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MULTIPLICITY AND CONCENTRATION OF POSITIVE SOLUTIONS FOR THE FRACTIONAL SCHRÖDINGER–POISSON SYSTEMS WITH CRITICAL GROWTH

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Abstract. In this paper, we study the multiplicity and concentration of solutions for the following critical fractional Schrödinger–Poisson system:

$$\begin{cases} \epsilon^{2s}(-\triangle)^s u + V(x)u + \phi u = f(u) + |u|^{2_s^*-2}u & \text{in } \mathbb{R}^3, \\ \epsilon^{2t}(-\triangle)^t \phi = u^2 & \text{in } \mathbb{R}^3, \end{cases}$$

where $\epsilon > 0$ is a small parameter, $(-\triangle)^\alpha$ denotes the fractional Laplacian of order $\alpha = s, t \in (0,1)$, where $2_\alpha^* = \frac{6}{3-2\alpha}$ is the fractional critical exponent in Dimension 3; $V \in C^1(\mathbb{R}^3, \mathbb{R}^+)$ and f is subcritical. We first prove that for $\epsilon > 0$ sufficiently small, the system has a positive ground state solution. With minimax theorems and Ljusternik–Schnirelmann theory, we investigate the relation between the number of positive solutions and the topology of the set where V attains its minimum for small ϵ. Moreover, each positive solution u_ϵ converges to the least energy solution of the associated limit problem and concentrates around a global minimum point of V.

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1. Introduction

In this paper, we are concerned with the following fractional nonlinear Schrödinger–Poisson system with critical growth

$$\begin{cases} \epsilon^{2s}(-\triangle)^s u + V(x)u + \phi u = f(u) + |u|^{2_s^*-2}u & \text{in } \mathbb{R}^3, \\ \epsilon^{2t}(-\triangle)^t \phi = u^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{1.1}$$

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where $\epsilon > 0$ is a small parameter, $(-\triangle)^\alpha$ denotes the fractional Laplacian of order $\alpha = s, t \in (0,1)$, $2^*_\alpha = \frac{6}{3-2\alpha}$ is the fractional critical exponent in Dimension 3. For the potential V, we impose the following assumption:

(V) $V \in C^1(\mathbb{R}^3, \mathbb{R}^+)$ and $V_\infty := \liminf_{|x|\to\infty} V(x) > V_0 = \inf_{x\in\mathbb{R}^3} V(x) > 0$,

which is firstly introduced by Rabinowitz [32] in the study of the nonlinear Schrödinger equations. The nonlinearity $f : \mathbb{R} \to \mathbb{R}$ is of C^1 class and satisfies the following conditions:

(f_1) $f(u) = 0$ for all $u < 0$ and $f(u) = o(u^3)$ as $u \to 0^+$;
(f_2) there exists $4 < q < 2^*_s = \frac{6}{3-2s}$ such that

$$\lim_{u\to\infty} \frac{f(u)}{u^{q-1}} = 0,$$

where $s \in (\frac{3}{4}, 1)$;
(f_3) the function $u \to \frac{f(u)}{u^3}$ is increasing in $(0, \infty)$;
(f_4) $f(u) \geq \rho u^\sigma$ for all $u > 0$ with some $\rho > 0$ and $\sigma \in (3, q-1)$.

Obviously, it follows from (f_1)–(f_3) that

$$0 \leq 3f(u) \leq f'(u)u,\ 0 \leq 4F(u) \leq f(u)u, \quad \forall u \in \mathbb{R},$$

where $F(u) = \int_0^u f(s)\mathrm{d}s$.

Remark 1.1. Let $p \in (4, 2^*_s)$ and $f(u) = u^{p-1}$ when $u \in [0, +\infty)$ and $f(u) \equiv 0$ when $u \in (-\infty, 0)$, then it is easy to check that f satisfies the above conditions (f_1)–(f_4). In view of (f_2), we have $4 < \frac{6}{3-2s}$, which implies that $s \in (\frac{3}{4}, 1)$.

We remark that, if $\phi \equiv 0$ then (1.1) reduces to the fractional Schrödinger equation of the form

$$(-\epsilon^2\triangle)^s u + V(x)u = f(x,u), \quad x \in \mathbb{R}^N. \tag{1.2}$$

Solutions of equations like (1.2) are related to the existence of standing wave solutions for the following fractional Schrödinger equations

$$i\epsilon\frac{\partial\psi}{\partial h} = (-\epsilon^2\triangle)^s\psi + V(x)\psi - f(x, |\psi|), \quad x \in \mathbb{R}^N, \tag{1.3}$$

where the standing wave solutions have the form

$$\psi(x,h) = u(x)e^{-\frac{iEh}{\epsilon}} \quad x \in \mathbb{R}^N, h \in \mathbb{R},$$

where E is a constant, $u(x)$ is a solution of (1.2). The fractional Schrödinger equation was introduced by Laskin [28] and arises in fractional quantum mechanics in the study of particles on stochastic fields modeled by Lévy processes. The operator $(-\triangle)^\alpha$ can be seen as the infinitesimal generators of Lévy stable diffusion processes [1]. Recently, many mathematical investigations on problem (1.2) have been devoted to the case where either $\epsilon \equiv 1$ or V and f satisfy some various conditions. See for example [12, 13, 22, 23, 28, 37]. In particular, when f is of subcritical growth and satisfies the following conditions:

$$0 < \mu F(u) \leq uf(u), \qquad |f'(u)| \leq a_1 + a_2|u|^{p-1}$$

and the map $u \mapsto u^{-1}hf(uh)$ is increasing on $(0,\infty)$, $h \in \mathbb{R}$, where $a_1, a_2 > 0$, $p \in (1, 2^*_s)$, and $\mu > 2$, Secchi [36] made use of minimax methods to prove the existence of nontrivial solutions for (1.2) with $f(x,u) = f(u)$. By using the Lyapunov–Schmidt reduction method, Dávila, Pino and Wei [15] studied the existence and concentration phenomenon of solutions for (1.2) with $f(x,u) = u^p$, $p \in (1, 2^*_s)$ (see also [13,16]). Notice that the arguments of [13,15,16,21] heavily rely on the uniqueness and non-degeneracy of ground state solutions to the corresponding autonomous problems which have been proved in [22,23]. In [42], the concentration phenomenon of ground state solutions was investigated around the non-degenerate critical point of the potential V. Using the penalization method, Alves and Miyagaki [2] also considered problem (1.2) in the subcritical case and obtained a positive solution concentrating around a local minimum point of V as ϵ goes to zero. As for critical problems driven by the fractional Laplace operator $(-\triangle)^{\alpha}$, Shang and Zhang [41] investigated (1.2) with $f(x,u) = \lambda f(u) + |u|^4 u$ and obtained some interesting results, where $\lambda > 0$ is large and $f \in C(\mathbb{R}^3, \mathbb{R})$ satisfies the following conditions

$$\begin{aligned} &f(u) = o(u) \text{ as } u \to 0, \qquad f(u)u > 0 \text{ for } u \neq 0,\ f(u) = 0 \text{ for } u \leq 0, \\ &\frac{f(u)}{u} \text{ is strictly increasing on } (0,\infty), \quad |f(u)| \leq c(1+|u|^{p-1}),\ c>0,\ p \in (2, 2_s^*). \end{aligned} \tag{1.4}$$

As shown in [41], ground state solutions and multiple nonnegative solutions exist for large $\lambda > 0$ and V satisfying (V), and the number of solutions are related to the topology of the set where V attains its minimum. We have to point out here that to overcome the obstacle due to the appearance of the critical nonlinearity term, the parameter $\lambda > 0$ should be large enough in [41]. When $\epsilon = 1$, Servadei and Valdinoci [38] showed that the famous result by Brezis and Nirenberg for the Laplace equations also hold for the nonlocal setting of the following problem:

$$\begin{cases} (-\triangle)^s u - \lambda u = |u|^{2_s^*-2}u, & x \in \Omega, \\ u = 0, & x \in \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1.5}$$

where λ is a positive constant, $\Omega \in \mathbb{R}^N$ is an open bounded domain with a Lipschitz boundary. For more related results, we refer the readers to [38,44] and the references therein.

Observe that, taking formally $s = t = 1$, then system (1.1) reduces to the classical Schrödinger–Poisson system:

$$\begin{cases} -\epsilon^2 \triangle u + V(x)u + \lambda \phi u = f(u) & \text{in } \mathbb{R}^3, \\ -\epsilon^2 \triangle \phi = u^2, & \text{in } \mathbb{R}^3, \end{cases} \tag{1.6}$$

whose existence, nonexistence and multiplicity for both bound states and ground states have already been widely studied since it was introduced in [8]. For instance, we refer the readers to [4–6,29,34] and the references therein. For (1.6) with $f(u) = u^p$ $(1 < p < 5)$ and λ contained in some intervals, Ruiz [34] obtained some general results about the existence, nonexistence of ground state solutions, while Ambrosetti and Ruiz [6] obtained the existence of multiple bound state solutions. In the case where $-u + f(u)$ satisfies Berestycki–Lions' assumptions, Azzollini *et al.* [4] showed that the existence of nontrivial solutions is determined by the parameter λ. Recently, there are also some results on the semiclassical state of system (1.6) when the potential V and nonlinearity f satisfy different conditions. For example, He [25] studied the multiplicity and concentration of positive solutions and proved that positive solutions concentrate around the global minimum of the potential V in the semi-classical limit. For system (1.6) with $f(u)$ replaced by $b(x)f(u)$, Wang *et al.* [45] studied the existence of the least energy solutions, and also investigated the concentration behavior of ground state solutions. He [25] and Wang *et al.* [45] only considered the subcritical case. For the critical case, He and Zou [26] proved that system (1.6) admits a positive ground state solution concentrating around the global minimum of the potential V and also studied the exponential decay of ground state solutions. For more results on the semiclassical states we refer the readers to [3,27,30,33,48] and the references therein.

Very recently, Giammetta [24] cosidered the evolution equation associated with the following system in Dimension one

$$\begin{cases} (-\triangle)u + \lambda \phi u = g(u) & \text{in } \mathbb{R}, \\ (-\triangle)^t \phi = \lambda u^2 & \text{in } \mathbb{R}. \end{cases} \tag{1.7}$$

In this case, the diffusion is fractional only in the Poisson equation and local and global well-posedness of the Cauchy problem associated with the above system were obtained in [24]. Soon, Zhang, Do Ó and Squassina [49] investigated the more general system

$$\begin{cases}(-\triangle)^s u+\lambda\phi u=g(u) \text{ in } \mathbb{R}^3,\\ (-\triangle)^t\phi=\lambda u^2 \qquad\quad \text{ in } \mathbb{R}^3.\end{cases} \tag{1.8}$$

Precisely, in [49] they considered the following subcritical case with Berestycki–Lions conditions which were firstly introduced in [9] and critical case with a general nonlinear term:

(H1) $g\in C(\mathbb{R},\mathbb{R})$ is odd;
(H2) $-\infty<\liminf_{s\to0^+}\frac{g(u)}{u}\le\limsup_{s\to0^+}\frac{G(u)}{u}=-m<0$;
(H3) $-\infty\le\limsup_{u\to0^+}\frac{g(u)}{u^l}\le0$, where $l=\frac{3+2s}{3-2s}$;
(H4) there exists $\zeta>0$ such that $G(\zeta):=\int_0^\zeta g(u)du>0$,

and

$(H2)'$ $\lim\limits_{u\to0}\frac{g(u)}{u}=-a<0$;
$(H3)'$ $\lim\limits_{|s|\to+\infty}\frac{g(u)}{u^{2_s^*-1}}=b>0$;
$(H4')$ there exist $D>0$ and $q\in(2,2_s^*)$ such that $g(u)+au\ge bu^{2_s^*-1}+Du^{q-1}$ for all $u>0$,

respectively. The authors in [49] proved that (1.8) admits a positive radial solution if $\lambda>0$ small enough.

To the best of our knowledge, the existence and concentration behavior of the positive solutions to (1.1) have not ever been studied by variational methods. Motivated by the above facts, the main purpose of this paper is to investigate the multiplicity and concentration of positive solutions to problem (1.1) involving critical growth. Notice that, in [49], the ground state solutions are obtained in the radially symmetric space $H_r^s(\mathbb{R}^3)$, where $H_r^s(\mathbb{R}^3):=\{u\in H^s(\mathbb{R}^3): u \,\text{is}\, \text{radial}\}$, because the embedding $H_r^s(\mathbb{R}^3)\hookrightarrow L^p(\mathbb{R}^3)$ $(2<p<2_s^*)$ is compact. Notice that $H^s(\mathbb{R}^3)$ will be defined in Section 2. In this paper, however, we do not require that the function V is radial in $\mathbb{R}^3$. Therefore, we have to use the standard space $H^s(\mathbb{R}^3)$ to take the place of $H_r^s(\mathbb{R}^3)$. Due to the lack of compactness of the embedding $H^s(\mathbb{R}^3)\hookrightarrow L^p(\mathbb{R}^3)$, $p\in(2,2_s^*]$, some difficulties arise in using the variational methods in a standard way. Thus, some new estimates for (1.1) involving Palais–Smale sequences, which play a crucial role in the variational approach, are needed to be re-established and some more tricks are needed. To describe our main result, we set

$$\Theta:=\{x\in\mathbb{R}^3\,|\,V(x)=V_0\}.$$

In view of the assumption (V), we can easily see that the set Θ is compact. For every $\delta>0$, define $\Theta_\delta:=\{x\in\mathbb{R}^3:\ \text{dist}(x,\Theta)\le\delta\}$. We recall that, if Y is a closed subset of a topological space X, the Ljusternik–Schnirelmann category $cat_X(Y)$ is the least number of closed and contractible sets in X which cover Y. Now we state our main results.

Theorem 1.2. *Assume that* (V) *and* (f$_1$)–(f$_4$) *hold and let* $s\in(\frac34,1)$, *then there exists* $\epsilon_0>0$ *such that, for any* $\epsilon\in(0,\epsilon_0)$, *system* (1.1) *has one positive ground state solution* $(u_\epsilon,\phi_\epsilon)\in H^s(\mathbb{R}^3)\times D^{t,2}(\mathbb{R}^3)$, *where* $D^{t,2}(\mathbb{R}^3)$ *will be defined in Section* 2.

Theorem 1.3. *Assume that* (V) *and* (f$_1$)–(f$_4$) *hold. Let* $s\in(\frac34,1)$, *then for any* $\delta>0$, *there exists* $\epsilon_\delta>0$ *such that, for any* $\epsilon\in(0,\epsilon_\delta)$, *system* (1.1) *has at least* $cat_{\Theta_\delta}(\Theta)$ *positive solutions in* $H^s(\mathbb{R}^3)\times D^{t,2}(\mathbb{R}^3)$. *Moreover, if* $(u_\epsilon,\phi_\epsilon)$ *denotes such a solution and* $\eta_\epsilon\in\mathbb{R}^3$ *is a global maximum point of* u_ϵ, *then*

$$\lim_{\epsilon\to0}V(\eta_\epsilon)=V_0.$$

For such an η_ϵ, $v_\epsilon(x)\equiv u_\epsilon(\epsilon x+\eta_\epsilon)$ *converges to a positive ground state solution of*

$$(-\triangle)^s u+V_0u+\phi_{u,t}u=f(u)+|u|^{2_s^*-1}u,\quad u\in H^s(\mathbb{R}^3),$$

where $\phi_{u,t}$ *will be defined later.*

Remark 1.4.

(1) Indeed, in our arguments it is not hard to see that the positive ground state u_ϵ in Theorem 1.2 has a similar concentration behavior as in Theorem 1.3.
(2) We observe that, in [40, 41] the authors used Ljusternik–Schnirelmann theory to investigate the multiplicity of nonnegative solutions for fractional Schrödinger equations with critical growth. However, in the present paper, we not only obtain the multiplicity of positive solutions for the critical case of the fractional Schrödinger–Poisson system but also investigate the concentration behavior of each positive solution as ϵ goes to zero. Indeed, the arguments of [40,41] are difficult to be used to get positive concentrating solutions.
(3) It is worth pointing out that, in [49], authors studied a class of fractional Schrödinger–Poisson systems with more general critical nonlinearities than that in the present paper, and obtained the existence of positive ground state solutions in the radial symmetric space for $\lambda > 0$ small enough. However, our results still hold for any $\lambda > 0$ even if the term ϕu is replaced by $\lambda\phi u$ in system (1.1).

Remark 1.5. Compared with the classical Schrödinger equations, there are only few references on the concentration phenomena for fractional nonlinear equations, because the different definitions of local and nonlocal operators cause that some techniques developed for the local case can not be adapted immediately to non-local case. In our arguments, we summarize three points as follows to illustrate this fact.

(1) Additivity. The nonlocal operator does not satisfy the following identity:

$$\int_{\mathbb{R}^3} |(-\triangle)^s u|^2 \mathrm{d}x = \int_{\mathbb{R}^3} |(-\triangle)^s u^-|^2 \mathrm{d}x + \int_{\mathbb{R}^3} |(-\triangle)^s u^+|^2 \mathrm{d}x,$$

where $u^- = \min\{u, 0\}$ and $u^+ = \max\{u, 0\}$, which is different from the classical local operator. We need extra arguments based on the comparison of the critical energy value of I_μ with the best fractional critical Sobolev constant in order to obtain the non-negativity of weak solution u (see Prop. 3.4).
(2) L^∞ bound. Integration by parts of the nonlocal operator is different from that of the local operator. So the classical Moser iterative method is very difficult to be used to get L^∞ bound of nonnegative weak solutions for the non-local case. So we borrow the idea from Barrios *et al.* [10] to obtain the boundedness of the weak solutions.
(3) Hölder estimates. In order to get the concentration properties, we need to use some local Hölder estimates of positive solutions u_n together with the regularity of the solutions to obtain decay estimates (that is, $u_n(x) \to 0$ as $n \to \infty$ uniformly for n). However, the classical results on Hölder estimates of Schrödinger equations are not adapted to the fractional case. To overcome this difficulty, together with the regularity, we use a Hölder estimate result of the non-local case developed by Silvestre (see Thm. 5.4 of [39]) to obtain decay estimates of positive solutions u_n. It is a key point in studying concentration properties.

