15), (3.16) and (3.17),

$$-v_t + \beta(v - yv_y) - \frac{\mu^2}{2\sigma^2} y^2 v_{yy} + ryv_y = \begin{cases} \frac{1-p}{p} y^{\frac{p}{p-1}}, & y \geq \ell^{p-1}, \ 0 < t < T, \\ \frac{1}{p}\ell^p - \ell y, & 0 < y < \ell^{p-1}, \ 0 < t < T, \end{cases} \tag{3.18}$$

$$v(y,T) = \frac{1-p}{p} y^{\frac{p}{p-1}}, \quad y > 0. \tag{3.19}$$

(3.18) is a linear parabolic equation on $(0,+\infty) \times (0,T)$.

4. Finding free boundary $x = g(t)$

Recall (3.6), the free boundary $x = g(t)$ is determined by $V_x(g(t),t) = \ell^{p-1}$. Thus for a given t,

$$g(t) = (the \ \ inverse \ \ of \ \ V_x)(\ell^{p-1}, t).$$

From (3.9) and (3.11), for each fixed t,

$$(the \ \ inverse \ \ of \ \ V_x)(y,t) = I(y,t) = -v_y(y,t),$$

it follows that

$$g(t) = -v_y(\ell^{p-1}, t).$$

In the following we will find the explicit formula for $-v_y(\ell^{p-1},t)$. Let $\tau = T - t$, denote

$$u(y,\tau) = -v_y(y,t), \tag{4.1}$$

then

$$g(t) = u(\ell^{p-1}, \tau) = u(\ell^{p-1}, T - t). \tag{4.2}$$

Differentiating (3.18) and (3.19) to y, applying (4.1),

$$u_\tau - \frac{\mu^2}{2\sigma^2} y^2 u_{yy} + \left(r - \beta - \frac{\mu^2}{\sigma^2} \right) yu_y + ru = \begin{cases} y^{\frac{1}{p-1}}, & y \geq \ell^{p-1}, \ 0 < \tau < T, \\ \ell, & 0 < y < \ell^{p-1}, \ 0 < \tau < T. \end{cases} \tag{4.3}$$

$$u(y,0) = y^{\frac{1}{p-1}}, \quad y > 0. \tag{4.4}$$

Equation (4.3) is degenerate on $y=0$. Let $y=\mathrm{e}^z$ and

$$u(y,\tau)=w(z,\tau), \tag{4.5}$$

then $yu_y=w_z,\ y^2u_{yy}=w_{zz}-w_z$, from (4.3) and (4.4), $w(z,\tau)$ satisfies

$$w_\tau-\frac{\mu^2}{2\sigma^2}w_{zz}+\left(r-\beta-\frac{\mu^2}{2\sigma^2}\right)w_z+rw=\begin{cases}\mathrm{e}^{\frac{1}{p-1}z}, & z\ge(p-1)\ln\ell,\ 0<\tau<T,\\ \ell, & z<(p-1)\ln\ell,\ 0<\tau<T,\end{cases} \tag{4.6}$$

$$w(z,0)=\mathrm{e}^{\frac{1}{p-1}z},\qquad z\in I\!R. \tag{4.7}$$

In order to simplify the calculations for $w(z,\tau)$, note that function

$$\overline{w}(z,\tau)=\left[\mathrm{e}^{-\lambda\tau}+\frac{1}{\lambda}(1-\mathrm{e}^{-\lambda\tau})\right]\mathrm{e}^{\frac{1}{p-1}z}$$

(λ was defined in (2.7)) satisfies

$$\overline{w}_\tau-\frac{\mu^2}{2\sigma^2}\overline{w}_{zz}+\left(r-\beta-\frac{\mu^2}{2\sigma^2}\right)\overline{w}_z+r\overline{w}=\mathrm{e}^{\frac{1}{p-1}z},\quad z\in I\!R,\ \tau>0, \tag{4.8}$$

$$\overline{w}(z,0)=\mathrm{e}^{\frac{1}{p-1}z},\qquad z\in I\!R. \tag{4.9}$$

Let

$$\begin{aligned}w(z,\tau)&=\overline{w}(z,\tau)+\tilde{w}(z,\tau)\\&=\left[\mathrm{e}^{-\lambda\tau}+\frac{1}{\lambda}(1-\mathrm{e}^{-\lambda\tau})\right]\mathrm{e}^{\frac{1}{p-1}z}+\tilde{w}(z,\tau),\end{aligned} \tag{4.10}$$

