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THE REGULARITY OF SOLUTIONS TO SOME VARIATIONAL PROBLEMS, INCLUDING THE p-LAPLACE EQUATION FOR $2 \le p < 3$

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Abstract. We consider the higher differentiability of solutions to the problem of minimizing

$$\int_\Omega [L(\nabla v(x)) + g(x, v(x))]\mathrm{d}x \quad \text{on} \quad u^0 + W_0^{1,p}(\Omega)$$

where $L(\xi) = l(|\xi|) = \frac{1}{p}|\xi|^p$ and $u^0 \in W^{1,p}(\Omega)$. We show that, for $2 \le p < 3$, under suitable regularity assumptions on g, there exists a solution u to the Euler–Lagrange equation associated to the minimization problem, such that

$$\nabla u \in W^{1,2}_{\mathrm{loc}}(\Omega).$$

In particular, for $g(x,u) = f(x)u$ with $f \in W^{1,2}(\Omega)$ and $2 \le p < 3$, any $W^{1,p}(\Omega)$ weak solution to the equation

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) = f$$

is in $W^{2,2}_{\mathrm{loc}}(\Omega)$.

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1. Introduction

We consider the regularity, in the sense of the higher differentiability, of solutions to the problem of minimizing

$$\int_\Omega [L(\nabla v(x)) + g(x, v(x))]\mathrm{d}x \quad \text{on} \quad u^0 + W_0^{1,p}(\Omega) \tag{1.1}$$

where $L(\xi) = l(|\xi|) = \frac{1}{p}|\xi|^p$ and $u^0 \in W^{1,p}(\Omega)$. As it is well known, for $p = 2$, under reasonable assumptions on the term g, there exists a solution u to the Euler–Lagrange equation associated to (1.1) that belongs to $W^{2,2}_{\mathrm{loc}}(\Omega)$; however, whenever $p > 2$, the matrix of the second derivatives of L vanishes at 0, an obstacle to establishing that the gradient of a solution u is a Sobolev function.

The regularity of solutions to minimization problems with a Lagrangian L growing like $\frac{1}{p}|\xi|^p$ or, more precisely, to their Euler–Lagrange equations, has been known since the work of Ladyzhenskaya and Uraltseva [10]; however,

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in their work, the Lagrangians are defined near the origin so as to satisfy a strict ellipticity condition of the kind

$$\sum_{i,j} \frac{\partial L^i}{\partial \xi_j} v_i v_j \geq \nu |v|^2$$

with $\nu > 0$, thus removing the degeneracy at the origin ($L(\xi) = \frac{1}{p}|\xi|^p$ does not satisfy this condition for $p > 2$); the same approach has been used since (*cf.*, for instance, [7] and, for different regularity results, [8, 15, 16]); hence, the assumptions on L used in establishing the regularity of solutions to problem (1.1) in the case of $p-$growth, do not apply to $L(\xi) = \frac{1}{p}|\xi|^p$, unless $p = 2$.

The purpose of the present paper is to show that, for $L(\xi) = \frac{1}{p}|\xi|^p$ and $2 \leq p < 3$, under suitable regularity assumptions on g (Assumption 2.1 below), there exists a solution u to to the Euler–Lagrange equation associated to the minimization of (1.1), such that

$$\nabla u \in W^{1,2}_{\mathrm{loc}}(\Omega).$$

In particular, for $g(x, u) = f(x)u$ with $f \in W^{1,2}(\Omega)$ and $2 \leq p < 3$, any $W^{1,p}(\Omega)$ weak solution to the equation

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) = f \tag{1.2}$$

is in $W^{2,2}_{\mathrm{loc}}(\Omega)$. Our assumption on f is stronger than the assumption required for the case $p = 2$ [13]. Hence, to extend the regularity result to $2 < p < 3$, we pay the price of assuming more regularity on f.

When $f = 0$, a solution to (1.2) is a p-harmonic function, and properties of such functions have been extensively studied (se, *e.g.*, [1, 2, 4, 5, 12, 14]). Still. it seems that the regularity we present in this paper is not known even in this case.