Throughout this paper, $C > 0$ denotes various positive generic constants. We denote by $\|\cdot\|_r$ the L^r -norm and $o(1)$ by the quantity which tends to zero when $n \to \infty$. For any $\rho > 0$ and $z \in \mathbb{R}^3$, $B_\rho(z) := \{x \in \mathbb{R}^3 : |x-z| \leq \rho\}$. The symbol $' \rightharpoonup '$ stands for the weak convergence in $H^s(\mathbb{R}^3)$.

The remainder of this paper is organized as follows.

- In Section 2, some notations and preliminaries are presented.
- In Section 3, we prove the existence of positive ground state solutions to the limit equation associated with (1.1).
- Sections 4 is denoted to proving Theorem 1.2. By virtue of the Mini-max approach (see [43]) and a new compactness lemma, we recover the compactness of (PS) sequence and get the existence of positive ground state solutions.
- In Section 5, we borrow an idea from Wang [46] to obtain the concentration phenomenon(see also [26, 45]), and the proof of multiplicity relies on the standard Ljusternik–Schnirelmann category theory (see [7]).

2. Preliminary

In this section we outline the variational framework for problem (1.1) and give some preliminary lemmas. We first give some useful facts of the fractional order Sobolev spaces (see [17]).

For any $\alpha \in (0,1)$, the fractional Sobolev space $H^\alpha(\mathbb{R}^3)$ is defined by

$$H^\alpha(\mathbb{R}^3) := \left\{ u \in L^2(\mathbb{R}^3) : \frac{|u(x)-u(y)|}{|x-y|^{\frac{3+2\alpha}{2}}} \in L^2(\mathbb{R}^3 \times \mathbb{R}^3) \right\},$$

endowed with the natural norm

$$\|u\|_{H^\alpha} = \left(\int_{\mathbb{R}^3} |u|^2 \mathrm{d}x + \int_{\mathbb{R}^6} \frac{|u(x)-u(y)|^2}{|x-y|^{3+2\alpha}} \mathrm{d}x\mathrm{d}y \right)^{\frac{1}{2}},$$

where the term

$$[u]_\alpha = \left(\int_{\mathbb{R}^6} \frac{|u(x)-u(y)|^2}{|x-y|^{3+2\alpha}} \mathrm{d}x\mathrm{d}y \right)^{\frac{1}{2}}$$

is the so called Gagliardo semi-norm of u. It is well-known that the fractional Laplacian $(-\triangle)^\alpha$ of a function $u : \mathbb{R}^3 \to \mathbb{R}$ is defined by

$$(-\triangle)^\alpha u(x) = \mathcal{F}^{-1}(|\xi|^{2\alpha}(\mathcal{F}u))(x), \quad \forall \xi \in \mathbb{R}^3,$$

where $\mathcal{F}$ is the Fourier transform, *i.e.*,

$$\mathcal{F}(u)(\xi) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} \exp(-2\pi i \xi \cdot x) u(x) \mathrm{d}x,$$

i is the image unit. If u is smooth enough, it can be computed by the following singular integral

$$(-\triangle)^\alpha u(x) = c_\alpha \mathrm{P.V.} \int_{\mathbb{R}^3} \frac{u(x)-u(y)}{|x-y|^{3+2\alpha}} \mathrm{d}y, \quad x \in \mathbb{R}^3,$$

where c_α is a normalization constant and P.V. stands for the principle value. Now one can get an alternative definition of the fractional Sobolev space $H^\alpha(\mathbb{R}^3)$ *via* the Fourier transform as follows:

$$H^\alpha(\mathbb{R}^3) := \left\{ u \in L^2(\mathbb{R}^3) : \int_{\mathbb{R}^3} |\xi|^{2\alpha} |\hat{u}|^2 \mathrm{d}\xi < \infty \right\},$$

endowed with the norm

$$\|u\|_\alpha = \left(\int_{\mathbb{R}^3} (1+|\xi|^{2\alpha}) |\hat{u}|^2 \mathrm{d}\xi \right)^{\frac{1}{2}},$$

where $\hat{u} \equiv \mathcal{F}(u)$ denotes the Fourier transform of u. It is easy to know that $\|\cdot\|_{H^\alpha}$ is equivalent to $\|\cdot\|_\alpha$.

The homogeneous Sobolev space $D^{\alpha,2}(\mathbb{R}^3)$ is defined by

$$D^{\alpha,2}(\mathbb{R}^3) = \left\{ u \in L^{2^*_\alpha}(\mathbb{R}^3) : |\xi|^\alpha \hat{u} \in L^2(\mathbb{R}^3) \right\},$$

which is the completion of $C_0^\infty(\mathbb{R}^3)$ under the norm

$$\|u\|_{D^{\alpha,2}} := \left(\int_{\mathbb{R}^3} |(-\triangle)^{\alpha/2} u|^2 \mathrm{d}x \right)^{\frac{1}{2}} = \left(\int_{\mathbb{R}^3} |\xi|^{2\alpha} |\hat{u}|^2 \mathrm{d}\xi \right)^{\frac{1}{2}}.$$

Now we introduce the following Sobolev embedding theorems.

Lemma 2.1 (see [35]).
For any $\alpha \in (0,1)$, $H^\alpha(\mathbb{R}^3)$ is continuously embedded into $L^p(\mathbb{R}^3)$ for $p \in [2, 2^*_\alpha]$ and compactly embedded into $L^p_{loc}(\mathbb{R}^3)$ for $p \in [1, 2^*_\alpha)$.

Lemma 2.2 (see [14]).
For any $\alpha \in (0,1)$, $D^{\alpha,2}(\mathbb{R}^3)$ is continuously embedded into $L^{2^*_\alpha}(\mathbb{R}^3)$ and define

$$S_\alpha := \inf_{u \in D^{\alpha,2}(\mathbb{R}^3)\setminus\{0\}} \frac{\int_{\mathbb{R}^3} |(-\triangle)^{\alpha/2} u|^2 \mathrm{d}x}{(\int_{\mathbb{R}^3} u^{2^*_\alpha} \mathrm{d}x)^{2/2^*_\alpha}}.$$

Lemma 2.3 (see [36]).
If $\{u_n\}$ is bounded in $H^s(\mathbb{R}^3)$ with $s \in (0,1)$ and

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^3} \int_{B_r(y)} |u_n|^2 \mathrm{d}x = 0,$$

where $r > 0$, then we have $u_n \to 0$ in $L^\nu(\mathbb{R}^3)$ for $\nu \in (2, 2^*_s)$.

It follows from Lemma 2.1 that $H^s(\mathbb{R}^3)$ is continuously embedded into $L^{\frac{12}{3+2t}}(\mathbb{R}^3)$ if $2t+4s \geq 3$. For $u \in H^s(\mathbb{R}^3)$ and $\epsilon > 0$ fixed, define a linear operator $T_u : D^{t,2}(\mathbb{R}^3) \to \mathbb{R}$ as follow:

$$T_u(v) := \frac{1}{\epsilon^{2t}} \int_{\mathbb{R}^3} u^2 v \mathrm{d}x.$$

So, from Lemma 2.2 we deduce that $T_u(v) \leq \frac{1}{\epsilon^{2t}} \|u\|^2_{12/(3+2t)} \|v\|_{2^*_t} \leq \frac{1}{\epsilon^{2t}} C \|u\|^2_s \|v\|_{D^{t,2}}$, which implies that T_u is well defined and continuous in $D^{t,2}(\mathbb{R}^3)$. It follows from the Lax–Milgram theorem that, for every $u \in H^s(\mathbb{R}^3)$, there exists a unique $\bar{\phi}_{u,t} \in D^{t,2}(\mathbb{R}^3)$ such that $\epsilon^{2t}(-\triangle)^t \bar{\phi}_{u,t} = u^2$. Moreover, for $x \in \mathbb{R}^3$,

$$\bar{\phi}_{u,t} := \frac{1}{\epsilon^{2t}} c_t \int_{\mathbb{R}^3} \frac{u^2(y)}{|x-y|^{3-2t}} \mathrm{d}y := \frac{1}{\epsilon^{2t}} \phi_{u,t}, \tag{2.1}$$

where $c_t := \frac{\Gamma(\frac{3}{2}-2t)}{\pi^{\frac{3}{2}} 2^{2t} \Gamma(t)}$. Notice that formula (2.1) is called as the t-Riesz potential. It follows from (2.1) that (1.1) can be rewritten as the following equivalent form

$$\epsilon^{2s}(-\triangle)^s u + V(x)u + \frac{1}{\epsilon^{2t}} \phi_{u,t} u = f(u) + |u|^{2^*_s - 1} u, \quad u \in H^s(\mathbb{R}^3). \tag{2.2}$$

Now we summarize some properties of $\phi_{u,t}$, which will be used later.

Lemma 2.4. *For any $u \in H^s(\mathbb{R}^3)$ with $s \in (\frac{3}{4}, 1)$, we have*

(1) $\|\phi_{u,t}\|^2_{D^{t,2}} = \int_{\mathbb{R}^3} \phi_{u,t} u^2 \mathrm{d}x \leq C \|u\|^4_{\frac{12}{3+2t}}$;
(2) $\phi_{u,t} \geq 0$, $x \in \mathbb{R}^3$;
(3) *if* $y \in \mathbb{R}^3$ *and* $\bar{u}(x) = u(x+y)$, *then* $\phi_{\bar{u},t}(x) = \phi_{u,t}(x+y)$ *and* $\int_{\mathbb{R}^3} \phi_{\bar{u},t} \bar{u}^2 \mathrm{d}x = \int_{\mathbb{R}^3} \phi_{u,t} u^2 \mathrm{d}x$;
(4) *if* $u_n \rightharpoonup u$ *in* $H^s(\mathbb{R}^3)$, *then* $\phi_{u_n} \rightharpoonup \phi_u$ *in* $D^{t,2}(\mathbb{R}^3)$.
(5) *if* $u_n \rightharpoonup u$ *in* $H^s(\mathbb{R}^3)$, *then* $\int_{\mathbb{R}^3} \phi_{u_n,t} |u_n|^2 \mathrm{d}x = \int_{\mathbb{R}^3} \phi_{(u_n-u),t} (u_n-u)^2 \mathrm{d}x + \int_{\mathbb{R}^3} \phi_{u,t} |u|^2 \mathrm{d}x + o(1)$.

Proof. The proof is similar as those in [34, 47], so we omit the details here. □

The lemma below provides a way to manipulate smooth truncations for the Laplacian see [31].

Lemma 2.5. *Suppose that* $0 < 2\alpha < 3$ *and* $u \in D^{\alpha,2}(\mathbb{R}^3)$. *Let* $\varphi \in C_0^\infty(\mathbb{R}^3)$ *and for each* $r > 0$, $\varphi_r(x) = \varphi(x/r)$. *Then*

$$u\varphi_r \to 0 \quad \text{in } D^{\alpha,2}(\mathbb{R}^3) \text{ as } r \to 0.$$

If, in addition, $\varphi \equiv 1$ *in a neighborhood of the origin, then*

$$u\varphi_r \to u \quad \text{in } D^{\alpha,2}(\mathbb{R}^3) \text{ as } r \to \infty.$$

From now on, assume (V) and (f_1)–(f_4) hold. Making the change of variables $x \mapsto \epsilon x$, we rewrite (2.2) as follows:

$$(-\triangle)^s u + V(\epsilon x)u + \phi_{u,t}u = f(u) + |u|^{2_s^*-1}u, \quad u \in H^s(\mathbb{R}^3). \tag{2.3}$$

Then to study problem (1.1), it suffices to consider problem (2.3). Let H_ϵ be the Hilbert subspace of $H^s(\mathbb{R}^3)$ under the norm

$$\|u\|_\epsilon = \left(\int_{\mathbb{R}^3} |\xi|^{2s}|\hat{u}(\xi)|^2 \mathrm{d}\xi + \int_{\mathbb{R}^3} V(\epsilon x)|u|^2 \mathrm{d}x\right)^{\frac{1}{2}}.$$

Define the energy functional associated with (2.3) by

$$I_\epsilon(u) = \frac{1}{2}\|u\|_\epsilon^2 + \frac{1}{4}\int_{\mathbb{R}^3} \phi_{u,t}u^2 \mathrm{d}x - \int_{\mathbb{R}^3} F(u)\mathrm{d}x - \frac{1}{2_s^*}\int_{\mathbb{R}^3} |u|^{2_s^*}\mathrm{d}x, \tag{2.4}$$

which is of C^1 class and whose derivative is given by

$$I'_\epsilon(u)v = \int_{\mathbb{R}^3} ((-\triangle)^{s/2}u(-\triangle)^{s/2}v + V(\epsilon x)uv + \phi_{u,t}uv)\mathrm{d}x - \int_{\mathbb{R}^3} (f(u) + |u|^{2_s^*-2}u)v\mathrm{d}x \tag{2.5}$$

for all $v \in H^s(\mathbb{R}^3)$. Critical points of I are called as weak solutions of system (2.3).

In the sequel, we need a compactness lemma to handle the difficulty due to the lack of compactness in the embedding $H^s(\mathbb{R}^3) \hookrightarrow L^p(\mathbb{R}^3)$ for $p \in (2, 2_s^*)$.

Lemma 2.6. *If* (V) *holds with* $V_\infty = \infty$, *then* $H_\epsilon \hookrightarrow L^p(\mathbb{R}^3)$ *is compact for* $p \in [2, 2_s^*)$.

Proof. For any fixed $\epsilon > 0$ and assume that $\{u_n\}$ is a bounded sequence in H_ϵ. It follows from Lemma 2.1 that there exists $u_0 \in H_\epsilon$ such that $u_n \rightharpoonup u_0$ in H_ϵ and $u_n \to u_0$ in $L^p_{loc}(\mathbb{R}^3)$ for $p \in [1, 2_s^*)$. We claim that for every $\varepsilon > 0$ there exists $R'_\varepsilon > 0$ such that

$$\int_{|x|\geq R'_\varepsilon} |u_n|^2 \mathrm{d}x \leq \varepsilon. \tag{2.6}$$

Indeed, for any fixed $\varepsilon > 0$, choosing $M > \frac{\|u\|_\epsilon^2}{\varepsilon}$, by (V) there exists $R'_\varepsilon > 0$ such that $V(\epsilon x) \geq M$ for $|x| \geq R'_\varepsilon$. Then we have

$$\int_{|x|\geq R'_\varepsilon} |u_n|^2 \mathrm{d}x \leq \int_{|x|\geq R'_\varepsilon} \frac{V(\epsilon x)}{M}|u_n|^2 \mathrm{d}x \leq \frac{\|u\|_\epsilon^2}{M} < \varepsilon.$$

From $u_0 \in H_\epsilon$, we can find $R_\varepsilon > R'_\varepsilon$ such that $\int_{|x|\geq R_\varepsilon} |u_0|^2 \mathrm{d}x < \varepsilon$. Then

$$\int_{\mathbb{R}^3} |u_n - u_0|^2 \mathrm{d}x = \int_{|x|\geq R_\varepsilon} |u_n - u_0|^2 \mathrm{d}x + \int_{|x|\leq R_\varepsilon} |u_n - u_0|^2 \mathrm{d}x \leq 3\varepsilon,$$

for large n, which implies that $u_n \to u_0$ in $L^2(\mathbb{R}^3)$. Let $v_n := u_n - u_0$. It follows from Lemma 2.3 that $u_n \to u_0$ in $L^p(\mathbb{R}^3)$ for $p \in [2, 2_s^*)$. □

Let

$$\mathcal{N}_\epsilon := \{u \in H_\epsilon \setminus \{0\} : I'_\epsilon(u)u = 0\},$$

then $\mathcal{N}_\epsilon$ is a Nehari manifold associated to I_ϵ. It follows from the Implicit Function Theorem that $\mathcal{N}_\epsilon$ is a C^1-manifold. Moreover, I_ϵ is bounded below on $\mathcal{N}_\epsilon$. So we can consider the following minimization problem:

$$c^*_\epsilon := \inf_{u\in\mathcal{N}_\epsilon} I_\epsilon(u).$$

Recall that I_ϵ satisfies the Palais–Smale condition at level c ($(\mathrm{PS})_c$ for short) if every sequence $\{u_n\} \subset H_\epsilon$ satisfying $I_\epsilon(u_n) \to c$ and $I'_\epsilon(u_n) \to 0$ in H_ϵ^{-1} possesses a convergent subsequence in H_ϵ.

Now we state some properties of c^*_ϵ, I_ϵ and $\mathcal{N}_\epsilon$.

Lemma 2.7. *If the assumptions* (V), (f$_1$)–(f$_4$) *hold, then the following statements hold.*

(i) *If $\{u_n\}$ is a (PS)$_c$ sequence in H_ϵ, then $u_n \rightharpoonup u$ for some $u \in H_\epsilon$ and $I'_\epsilon(u) = 0$.*

(ii) *For every $u \in H_\epsilon \setminus \{0\}$, there is a unique $h_u > 0$ such that $h_u u \in \mathcal{N}_\epsilon$ and*

$$I_\epsilon(h_u u) = \max_{h\geq 0} I_\epsilon(hu).$$

(iii) *For every $u \in \mathcal{N}_\epsilon$, there exists $C > 0$ such that $\|u\|_\epsilon \geq C$.*

(iv) *Let $\{u_n\} \subset H_\epsilon$ satisfying $I'_\epsilon(u_n)u_n \to 0$ and $\int_{\mathbb{R}^3}(f(u_n)u_n + |u_n|^{2^*_s})\mathrm{d}x \to \tau$ as $n \to \infty$, where $\tau > 0$, then up to a subsequence, there exists $h_n > 0$ such that $I'_\epsilon(h_n u_n)h_n u_n = 0$, and $h_n \to 1$ as $n \to \infty$.*

Proof.

(i) From the conclusion (4) of Lemma 2.4 we can easily prove (i).