from (4.8) and (4.9), $\tilde{w}(z,\tau)$ satisfies

$$\tilde{w}_\tau-\frac{\mu^2}{2\sigma^2}\tilde{w}_{zz}+\left(r-\beta-\frac{\mu^2}{2\sigma^2}\right)\tilde{w}_z+r\tilde{w}=\begin{cases}0, & z\ge(p-1)\ln\ell,\ 0<\tau<T,\\ \ell-\mathrm{e}^{\frac{1}{p-1}z}, & z<(p-1)\ln\ell,\ 0<\tau<T,\end{cases} \tag{4.11}$$

$$\tilde{w}(z,0)=0,\qquad z\in I\!R. \tag{4.12}$$

The solution to problem (4.11) and (4.12) (see Appendix A) is

$$\tilde{w}(z,\tau)=\ell\int_0^\tau \mathrm{e}^{-r\eta}N\left(\frac{-z+(p-1)\ln\ell+2bk^2\eta}{k\sqrt{2\eta}}\right)\mathrm{d}\eta-\mathrm{e}^{\frac{1}{p-1}z}\int_0^\tau \mathrm{e}^{-\lambda\eta}N\left(\frac{-z+(p-1)\ln\ell+2k^2(b+\frac{1}{1-p})\eta}{k\sqrt{2\eta}}\right)\mathrm{d}\eta, \tag{4.13}$$

where λ was defined in (2.7) and (see (A.4) and (A.8))

$$b=(r-\beta)\frac{\sigma^2}{\mu^2}-\frac{1}{2},\qquad k=\frac{\mu}{\sqrt{2}\,\sigma},\qquad N(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^x \mathrm{e}^{-\frac{\zeta^2}{2}}\mathrm{d}\zeta.$$

It follows from (4.10),

$$\begin{aligned}w(z,\tau)=&\left[\mathrm{e}^{-\lambda\tau}+\frac{1}{\lambda}(1-\mathrm{e}^{-\lambda\tau})\right]\mathrm{e}^{\frac{1}{p-1}z}\\&+\ell\int_0^\tau \mathrm{e}^{-r\eta}N\left(\frac{-z+(p-1)\ln\ell+2bk^2\eta}{k\sqrt{2\eta}}\right)\mathrm{d}\eta\\&-\mathrm{e}^{\frac{1}{p-1}z}\int_0^\tau \mathrm{e}^{-\lambda\eta}N\left(\frac{-z+(p-1)\ln\ell+2k^2(b+\frac{1}{1-p})\eta}{k\sqrt{2\eta}}\right)\mathrm{d}\eta.\end{aligned}$$

Applying (4.5), replacing z by lny,

$$u(y,\tau) = \left[\mathrm{e}^{-\lambda\tau} + \frac{1}{\lambda}(1-\mathrm{e}^{-\lambda\tau})\right] y^{\frac{1}{p-1}} + \ell\int_0^\tau \mathrm{e}^{-r\eta} N\left(\frac{-\ln y + (p-1)\ln\ell + 2bk^2\eta}{k\sqrt{2\eta}}\right)\mathrm{d}\eta$$
$$-y^{\frac{1}{p-1}}\int_0^\tau \mathrm{e}^{-\lambda\eta} N\left(\frac{-\ln y + (p-1)\ln\ell + 2k^2(b+\frac{1}{1-p})\eta}{k\sqrt{2\eta}}\right)\mathrm{d}\eta. \tag{4.14}$$

According to (4.2), we obtain

$$\begin{aligned} g(t) &= u(\ell^{p-1},\tau) \\ &= \ell\left[\mathrm{e}^{-\lambda\tau} + \frac{1}{\lambda}(1-\mathrm{e}^{-\lambda\tau})\right] + \ell\int_0^\tau \mathrm{e}^{-r\eta} N\left(kb\sqrt{2\eta}\right)\mathrm{d}\eta \\ &\quad -\ell\int_0^\tau \mathrm{e}^{-\lambda\eta} N\left(k(b+\frac{1}{1-p})\sqrt{2\eta}\right)\mathrm{d}\eta, \end{aligned} \tag{4.15}$$

where $\tau = T-t$. More precisely, two integrations in (4.15) can be calculated (see Appendix B), and we have

$$\begin{aligned} g(t) &= u(\ell^{p-1},\tau) \\ &= \ell\left[\mathrm{e}^{-\lambda\tau} + \frac{1}{\lambda}(1-\mathrm{e}^{-\lambda\tau})\right] + \ell\frac{1}{r}\left[\frac{1}{2} - \mathrm{e}^{-r\tau}N(kb\sqrt{2\tau}) + \frac{kb}{\sqrt{-a}}\left(N(\sqrt{-2a\tau}) - \frac{1}{2}\right)\right] \\ &\quad -\ell\frac{1}{\lambda}\left[\frac{1}{2} - \mathrm{e}^{-\lambda\tau}N\left(k(b+\frac{1}{1-p})\sqrt{2\tau}\right) + \frac{k(b+\frac{1}{1-p})}{\sqrt{-a}}\left(N(\sqrt{-2a\tau}) - \frac{1}{2}\right)\right]\cdot \end{aligned} \tag{4.16}$$

From (4.2), the main theorem is the following.