Our proof is based on a variation that is a nonlinear function of $|\nabla u|$, suited to contrast the fact that the matrix of the second derivatives of L tends to zero. The proof fails at $p = 3$, since in this case the required function of $|\nabla u|$ would have to grow like the logarithm, that is unbounded near the origin.

For semilinear problems, in particular for $L(\xi) = \frac{1}{p}|\xi|^p$, a recent comprehensive survey of pointwise estimates for the gradient of a solution (hence, different from our results) is presented in [9].

2. Notations and preliminary results

The transpose of a is a^T; the dimension of the space is N and $\Omega \subset \mathbb{R}^N$; $|\Omega|$ is the measure of Ω. For a fixed coordinate direction e_s, we set $\delta_{he_s}u$ to be the difference quotient of the function u, defined by $\delta_{he_s}u(x) = \frac{u(x+he_s)-u(x)}{h}$. For a variation η to be defined, D_η is such that $|\nabla\eta(x)| \leq D_\eta$ and $\operatorname{supp}\eta$ is the support of η. By H_v we mean the Hessian matrix of the function v.

The assumptions on g are:

Assumption 2.1. There exist $\tau \in L^1(\Omega)$ and a non-negative $\lambda_g \in L^2(\Omega)$ such that, for a.e. $x \in \Omega$ and every u, we have

(i) $g(x, u) \geq \tau(x) - \lambda_g(x)|u|$
(ii) $g_u(x, u) = f(x) + G(u)$, with $f \in W^{1,2}(\Omega)$ and G uniformly Lipschitzian of Lipschitz constant Λ_G and differentiable except at most finitely many values.

Under the assumptions on g, the composition of g_u with a Sobolev function v is a Sobolev function and $\nabla g_u(x, v(x)) = \nabla f(x) + G'(v(x)) \cdot \nabla v(x)$.

The purpose of this paper is to prove the following result:

Theorem 2.2. *Let Ω be a bounded open subset of $\mathbb{R}^N$, let $L(\xi) = \frac{1}{p}|\xi|^p$ for $2 \le p < 3$.*

i) *Let g satisfy Assumption 2.1; then, there exist u, a solution to the Euler–Lagrange equation,* i.e. *such that*

$$\int_\Omega [\langle \nabla L(\nabla u(x)), \nabla \eta(x)\rangle + g_u(x, u(x))\eta(x)]\mathrm{d}x = 0$$

for every $\eta \in C^1_c(\Omega)$, such that $\nabla u \in W^{1,2}_{\mathrm{loc}}(\Omega)$

ii) *Let f be in $W^{1,2}(\Omega)$ and let u be a $W^{1,p}(\Omega)$ weak solution to the equation*

$$\mathrm{div}(|\nabla u|^{p-2}\nabla u) = f;$$

then u is in $W^{2,2}_{\mathrm{loc}}(\Omega)$.

Under some additional assumptions, mainly when g is convex in the variable v (in particular, g linear in v), a solution to the Euler–Lagrange equation is actually a solution to the minimization problem (1.1).

The Proof is based on (a modification of) Nirenberg's method and will also depend on approximating L by the following auxiliary Lagrangian. For $\tau < 1 < T$ and $t \ge 0$, set

$$l_{\tau,T}(t) = \begin{cases} \frac{1}{2}\tau^{p-2}t^2 + \left(\frac{1}{p} - \frac{1}{2}\right)\tau^p & \text{for } 0 \le t < \tau \\ \frac{1}{p}|t|^p & \text{for } 0 \le t < T \\ \frac{1}{p}T^p + T^{p-1}(t-T) + \frac{1}{2}(p-1)T^{p-2}(t-T)^2 & \text{for } t \ge T \end{cases}$$

so that, for $t \ge 0$,

$$\frac{1}{2}(p-1)t^2 - (p-2)t - \frac{3-p}{2} + \frac{1}{p} = \frac{1}{p} + (t-1) + \frac{1}{2}(p-1)(t-1)^2 \le l_{\tau,T}(t) \le l(t) \tag{2.1}$$

and

$$l'_{\tau,T}(t) = \begin{cases} \tau^{p-2}t & \text{for} \quad 0 \le t < \tau \\ t^{p-1} & \text{for} \quad \tau \le t \le T \\ T^{p-1}(2-p) + (p-1)T^{p-2}t & \text{for} \quad t \ge T; \end{cases}$$

set also $L_{\tau,T}(\xi) = l_{\tau,T}(|\xi|)$.