(ii) Let $u \in H_\epsilon \setminus \{0\}$ be fixed and let $g(h) := I_\epsilon(hu)$ for $h \geq 0$. We observe that $g'(h) = I'_\epsilon(hu)u = 0$ for $h > 0$ if and only if $hu \in \mathcal{N}_\epsilon$. Observe that $g'(h) = 0$ is equivalent to

$$\int_{\mathbb{R}^3} \phi_{u,t}|u|^2\mathrm{d}x = -\frac{\|u\|^2_\epsilon}{h^2} + \int_{\mathbb{R}^3} \frac{f(hu)u^4}{(hu)^3}\mathrm{d}x + h^{2^*_s-4}\int_{\mathbb{R}^3} |u|^{2^*_s}\mathrm{d}x, \tag{2.7}$$

which immediately implies that the right side of (2.7) is an increasing function of $h > 0$ by (f_4). It is easy to see that $g(0) = 0, g(h) > 0$ for $h > 0$ small and $g(h) < 0$ for h large. Hence, there exists a unique $h(u) > 0$ such that $g'(h(u)) = 0$, *i.e.*, $h(u)u \in \mathcal{N}_\epsilon$. Moreover, $I_\epsilon(h(u)u) = \max_{h\geq 0} I_\epsilon(hu)$.

(iii) For any $\varepsilon > 0$, it follows from (f_1) and (f_2) that there exists $C_\varepsilon > 0$ such that

$$|f(u)| \leq \varepsilon|u| + C_\varepsilon|u|^{2^*_s-1}, \quad |F(u)| \leq \frac{\varepsilon}{2}|u|^2 + \frac{C_\varepsilon}{2^*_s}|u|^{2^*_s}. \tag{2.8}$$

Notice that for every $u \in \mathcal{N}_\epsilon$, we have $I'_\epsilon(u)u = 0$. Let $\varepsilon < \frac{1}{2}$, then by Lemma 2.1

$$\begin{aligned}0 &= \|u\|^2_\epsilon + \int_{\mathbb{R}^3} \phi_{u,t}|u|^2\mathrm{d}x - \int_{\mathbb{R}^3} f(u)u\mathrm{d}x - \int_{\mathbb{R}^3} |u|^{2^*_s}\mathrm{d}x\\ &\geq \frac{1}{2}\|u\|^2_\epsilon - C_\varepsilon C\|u\|^{2^*_s}_\epsilon - C\|u\|^{2^*_s}_\epsilon,\end{aligned}$$

which implies that $\|u\|_\epsilon \geq C$, where C is independent of u.

(iv) Let $\{u_n\} \subset H_\epsilon$ and satisfy $I'_\epsilon(u_n)u_n \to 0$ and $\int_{\mathbb{R}^3}(f(u_n)u_n + |u_n|^{2^*_s})\mathrm{d}x \to \tau$ as $n \to \infty$. It is easy to get that $\{u_n\} \subset H_\epsilon$ is bounded and

$$\liminf_{n\to\infty} \|u_n\|_\epsilon > 0 \text{ and } \liminf_{n\to\infty} \|u_n\|_{2^*_s} > 0. \tag{2.9}$$

It follows from (ii) that, for each $n \in \mathbb{N}$, there exists $h_n > 0$ such that $h_n u_n \in \mathcal{N}_\epsilon$. Then,

$$\frac{\|u_n\|_\epsilon^2}{h_n^2} + \int_{\mathbb{R}^3} \phi_{u_n,t}|u_n|^2 \mathrm{d}x = \int_{\mathbb{R}^3} \frac{f(h_n u_n)u_n^4}{(h_n u_n)^3}\mathrm{d}x + h_n^{2_s^*-4}\int_{\mathbb{R}^3}|u_n|^{2_s^*}\mathrm{d}x, \tag{2.10}$$

which implies that $\{h_n\} \subset \mathbb{R}^+$ is bounded. Up to a subsequence, we assume that $h_n \to A$ for some $A \geq 0$. Obviously, $A \neq 0$. Otherwise, there exists a subsequence of $\{h_n\}$(still denoted by $\{h_n\}$) such that $h_n \to 0$ as $n \to \infty$. By (f_1)–(f_2), there exist $C_1, C_2 > 0$ such that

$$\begin{aligned}\frac{\|u_n\|_\epsilon^2}{h_n^2} + \int_{\mathbb{R}^3} \phi_{u_n,t}|u_n|^2 \mathrm{d}x &= \int_{\mathbb{R}^3} \frac{f(h_n u_n)u_n^4}{(h_n u_n)^3}\mathrm{d}x + h_n^{2_s^*-4}\int_{\mathbb{R}^3}|u_n|^{2_s^*}\mathrm{d}x\\ &\leq \int_{\mathbb{R}^3}(C_1|u_n|^4 + C_2 h_n^{q-4}|u_n|^q)\mathrm{d}x + h_n^{2_s^*-4}\int_{\mathbb{R}^3}|u_n|^{2_s^*}\mathrm{d}x,\end{aligned}$$

which yields by (2.9) a contradiction for large n. So $A > 0$. In view of (f_1)–(f_2), we infer that for any $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(s)| \leq \varepsilon|s|^{2^*-1}$, for $|s| \geq A\delta/2$. Let $\Omega_n(\delta) := \{x \in \mathbb{R}^3 : |u_n(x)| \geq \delta\}$, then for n large, $h_n \in [A/2, 2A]$ and

$$\begin{aligned}\left|\int_{\mathbb{R}^3}[f(h_n u_n) - f(Au_n)]u_n \mathrm{d}x\right| &\leq \left|\int_{\Omega_n(\delta)}[f(h_n u_n) - f(Au_n)]u_n \mathrm{d}x\right| + \left|\int_{\mathbb{R}^3\setminus\Omega_n(\delta)}[f(h_n u_n) - f(Au_n)]u_n \mathrm{d}x\right|\\ &\leq (1+2^{2^*-1})A^{2^*-1}\varepsilon\int_{\mathbb{R}^3}|u_n|^{2^*}\mathrm{d}x + |(h_n - A)|\max_{|s|\leq 2A\delta}|f'(s)|\int_{\mathbb{R}^3}|u_n|^2\mathrm{d}x\\ &= (1+2^{2^*-1})A^{2^*-1}\varepsilon\int_{\mathbb{R}^3}|u_n|^{2^*}\mathrm{d}x + o_n(1).\end{aligned}$$

By the arbitrary choice of ε, we get $\int_{\mathbb{R}^3}[f(h_n u_n) - f(Au_n)]u_n \to 0$ as $n \to \infty$, which yields that

$$\int_{\mathbb{R}^3}[f(h_n u_n)h_n u_n - f(Au_n)Au_n]\mathrm{d}x \to 0,\ n \to \infty.$$

Then, by (2.10) we have

$$\frac{\|u_n\|_\epsilon^2}{A^2} + \int_{\mathbb{R}^3} \phi_{u_n,t}|u_n|^2 \mathrm{d}x = \int_{\mathbb{R}^3} \frac{f(Au_n)u_n^4}{(Au_n)^3}\mathrm{d}x + A^{2_s^*-4}\int_{\mathbb{R}^3}|u_n|^{2_s^*}\mathrm{d}x + o(1),$$

which, together with $I_\epsilon'(u_n)u_n = o(1)$, implies that

$$\left(1 - \frac{1}{A^2}\right)\|u_n\|_\epsilon^2 = \int_{\mathbb{R}^3}\left(\frac{f(u_n)u_n^4}{u_n^3} - \frac{f(Au_n)u_n^4}{(Au_n)^3}\right)\mathrm{d}x + (1 - A^{2_s^*-4})\int_{\mathbb{R}^3}|u_n|^{2_s^*}\mathrm{d}x + o(1). \tag{2.11}$$

If $A > 1$, then by (f_3) we have

$$0 < \left(1 - \frac{1}{A^2}\right)\|u_n\|_\epsilon^2 < (1 - A^{2_s^*-4})\int_{\mathbb{R}^3}|u_n|^{2_s^*}\mathrm{d}x < 0$$

for large n, which is impossible. If $A < 1$, then by (f_3) we also have

$$0 > \left(1 - \frac{1}{A^2}\right)\|u_n\|_\epsilon^2 < (1 - A^{2_s^*-4})\int_{\mathbb{R}^3}|u_n|^{2_s^*}\mathrm{d}x > 0$$

for large n, which is impossible. Hence, $A = 1$. The proof is complete. □

The functional I_ϵ satisfies the mountain pass geometry.

Lemma 2.8. *The functional I_ϵ has the following properties.*

(i) *There exist α, $\rho > 0$ such that $I_\epsilon(u) \geq \alpha$ for $\|u\|_\epsilon = \rho$;*
(ii) *There exists $e \in H_\epsilon$ satisfying $\|e\|_\epsilon > \rho$ such that $I_\epsilon(e) < 0$.*

The proof of Lemma 2.8 is standard and hence is also omitted. Let

$$c_\epsilon = \inf_{\gamma\in\Gamma} \sup_{h\in[0,1]} I_\epsilon(\gamma(h)) > 0,$$

where $\Gamma := \{\gamma \in C^1([0,1], H_\epsilon) : I_\epsilon(\gamma(0)) = 0, I_\epsilon(\gamma(1)) < 0\}$. By Lemma 2.8 and the mountain pass theorem without (PS) condition (see [43]), there exists a $(\mathrm{PS})_{c_\epsilon}$ sequence $\{u_n\} \subset H_\epsilon$ such that $I_\epsilon(u_n) \to c_\epsilon$ and $I'_\epsilon(u_n) \to 0$ in H_ϵ^{-1}. Motivated by [32], we also have the following equivalent characterization of c_ϵ, which is more adequate to our purpose.

Lemma 2.9.

$$c_\epsilon = c_\epsilon^* = c_\epsilon^{**} := \inf_{u\in H_\epsilon\setminus\{0\}} \max_{h\geq 0} I_\epsilon(hu) > 0. \tag{2.12}$$

Proof. Indeed, it follows form Lemma 2.7(ii) that $c_\epsilon^* = c_\epsilon^{**}$. Notice that for any $u \in H_\epsilon \setminus \{0\}$, there exists some $h_0 > 0$ large, such that $I_\epsilon(h_0 u) < 0$. Define a path $\gamma : [0,1] \to H_\epsilon$ by $\gamma(h) = hh_0u$. Clearly, $\gamma \in \Gamma$ and consequently, $c_\epsilon \leq c_\epsilon^{**}$. On the other hand, for every path $\gamma \in \Gamma$, let $g(t) := I'_\epsilon(\gamma(t))\gamma(t)$, then $g(0) = 0$ and $g(t) > 0$ for $t > 0$ small. Meanwhile, by (f_3)

$$4I_\epsilon(\gamma(1)) - I'_\epsilon(\gamma(1))\gamma(1) \geq 0,$$

which implies that $g(1) \leq 4I_\epsilon(\gamma(1)) < 0$. Then there exists $t_0 > 0$ such that $g(t_0) = 0$, *i.e.*, $\gamma(t_0) \in \mathcal{N}_\epsilon$. So $c_\epsilon^* \leq c_\epsilon$. □

3. The limit problem

In this section, we consider the existence of ground state solutions to the following equation

$$(-\triangle)^s u + \mu u + \phi_{u,t} u = f(u) + |u|^{2_s^*-1}u, \quad u \in H^s(\mathbb{R}^3), \tag{3.1}$$

where $\mu > 0$ and the associated energy functional is

$$I_\mu(u) = \frac{1}{2}\int_{\mathbb{R}^3} |\xi|^{2s}|\hat{u}(\xi)|^2 \mathrm{d}\xi + \int_{\mathbb{R}^3} \left(\frac{1}{2}\mu u^2 + \frac{1}{4}\phi_{u,t}u^2\right) \mathrm{d}x - \int_{\mathbb{R}^3} \left(F(u) + \frac{1}{2_s^*}|u|^{2_s^*}\right) \mathrm{d}x. \tag{3.2}$$

The Nehari manifold corresponding to I_μ is defined by

$$\mathcal{N}_\mu = \{u \in H^\mu \setminus \{0\} : I'_\mu(u)u = 0\},$$

where $H^\mu = H^s(\mathbb{R}^3)$ with the norm $\|u\|_{H^\mu}^2 = \int_{\mathbb{R}^3} |\xi|^{2s}|\hat{u}(\xi)|^2 \mathrm{d}\xi + \int_{\mathbb{R}^3} \mu u^2 \mathrm{d}x$. Define the ground state energy m_μ by

$$m_\mu := \inf_{u\in\mathcal{N}_\mu} I_\mu(u).$$

It is easy to check that m_μ and $\mathcal{N}_\mu$ have properties similar to those of c_ϵ^* and $\mathcal{N}_\epsilon$ which have been defined in Section 2.

In the following lemma, we give an upper estimate of the minimax level m_μ.

Lemma 3.1. *For any $\mu > 0$, there exists some $v \in H^\mu \setminus \{0\}$ such that*

$$\max_{t\in[0,\infty)} I_\mu(tv) < \frac{s}{3} S_s^{\frac{3}{2s}}.$$

In particular, $m_\mu < \frac{s}{3} S_s^{\frac{3}{2s}}$.

Proof. It follows from [38] that S_s can be achieved by

$$u_\varepsilon(x) := \frac{\kappa\varepsilon^{-\frac{3-2s}{2}}}{(\theta^2 + |\frac{x}{\varepsilon S_s^{1/2s}}|)^{-\frac{3-2s}{2}}}$$

for any fixed $\varepsilon > 0$, where $\kappa \in \mathbb{R}$, $\theta > 0$ are fixed constants. Let $\eta \in C_0^\infty(\mathbb{R}^3)$ be a cut-off function with support $B_2(0)$ such that $\eta \equiv 1$ on $B_1(0)$ and $\eta \in [0,1]$ on $B_2(0)$. Define $U_\varepsilon(x) = \eta(x)u_\varepsilon(x)$, from [38] we have

$$\int_{\mathbb{R}^3} |(-\triangle)^{s/2} U_\varepsilon(x)|^2 \mathrm{d}x = S_s^{3/(2s)} + O(\varepsilon^{3-2s}), \quad \int_{\mathbb{R}^3} |U_\varepsilon|^{2_s^*} \mathrm{d}x = S_s^{3/(2s)} + O(\varepsilon^3), \tag{3.3}$$

and

$$\int_{\mathbb{R}^3} |U_\varepsilon|^2 \mathrm{d}x = C\varepsilon^{3-2s} + O(\varepsilon^{2s}), \tag{3.4}$$

$$\int_{\mathbb{R}^3} |U_\varepsilon|^q \mathrm{d}x = O(\varepsilon^{3-\frac{(3-2s)q}{2}}) \text{ for } q \in \left(4, \frac{6}{3-2s}\right). \tag{3.5}$$

Let $v_\varepsilon = \frac{U_\varepsilon}{\|U_\varepsilon\|_{2_s^*}}$ be such that $\int_{\mathbb{R}^3} |(-\triangle)^{s/2} v_\varepsilon(x)|^2 \mathrm{d}x \le S_s + O(\varepsilon^{3-2s})$. Moreover,

$$\int_{\mathbb{R}^3} |v_\varepsilon|^2 \mathrm{d}x = O(\varepsilon^{3-2s}), \tag{3.6}$$

$$\int_{\mathbb{R}^3} |v_\varepsilon|^q \mathrm{d}x = O(\varepsilon^{3-\frac{(3-2s)q}{2}}) \text{ for } q \in \left(4, \frac{6}{3-2s}\right). \tag{3.7}$$

In view of the definition of m_μ, we infer that $m_\mu \le \max\limits_{\lambda\ge 0} I_\mu(\lambda v_\varepsilon)$. Define

$$y(\lambda) := \frac{\lambda^2}{2} \|v_\varepsilon\|^2_{D^{s,2}} - \frac{\lambda^{2_s^*}}{2_s^*} \int_{\mathbb{R}^3} |v_\varepsilon|^{2_s^*} \mathrm{d}x.$$

It is clear that $y(\lambda)$ attains its maximum at

$$\lambda_0 = \left(\frac{\|v_\varepsilon\|^2_{D^{s,2}}}{\int_{\mathbb{R}^3} |v_\varepsilon|^{2_s^*} \mathrm{d}x} \right)^{\frac{1}{2_s^*-2}} = \|v_\varepsilon\|_{D^{s,2}}^{\frac{2}{2_s^*-2}}$$

and $y(\lambda_0) = \frac{1}{2} \|v_\varepsilon\|_{D^{s,2}}^{\frac{4}{2_s^*-2}} \|v_\varepsilon\|^2_{D^{s,2}} - \frac{1}{2_s^*} \|v_\varepsilon\|_{D^{s,2}}^{\frac{22_s^*}{2_s^*-2}}$. It is easy to see that $y(\lambda_0) \le \frac{s}{3} S_s^{\frac{3}{2s}} + O(\varepsilon^{3-2s})$. Observe that there exists $\lambda' \in (0,1)$ such that for $\varepsilon < 1$,

$$\max_{\lambda\in[0,\lambda']} I_\mu(\lambda v_\varepsilon) \le \max_{\lambda\in[0,\lambda']} \left[\frac{\lambda^2 \|v_\varepsilon\|^2_{H^\mu}}{2} + \frac{\lambda^4}{4} \int_{\mathbb{R}^3} \phi_{v_\varepsilon,t} |v_\varepsilon|^2 \mathrm{d}x \right] < \frac{s}{3} S_s^{\frac{3}{2s}}. \tag{3.8}$$

It follows from (f_4) that

$$I_\mu(\lambda v_\varepsilon) \le \frac{\lambda^2 \|v_\varepsilon\|^2_{H^\mu}}{2} + \frac{\lambda^4}{4} \int_{\mathbb{R}^3} \phi_{v_\varepsilon,t} |v_\varepsilon|^2 \mathrm{d}x - \frac{\rho\lambda^{\sigma+1}}{\sigma+1} \int_{\mathbb{R}^3} |v_\varepsilon|^{\sigma+1} \mathrm{d}x - \frac{\lambda^{2_s^*}}{2_s^*} \int_{\mathbb{R}^3} |v_\varepsilon|^{2_s^*} \mathrm{d}x,$$

which, together with (3.6) and (3.7), implies that there exists $\varepsilon_0 \in (0,1)$ such that $\lim_{\lambda\to\infty} I_\mu(\lambda v_\varepsilon) = -\infty$ uniformly for $\varepsilon \in (0,\varepsilon_0)$. Thus, there exists $\lambda'' > 0$ such that for $\varepsilon \in (0,\varepsilon_0)$,

$$\max_{\lambda\in[\lambda'',+\infty)} I_\mu(\lambda v_\varepsilon) < \frac{s}{3} S_s^{\frac{3}{2s}}. \tag{3.9}$$

On the other hand, from the definition of I_μ, Lemma 2.4, (f_4) and (3.6)–(3.7), we derive that

$$\begin{aligned}\max_{\lambda\in[\lambda',\lambda'']} I_\mu(\lambda v_\varepsilon) &\le \max_{\lambda\in(0,\infty)} y(\lambda) + \int_{\mathbb{R}^3} (\frac{\lambda^4}{4}\phi_{u_\varepsilon,t}|u_\varepsilon|^2 + \frac{\lambda^2}{2}|v_\varepsilon|^2)\mathrm{d}x - \int_{\mathbb{R}^3} F(\lambda v_\varepsilon)\mathrm{d}x \\ &\le \frac{s}{3} S_s^{\frac{3}{2s}} + O(\varepsilon^{3-2s}) + C\|v_\varepsilon\|^4_{\frac{12}{3+2t}} - \frac{\rho\lambda^{\sigma+1}}{\sigma+1}\int_{\mathbb{R}^3} |v_\varepsilon|^{\sigma+1}\mathrm{d}x + \frac{\lambda^2}{2}\int_{\mathbb{R}^3}|v_\varepsilon|^2\mathrm{d}x \\ &\le \frac{s}{3} S_s^{\frac{3}{2s}} + O(\varepsilon^{3-2s}) + O\left(\varepsilon^{2t+4s-3}\right) - O\left(\varepsilon^{3-\frac{(3-2s)(\sigma+1)}{2}}\right) \\ &\le \frac{s}{3} S_s^{\frac{3}{2s}}\end{aligned} \tag{3.10}$$

for small $\varepsilon > 0$ since $3 - \frac{(3-2s)(\sigma+1)}{2} < \min\{3-2s, 2t+4s-3\}$. Therefore, it follows from (3.8)–(3.10) that $m_\mu < \frac{s}{3} S_s^{\frac{3}{2s}}$. □

Remark 3.2. Notice that from the lemma above, in case $V_\infty < \infty$ we have $m_{V_\infty} < \frac{s}{3} S_s^{\frac{3}{2s}}$.