Theorem 4.1. *The free boundary, $x = g(t)$, has an explicit formula (4.15) or (4.16), which is proportional to the constraint ℓ and is infinitely differentiable.*

Remark 4.2. It should be noted that the consumption constraint represents credit market frictions. Those frictions in the credit market manifest in the model as a welfare loss. This is seen in the Proposition 3.1 which says, in general, the value function in the constraint case will be lower than the unconstrained case. This welfare loss can also be seen through the region where the constraint becomes binding. As the credit limit ℓ increases, the free boundary $x = g(t)$ shifts to the right (as seen from the explicit formula of $g(t)$); thus reducing the area of the constrained region (right hand side of the free boundary).

5. Confirming strict concavity of $V(x,t)$ in x

Proposition 3.1 says that $V(\cdot,t)$ is concave on $x \in [0,+\infty)$, and transformation (3.8) required that $V(\cdot,t)$ is strict concave on $x \in [0,+\infty)$. Now we confirm this condition. By the use of (3.12), (4.1) and (4.5),

$$V_{xx}(x,t) = -\frac{1}{v_{yy}(y,t)} = \frac{1}{u_y(y,\tau)} = \frac{\mathrm{e}^z}{w_z(z,\tau)}, \tag{5.1}$$

so it is enough to prove $w_z(z,\tau) < 0$. In fact, from (4.10),

$$\begin{aligned} w_z(z,\tau) &= \overline{w}_z(z,\tau) + \tilde{w}_z(z,\tau) \\ &= \frac{1}{p-1}\left[\mathrm{e}^{-\lambda\tau} + \frac{1}{\lambda}(1-\mathrm{e}^{-\lambda\tau})\right]\mathrm{e}^{\frac{1}{p-1}z} + \tilde{w}_z(z,\tau). \end{aligned} \tag{5.2}$$

Now we pay attention to the last term $\tilde{w}_z(z,\tau)$ in (5.2). Differentiating (4.11) and (4.12) to z,

$$(\tilde{w}_z)_\tau - \frac{\mu^2}{2\sigma^2}(\tilde{w}_z)_{zz} + \left(r - \beta - \frac{\mu^2}{2\sigma^2}\right)(\tilde{w}_z)_z + r\tilde{w}_z = \begin{cases} 0, & z \ge (p-1)\ln\ell, \\ \frac{1}{1-p}\mathrm{e}^{\frac{1}{p-1}z}, & z < (p-1)\ln\ell, \end{cases} \tag{5.3}$$

$$\tilde{w}_z(z,0) = 0, \qquad z \in I\!R. \tag{5.4}$$

Corresponding to (A.9), the solution of (5.3) and (5.4)

$$\begin{aligned}\tilde{w}_z(z,\tau) &= \frac{1}{1-p}I_2 \\ &= \frac{1}{1-p}\mathrm{e}^{\frac{1}{p-1}z}\int_0^\tau \mathrm{e}^{-\lambda\eta}N\left(\frac{-z+(p-1)\ln\ell + 2k^2(b+\frac{1}{1-p})\eta}{k\sqrt{2\eta}}\right)\mathrm{d}\eta \\ &\le \frac{1}{1-p}\mathrm{e}^{\frac{1}{p-1}z}\int_0^\tau \mathrm{e}^{-\lambda\eta}\mathrm{d}\eta \\ &= \frac{1}{1-p}\mathrm{e}^{\frac{1}{p-1}z}\,\frac{1}{\lambda}(1-\mathrm{e}^{-\lambda\tau})\end{aligned} \tag{5.5}$$

back to (5.2), we obtain that

$$\begin{aligned}w_z(z,\tau) &\le \frac{1}{p-1}\left[\mathrm{e}^{-\lambda\tau} + \frac{1}{\lambda}(1-\mathrm{e}^{-\lambda\tau})\right]\mathrm{e}^{\frac{1}{p-1}z} + \frac{1}{1-p}\mathrm{e}^{\frac{1}{p-1}z}\,\frac{1}{\lambda}(1-\mathrm{e}^{-\lambda\tau}) \\ &= \mathrm{e}^{-\lambda\tau}\frac{1}{p-1}\mathrm{e}^{\frac{1}{p-1}z} < 0. \qquad (\text{by } p<1)\end{aligned} \tag{5.6}$$

It follows that, by (5.1), $V(\cdot,t)$ is strict concave on $x \in [0,+\infty)$.

6. Monotonicity of the free boundary

Differentiating (4.15) to t yields

$$g'(t) = -u_\tau(\ell^{p-1},\tau) \tag{6.1}$$

$$= \ell(\lambda-1)\mathrm{e}^{-\lambda\tau} - \ell\mathrm{e}^{-r\tau}N(kb\sqrt{2\tau}) + \ell\mathrm{e}^{-\lambda\tau}N\left(k(b+\frac{1}{1-p})\sqrt{2\tau}\right), \tag{6.2}$$

where $\tau = T-t$.

Now we look for a condition under which $g'(t) \le 0$. From (6.2),

$$g'(t) \le \ell(\lambda-1)\mathrm{e}^{-\lambda\tau} + \ell\mathrm{e}^{-\lambda\tau}N\left(k(b+\frac{1}{1-p})\sqrt{2\tau}\right).$$

If $b+\frac{1}{1-p} \le 0$, then $N\left(k(b+\frac{1}{1-p})\sqrt{2\tau}\right) \le \frac{1}{2}$ and

$$g'(t) \le \ell(\lambda-1)\mathrm{e}^{-\lambda\tau} + \frac{1}{2}\ell\mathrm{e}^{-\lambda\tau} = \ell\left(\lambda - \frac{1}{2}\right)\mathrm{e}^{-\lambda\tau}.$$

So if $\lambda \le \frac{1}{2}$, then $g'(t) \le 0$.

In the following we take a different approach to find a condition under which $g'(t) \ge 0$. From (4.5),

$$u_\tau(y,\tau) = w_\tau(z,\tau). \tag{6.3}$$

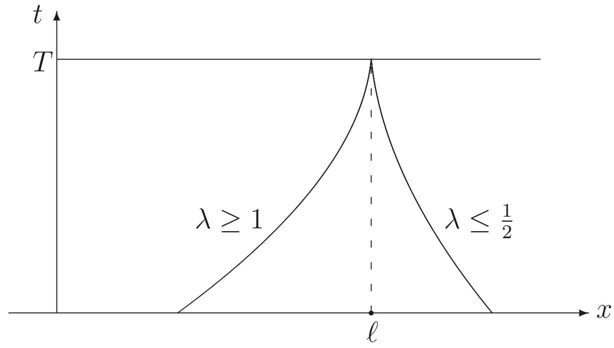

FIGURE 1. Monotonicity of free boundary.

And applying (4.10),

$$w_\tau(z,\tau) = (1-\lambda)\mathrm{e}^{-\lambda\tau}\mathrm{e}^{\frac{1}{p-1}z} + \tilde{w}_\tau(z,\tau). \tag{6.4}$$

Now we prove $\tilde{w}_\tau(z,\tau) \leq 0$. Differentiating (4.11) to τ,

$$(\tilde{w}_\tau)_\tau - \frac{\mu^2}{2\sigma^2}(\tilde{w}_\tau)_{zz} + \left(r - \beta - \frac{\mu^2}{2\sigma^2}\right)(\tilde{w}_\tau)_z + r\tilde{w}_\tau = 0, \quad z \in I\!R,\ 0 < \tau < T. \tag{6.5}$$

let $\tau = 0$ in (4.11) and applying condition (4.12), we have

$$\tilde{w}_\tau(z,0) = \begin{cases} 0, & z \geq (p-1)\ln\ell, \\ \ell - \mathrm{e}^{\frac{1}{p-1}z}, & z < (p-1)\ln\ell. \end{cases}$$

It follows that

$$\tilde{w}_\tau(z,0) \leq 0, \quad z \in I\!R, \tag{6.6}$$

applying maximum principle to the system (6.5) and (6.6), we obtain

$$\tilde{w}_\tau(z,\tau) \leq 0$$

Backing to (6.4), we see that $w_\tau(z,\tau) \leq 0$ if $\lambda \geq 1$. Hence $u_\tau(y,\tau) \leq 0$ by (6.3). Moreover $g'(t) \geq 0$ by (6.1).

Summarizing above results, we have

Theorem 6.1.