Lemma 3.3. *Let $\{u_n\} \subset H^\mu$ be a $(PS)_{m_\mu}$ sequence for I_μ, where $m_\mu < \frac{s}{3} S_s^{\frac{3}{2s}}$. Then one of the following conclusions holds:*

(a) *$u_n \to 0$ in H^μ;*
(b) *there exists a sequence $\{y_n\} \subset \mathbb{R}^3$ and constants $R, \tau > 0$ such that*

$$\liminf_{n\to\infty} \int_{B_R(y_n)} u_n^2 \mathrm{d}x \ge \tau > 0.$$

Proof. Suppose that (b) does not occur. It follows from Lemma 2.3 that $u_n \to 0$ in $L^\nu(\mathbb{R}^3)$ for $\nu \in (2, 2_s^*)$. We observe that, by (f_1) and (f_2), for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that $f(u_n) \le \varepsilon|u_n| + C_\varepsilon|u_n|^q$. Then, we have $\int_{\mathbb{R}^3} F(u_n)\mathrm{d}x \to 0$, $\int_{\mathbb{R}^3} f(u_n)u_n\mathrm{d}x \to 0$, as $n\to\infty$. On the other hand, from Lemma 2.4 we derive that $\int_{\mathbb{R}^3}\phi_{u_n,t}|u_n|^2\mathrm{d}x \to 0$, as $n\to\infty$. In view of the above facts and the behavior of $(PS)_c$ sequence, we can easily know that

$$o(1) = I_\mu(u_n)u_n = \|u_n\|^2_{H^\mu} - \int_{\mathbb{R}^3} |u_n|^{2_s^*}\mathrm{d}x,$$
$$m_\mu + o(1) = I_\mu(u_n) = \frac{1}{2}\|u_n\|^2_{H^\mu} - \frac{1}{2_s^*}\int_{\mathbb{R}^3} |u_n|^{2_s^*}\mathrm{d}x.$$

Let $l := \|u_n\|^2_{H^\mu} + o(1)$, then $l = \int_{\mathbb{R}^3}|u_n|^{2_s^*}\mathrm{d}x + o(1)$ and $l \ge 0$. Furthermore, $m_\mu = \frac{s}{3}l$. If $l > 0$, then from the definition of S_s, we get $S_s \le l^{\frac{2_s^*-2}{2_s^*}}$, which contradicts $m_\mu = \frac{s}{3}l < \frac{s}{3}S_s^{\frac{3}{2s}}$. Therefore, $l = 0$. This implies that $u_n \to 0$ in H^μ. □

Proposition 3.4. *For any $\mu > 0$, problem (3.1) has a positve ground state solution in $H^s(\mathbb{R}^3)$.*

Proof. We divide the proof into three steps.
Step 1. *Existence.* It is easy to check that I_μ satisfies the mountain pass geometry. Thus, there exists a sequence $u_n \subset H^\mu$ such that $I_\mu(u_n) \to m_\mu$ and $I'_\mu(u_n) \to 0$. Moreover, $\{u_n\}$ is a bounded sequence in H^μ. Then, up to a

subsequence, $u_n \rightharpoonup u$ in H^μ and $u_n \to u$ a.e. in $\mathbb{R}^3$ for some $u \in H^\mu$ as $n \to \infty$. The weak convergence of $\{u_n\}$ implies that $I'_\mu(u) = 0$. We remark that $u \in \mathcal{N}_\mu$ if $u \neq 0$. Using the Fatou's lemma we get

$$\begin{aligned} m_\mu &\le I_\mu(u) - \frac{1}{4} I'_\mu(u)u \\ &\le \frac{1}{4}\|u\|^2_{H^\mu} + \int_{\mathbb{R}^3} \left[\frac{1}{4} f(u)u - F(u)\right] \mathrm{d}x + \left(\frac{1}{4} - \frac{1}{2^*_s}\right) \int_{\mathbb{R}^3} |u|^{2^*_s} \mathrm{d}x \\ &\le \liminf_{n\to\infty} \left(\frac{1}{4}\|u_n\|^2_{H^\mu} + \int_{\mathbb{R}^3} \left(\frac{1}{4} f(u_n)u_n - F(u_n)\right) \mathrm{d}x + \left(\frac{1}{4} - \frac{1}{2^*_s}\right) \int_{\mathbb{R}^3} |u_n|^{2^*_s} \mathrm{d}x\right) \\ &= m_\mu. \end{aligned}$$

Then, $I_\mu(u) = m_\mu$. Now we consider the case $u = 0$. Since $m_\mu > 0$ and I_μ is continuous in H^μ, we deduce that $\|u_n\|_{H^\mu} \not\to 0$. It follows from Lemma 3.1 that there exist a sequence $\{y_n\} \subset \mathbb{R}^3$ and constants $R, \tau > 0$ such that $\liminf_{n\to\infty} \int_{B_R(y_n)} u_n^2 \mathrm{d}x \ge \tau > 0$. Set $v_n(x) = u_n(x + y_n)$, then we use the invariance of $\mathbb{R}^3$ by translation to conclude that $I_\mu(v_n) \to m_\mu$ and $I'_\mu(v_n) \to 0$. Moreover, there exists a critical point $v \in H^\mu$ of I_μ such that, up to a subsequence, $v_n \rightharpoonup v$ in H^μ and $v_n \to v$ in $L^2(B_R(0))$. Observe that

$$\int_{B_R(0)} v^2 \mathrm{d}x = \liminf_{n\to\infty} \int_{B_R(0)} v_n^2 \mathrm{d}x = \liminf_{n\to\infty} \int_{B_R(y_n)} u_n^2 \mathrm{d}x \ge \tau > 0,$$

we know $v \not\equiv 0$ and similar as above, $I_\mu(v) = m_\mu$. Let $u \in H^\mu$ be a ground state solution of problem (3.1). Now, we claim that $u \ge 0$. Using $u^- = \min\{u, 0\}$ as a test function in equation (3.1), and integrating by parts, by Lemma 2.4 we obtain

$$\int_{\mathbb{R}^3} (-\triangle)^s u u^- \mathrm{d}x + \mu \int_{\mathbb{R}^3} |u^-|^2 \mathrm{d}x + \int_{\mathbb{R}^3} \phi_{u,t} |u^-|^2 \mathrm{d}x = \int_{\mathbb{R}^3} |u^-|^{2^*_s}. \tag{3.11}$$

It is easy to check that for any $x, y \in \mathbb{R}^3$,

$$\begin{aligned} [u^-(x) - u^-(y)] =& [u^+(x) - u^+(y)][u^-(x) - u^-(y)] + [u^-(x) - u^-(y)]^2 \\ \ge& [u^-(x) - u^-(y)]^2, \end{aligned}$$

which yields that

$$\begin{aligned} \int_{\mathbb{R}^3} (-\triangle)^s u u^- \mathrm{d}x &= c_s \int_{\mathbb{R}^6} \frac{[u(x) - u(y)][u^-(x) - u^-(y)]}{|x - y|^{3+2s}} \mathrm{d}x\mathrm{d}y \\ &\ge c_s \int_{\mathbb{R}^6} \frac{[u^-(x) - u^-(y)]^2}{|x - y|^{3+2s}} \mathrm{d}x\mathrm{d}y = \|u^-\|^2_{D^{s,2}}. \end{aligned}$$

Then it follows from (3.11) that if $u^- \not\equiv 0$, $\|u^-\|^{2^*_s}_{2^*_s} \ge \|u^-\|^2_{D^{s,2}} \ge S_s^{\frac{3}{2s}}$. Similarly, for any $x, y \in \mathbb{R}^3$, $[u(x) - u(y)][u^+(x) - u^+(y)] \ge [u^+(x) - u^+(y)]^2$. Then we get $\int_{\mathbb{R}^3} (-\triangle)^s u u^+ \mathrm{d}x \ge \|u^+\|^2_{D^{s,2}}$ and

$$\begin{aligned} \frac{s}{3} S_s^{\frac{3}{2s}} > I_\mu(u) &= I_\mu(u) - \frac{1}{4} I'_\mu(u)u \\ &\ge \frac{1}{4}\|u\|^2_\mu + \left(\frac{1}{4} - \frac{1}{2^*_s}\right) \|u\|^{2^*_s}_{2^*_s} \\ &\ge \frac{1}{4}\|u^-\|^2_{D^{s,2}} + \left(\frac{1}{4} - \frac{1}{2^*_s}\right) \|u^-\|^{2^*_s}_{2^*_s} \ge \frac{s}{3} S_s^{\frac{3}{2s}}, \end{aligned}$$

which is a contradiction. So $u^- \equiv 0$ and $u \ge 0$.

Step 2. L^∞*-estimates.* We use an iteration method which was firstly introduced in [10] to prove $u \in L^\infty(\mathbb{R}^3)$. Our argument is similar to that developed in Proposition 3.2 of [10] but we give the details for the reader's convenience.

Define

$$\psi(t) = \psi_{T,\kappa}(t) = \begin{cases} 0, & t \leq 0 \\ t^\kappa & 0 < t < T, \\ \kappa T^{\kappa-1}(t-T) + T^\kappa & t \geq T. \end{cases}$$

Observe that ψ is a convex and differentiable function, then $(-\triangle)^s \psi(u) \leq \psi'(u)(-\triangle)^s u$. Moreover, $\|\psi(u)\|_{D^{s,2}} \leq \kappa T^{\kappa-1}\|u\|_{D^{s,2}}$. From Lemma 2.2 we have $\int_{\mathbb{R}^3} \psi(u)(-\triangle)^s\psi(u)\mathrm{d}x = \|\psi(u)\|^2_{D^{s,2}} \geq S_s\|\psi(u)\|^2_{2^*_s}$. On the other hand, since $u \geq 0$ solves (3.1), it follows from (f$_1$) and (f$_2$) that there exists $C_1 > 0$ such that $(-\triangle)^s u \leq C_1 u^{2^*_s-1}$. Then, we have

$$\int_{\mathbb{R}^3} \psi(u)(-\triangle)^s\psi(u)\mathrm{d}x \leq \int_{\mathbb{R}^3} \psi(u)\psi'(u)(-\triangle)^s u\mathrm{d}x \leq C_1\int_{\mathbb{R}^3} \psi(u)\psi'(u)u^{2^*_s-1}\mathrm{d}x,$$

which, together with Lemma 2.2, implies that $\|\psi(u)\|^2_{2^*_s} \leq \frac{C_1}{S_s}\int_{\mathbb{R}^3} \psi(u)\psi'(u)u^{2^*_s-1}\mathrm{d}x$. It follows from $u\psi'(u) \leq \kappa\psi(u)$ that

$$\|\psi(u)\|^2_{2^*_s} \leq C\kappa\int_{\mathbb{R}^3} \psi^2(u)u^{2^*_s-2}\mathrm{d}x. \tag{3.12}$$

Let $\kappa_1 = 2^*_s/2$, we claim that $u \in L^{\kappa_1 2^*_s}$. Indeed, by Hölder's inequality we infer that

$$\begin{aligned}\int_{\mathbb{R}^3} \psi^2(u)u^{2^*_s-2}\mathrm{d}x &= \int_{\{u\leq R\}} \psi^2(u)u^{2^*_s-2}\mathrm{d}x + \int_{\{u\geq R\}} \psi^2(u)u^{2^*_s-2}\mathrm{d}x \\ &\leq \int_{\{u\leq R\}} \psi^2(u)R^{2^*_s-2}\mathrm{d}x + \|\psi(u)\|^2_{2^*_s}\left(\int_{\{u\geq R\}} u^{2^*_s}\mathrm{d}x\right)^{\frac{2^*_s-2}{2^*_s}},\end{aligned} \tag{3.13}$$

where $R > 0$. Clearly, taking R large enough, we get $\int_{\{u\geq R\}} u^{2^*_s}\mathrm{d}x \leq (C\kappa_1)^{\frac{-2^*_s}{2^*_s-2}}$. Then, combining (3.12) and (3.13) we have $\|\psi(u)\|^2_{2^*_s} \leq 2C\kappa_1\int_{\{u\leq R\}} \psi^2(u)R^{2^*_s-2}\mathrm{d}x$. Using that $\psi_{T,\kappa_1} \leq u^{\kappa_1}$ in the right hand side of the above formula and then letting $T \to \infty$ in the left hand side, by $2\kappa_1 = 2^*_s$ we have $\|u\|^{2\kappa_1}_{\kappa_1 2^*_s} \leq 2C\kappa_1 R^{2^*_s-2}\|u\|^{2^*_s}_{2^*_s} < \infty$. Our claim is true. Using that $\psi_{T,\kappa_1} \leq u^{\kappa_1}$ in the right hand side of (3.12) and then letting $T \to \infty$ in the left hand side, we have $\|u\|^{2\kappa}_{\kappa 2^*_s} \leq C\kappa\int_{\mathbb{R}^3} u^{2\kappa+2^*_s-2}\mathrm{d}x$. So, let $C_\kappa = C\kappa$,

$$\left(\int_{\mathbb{R}^3} u^{\kappa 2^*_s}\right)^{\frac{1}{(\kappa-1)2^*_s}} \leq C_\kappa^{\frac{1}{2(\kappa-1)}}\left(\int_{\mathbb{R}^3} u^{2\kappa+2^*_s-2}\mathrm{d}x\right)^{\frac{1}{2(\kappa-1)}}.$$

For $m \geq 1$, we define κ_{m+1} inductively so that $2\kappa_{m+1} + 2^*_s - 2 = 2^*_s\kappa_m$ and $\kappa_1 = \frac{2^*_s}{2}$. So we have

$$\left(\int_{\mathbb{R}^3} u^{\kappa_{m+1}2^*_s}\right)^{\frac{1}{(\kappa_{m+1}-1)2^*_s}} \leq C_{\kappa_{m+1}}^{\frac{1}{2(\kappa_{m+1}-1)}}\left(\int_{\mathbb{R}^3} u^{2^*_s\kappa_m}\mathrm{d}x\right)^{\frac{1}{2^*_s(\kappa_m-1)}}.$$

Then, define for $m \geq 1$, $D_m := \left(\int_{\mathbb{R}^3} u^{2^*_s\kappa_m}\mathrm{d}x\right)^{\frac{1}{2^*_s(\kappa_m-1)}}$. By using the iteration technique, we conclude that there exists $C_0 > 0$, independent of m, such that

$$D_{m+1} \leq \prod_{k=1}^m C_{\kappa_{k+1}}^{\frac{1}{2(\kappa_{k+1}-1)}} \cdot D_1 \leq C_0 D_1.$$

Letting $m \to 0$, we are going to obtain $\|u\|_\infty \le C_0 D_1$.

Step 3. *Positivity, i.e.,* $u > 0$. Observing that $u \in L^\infty(\mathbb{R}^3)$, by the definition of $\phi_{u,t}(x)$, there exists $C > 0$ such that

$$\begin{aligned}\phi_{u,t}(x) &\le \int_{|x-y|\ge 1} \frac{u^2(y)}{|x-y|^{3-2t}}\mathrm{d}y + \int_{|x-y|<1} \frac{u^2(y)}{|x-y|^{3-2t}}\mathrm{d}y \\ &\le \|u\|_2^2 + C\int_{|x-y|<1} \frac{1}{|x-y|^{3-2t}}\mathrm{d}y < +\infty\end{aligned}$$

and $|g| \le C(|u| + |u|^{q-1})$, where $g(x) = f(u(x)) + |u(x)|^{2_s^*-2}u(x) - \mu u(x) - \phi_{u,t}(x)u(x)$. Then it follows from Theorem 3.4 in [19] that there exists $\sigma \in (0,1)$ such that $u \in C^{0,\sigma}$. Let w satisfy $-\triangle w = -\mu u - \phi_{u,t}u + f(u) + |u|^{2_s^*-2}u \in C^{0,\sigma}$. By the Hölder regularity theory for the Laplacian, we have $w \in C^{2,\sigma}$. It follows from $2s+\sigma > 1$ that $(-\triangle)^{1-s}w \in C^{1,2s+\sigma-1}$. Then, since $(-\triangle)^s(u - (-\triangle)^{1-s}w) = 0$, the function $u - (-\triangle)^{1-s}w$ is harmonic and we get u has the same regularity as $(-\triangle)^{1-s}w$. That is, $u \in C^{1,2s+\sigma-1}$. The regularity obtained above implies that

$$(-\triangle)^s u = -\int_{\mathbb{R}^3} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{3+2s}}\mathrm{d}y.$$

Assume there exists $x_0 \in \mathbb{R}^3$ such that $u(x_0) = 0$, then by $u \not\equiv 0$ and $u \ge 0$,

$$(-\triangle)^s u(x_0) = -\int_{\mathbb{R}^3} \frac{u(x_0+y) + u(x_0-y)}{|y|^{3+2s}}\mathrm{d}y < 0.$$

However, noting that $(-\triangle)^s u = -\mu u - \phi_{u,t}u + f(u) + |u|^{2_s^*-2}u$ we get $(-\triangle)^s u(x_0) = 0$, which is a contradiction. Therefore, $u > 0$. The proof is complete. ☐

4. Existence of positive solutions

Let $u_0 \in H^s$ be a ground state solution of the following equation:

$$(-\triangle)^s u + V_0 u + \phi_{u,t}u = f(u) + |u|^{2_s^*-2}u \quad \text{in } \mathbb{R}^3, \tag{4.1}$$

and $I_{V_0}(u_0) = m_{V_0}$, where I_{V_0} and m_{V_0} are given in Section 3 by replacing μ by V_0. Similar for m_{V_∞} and I_{V_∞}.