(1) *If* $b + \frac{1}{1-p} \leq 0$ *and* $\lambda \leq \frac{1}{2}$, *then the free boundary* $x = g(t)$ *is decreasing* (*Fig.* 1, *the curve in right hand side*),

(2) *If* $\lambda \geq 1$, *then* $x = g(t)$ *is increasing* (*Fig.* 1, *the curve in left hand side*).

7. Regularity of $V(x,t)$

Denote

$$Q = (0,+\infty) \times [0,T]).$$

In this section we prove the value function $V(x,t) \in C^{2,1}(Q) \cap C(\overline{Q})$, *i.e.*, $V(x,t)$ is a classical solution of HJB equation (3.2)–(3.4). This regularity guarantees the applicability of Ito formula which is the key in the proof of verification theorem.

In fact, since the right hand sides of (3.18) is Hölder continuous, applying regularity theory of parabolic equation [3] we see that

$$v(y,t) \in C^{2,1}(Q)$$

which implies that

$$v(y,t),\ v_y(y,t),\ v_{yy}(y,t),\ v_t(y,t) \in C(Q). \tag{7.1}$$

It follows that, by (3.11),

$$I(y,t) = -v_y(y,t) \in C(Q). \tag{7.2}$$

Hence inverse function of $x = I(y,t)$

$$V_x(x,t) \in C(Q). \tag{7.3}$$

Moreover, by (3.14), (3.12) and (3.13),

$$V(x,t) = v(V_x(x,t),t) + xV_x(x,t) \in C(Q), \tag{7.4}$$

$$V_{xx}(x,t) = -\frac{1}{v_{yy}(y,t)} = -\frac{1}{v_{yy}(V_x(x,t),t)} \in C(Q), \tag{7.5}$$

$$V_t(x,t) = v_t(y,t) = v_t(V_x(x,t),t) \in C(Q). \tag{7.6}$$

(7.3)–(7.6) reveal $V(x,t) \in C^{2,1}(Q)$, recalling that Proposition 3.1 says $V(\cdot,t)$ is continuous on $x \in [0,+\infty)$ with $V(0,t) = 0$. Thus the value function $V(x,t)$ is a classical solution of HJB equation (3.2)–(3.4).

8. Optimal strategy and concluding remark

Let $g(t)$ be defined as in (4.15) or (4.16). The optimal investment-consumption strategy (π_t^*, c_t^*) for problem (1.5) is given in a feedback form $(\pi_t^*, c_t^*) = (\pi^*(X_t,t), c^*(X_t,t))$, where

$$\pi^*(x,t) = -\frac{\mu}{\sigma^2}\frac{V_x(x,t)}{V_{xx}(x,t)}, \quad x > 0,$$

$$c^*(x,t) = \begin{cases} V_x(x,t)^{\frac{1}{p-1}}, & 0 < x < g(t); \\ \ell, & x \geqslant g(t), \end{cases}$$

Utilize the dual relationship of (3.7), the optimal value function

$$V(x,t) = \max_{y>0}\,(v(y,t) + xy), \quad x > 0. \tag{8.1}$$

9. Conclusion

The main purpose of this paper is to study finite horizon investment-consumption problem with credit market frictions. We model the credit market frictions by imposing a credit limit on investors. Using dual transformation techniques, we are able to explicitly characterize the free boundary thus characterizing the region in which the credit limit will be binding. This free boundary is shown to be proportional to the credit limit. The area in the right hand side of the free boundary indicates how severe the frictions are in the credit market. We are also able to find out the optimal invest strategies which is negatively proportional to market volatility, proportional to market expected rate of return, Moreover, it is shown to be negatively proportional to absolute risk aversion of the value function. Due to the assumption of finite horizon, this value function depends on time explicitly. Thus the optimal investment strategy will be explicitly time dependent. On the other hand, the optimal consumption strategy is given by the marginal value of wealth (in the region where credit limit is not binding). This also depends on time explicitly (unlike the in.nite horizon case).