Here, we give an upper estimate of the minimax level c_ϵ, which is defined in Section 2.

Lemma 4.1. *Assume that* (V) *and* (f_1)*–*(f_4) *hold. Then there exists* $\epsilon_0 > 0$ *such that for* $\epsilon \in (0, \epsilon_0)$*,*

$$c_\epsilon < \frac{s}{3}S_s^{\frac{3}{2s}}.$$

Moreover, if $V_\infty < \infty$*, then* $c_\epsilon < m_{V_\infty}$ *for* $\epsilon \in (0, \epsilon_0)$*.*

Proof. Firstly, we claim that $c_\epsilon \to m_{V_0}$ as $\epsilon \to 0^+$. For each $R > 0$, set $u_R(x) = \psi_R(x)u_0$, here u_0 is a positive ground state solution of problem (4.1) and $\psi_R(x) = \phi(x/R)$, where $\phi \in C^\infty(\mathbb{R}^3, [0,1])$ is such that $\phi(x) = 1$ if $|x| \le \frac{1}{2}$, and $\phi(x) = 0$ if $|x| \ge 1$. From Lemma 2.5 we derive that

$$u_R \to u_0 \quad \text{in } H^s(\mathbb{R}^3) \quad \text{as } R \to \infty. \tag{4.2}$$

For each $\epsilon, R > 0$, there exists $h_{\epsilon,R} > 0$ such that $I_\epsilon(h_{\epsilon,R}u_R) = \max_{h\ge 0} I_\epsilon(hu_R)$. Thus, $I'_\epsilon(h_{\epsilon,R}u_R) = 0$ and

$$\frac{1}{h_{\epsilon,R}^2}\int_{B_R(0)} (|(-\triangle)^s u_R|^2 + V(\epsilon x)u_R^2)\mathrm{d}x + \int_{B_R(0)} \phi_{u_R,t}|u_R|^2\mathrm{d}x = \int_{B_R(0)} \frac{f(h_{\epsilon,R}u_R)}{h_{\epsilon,R}^3 u_R^3}u_R^4\mathrm{d}x + h_{\epsilon,R}^{2_s^*-4}\int_{B_R(0)} |u_R|^{2_s^*}\mathrm{d}x, \tag{4.3}$$

which implies that

$$\frac{1}{h_{\epsilon,R}^2}\int_{B_R(0)}(|(-\triangle)^s u_R|^2+\|V\|_{\infty(|x|<R)}u_R^2)\mathrm{d}x+\int_{B_R(0)}\phi_{u_R,t}|u_R|^2\mathrm{d}x \geq \int_{B_R(0)}\frac{f(h_{\epsilon,R}u_R)}{h_{\epsilon,R}^3u_R^3}u_R^4\mathrm{d}x+h_{\epsilon,R}^{2_s^*-4}\int_{B_R(0)}|u_R|^{2_s^*}\mathrm{d}x.$$

Thus, from (4.3), (f_3) and the last inequality, for each $R>0$ we deduce that

$$0<\lim_{\epsilon\to 0^+}h_{\epsilon,R}=h_R<\infty.$$

Passing to the limit as $\epsilon\to 0^+$ in (4.3), we get

$$\frac{1}{h_R^2}\int_{B_R(0)}(|(-\triangle)^s u_R|^2+V_0u_R^2)\mathrm{d}x+\int_{B_R(0)}\phi_{u_R,t}|u_R|^2\mathrm{d}x=\int_{B_R(0)}\frac{f(h_Ru_R)}{h_R^3u_R^3}u_R^4\mathrm{d}x+h_R^{2_s^*-4}\int_{B_R(0)}|u_R|^{2_s^*}\mathrm{d}x. \quad (4.4)$$

It follows from (4.2) and (4.4) that $\lim_{R\to\infty}h_R=1$, and $I_{V_0}(h_Ru_R)=\max_{t\geq 0}I_{V_0}(hu_R)$. Then, by (4.4), $c_\epsilon\leq\max_{h\geq 0}I_\epsilon(hu_R)=I_\epsilon(h_{\epsilon,R}u_R)$ and $\limsup_{\epsilon\to 0^+}c_\epsilon\leq I_{V_0}(h_Ru_R)$ uniformly for $R>0$. From (4.2), we deduce that $\limsup_{\epsilon\to 0^+}c_\epsilon\leq m_{V_0}$. Now it suffices to verify that

$$c_\epsilon\geq m_{V_0}\quad \text{for all } \epsilon>0. \quad (4.5)$$

In fact, we assume on the contrary that $c_{\epsilon^*}<m_{V_0}$ for some $\epsilon^*>0$. From Lemma 2.9 and the definition of c_{ϵ^*}, we know that there exists $u_{\epsilon^*}\in H^s(\mathbb{R}^3)\setminus\{0\}$ such that $c_{\epsilon^*}\leq\max_{h>0}I_{\epsilon^*}(hu_{\epsilon^*})<m_{V_0}$. Again by the definition of m_{V_0}, we know that $m_{V_0}\leq\max_{h>0}I_{V_0}(hu_{\epsilon^*})$. It follows from $V_0\leq V(\epsilon x)$ for all $\epsilon>0$ and $x\in\mathbb{R}^3$ that $m_{V_0}>\max_{h>0}I_{\epsilon^*}(hu_{\epsilon^*})\geq\max_{h>0}I_{V_0}(hu_{\epsilon^*})\geq m_{V_0}$, which is a contradiction. So (4.5) holds. It is easy to see $\liminf_{\epsilon\to 0^+}c_\epsilon\geq m_{V_0}$. Hence, $\lim_{\epsilon\to 0^+}c_\epsilon=m_{V_0}$.

If $V_\infty<\infty$, then $m_{V_0}<m_{V_\infty}$. It follows from $m_{V_0}<\min\{\frac{s}{3}S_s^{\frac{3}{2s}},m_{V_\infty}\}$ that there exists $\epsilon_0>0$ such that $c_\epsilon<\min\{\frac{s}{3}S_s^{\frac{3}{2s}},m_{V_\infty}\}$ for $\epsilon\in(0,\epsilon_0)$. □

Lemma 4.2. *Assume that* (V) *and* (f_1)–(f_4) *hold and for* $\epsilon\in(0,\epsilon_0)$ *there exists a sequence* $\{u_n\}\subset H_\epsilon$ *satisfying*

$$I_\epsilon(u_n)\to c_\epsilon,\qquad I'_\epsilon(u_n)\to 0\quad \text{as } n\to\infty, \quad (4.6)$$

then $\{u_n\}$ *has a subsequence, still denoted by* $\{u_n\}$*, satisfying* $u_n\to u_0$ *in* H_ϵ *as* $n\to\infty$.

Proof. If $\epsilon\in(0,\epsilon_0)$, then by Lemma 4.1, we have $c_\epsilon<\frac{s}{3}S_s^{\frac{3}{2s}}$. Moreover, if $V_\infty<\infty$, we have $c_\epsilon<c_{V_\infty}$ for $\epsilon\in(0,\epsilon_0)$. It is easy to see that $\{u_n\}$ is bounded in H_ϵ, we assume that, up to a subsequence, $u_n\rightharpoonup u_0$ in H_ϵ and $u_n\to u_0$ a.e. in $\mathbb{R}^3$ as $n\to\infty$, for some $u_0\in H_\epsilon$. Then, $I'_\epsilon(u_0)=0$. Now we use two steps to complete the proof.

Step 1. $u_0\not\equiv 0$. We observe that if there exist constants $\beta,R>0$ such that

$$\liminf_{n\to\infty}\int_{B_R(0)}|u_n|^2\mathrm{d}x\geq\beta, \quad (4.7)$$

then $u_0\not\equiv 0$. Suppose on the contrary that $u_0\equiv 0$. Then there exists a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$, such that for every $R_1>0$,

$$\limsup_{n\to\infty}\int_{B_{R_1}(0)}|u_n|^2\mathrm{d}x=0. \quad (4.8)$$

Case 1: $V_\infty < \infty$. Let μ be such that

$$\inf_{x\in\mathbb{R}^3} V(x) < \mu < \liminf_{|x|\to\infty} V(\epsilon x) = V_\infty$$

and take $R > 0$ such that $V(\epsilon x) \geq \mu, \quad \forall x \in \mathbb{R}^3 \setminus B_R(0)$. Observe that $\|u_n\|_\epsilon^2 \to C$ for some $C > 0$. Indeed, assume on the contrary that $\|u_n\|_\epsilon^2 \to 0$ after extracting a subsequence, then we have $I_\epsilon(u_n) \to 0$, which is impossible. Then, there exists $C > 0$ such that $\int_{\mathbb{R}^3}(f(u_n)u_n + |u|^{2_s^*})\mathrm{d}x \to C > 0$. It follows from (iv) of Lemma 2.7 that there exists a sequence $\{\bar{h}_n\} \subset \mathbb{R}^+$ with $\bar{h}_n \to 1$ and $\bar{h}_n u_n \in \mathcal{N}_\epsilon$. Consequently,

$$\begin{aligned}
I_\epsilon(u_n) &= I_\epsilon(\bar{h}_n u_n) + o(1) \geq I_\epsilon(hu_n) + o(1)\\
&= \frac{h^2}{2}\|u_n\|_\epsilon^2 + \frac{h^4}{4}\int_{\mathbb{R}^3}\phi_{u_n,t}|u_n|^2\mathrm{d}x - \int_{\mathbb{R}^3}F(hu_n) - \frac{h^{2_s^*}}{6}\int_{\mathbb{R}^3}|u_n|^{2_s^*}\mathrm{d}x\\
&\quad + I_\mu(hu_n) - I_\mu(hu_n) + o(1)\\
&\geq \frac{h^2}{2}\int_{B_R(0)}(V(\epsilon x) - \mu)|u_n|^2\mathrm{d}x + I_\mu(hu_n) + o(1)
\end{aligned} \tag{4.9}$$

for any $h > 0$. Let $h_n \geq 0$ be such that $I_\mu(h_n u_n) = \max_{h\geq 0} I_\mu(hu_n)$, then $\{h_n\}$ is bounded. If not, there exists a subsequence of $\{h_n\}$, still denoted by $\{h_n\}$, such that $h_n \to \infty$ as $n \to \infty$. We observe that

$$h_n^2\int_{\mathbb{R}^3}(|(-\triangle)^{s/2}u_n|^2 + \mu u_n^2)\mathrm{d}x + h_n^4\int_{\mathbb{R}^3}\phi_{u_n,t}|u_n|^2\mathrm{d}x \geq |h_n|^{2_s^*}\int_{\mathbb{R}^3}|u_n|^{2_s^*}\mathrm{d}x. \tag{4.10}$$

Now we show that there exists $\delta > 0$ such that

$$\|u_n\|_{2_s^*}^{2_s^*} \geq \delta > 0. \tag{4.11}$$

Otherwise, there exists a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$, such that $\|u_n\|_{2_s^*}^{2_s^*} \to 0$ as $n \to \infty$. Then, we have for any $r > 0$,

$$\lim_{n\to\infty}\sup_{y\in\mathbb{R}^3}\int_{B_r(y)}|u_n|^{2_s^*}\mathrm{d}x = 0.$$

Lemma 2.3 implies $u_n \to 0$ in $L^\nu(\mathbb{R}^3)$ for $\nu \in (2, 2_s^*]$. So, from (f_1) and (f_2) we deduce that $\int_{\mathbb{R}^3}F(u_n)\mathrm{d}x \to 0$ as $n \to \infty$. In view of Lemma 2.4, we can easily see that

$$\int_{\mathbb{R}^3}\phi_{u_n,t}u_n^2\mathrm{d}x \leq C\|u_n\|_{\frac{12}{3+2t}}^4 \to 0,$$

which, together with the definition of I_ϵ, implies that

$$\begin{aligned}
c_\epsilon &= I_\epsilon(u_n) + o(1)\\
&= \frac{1}{2}\|u_n\|_\epsilon^2 + \frac{1}{4}\int_{\mathbb{R}^3}\phi_{u_n,t}u_n^2\mathrm{d}x - \int_{\mathbb{R}^3}F(u_n)\mathrm{d}x - \frac{1}{2_s^*}\int_{\mathbb{R}^3}|u_n|^{2_s^*}\mathrm{d}x + o(1)\\
&= \frac{1}{2}\|u_n\|_\epsilon^2 + o(1).
\end{aligned}$$

Thus, there exists some constant $C > 0$ such that $\|u_n\|_\epsilon^2 > C$. Based on the above facts, we infer from the definition of I_ϵ and Lemma 2.9 that

$$\begin{aligned}
c_\epsilon + o(1) = I_\epsilon(u_n) &\geq I_\epsilon(Ku_n)\\
&= \frac{K^2}{2}\|u_n\|_\epsilon^2 + \frac{K^4}{4}\int_{\mathbb{R}^3}\phi_{u_n,t}u_n^2\mathrm{d}x - \int_{\mathbb{R}^3}F(Ku_n)\mathrm{d}x - \frac{K^{2_s^*}}{2_s^*}\int_{\mathbb{R}^3}|u_n|^{2_s^*}\mathrm{d}x\\
&\geq \frac{K^2C}{2} - \int_{\mathbb{R}^3}F(Ku_n)\mathrm{d}x - \frac{K^{2_s^*}}{2_s^*}\int_{\mathbb{R}^3}|u_n|^{2_s^*}\mathrm{d}x \to \frac{K^2C}{2}
\end{aligned}$$

as $n \to \infty$. Now we arrive a contradiction if K is large enough. Therefore, (4.11) holds. Combining (4.10), (4.11) and the boundedness of $\{u_n\}$ in $H^s(\mathbb{R}^3)$, we obtain a contradiction. Consequently, $\{h_n\}$ is bounded. Letting $h = h_n$ in (4.9), we have

$$I_\epsilon(u_n) \geq \frac{h_n^2}{2} \int_{B_R(0)} (V(\epsilon x) - \mu) u_n^2 \mathrm{d}x + m_\mu + o(1).$$

Taking the limit as $n \to +\infty$, from (4.8) and the Sobolev embedding inequality, we have $c_\epsilon \geq m_\mu$. Next, letting $\mu \to V_\infty$, we obtain $c_\epsilon \geq m_{V_\infty}$, which is a contradiction.

Case 2: $V_\infty = \infty$. It follows from Lemma 2.6 that the embedding $H_\epsilon \hookrightarrow L^s(\mathbb{R}^3)$ is compact for $2 \leq s < 6$. Hence, using (4.8), up to a subsequence, $u_n \to 0$ in $L^s(\mathbb{R}^3)$ as $n \to \infty$. Furthermore, by (f_1)-(f_2) and $I'_\epsilon(u_n) = o(1)$, we have $\|u_n\|_\epsilon^2 = \int_{\mathbb{R}^3} |u_n|^{2_s^*} \mathrm{d}x + o(1)$. Since $\{u_n\}$ is bounded in H_ϵ, up to a subsequence, we can assume $\|u_n\|_\epsilon^2 \to l > 0$ and $\|u_n\|_{2_s^*}^{2_s^*} \to l$. In view of $I_\epsilon(u_n) = c_\epsilon + o(1)$, we have $\frac{l}{2} - \frac{l}{2_s^*} = c_\epsilon$. Noting that $l \geq S_s^{\frac{3}{2s}}$, $S_s^{\frac{3}{2s}} \leq l = \frac{3}{s} c_\epsilon < S_s^{\frac{3}{2s}}$ which is a contradiction. Therefore, $u_0 \not\equiv 0$.