Appendix A. The solution of (4.11) and (4.12)

We transfer equation (4.11) into heat equation, and then use Poisson formula. Making transformation

$$\tilde{w}(z,\tau) = \mathrm{e}^{a\tau+bz} W(z,\tau), \tag{A.1}$$

where a, b will be determined later on. Then

$$\begin{aligned}
\tilde{w}_\tau &= \mathrm{e}^{a\tau+bz}(W_\tau + aW),\\
\tilde{w}_z &= \mathrm{e}^{a\tau+bz}(W_z + bW),\\
\tilde{w}_{zz} &= \mathrm{e}^{a\tau+bz}(W_{zz} + 2bW_z + b^2 W),
\end{aligned}$$

from (4.11) and (4.12), $W(z,\tau)$ satisfies

$$W_\tau - \frac{\mu^2}{2\sigma^2} W_{zz} + \left(r - \beta - \frac{\mu^2}{2\sigma^2} - b\frac{\mu^2}{\sigma^2}\right) W_z + \left(r + b(r - \beta - \frac{\mu^2}{2\sigma^2}) - b^2\frac{\mu^2}{2\sigma^2} + a\right) W$$
$$= \begin{cases} 0, & z \geq (p-1)\ln\ell,\ 0 < \tau < T,\\ (\ell - \mathrm{e}^{\frac{1}{p-1}z})\mathrm{e}^{-a\tau-bz}, & z < (p-1)\ln\ell,\ 0 < \tau < T, \end{cases} \tag{A.2}$$

$$W(z,0) = 0, \quad -\infty < z < +\infty. \tag{A.3}$$

let

$$b = (r-\beta)\frac{\sigma^2}{\mu^2} - \frac{1}{2}, \tag{A.4}$$

$$\begin{aligned}
a &= b^2\frac{\mu^2}{2\sigma^2} - b\left(r - \beta - \frac{\mu^2}{2\sigma^2}\right) - r = b^2\frac{\mu^2}{2\sigma^2} - b\cdot\frac{\mu^2}{\sigma^2}b - r\\
&= -b^2\frac{\mu^2}{2\sigma^2} - r.
\end{aligned} \tag{A.5}$$

Thus

$$W_\tau - k^2 W_{zz} = \begin{cases} 0, & z \geq (p-1)\ln\ell,\ 0 < \tau < T,\\ (\ell - \mathrm{e}^{\frac{1}{p-1}z})\mathrm{e}^{-a\tau-bz}, & z < (p-1)\ln\ell,\ 0 < \tau < T, \end{cases} \tag{A.6}$$

$$W(z,0) = 0, \quad -\infty < z < +\infty. \tag{A.7}$$

where

$$k^2 = \frac{\mu^2}{2\sigma^2}. \tag{A.8}$$

Applying Poisson formula of heat equation to (A.6) and (A.7),

$$W(z,\tau) = \int_0^\tau \mathrm{d}\eta \int_{-\infty}^{(p-1)\ln\ell} \frac{1}{2k\sqrt{\pi(\tau-\eta)}} \mathrm{e}^{-\frac{(z-\xi)^2}{4k^2(\tau-\eta)}} \left(\ell - \mathrm{e}^{\frac{1}{p-1}\xi}\right) \mathrm{e}^{-a\eta-b\xi}\mathrm{d}\xi.$$

Backing to (A.1),

$$\tilde{w}(z,\tau) = \mathrm{e}^{a\tau+bz}W(z,\tau) = I_1 - I_2, \tag{A.9}$$

where

$$\begin{aligned}
I_1 &= \mathrm{e}^{a\tau+bz}\int_0^\tau \mathrm{d}\eta \int_{-\infty}^{(p-1)\ln\ell} \frac{1}{2k\sqrt{\pi(\tau-\eta)}}\mathrm{e}^{-\frac{(z-\xi)^2}{4k^2(\tau-\eta)}}\left(\ell \mathrm{e}^{-a\eta-b\xi}\right)\mathrm{d}\xi \\
&= \ell\int_0^\tau \mathrm{e}^{a(\tau-\eta)}\mathrm{d}\eta \int_{-\infty}^{(p-1)\ln\ell} \frac{1}{2k\sqrt{\pi(\tau-\eta)}}\mathrm{e}^{-\frac{(z-\xi)^2}{4k^2(\tau-\eta)}}\mathrm{e}^{b(z-\xi)}\mathrm{d}\xi, \\
I_2 &= \mathrm{e}^{a\tau+bz}\int_0^\tau \mathrm{d}\eta \int_{-\infty}^{(p-1)\ln\ell} \frac{1}{2k\sqrt{\pi(\tau-\eta)}}\mathrm{e}^{-\frac{(z-\xi)^2}{4k^2(\tau-\eta)}}\left(\mathrm{e}^{\frac{1}{p-1}\xi}\mathrm{e}^{-a\eta-b\xi}\right)\mathrm{d}\xi \\
&= \mathrm{e}^{\frac{1}{p-1}z}\int_0^\tau \mathrm{e}^{a(\tau-\eta)}\mathrm{d}\eta \int_{-\infty}^{(p-1)\ln\ell} \frac{1}{2k\sqrt{\pi(\tau-\eta)}}\mathrm{e}^{-\frac{(z-\xi)^2}{4k^2(\tau-\eta)}}\mathrm{e}^{\frac{1}{1-p}(z-\xi)}\mathrm{e}^{b(z-\xi)}\mathrm{d}\xi.
\end{aligned}$$