Step 2. We prove that $u_n \to u_0$ in H_ϵ as $n \to \infty$. Set $v_n = u_n - u_0$. Assume on the contrary that $\|v_n\|_\epsilon \geq \theta$ for large n and some constant $\theta > 0$. Noting that $I'_\epsilon(u_0) = 0$, we get

$$\|u_0\|_\epsilon^2 + \int_{\mathbb{R}^3} \phi_{u_0,t} u_0^2 \mathrm{d}x = \int_{\mathbb{R}^3} f(u_0) u_0 \mathrm{d}x + \int_{\mathbb{R}^3} |u_0|^{2_s^*} \mathrm{d}x \tag{4.12}$$

and

$$\|u_n\|_\epsilon^2 + \int_{\mathbb{R}^3} \phi_{u_n,t} u_n^2 \mathrm{d}x = \int_{\mathbb{R}^3} f(u_n) u_n \mathrm{d}x + \int_{\mathbb{R}^3} |u_n|^{2_s^*} \mathrm{d}x + o(1), \tag{4.13}$$

respectively. Using the Brezis–Lieb lemma [11], we obtain

$$\begin{aligned} \int_{\mathbb{R}^6} \frac{|v_n(x) - v_n(y)|^2}{|x-y|^{3+2s}} \mathrm{d}x\mathrm{d}y &= \int_{\mathbb{R}^6} \frac{|u_n(x) - u_n(y)|^2}{|x-y|^{3+2s}} \mathrm{d}x\mathrm{d}y \\ &\quad - \int_{\mathbb{R}^6} \frac{|u_0(x) - u_0(y)|^2}{|x-y|^{3+2s}} \mathrm{d}x\mathrm{d}y + o(1), \\ \int_{\mathbb{R}^3} f(v_n) v_n \mathrm{d}x &= \int_{\mathbb{R}^3} f(u_n) u_n \mathrm{d}x - \int_{\mathbb{R}^3} f(u_0) u_0 \mathrm{d}x, \\ \int_{\mathbb{R}^3} |v_n|^p \mathrm{d}x &= \int_{\mathbb{R}^3} |u_n|^p \mathrm{d}x - \int_{\mathbb{R}^3} |u_0|^p \mathrm{d}x + o(1), \quad p \in [2, 2_s^*]. \end{aligned}$$

Then, by (4.12), (4.13), and Lemma 2.4, we obtain

$$\|v_n\|_\epsilon^2 + \int_{\mathbb{R}^3} \phi_{v_n,t} v_n^2 \mathrm{d}x = \int_{\mathbb{R}^3} f(v_n) v_n \mathrm{d}x + \int_{\mathbb{R}^3} |v_n|^{2_s^*} \mathrm{d}x + o(1).$$

Since $\|v_n\|_\epsilon \geq \theta$ for large n, similar as above, there exists a sequence $\{\tau_n\} \subset \mathbb{R}^+$ with $\tau_n \to 1$ as $n \to \infty$ and $\tau_n v_n \in \mathcal{N}_\epsilon$, that is,

$$\|\tau_n v_n\|_\epsilon^2 + \int_{\mathbb{R}^3} \phi_{\tau_n v_n,t} (\tau_n v_n)^2 \mathrm{d}x = \int_{\mathbb{R}^3} f(\tau_n v_n) \tau_n v_n \mathrm{d}x + \int_{\mathbb{R}^3} (\tau_n v_n)^{2_s^*} \mathrm{d}x.$$

So

$$I_\epsilon(v_n) = I_\epsilon(\tau_n v_n) + o(1) \geq c_\epsilon + o(1). \tag{4.14}$$

Similarly, it follows from (4.12) that

$$I_\epsilon(u_0) \geq c_\epsilon. \tag{4.15}$$

Thus, by (4.14), (4.15), the Brezis–Lieb lemma [11] and Lemma 2.4,

$$\begin{aligned} I_\epsilon(u_n) =&\frac{1}{2}\|u_n\|_\epsilon^2+\frac{1}{4}\int_{\mathbb{R}^3}\phi_{u_n,t}u_n^2\mathrm{d}x-\int_{\mathbb{R}^3}F(u_n)\mathrm{d}x+\frac{1}{2_s^*}\int_{\mathbb{R}^3}|u_n|^{2_s^*}\mathrm{d}x\\ =&\frac{1}{2}\|u_0\|_\epsilon^2+\frac{1}{4}\int_{\mathbb{R}^3}\phi_{u_0,t}u_0^2\mathrm{d}x-\int_{\mathbb{R}^3}F(u_0)\mathrm{d}x+\frac{1}{2_s^*}\int_{\mathbb{R}^3}|u_0|^{2_s^*}\mathrm{d}x\\ &+\frac{1}{2}\|v_n\|_\epsilon^2+\frac{1}{4}\int_{\mathbb{R}^3}\phi_{v_n,t}v_n^2\mathrm{d}x-\int_{\mathbb{R}^3}F(v_n)\mathrm{d}x+\frac{1}{2_s^*}\int_{\mathbb{R}^3}|v_n|^{2_s^*}\mathrm{d}x+o(1)\\ \geq&2c_\epsilon+o(1), \end{aligned}$$

which implies that $\lim\limits_{n\to\infty}I_\epsilon(u_n)=c_\epsilon\geq 2c_\epsilon$, which contradicts $c_\epsilon>0$ for any $\epsilon>0$. Therefore, up to a subsequence, $u_n\to u_0$ in H_ϵ as $n\to\infty$. □

Proof of Theorem 1.2. Combining Lemmas 2.8, 4.1 and 4.2, we conclude that for any $\epsilon\in(0,\epsilon_0)$, I_ϵ admits a nontrivial critical point $u_\epsilon\in H_\epsilon$. It follows from Lemma 2.9 that u_ϵ is a ground state solution of (2.3). Similarly as in Proposition 3.4, we have $u_\epsilon\in L^\infty(\mathbb{R}^3)$, $u_\epsilon\in C^{1,2s+\sigma-1}$ and $u_\epsilon(x)>0$ for $x\in\mathbb{R}^3$. Therefore, $(w_\epsilon,\phi_{w_\epsilon,t})$ is a positive solution of system (1.1), where $w_\epsilon(x)=u_\epsilon(\frac{x}{\epsilon})$. □

5. Multiplicity and concentration of positive solutions

This section is devoted to the multiplicity and concentration of positive solutions of (1.1). For this purpose, we first give the following compactness lemma.

Lemma 5.1. *Let $\{u_n\}\subset\mathcal{N}_{V_0}$ be a sequence satisfying $I_{V_0}(u_n)\to m_{V_0}$. Then either $\{u_n\}$ has a strongly convergent subsequence in $H^s(\mathbb{R}^3)$ or there exists $\{y_n\}\subset\mathbb{R}^3$ such that $w_n(x)=u_n(x+y_n)$ converges strongly in $H^s(\mathbb{R}^3)$. In particular, there exists a minimizer of m_{V_0}.*

Proof. It is easy to see that $\{u_n\}$ is a bounded sequence in $H^s(\mathbb{R}^3)$. Up to a subsequence, we assume that there exists $u\in H^s(\mathbb{R}^3)$ such that $u_n\rightharpoonup u$ in $H^s(\mathbb{R}^3)$. Now we **claim** that

$$I_{V_0}(u_n)\to m_{V_0}\quad I'_{V_0}(u_n)\to 0\quad\text{in } H^s(\mathbb{R}^3). \tag{5.1}$$

Due to the Ekeland's variational principle in [20], there exists a sequence $\{\bar{u}_n\}\subset\mathcal{N}_{V_0}$ such that

$$\|\bar{u}_n-u_n\|_s=o(1),\quad I_{V_0}(\bar{u}_n)=m_{V_0}+o(1),\quad I'_{V_0}(\bar{u}_n)-\gamma_nJ'_{V_0}(\bar{u}_n)=o(1),$$

where γ_n is a real number and $J_{V_0}(u)=I'_{V_0}(u)u$ for all $u\in H^s(\mathbb{R}^3)$. We show that there is $b>0$ such that $|J'_{V_0}(\bar{u}_n)\bar{u}_n|\geq b$ for all $n\in\mathbb{N}$. Indeed, by (f_3) and (f_4), and the definition of $J_{V_0,0}$, we have

$$\begin{aligned} -J'_{V_0}(\bar{u}_n)\bar{u}_n=&-2\int_{\mathbb{R}^3}(|(-\triangle)^{s/2}\bar{u}_n|^2+V_0\bar{u}_n^2)\mathrm{d}x-4\int_{\mathbb{R}^3}\phi_{\bar{u}_n,t}\bar{u}_n^2\mathrm{d}x\\ &+\int_{\mathbb{R}^3}(f(\bar{u}_n)\bar{u}_n+f'(\bar{u}_n)\bar{u}_n^2+2_s^*|\bar{u}_n|^{2_s^*})\mathrm{d}x\\ \geq& 2\int_{\mathbb{R}^3}(|(-\triangle)^{s/2}\bar{u}_n|^2+V_0\bar{u}_n^2)\mathrm{d}x-3\int_{\mathbb{R}^3}f(\bar{u}_n)\bar{u}_n\mathrm{d}x\\ &+\int_{\mathbb{R}^3}f'(\bar{u}_n)\bar{u}_n^2\mathrm{d}x+(2_s^*-4)\int_{\mathbb{R}^3}|\bar{u}_n|^{2_s^*}\mathrm{d}x\\ \geq& 2\int_{\mathbb{R}^3}(|(-\triangle)^{s/2}\bar{u}_n|^2+V_0\bar{u}_n^2)\mathrm{d}x, \end{aligned} \tag{5.2}$$

which follows from $3f(\tau)-f'(\tau)\tau\le 0$ for any $\tau\in\mathbb{R}$. Assume by contradiction that $J'_{V_0,0}(\bar{u}_n)\bar{u}_n\to 0$, then one has $\int_{\mathbb{R}^3}(|(-\triangle)^{s/2}\bar{u}_n|^2+V_0\bar{u}_n^2)\mathrm{d}x\to 0$ as $n\to\infty$. Consequently, we can deduce that $\bar{u}_n\to 0$ in $H^s(\mathbb{R}^3)$ as $n\to\infty$. However, this is impossible because $\{\bar{u}_n\}\subset\mathcal{N}_{V_0}$. So $|J'_{V_0}(\bar{u}_n)\bar{u}_n|\ge b$ for all $n\in\mathbb{N}$. By $I'_{V_0,0}(\bar{u}_n)\bar{u}_n=o(1)$, we have $\gamma_n J'_{V_0,0}(\bar{u}_n)\bar{u}_n=o(1)$. Then, from (5.2), we see that $\gamma_n=o(1)$, which yields that $\{\bar{u}_n\}$ is a $(PS)_{m_{V_0}}$ sequence in $H^s(\mathbb{R}^3)$ for I_{V_0}. Hence, it is easy to check that (5.1) holds.

Since I'_{V_0} is weakly sequentially continuous, we know that $I'_{V_0}(u)=0$ in $H^s(\mathbb{R}^3)$. If $u\ne 0$, then from the definition of I_{V_0} we deduce that

$$\begin{aligned}
m_{V_0}\le & I_{V_0}(u)=I_{V_0}(u)-\frac{1}{4}I'_{V_0}(u)u\\
=&\frac{1}{4}\int_{\mathbb{R}^3}(|(-\triangle)^{s/2}u|^2+V_0u^2)\mathrm{d}x+\int_{\mathbb{R}^3}\left(\frac{1}{4}f(u)u-F(u)\right)\mathrm{d}x+\left(\frac{1}{4}-\frac{1}{2_s^*}\right)\int_{\mathbb{R}^3}|u|^{2_s^*}\mathrm{d}x\\
\le&\liminf_{n\to\infty}\left\{\frac{1}{4}\int_{\mathbb{R}^3}(|(-\triangle)^{s/2}u_n|^2+V_0u_n^2)\mathrm{d}x+\int_{\mathbb{R}^3}\left(\frac{1}{4}f(u_n)u_n-F(u_n)\right)\mathrm{d}x\right.\\
&\left.+\left(\frac{1}{4}-\frac{1}{2_s^*}\right)\int_{\mathbb{R}^3}|u_n|^{2_s^*}\mathrm{d}x\right\}\\
=&\liminf_{n\to\infty}\left(I_{V_0}(u_n)-\frac{1}{4}I'_{V_0}(u_n)u_n\right)\le m_{V_0}.
\end{aligned}$$

Then $\lim_{n\to\infty}\frac{1}{4}\int(|(-\triangle)^{s/2}u_n|^2+V_0u_n^2)\mathrm{d}x=\int_{\mathbb{R}^3}(|(-\triangle)^{s/2}u|^2+V_0u^2)\mathrm{d}x$. That is, $u_n\to u$ in $H^s(\mathbb{R}^3)$.

Now we consider the case $u=0$. We claim that there exist $r,\delta>0$ and $\{y_n\}\subset\mathbb{R}^3$ such that

$$\liminf_{n\to\infty}\int_{B_r(y_n)}u_n^2\mathrm{d}x\ge\delta>0. \tag{5.3}$$

To this end, we assume on the contrary that for all $R>0$ $\lim_{n\to\infty}\sup_{y\in\mathbb{R}^3}\int_{B_R(y)}u_n^2\mathrm{d}x=0$, then from Lemma 2.3 we deduce that $u_n\to 0$ in $L^\nu(\mathbb{R}^3)$ for $\nu\in(2,2_s^*)$. Then $\int_{\mathbb{R}^3}f(u_n)u_n\mathrm{d}x\to 0$ as $n\to\infty$. Moreover, it follows from $I'_{V_0}(u_n)u_n=0$ that

$$\int_{\mathbb{R}^3}|(-\triangle)^{s/2}u_n|^2\mathrm{d}x+\int_{\mathbb{R}^3}V_0u_n^2\mathrm{d}x=\int_{\mathbb{R}^3}|u_n|^{2_s^*}\mathrm{d}x. \tag{5.4}$$

We also assume that as $n\to\infty$, there exists $l\ge 0$ such that

$$\int_{\mathbb{R}^3}|(-\triangle)^{s/2}u_n|^2\mathrm{d}x+\int_{\mathbb{R}^3}V_0u_n^2\mathrm{d}x\to l,\quad \int_{\mathbb{R}^3}|u_n|^{2_s^*}\mathrm{d}x\to l,$$

then from the definition of S_s, it is easy to know $l\ge S_s^{\frac{3}{2s}}$. Thus, $m_{V_0}=(\frac{1}{2}-\frac{1}{2_s^*})l\ge\frac{s}{3}S_s^{\frac{3}{2s}}$, which is impossible. Hence, (5.3) holds. Set $v_n(x)=u_n(x+y_n)$, then $I_{V_0}(v_n)\to m_{V_0}$ and $I'_{V_0}(v_n)\to 0$ in H^{-1}, where H^{-1} is the dual space of $H^s(\mathbb{R}^3)$. So there exists $v\in H^s(\mathbb{R}^3)$ with $v\ne 0$ such that $v_n\rightharpoonup v$ in $H^s(\mathbb{R}^3)$. Then the proof follows from the arguments used in the case $u\ne 0$. □

Lemma 5.2. *Let $u_\epsilon\in\mathcal{N}_\epsilon$ satisfy $I_\epsilon(u_\epsilon)\to m_{V_0}$ as $\epsilon\to 0^+$, then there exist $\{\tilde{y}_\epsilon\}\subset\mathbb{R}^3$ and $R,\beta>0$ such that*

$$\int_{B_R(\tilde{y}_\epsilon)}u_\epsilon^2\mathrm{d}x\ge\beta \tag{5.5}$$

for small $\epsilon>0$. Moreover, let $\epsilon_n\to 0$ and $\{u_{\epsilon_n}\}$ be such that $I_{\epsilon_n}(u_{\epsilon_n})\to m_{V_0}$, then $v_n(x)=u_{\epsilon_n}(x+\tilde{y}_{\epsilon_n})$ has a strongly convergent subsequence in $H^s(\mathbb{R}^3)$. Moreover, up to a subsequence, $y_n:=\epsilon_n\tilde{y}_{\epsilon_n}\to y^*\in\Theta$ and the limit of $\{v_n\}$ is a ground state solution of problem (4.1).

Proof. Suppose by contradiction that (5.5) does not hold. Then, there is a sequence ϵ_n converging to zero such that for all $R>0$, $\lim_{n\to\infty}\sup_{y\in\mathbb{R}^3}\int_{B_R(y)}|u_{\epsilon_n}|^2\mathrm{d}x=0$. Using a similar argument as we have done in the proof of Lemma 5.1, we can deduce that (5.5) holds. Let $u_n:=u_{\epsilon_n}$ and $\tilde{y}_n:=\tilde{y}_{\epsilon_n}$. Then, it follows from (5.5) that there exist $R,\beta>0$ and a sequence $\{\tilde{y}_n\}\subset\mathbb{R}^3$ such that $\int_{B_R(\tilde{y}_n)}u_n^2\mathrm{d}x\geq\beta>0$. Moreover, we know that $\{u_n\}$ is bounded in $H^s(\mathbb{R}^3)$. Take $v_n=u_n(x+\tilde{y}_n)$ such that $v_n\rightharpoonup v\neq 0$ in $H^s(\mathbb{R}^3)$ and $v_n(x)\to v(x)$ a.e., in $\mathbb{R}^3$. Then, $v_n:=u_n(x+\tilde{y}_n)$ is a solution of the following equation

$$\begin{cases}(-\triangle)^s v_n+V_n(x)v_n+\phi_{v_n,t}v_n=f(v_n)+|v_n|^{2_s^*-2}v_n & \text{in } \mathbb{R}^3,\\ v_n\in H^s(\mathbb{R}^3), v_n>0,\end{cases}$$

where $V_n(x)=V(\epsilon_n x+\epsilon_n\tilde{y}_n)$ and the associated energy functional is

$$L_{\epsilon_n}(v_n):=\frac{1}{2}\int_{\mathbb{R}^3}(|(-\triangle)^{s/2}v_n|^2+V_n(x)|v_n|^2)\mathrm{d}x+\frac{1}{4}\int_{\mathbb{R}^3}\phi_{v_n,t}|v_n|^2\mathrm{d}x-\int_{\mathbb{R}^3}F(v_n)\mathrm{d}x-\frac{1}{6}\int_{\mathbb{R}^3}|v_n|^{2_s^*}\mathrm{d}x.$$

Choosing $h_n>0$ such that $h_nv_n\in\mathcal{N}_{V_0}$, we deduce from $u_n\in\mathcal{N}_{\epsilon_n}$ that

$$\begin{aligned}I_{V_0}(h_nv_n)\leq&\frac{h_n^2}{2}\int_{\mathbb{R}^3}|(-\triangle)^{s/2}v_n|^2+V(\epsilon_nx+\epsilon_n\tilde{y}_n)|v_n|^2)\mathrm{d}x+\frac{h_n^4}{4}\int_{\mathbb{R}^3}\phi_{v_n,t}|v_n|^2\mathrm{d}x\\&-\int_{\mathbb{R}^3}F(h_nv_n)\mathrm{d}x-\frac{h_n^{2_s^*}}{6}\int_{\mathbb{R}^3}|v_n|^{2_s^*}\mathrm{d}x\\=&I_{\epsilon_n}(h_nu_n)\leq I_{\epsilon_n}(u_n)=m_{V_0}+o(1).\end{aligned}$$

So it follows from $I_{V_0}(h_nv_n)\geq m_{V_0}$ that $\lim_{n\to\infty}I_{V_0}(h_nv_n)=m_{V_0}$. We first claim that $\{h_n\}$ is bounded. If not, then $h_n\to+\infty$ and $I_{V_0}(h_nv_n)\to-\infty$, which contradicts $I_{V_0}(h_nv_n)>m_{V_0}$ for all $n\in\mathbb{N}$. Without loss of generality we assume that $h_n\to h\geq 0$. If $h=0$, by the boundness of sequence $\{v_n\}$ in $H^s(\mathbb{R}^3)$, we have $h_nv_n\to 0$ in $H^s(\mathbb{R}^3)$. Hence $I_{V_0}(h_nv_n)\to 0$ as $n\to\infty$, which contradicts $m_{V_0}>0$. So $h>0$ and the weak limit of h_nv_n is nontrivial. Let $\bar{v}_n:=h_nv_n\rightharpoonup\bar{v}$ in $H^s(\mathbb{R}^3)$. By the uniqueness of the weak limit, $\bar{v}=hv$. From Lemma 5.1, $\bar{v}\in\mathcal{N}_{V_0}$ and $\bar{v}_n\to\bar{v}$ in $H^s(\mathbb{R}^3)$, and so $v_n\to v$ in $H^s(\mathbb{R}^3)$.