We calculate I_1 first. Note that

$$\begin{aligned}
-\frac{(z-\xi)^2}{4k^2(\tau-\eta)} + b(z-\xi) &= -\frac{1}{4k^2(\tau-\eta)}\left[(z-\xi)^2 - 4bk^2(\tau-\eta)(z-\xi)\right] \\
&= -\frac{1}{4k^2(\tau-\eta)}\left\{\left[z-\xi-2bk^2(\tau-\eta)\right]^2 - 4b^2k^4(\tau-\eta)^2\right\} \\
&= -\frac{1}{4k^2(\tau-\eta)}\left[z-\xi-2bk^2(\tau-\eta)\right]^2 + b^2k^2(\tau-\eta),
\end{aligned} \tag{A.10}$$

so let

$$\zeta = \frac{-z+\xi+2bk^2(\tau-\eta)}{k\sqrt{2(\tau-\eta)}},$$

then

$$\begin{aligned}
I_1 &= \ell\int_0^\tau \mathrm{e}^{a(\tau-\eta)}\mathrm{d}\eta\,\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\frac{-z+(p-1)\ln\ell+2bk^2(\tau-\eta)}{k\sqrt{2(\tau-\eta)}}} \mathrm{e}^{-\frac{\zeta^2}{2}}\mathrm{e}^{b^2k^2(\tau-\eta)}\mathrm{d}\zeta \\
&= \ell\int_0^\tau \mathrm{e}^{a(\tau-\eta)}\mathrm{e}^{b^2k^2(\tau-\eta)}N\left(\frac{-z+(p-1)\ln\ell+2bk^2(\tau-\eta)}{k\sqrt{2(\tau-\eta)}}\right)\mathrm{d}\eta \\
&= \ell\int_0^\tau \mathrm{e}^{-r(\tau-\eta)}N\left(\frac{-z+(p-1)\ln\ell+2bk^2(\tau-\eta)}{k\sqrt{2(\tau-\eta)}}\right)\mathrm{d}\eta \qquad \text{(by (A.5))} \\
&= \ell\int_0^\tau \mathrm{e}^{-r\eta}N\left(\frac{-z+(p-1)\ln\ell+2bk^2\eta}{k\sqrt{2\eta}}\right)\mathrm{d}\eta,
\end{aligned} \tag{A.11}$$

Now we calculate I_2. Note that, similar to (A.10),

$$-\frac{(z-\xi)^2}{4k^2(\tau-\eta)} + \left(b+\frac{1}{1-p}\right)(z-\xi) = -\frac{1}{4k^2(\tau-\eta)}\left[z-\xi-2k^2\left(b+\frac{1}{1-p}\right)(\tau-\eta)\right]^2 + k^2\left(b+\frac{1}{1-p}\right)^2(\tau-\eta),$$

so let

$$\zeta = \frac{-z+\xi+2k^2(b+\frac{1}{1-p})(\tau-\eta)}{k\sqrt{2(\tau-\eta)}},$$

then

$$\begin{aligned}
I_2 &= \mathrm{e}^{\frac{1}{p-1}z}\int_0^\tau \mathrm{e}^{a(\tau-\eta)}\mathrm{d}\eta\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\frac{-z+(p-1)\ln\ell+2k^2(b+\frac{1}{1-p})(\tau-\eta)}{k\sqrt{2(\tau-\eta)}}}\mathrm{e}^{-\frac{\zeta^2}{2}}\mathrm{e}^{k^2(b+\frac{1}{1-p})^2(\tau-\eta)}\mathrm{d}\zeta\\
&= \mathrm{e}^{\frac{1}{p-1}z}\int_0^\tau \mathrm{e}^{a(\tau-\eta)}\mathrm{e}^{k^2(b+\frac{1}{1-p})^2(\tau-\eta)}N\left(\frac{-z+(p-1)\ln\ell+2k^2(b+\frac{1}{1-p})(\tau-\eta)}{k\sqrt{2(\tau-\eta)}}\right)\mathrm{d}\eta\\
&= \mathrm{e}^{\frac{1}{p-1}z}\int_0^\tau \mathrm{e}^{-\lambda(\tau-\eta)}N\left(\frac{-z+(p-1)\ln\ell+2k^2(b+\frac{1}{1-p})(\tau-\eta)}{k\sqrt{2(\tau-\eta)}}\right)\mathrm{d}\eta \qquad (\textit{by (A.13)})\\
&= \mathrm{e}^{\frac{1}{p-1}z}\int_0^\tau \mathrm{e}^{-\lambda\eta}N\left(\frac{-z+(p-1)\ln\ell+2k^2(b+\frac{1}{1-p})\eta}{k\sqrt{2\eta}}\right)\mathrm{d}\eta, \qquad \text{(A.12)}
\end{aligned}$$