Now we show that $y_n:=\epsilon_n\tilde{y}_n$ is bounded. If not, we assume $|y_n|\to\infty$. We first consider the case $V_\infty=\infty$. From Fatou's Lemma that and $L'_{\epsilon_n}(v_n)v_n=0$, we infer that

$$\begin{aligned}\infty=\liminf_{n\to\infty}\int_{\mathbb{R}^3}V_n(x)|v_n|^2\mathrm{d}x\leq&\int_{\mathbb{R}^3}(|(-\triangle)^{s/2}v_n|^2+V_n(x)|v_n|^2)\mathrm{d}x+\int_{\mathbb{R}^3}\phi_{v_n,t}|v_n|^2\mathrm{d}x\\=&\int_{\mathbb{R}^3}f(v_n)v_n\mathrm{d}x+\int_{\mathbb{R}^3}|v_n|^{2_s^*}\mathrm{d}x<\infty,\end{aligned}$$

which is a contradiction. For the case $V_\infty<\infty$. It follows from $h_nv_n\in\mathcal{N}_{V_0}$ and Fatou's Lemma that

$$\begin{aligned}m_{V_0}\leq&I_{V_0}(hv)<I_{V_\infty}(hv)-\frac{1}{4}I'_{V_0}(hv)hv\\=&\int_{\mathbb{R}^3}\left[\frac{1}{4}|(-\triangle)^{s/2}hv|^2+\left(\frac{1}{2}V_\infty-\frac{1}{4}V_0\right)h^2v^2\right]\mathrm{d}x\\&+\int_{\mathbb{R}^3}\left(f(hv)hv-\frac{1}{4}F(hv)\right)\mathrm{d}x+\left(\frac{1}{4}-\frac{1}{2_s^*}\right)\int_{\mathbb{R}^3}|hv|^{2_s^*}\mathrm{d}x\\\leq&\liminf_{n\to\infty}\left(\frac{1}{4}\int_{\mathbb{R}^3}|(-\triangle)^{s/2}h_nv_n|^2\mathrm{d}x+\int_{\mathbb{R}^3}\left(\frac{1}{2}V(\epsilon_nx+y_n)-\frac{1}{4}V_0\right)h_n^2v_n^2\mathrm{d}x\right)\\&+\liminf_{n\to\infty}\left(\int_{\mathbb{R}^3}\left(f(h_nv_n)h_nv_n-\frac{1}{4}F(h_nv_n)\right)\mathrm{d}x+\left(\frac{1}{4}-\frac{1}{2_s^*}\right)\int_{\mathbb{R}^3}|h_nv_n|^{2_s^*}\mathrm{d}x\right)\\=&\liminf_{n\to\infty}I_{\epsilon_n}(h_nu_n)\leq\liminf_{n\to\infty}I_{\epsilon_n}(u_n)=m_{V_0},\end{aligned}$$

which gives a contradiction. So $\{y_n\}$ is bounded. Without loss of generality we may assume that $y_n \to y^*$. For each $\eta \in C_0^\infty(\mathbb{R}^3)$, we deduce from $v_n \to v$ in $H^s(\mathbb{R}^3)$ that

$$\begin{aligned}\lim_{n\to\infty} L'_{\epsilon_n}(v_n)\eta &= \lim_{n\to\infty}\int_{\mathbb{R}^3}((-\triangle)^{s/2}v_n(-\triangle)^{s/2}\eta + V_n(x)v_n\eta)\mathrm{d}x\\ &\quad + \lim_{n\to\infty}\left(\int_{\mathbb{R}^3}\phi_{v_n,t}v_n\eta\mathrm{d}x - \int_{\mathbb{R}^3} f(v_n)\eta\mathrm{d}x - \int_{\mathbb{R}^3}|v_n|^{2_s^*-2}v_n\eta\mathrm{d}x\right)\\ &= \int_{\mathbb{R}^3}((-\triangle)^{s/2}v(-\triangle)^{s/2}\eta + V(y^*)v\eta)\mathrm{d}x\int_{\mathbb{R}^3}\phi_{v,t}v\eta\mathrm{d}x - \int_{\mathbb{R}^3}(f(v)+|v|^{2_s^*-2}v)\eta\mathrm{d}x.\end{aligned}$$

Then, the limit v of the sequence $\{v_n\}$ solves the equation

$$(-\triangle)^s u + V(y^*)u + \phi_{u,t}u = f(u) + |u|^{2_s^*-2}u \quad \text{in}\, \mathbb{R}^3.$$

Define the functional

$$I_{y^*}(u) := \frac{1}{2}\int_{\mathbb{R}^3}(|(-\triangle)^{s/2}u|^2 + V(y^*)u^2)\mathrm{d}x + \frac{1}{4}\int_{\mathbb{R}^3}\phi_{u,t}u^2\mathrm{d}x - \int_{\mathbb{R}^3}F(u)\mathrm{d}x - \frac{1}{2_s^*}\int_{\mathbb{R}^3}|u|^{2_s^*}\mathrm{d}x.$$

If $y^* \notin \Theta$, then $V(y^*) > V_0$ and we can get a contradiction by similar arguments as above. So $y^* \in \Theta$ and $I_{y^*}(v) = m_{V_0}$. □

Let w be a ground state solution of problem (4.1) and η be a smooth non-increasing function defined in $[0,\infty)$ such that $\eta(s) = 1$ if $0 \le s \le 1$ and $\eta(s) = 0$ if $s \ge 2$. For every $y \in \Theta$, we define

$$\Psi_{\epsilon,y} = \eta\left(|\epsilon x - y|\right) w\left(x - \frac{y}{\epsilon}\right), \tag{5.6}$$

and then there exists $h_\epsilon > 0$ such that $\max_{h\ge 0} I_\epsilon(h\Psi_{\epsilon,y}) = I_\epsilon(h_\epsilon\Psi_{\epsilon,y})$. and $\Phi_\epsilon : \Theta \to \mathcal{N}_\epsilon$ Let $\Phi_\epsilon(y) := h_\epsilon\Psi_{\epsilon,y}$. It is easy to check that $\Phi_\epsilon(y)$ has a compact support for all $y \in \Theta$.

Lemma 5.3. *Assume* (V) *and* (f_1)–(f_4), *then* $\lim_{\epsilon\to 0} I_\epsilon(\Phi_\epsilon(y)) = m_{V_0}$ *uniformly for* $y \in \Theta$.

Proof. Assume on the contrary that there exist some $\tau_0 > 0$, $\{y_n\} \subset \Theta$ and $\epsilon_n \to 0$ such that

$$|I_{\epsilon_n}(\Phi_{\epsilon_n}(y_n)) - m_{V_0}| \ge \tau_0. \tag{5.7}$$

We show that $\lim_{n\to\infty} h_{\epsilon_n} = 1$. Indeed, it follows from the definition of h_{ϵ_n} and Lemma 2.7 that, there exists $\rho > 0$ such that

$$\begin{aligned}0 < \rho &\le \int_{\mathbb{R}^3}(|(-\triangle)^{s/2}h_{\epsilon_n}\Psi_{\epsilon_n,y_n}|^2 + V(\epsilon_n x)|h_{\epsilon_n}\Psi_{\epsilon_n,y_n}|^2)\mathrm{d}x\\ &\quad + \int_{\mathbb{R}^3}\phi_{h_{\epsilon_n}\Psi_{\epsilon_n,y_n},t}|h_{\epsilon_n}\Psi_{\epsilon_n,y_n}|^2\mathrm{d}x\\ &= \int_{\mathbb{R}^3} f(h_{\epsilon_n}\Psi_{\epsilon_n,y_n})h_{\epsilon_n}\Psi_{\epsilon_n,y_n}\mathrm{d}x + \int_{\mathbb{R}^3}|h_{\epsilon_n}\Psi_{\epsilon_n,y_n}|^{2_s^*}\mathrm{d}x,\end{aligned}$$

which implies that $\{h_{\epsilon_n}\}$ can not converge to zero, that is, $h_{\epsilon_n} \ge h_0 > 0$ for some $h_0 > 0$ and large n. If $h_{\epsilon_n} \to \infty$, then from the boundedness of Ψ_{ϵ_n,y_n} we deduce that

$$\begin{aligned}&\frac{1}{h_{\epsilon_n}^2}\int_{\mathbb{R}^3}(|(-\triangle)^{s/2}\Psi_{\epsilon_n,y_n}|^2 + V(\epsilon_n x)|\Psi_{\epsilon_n,y_n}|^2)\mathrm{d}x + \int_{\mathbb{R}^3}\phi_{\Psi_{\epsilon_n,y_n},t}|\Psi_{\epsilon_n,y_n}|^2\mathrm{d}x\\ &= \int_{\mathbb{R}^3}\frac{f(h_{\epsilon_n}\Psi_{\epsilon_n,y_n})}{h_{\epsilon_n}^3\Psi_{\epsilon_n,y_n}^3}\Psi_{\epsilon_n,y_n}^4\mathrm{d}x + h_{\epsilon_n}^{2_s^*-4}\int_{\mathbb{R}^3}|\Psi_{\epsilon_n,y_n}|^{2_s^*}\mathrm{d}x\\ &\ge h_{\epsilon_n}^{2_s^*-4}\int_{\mathbb{R}^3}(\eta(|\epsilon_n z|)w(z))^{2_s^*}\mathrm{d}z\\ &\ge h_{\epsilon_n}^{2_s^*-4}\int_{B_{\frac{1}{2}}(0)}|w(z)|^{2_s^*}\mathrm{d}z \to \infty \quad \text{as } n\to\infty,\end{aligned} \tag{5.8}$$

which contradicts $h_{\epsilon_n} \to \infty$ as $n \to \infty$. Hence, $\{h_{\epsilon_n}\}$ is bounded uniformly for n. Up to a subsequence, we assume that $h_{\epsilon_n} \to T$. It suffices to prove that $T = 1$. Using Lebesgue's theorem and Lemma 2.5, one can verify that

$$\begin{aligned}
&\lim_{n\to\infty} \|\varPsi_{\epsilon_n,y_n}\|^2_{\epsilon_n} = \int_{\mathbb{R}^3} (|(-\triangle)^{s/2} w|^2 + V_0 w^2)\mathrm{d}x,\\
&\lim_{n\to\infty} \int_{\mathbb{R}^3} \phi_{\varPsi_{\epsilon_n,y_n},t} |\varPsi_{\epsilon_n,y_n}|^2 \mathrm{d}x = \int_{\mathbb{R}^3} \phi_{w,t}|w|^2\mathrm{d}x,\\
&\lim_{n\to\infty} \int_{\mathbb{R}^3} \frac{f(h_{\epsilon_n}\varPsi_{\epsilon_n,y_n})}{h^3_{\epsilon_n}\varPsi^3_{\epsilon_n,y_n}} \varPsi^4_{\epsilon_n,y_n}\mathrm{d}x = \int_{\mathbb{R}^3} \frac{f(Tw)}{T^3w^3} w^4 \mathrm{d}x,\\
&\lim_{n\to\infty} \int_{\mathbb{R}^3} |\varPsi_{\epsilon_n,y_n}|^{2^*_s}\mathrm{d}x = \int_{\mathbb{R}^3} |w|^{2^*_s}\mathrm{d}x. \qquad (5.9)
\end{aligned}$$

Then, from (5.8) we have

$$\frac{1}{T^2}\int_{\mathbb{R}^3} (|(-\triangle)^{s/2} w|^2 + V_0 w^2)\mathrm{d}x + \int_{\mathbb{R}^3} \phi_{w,t}|w|^2\mathrm{d}x = \int_{\mathbb{R}^3} \frac{f(Tw)}{T^3w^3} w^4\mathrm{d}x + T^{2^*_s-2}\int_{\mathbb{R}^3} |w|^{2^*_s}\mathrm{d}x. \qquad (5.10)$$

On the other hand, since w is a ground state solution of (4.1), then we have

$$\int_{\mathbb{R}^3} (|(-\triangle)^{s/2} w|^2 + V_0 w^2)\mathrm{d}x + \int_{\mathbb{R}^3} \phi_{w,t}|w|^2\mathrm{d}x = \int_{\mathbb{R}^3} f(w)w\mathrm{d}x + \int_{\mathbb{R}^3} |w|^{2^*_s}\mathrm{d}x. \qquad (5.11)$$

It follows from (5.10) and (5.11) that $T = 1$. Notice that

$$\begin{aligned}
I_{\epsilon_n}(\varPhi_{\epsilon_n}(y_n)) =& \frac{h^2_{\epsilon_n}}{2}\int_{\mathbb{R}^3} (|(-\triangle)^{s/2}\varPsi_{\epsilon_n,y_n}|^2 + V(\epsilon_n x)|\varPsi_{\epsilon_n,y_n}|^2)\mathrm{d}x\\
&+ \frac{h^4_{\epsilon_n}}{4}\int_{\mathbb{R}^3} \phi_{\varPsi_{\epsilon_n,y_n},t}|\varPsi_{\epsilon_n,y_n}|^2\mathrm{d}x - \int_{\mathbb{R}^3} F(h_{\epsilon_n}\varPsi_{\epsilon_n,y_n})\mathrm{d}x - \frac{h^{2^*_s}_{\epsilon_n}}{2^*_s}\int_{\mathbb{R}^3} |\varPsi_{\epsilon_n,y_n}|^{2^*_s}\mathrm{d}x.
\end{aligned}$$

Then $\lim_{n\to\infty} I_{\epsilon_n}(\varPhi_{\epsilon_n}(y_n)) = I_{V_0}(w) = m_{V_0}$, which contradicts (5.7). This completes the proof. □

For any $\delta > 0$, take $\rho = \rho(\delta) > 0$ such that $\varTheta_\delta \subset B_\rho(0)$, where $\varTheta_\delta$ was given in Section 1. Define $\chi : \mathbb{R}^3 \to \mathbb{R}^3$ by $\chi(x) = x$ for $|x| \le \rho$ and $\chi(x) = \rho x/|x|$ for $|x| \ge \rho$. Set $\beta_\epsilon : \mathcal{N}_\epsilon \to \mathbb{R}^3$ as

$$\beta_\epsilon(u) = \frac{\int_{\mathbb{R}^3} \chi(\epsilon x)u^4\mathrm{d}x}{\int_{\mathbb{R}^3} u^4\mathrm{d}x},\ \epsilon > 0.$$

Lemma 5.4. *Assume* (V) *and* (f$_1$)–(f$_4$), *then* $\lim_{\epsilon\to 0}\beta_\epsilon(\varPhi_\epsilon(y)) = y$ *uniformly for* $y \in \varTheta$.

Proof. Suppose, by contradiction, there exist $\delta_0 > 0$, $\{y_n\} \subset \varTheta$ and $\epsilon_n \to 0$ such that $|\beta_{\epsilon_n}(\varPhi_{\epsilon_n}(y_n)) - y_n| \ge \delta_0$. Set $z = (\epsilon_n x - y_n)/\epsilon_n$, we then have

$$\beta_{\epsilon_n}(\varPhi_{\epsilon_n}(y_n)) = y_n + \frac{\int_{\mathbb{R}^3}(\chi(\epsilon_n z + y_n) - y_n)|w(z)\eta(|\epsilon z|)|^4\mathrm{d}z}{\int_{\mathbb{R}^3}|w(z)\eta(|\epsilon z|)|^4\mathrm{d}z}.$$

Since $\varTheta \subset B_\rho(0)$ and $\chi|_{B_\rho(0)} \equiv \mathrm{Id}$, it follows from Lebesgue's theorem that $|\beta_{\epsilon_n}(\varPhi_{\epsilon_n}(y_n)) - y_n| = o(1)$, which is a contradiction. The lemma is proved. □

Let $H : \mathbb{R}^+ \to \mathbb{R}^+$ be a positive function satisfying $H(\epsilon) \to 0^+$ as $\epsilon \to 0^+$ and set

$$\varSigma_\epsilon := \{u \in \mathcal{N}_\epsilon : I_\epsilon(u) \le m_{V_0} + H(\epsilon)\}.$$

Given $y \in \varTheta$, we can use Lemma 5.3 to conclude that $H(\epsilon) = |I_\epsilon(\varPhi_\epsilon(y)) - m_{V_0}| \to 0$ as $\epsilon \to 0$. Thus, $\varPhi_\epsilon(y) \in \varSigma_\epsilon$ and $\varSigma_\epsilon \ne \emptyset$ for all $\epsilon > 0$. Moreover, the following lemma holds.