where we used relationship

$$\begin{aligned}
a+k^2\left(b+\frac{1}{1-p}\right)^2 &= a+k^2\left(b^2+\frac{2b}{1-p}+\frac{1}{(1-p)^2}\right)\\
&= -r+k^2\left(\frac{2b}{1-p}+\frac{1}{(1-p)^2}\right)\\
&= -r+(1-p)\theta\left(\frac{\frac{r-\beta}{\theta(1-p)}-1}{1-p}+\frac{1}{(1-p)^2}\right)\\
&= -r+\frac{r-\beta}{1-p}-\theta+\frac{\theta}{1-p}\\
&= \frac{pr-\beta+p\theta}{1-p}=-\lambda. \qquad \text{(A.13)}
\end{aligned}$$

Hence (4.13) follows from (A.9), (A.11) and (A.12).

Appendix B. Calculating two integrations in (4.15)

We use integration by parts. The first integration in (4.15)

$$\begin{aligned}
&\int_0^\tau \mathrm{e}^{-r\eta}N\left(kb\sqrt{2\eta}\right)\mathrm{d}\eta = -\frac{1}{r}\int_0^\tau N(kb\sqrt{2\eta})\mathrm{d}\mathrm{e}^{-r\eta}\\
&= -\frac{1}{r}\left[\mathrm{e}^{-r\eta}N(kb\sqrt{2\eta})\Big|_0^\tau - \int_0^\tau \mathrm{e}^{-r\eta}\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-k^2b^2\eta}\frac{kb}{\sqrt{2\eta}}\mathrm{d}\eta\right]\\
&= -\frac{1}{r}\left[\mathrm{e}^{-r\tau}N(kb\sqrt{2\tau})-\frac{1}{2}-\frac{kb}{2\sqrt{\pi}}\int_0^\tau \mathrm{e}^{a\eta}\frac{1}{\sqrt{\eta}}\mathrm{d}\eta\right]\\
&= -\frac{1}{r}\left[\mathrm{e}^{-r\tau}N(kb\sqrt{2\tau})-\frac{1}{2}-\frac{kb}{2\sqrt{\pi}}\int_0^{\sqrt{-2a\tau}} \mathrm{e}^{-\frac{\mu^2}{2}}\sqrt{\frac{2}{-a}}\mathrm{d}\mu\right] \qquad (\mu=\sqrt{-2a\eta})\\
&= -\frac{1}{r}\left[\mathrm{e}^{-r\tau}N(kb\sqrt{2\tau})-\frac{1}{2}-\frac{kb}{\sqrt{-a}}\frac{1}{\sqrt{2\pi}}\int_0^{\sqrt{-2a\tau}} \mathrm{e}^{-\frac{\mu^2}{2}}\mathrm{d}\mu\right]\\
&= -\frac{1}{r}\left[\mathrm{e}^{-r\tau}N(kb\sqrt{2\tau})-\frac{1}{2}-\frac{kb}{\sqrt{-a}}\left(N(\sqrt{-2a\tau})-\frac{1}{2}\right)\right]\\
&= \frac{1}{r}\left[\frac{1}{2}-\mathrm{e}^{-r\tau}N(kb\sqrt{2\tau})+\frac{kb}{\sqrt{-a}}\left(N(\sqrt{-2a\tau})-\frac{1}{2}\right)\right].
\end{aligned}$$

In a same way, the second integration in (4.15)

$$\int_0^\tau \mathrm{e}^{-\lambda\eta} N\left(k(b+\frac{1}{1-p})\sqrt{2\eta}\right)\mathrm{d}\eta = \frac{1}{\lambda}\left[\frac{1}{2} - \mathrm{e}^{-\lambda\tau} N\left(k(b+\frac{1}{1-p})\sqrt{2\tau}\right) + \frac{k(b+\frac{1}{1-p})}{\sqrt{-a}}\left(N(\sqrt{-2a\tau}) - \frac{1}{2}\right)\right].$$

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