Lemma 5.5. *For any $\delta > 0$, $\lim_{\epsilon\to 0^+} \sup_{u\in\Sigma_\epsilon} \mathrm{dist}(\beta_\epsilon(u), \Theta_\delta) = 0$, where Θ_δ was given in Section 1.*

Proof. Let $\{\epsilon_n\} \subset \mathbb{R}$ and $\epsilon_n \to 0^+$, then there exists $\{u_n\} \subset \Sigma_{\epsilon_n}$ such that

$$\mathrm{dist}(\beta_{\epsilon_n}(u_n), \Theta_\delta) = \sup_{u\in\Sigma_{\epsilon_n}} \mathrm{dist}(\beta_{\epsilon_n}(u), \Theta_\delta) + o(1).$$

Thus, it suffices to find a sequence $\{y_n\} \subset \Theta_\delta$ such that

$$|\beta_{\epsilon_n}(u_n) - y_n| = o(1). \tag{5.12}$$

Since $\{u_n\} \subset \Sigma_{\epsilon_n} \subset \mathcal{N}_{\epsilon_n}$, $m_{V_0} \le c_{\epsilon_n} \le I_{\epsilon_n}(u_n) \le m_{V_0} + H(\epsilon_n)$, and $I_{\epsilon_n}(u_n) \to m_{V_0}$. Noting that $\{u_n\}$ satisfies the conditions of Lemma 5.2, then by Lemma 5.2, there exists $\{\tilde{y}_{\epsilon_n}\} \subset \mathbb{R}^3$ such that $y_n = \epsilon_n \tilde{y}_{\epsilon_n} \in \Theta_\delta$ for n large. Hence, we have

$$\beta_{\epsilon_n}(u_n) = y_n + \frac{\int_{\mathbb{R}^3} (\chi(\epsilon_n z + y_n) - y_n) u_n^4(z + \tilde{y}_{\epsilon_n}) \mathrm{d}z}{\int_{\mathbb{R}^3} u_n^4(z + \tilde{y}_{\epsilon_n}) \mathrm{d}z}.$$

Since $\epsilon_n z + y_n \to y^* \in \Theta$ and $\{u_n(x + \tilde{y}_{\epsilon_n})\}$ has a convergent subsequence by Lemma 5.2, we get $\beta_{\epsilon_n}(u_n) = y_n + o(1)$ and therefore $\{y_n\}$ is what we want. □

The proof of Theorem 1.3 will be done by applying the following Ljusternik–Schnirelmann abstract result (see [43]).

Proposition 5.6. *Let I be a C^1 functional defined on a C^1-Finsler manifold ν. If I is bounded from below and satisfies the (PS) condition, then I has at least $cat_\nu(\nu)$ distinct critical points.*

Proof of Theorem 1.3. We first show that (3.1) has at least $cat_{\Theta_\delta}(\Theta)$ positive solutions. Given $\delta > 0$, we can use Lemmas 5.3–5.5 to obtain some $\epsilon_\delta > 0$ such that, for every $\epsilon \in (0, \epsilon_\delta)$, the diagram $\Theta \xrightarrow{\Phi_\epsilon} \Sigma_\epsilon \xrightarrow{\beta_\epsilon} \Theta_\delta$ is well defined and $\beta_\epsilon \circ \Phi_\epsilon$ is homotopically equivalent to the embedding $\mathrm{Id} : \Theta \to \Theta_\delta$. Similarly to the proof of Lemmas 4.2 and 4.3 in [7], we obtain that $cat_{\Sigma_\epsilon}(\Sigma_\epsilon) \ge cat_{\Theta_\delta}(\Theta)$. Moreover, use the definition of Σ_ϵ and take ϵ_δ small if necessary such that $m_{V_0} + H(\epsilon) < \min\{m_{V_\infty}, \frac{s}{3} S_s^{\frac{3}{2s}}\}$. Thus, I_ϵ satisfies the Palais–Smale condition in Σ_ϵ. By the standard Ljusternik–Schnirelmann theory, I_ϵ restricted to $\mathcal{N}_\epsilon$ has at least $cat_{\Sigma_\epsilon}(\Sigma_\epsilon)$ critical points. We claim that any critical point u of I_ϵ restricted on $\mathcal{N}_\epsilon$ is a free critical point in H_ϵ. Let

$$\begin{aligned} G(u) = I'_\epsilon(u)u &= \int_{\mathbb{R}^3} (|(-\triangle)^{s/2} u|^2 + V(\epsilon x) u^2) \mathrm{d}x \\ &\quad + \int_{\mathbb{R}^3} \phi_{u,t} |u|^2 \mathrm{d}x - \int_{\mathbb{R}^3} f(u) u \mathrm{d}x + \int_{\mathbb{R}^3} |u|^{2_s^*} \mathrm{d}x. \end{aligned}$$

If u is a critical point of I_ϵ constrained on $\mathcal{N}_\epsilon$, then there exists $\nu \in \mathbb{R}$ such that $I'_\epsilon(u) = \nu G'(u)$. Hence,

$$0 = G(u) = I'_\epsilon(u)u = \nu G'(u)u. \tag{5.13}$$

By simple calculations, we have

$$\begin{aligned} G'(u)u &= 2\int_{\mathbb{R}^3} (|(-\triangle)^{s/2} u|^2 + V(\epsilon x) u^2) \mathrm{d}x + 4\int_{\mathbb{R}^3} \phi_{u,t} |u|^2 \mathrm{d}x \\ &\quad - \int_{\mathbb{R}^3} [f'(u) u^2 + f(u) u + 2_s^* |u|^{2_s^*}] \mathrm{d}x \\ &\le -2\int_{\mathbb{R}^3} (|(-\triangle)^{s/2} u|^2 + V(\epsilon x) u^2) \mathrm{d}x + (4 - 2_s^*) \int_{\mathbb{R}^3} |u|^{2_s^*} \mathrm{d}x < 0. \end{aligned}$$

Therefore, $\nu = 0$ and then $I'_\epsilon(u) = 0$ in H_ϵ. Consequently, we conclude that I_ϵ has at least $cat_{\Theta_\delta}(\Theta)$ critical points in H_ϵ. Similarly to the proof of Propositions 3.4, each critical point u_ϵ having been obtained belongs to space $L^\infty(\mathbb{R}^3)$ and $u_\epsilon \in C^{1,2s+\sigma-1}$ for some $\sigma \in (0,1)$ and u_ϵ is positive. Then, problem (2.3) has at least $cat_{\Theta_\delta}(\Theta)$ positive solutions in H_ϵ.

In the following, we show the concentration properties. Let $u_n \in \Sigma_{\epsilon_n}$ be a positive solution, where $\epsilon_n \to 0$. Then from the definition of Σ_ϵ we have $I_{\epsilon_n}(u_n) \to m_{V_0}$. Similarly to the arguments in Lemma 5.2, we deduce that there exists $\{\bar{y}_n\} \subset \mathbb{R}^3$ such that $w_n := u_n(\cdot + \bar{y}_n) \to \bar{u} \neq 0$ in $H^s(\mathbb{R}^3)$. Then for any $\varepsilon > 0$, there exists $R_\varepsilon > 0$ such that $\int_{|x|\geq R_\varepsilon} |w_n|^{2_s^*}\mathrm{d}x \leq \varepsilon$. Similarly to the iteration method in Proposition 3.4, from the above fact we deduce that there exists $C > 0$ independent of n such that $\|w_n\|_\infty \leq C$, and then $\|u_n\|_\infty \leq C$. Now, we claim that there exists $c > 0$ such that $\|u_n\|_\infty \geq c > 0$. If not, we suppose that $\|u_n\|_\infty \to 0$, by (f_1)-(f_3), for n large enough, we have

$$\int_{\mathbb{R}^3} (|(-\triangle)^{s/2}u_n|^2 + V_0|u_n|^2)\mathrm{d}x \leq \frac{V_0}{2}\int_{\mathbb{R}^3} |u_n|^2\mathrm{d}x,$$

which is a contradiction. So $c \leq \|u_n\|_\infty \leq C$ uniformly for n. It follows from Theorem 5.4 in [39] that for any $r > 0$ and any fixed $x_0 \in \mathbb{R}^3$, there exists $\tau \in (0,1)$ (independent of n, r) such that $|u_n(x) - u_n(y)| \leq C_r|x-y|^\tau$ for any $x, y \in B_r(x_0)$, where $C_r > 0$ only depends on r. Since $u_n \in L^\infty(\mathbb{R}^3) \cap L^2(\mathbb{R}^3) \cap C^{1,2s+\sigma-1}(\mathbb{R}^3)$ for some $\sigma \in (0,1)$, we get $u_n(x) \to 0$ as $|x| \to \infty$. Let $\tilde{y}_n$ be any global maximum point of u_n and $y = x_0 := \tilde{y}_n$ and $r = 1$, then $|u_n(\tilde{y}_n) - u_n(x)| \leq C_1|x - \tilde{y}_n|^\tau$ for any $x \in B_1(\tilde{y}_n)$. Taking $R > 0$ (independent of n) small enough, we have $u_n(x) \geq \frac{c}{2}$ for $x \in B_R(\tilde{y}_n)$ uniformly for n. Then it follows from Lemma 5.2 that $v_n := u_n(x + \tilde{y}_n)$ is a solution of the following equation

$$\begin{cases} (-\triangle)^s v_n + V_n(x)v_n + \phi_{v_n,t}v_n = f(v_n) + |v_n|^{2_s^*-2}v_n \quad \text{in } \mathbb{R}^3, \\ v_n \in H^s(\mathbb{R}^3), v_n > 0, \end{cases}$$

where $V_n(x) = V(\epsilon_n x + \epsilon_n\tilde{y}_n)$. Moreover, $v_n \to v \neq 0$ in $H^s(\mathbb{R}^3)$ and $y_n \to y \in \Theta$ with $y_n = \epsilon_n\tilde{y}_n$. Here we know that v is a positive ground state solution of problem (4.1). The proof is complete. □

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References

[1] N. Ackermann, Lévy processes-from probability theory to finance and quantum groups. *Notices Am. Math. Soc.* **51** (2004) 320–1331.

[2] C. Alves and O. Miyagaki, Existence and concentration of solution for a class of fractional elliptic equation in $\mathbb{R}^N$ via penalization method. To appear in *Cal. Var. Partial Differ. Equ.* (2016).

[3] T. D'Aprile and J. Wei, On bound states concentrating on spheres for the Maxwell-Schrödinger equation. *SIAM J. Math. Anal.* **37** (2005) 321–342.

[4] A. Azzollini, P. d'Avenia and A. Pomponio, On the Schrödinger-Maxwell equations under the effect of a general nonlinear term. *Ann. Inst. Henri Poincaré, Anal. Non Linéaire* **27** (2010) 779–791.

[5] A. Azzollini and A. Pomponio, Groud state solutions for nonlinear Schrödinger-Maxwell equations. *J. Math. Anal. Appl.* **345** (2008) 90–108.

[6] A. Ambrosetti and D. Ruiz, Multiple bound states for the Schrödinger–Poisson problem. *Commun. Contemp. Math.* **10** (2008) 391–404.

[7] V. Benci and G. Cerami, Multiple positive solutions of some elliptic problems via the Morse theory and the domain topology. *Cal. Var. Partial Differ. Equ.* **2** (1994) 29–48.

[8] V. Benci and D. Fortunato, An eigenvalue problem for the Schrödinger-Maxwell equations. *Topol. Methods Nonlinear Anal.* **11** (1998) 283–293.

[9] H. Berestycki and P. Lions, Nonlinear scalar field equations. *I.* Existence of a ground state. *Arch. Ration. Mech. Anal.* **82** (1983) 313–345.

[10] B. Barrios, E. Colorado, R. Servadei and F. Soria, A critical fractional equation with concave-convex power nonlinearities. *Ann. Inst. Henri Poincaré, Anal. Non Linéaire* **32** (2015) 875–900.

[11] H. Brezis and E.H. Lieb, A relation between pointwise convergence of functions and convergence of functionals. *Proc. Amer. Math. Soc.* **8** (1983) 486–490.

[12] X. Chang and Z. Wang, Ground state of scalar field equations involving a fractional Laplacian with general nonlinearity. *Nonlinearity* **26** (2013) 479–494.

[13] G. Chen and Y. Zheng, Concentration phenomenon for fractional nonlinear Schrödinger equations. *Commun. Pure Appl. Anal.* **13** (2014) 2359–2376.

[14] G. Cotsiolis and N. Tavoularis, Best constants for Sobolev inequalities for higher order fractional derivatives. *J. Math. Anal. Appl.* **295** (2004) 225–236.

[15] J. Dávila, M. Pino and J. Wei, Concentrating standing waves for the fractional nonlinear Schrödinger equation. *J. Differ. Equ.* **256** (2014) 858–892.

[16] J. Dávila, M. Pino, S. Dipierro and E. Valdinoci, Concentration phenomena for the nonlocal Schrödinger equation with Dirichlet datum. *Anal. Partial Differ. Equ.* **8** (2015) 1165–1235.

[17] E. Di Nezza, G. Palatucci and E. Valdinoci, Hitchhiker's guide to the fractional Sobolev spaces. *Bull. Sci. Math.* **136** (2012) 521–573.

[18] E. Fabes, C. Kenig and R. Serapioni, The local regularity of solutions of degenerate elliptic equations. *Commun. Partial Differ. Equ.* **7** (1982) 77–116.

[19] P. Felmer, A. Quaas and J. Tan, Positive solutions of the nonlinear Schrödinger equation with the fractional Laplacian. *Proc. R. Soc. Edinb. Sect. A* **142** (2012) 1237–1262.

[20] I. Ekeland, On the variational principle. *J. Math. Anal. Appl.* **47** (1974) 324–353.

[21] M. Fall, F. Mahmoudi and E. Valdinoci, Ground states and concentration phenomena for the fractional Schrödinger equation. *Nonlinearity* **28** (2015) 1937–1961.

[22] R. Frank and E. Lenzmann, Uniqueness and non degeneracy of ground states for $(-\triangle)^s + Q - Q^{\alpha+1} = 0$ in $\mathbb{R}$. *Acta Math.* **210** (2013) 261–318.

[23] R. Frank, E. Lenzmann and L. Silvestre, Uniqueness of radial solutions for the fractional laplacian. *Commun. Pure. Appl. Math.* **69** (2016) 1671–1726.

[24] A. Giammetta, Fractional Schrödinger–Poisson-Slater system in one dimension. Preprint arXiv:1405.2796 (2014).

[25] X. He, Multiplicity and concentration of positive solutions for the Schrödinger–Poisson equations. *Z. Angew. Math. Phys.* **62** (2011) 869–889.

[26] X. He and W. Zou, Existence and concentration of ground states for Schrödinger–Poisson equations with critical growth. *J. Math. Phys.* **53** (2012) 023702.

[27] I. Ianni and G. Vaira, On concentration of positive bound states for the Schrödinger–Poisson problem with potentials. *Adv. Nonlinear Stud.* **8** (2008) 573–595.

[28] N. Laskin, Fractional Schrödinger equation. *Phys. Rev. E* **66** (2002) 056108.

[29] Z. Liu and S. Guo, On ground state solutions for the Schrödinger–Poisson equations with critical growth. *J. Math. Anal. Appl.* **412** (2014) 435–448.

[30] Z. Liu, S. Guo and Y. Fang, Multiple semiclassical states for coupled Schrödinger–Poisson equations with critical exponential growth. *J. Math. Phys.* **56** (2015) 041505.

[31] G. Palatucci and A. Pisante, Improved Sobolev embeddings, profile decomposition, and concentration compactness for fractional Sobolev spaces. *Calc. Var.* **56** (2015) 041505.

[32] P. Rabinowitz, On a class of nonlinear Schrödinger equations. *Z. Angew. Math. Phys.* **43** (1992) 270–291.

[33] D. Ruiz, Semiclassical states for coupled Schrödinger-Maxwell equations: Concentration around a sphere. *Math. Models Methods Appl. Sci.* **15** (2005) 141–164.

[34] D. Ruiz, The Schrödinger–Poisson equation under the effect of a nonlinear local term. *J. Funct. Anal.* **237** (2006) 655–674.

[35] S. Secchi, On fractional Schrödinger equations in $\mathbb{R}^N$ without the Ambrosetti-Rabinowitz condition. *Topol. Methods Nonlinear Anal.* **47** (2016) 19–41.

[36] S. Secchi, Ground state solutions for nonlinear fractional Schrödinger equations in $\mathbb{R}^N$. *J. Math. Phys.* **54** (2013) 031501.

[37] R. Servadei and E. Valdinoci, A Brezis-Nirenberg result for non-local critical equations in low dimension. *Commun. Pure Appl. Anal.* **6** (2013) 2445–2464.

[38] R. Servadei and E. Valdinoci, The Brezis-Nirenberg result for the fractional Laplacian. *Trans. Am. Math. Soc.* **367** (2015) 67–102.

[39] L. Silvestre, Hölder estimates for solutions of integro-differential equations like the fractional Laplace. *Indiana Univ. Math. J.* **55** (2006) 1155–1174.

[40] X. Shang and J. Zhang, On fractional Schrödinger equations in $\mathbb{R}^N$ with critical growth. *J. Math. Phys.* **54** (2013) 121502.

[41] X. Shang and J. Zhang, Ground states for fractional Schrödinger equations with critical growth. *Nonlinearity* **27** (2014) 187–207.

[42] X. Shang and J. Zhang, Concentrating solutions of nonlinear fractional Schrödinger equation with potentials. *J. Differ. Equ.* **258** (2015) 1106–1128.

[43] M. Struwe, Variational Methods; Application to Nonlinear Partial Differential Equations and Hamiltonian Systems. Springer-Verlag (2007).

[44] J. Tan, The Brezis-Nirenberg type problem involving the square root of the Laplacian. *Calc. Var. Partial Differ. Equ.* **42** (2011) 21–41.

[45] J. Wang, L. Tian, J. Xu and F. Zhang, Existence and concentration of positive solutions for semilinear Schrödinger–Poisson systems in $\mathbb{R}^3$. *Calc. Var. Partial Differ. Equ.* **48** (2013) 243–273.

[46] X. Wang, On concentration of positive bound states of nonlinear Schrödinger equations. *Commun. Math. Phys.* **153** (1993) 229–244.

[47] L. Zhao and F. Zhao, On the existence of solutions for the Schrödinger–Poisson equations. *J. Math. Anal. Appl.* **346** (2008) 155–169.

[48] J. Zhang, The existence and concentration of positive solutions for a nonlinear Schrödinger–Poisson system with critical growth. *J. Math. Phys.* **55** (2014) 031507.

[49] J. Zhang, J.M. Do Ó and M. Squassina, Fractional Schrödinger–Poisson systems with a general subcritical or critical nonlinearity. *Adv. Nonlinear Stud.* **16** (2016) 15–30.