Let $u^{\tau,T}$ be a solution to the problem of minimizing

$$\int_\Omega [L_{\tau,T}(\nabla v(x)) + g(x, v(x))]\mathrm{d}x \quad \text{on} \quad u^0 + W^{1,2}_0(\Omega). \tag{2.2}$$

The Lagrangian $L_{\tau,T}(|\xi|)$ is of quadratic growth; moreover, it satisfies a "quadratic strict convexity condition" in the sense that, for a positive constant ν, we have

$$\langle \nabla L(z_1) - \nabla L(z_2), z_1 - z_2\rangle \ge \nu|z_1 - z_2|^2; \tag{2.3}$$

hence (see [6]) we have that, for each i, both $u^{\tau,T} \in W^{2,2}_{\mathrm{loc}}(\Omega)$ and $\frac{l'_{\tau,T}(|\nabla u^{\tau,T}|)}{|\nabla u^{\tau,T}|}u^{\tau,T}_{x_i} \in W^{1,2}_{\mathrm{loc}}(\Omega)$.

In order to prove Theorem 2.2 we shall need the following Lemmas.

Lemma 2.3. *Let Ω and g as in Theorem 2.2; let $u^{\tau,T}$ be a solution to the minimization of (2.2); let $\phi \in W^{1,2}(\Omega)$ with support compactly contained in Ω. Then, for $s = 1, \ldots, N$, we have*

$$\int_\Omega \left\langle \frac{\mathrm{d}}{\mathrm{d}x_s} \nabla L_{\tau,T}(\nabla u^{\tau,T}), \nabla\phi \right\rangle = \int_\Omega \left(\frac{\mathrm{d}}{\mathrm{d}x_s} g_u(\cdot, u^{\tau,T}) \right) \phi$$

Proof.

a) We have that $\nabla u^{\tau,T}$ is in $W^{1,2}_{\text{loc}}(\Omega)$, so that $\frac{\mathrm{d}}{\mathrm{d}x_i}|\nabla u^{\tau,T}| = \langle \frac{\nabla u^{\tau,T}}{|\nabla u^{\tau,T}|}, \nabla u^{\tau,T}_{x_i} \rangle$. The map

$$\frac{l'_{\tau,T}(t)}{t} = \begin{cases} \tau^{p-2} & \text{for} \quad 0 \le t < \tau \\ t^{p-2} & \text{for} \quad \tau \le t \le T \\ \dfrac{T^{p-1}(2-p)}{t} + (p-1)T^{p-2} & \text{for} \quad t \ge T \end{cases}.$$

is (uniformly) Lipschitzian and it is not differentiable only at at $t = \tau$ and $t = T$; then, as it is known, the map $x \to \frac{l'_{\tau,T}(|\nabla u^{\tau,T}(x)|)}{|\nabla u^{\tau,T}(x)|}$ is a Sobolev function with

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x_i} \frac{l'_{\tau,T}(|\nabla u^{\tau,T}(x)|)}{|\nabla u^{\tau,T}(x)|} &= \left[\left(\frac{l'_{\tau,T}(t)}{t} \right)' \circ |\nabla u^{\tau,T}(x)| \right] \left\langle \frac{\nabla u^{\tau,T}}{|\nabla u^{\tau,T}|}, \nabla u^{\tau,T}_{x_i} \right\rangle \\ &= \begin{cases} 0 & \text{for } |\nabla u^{\tau,T}(x)| \le \tau \\ (p-2)|\nabla u^{\tau,T}(x)|^{p-3} \left\langle \dfrac{\nabla u^{\tau,T}}{|\nabla u^{\tau,T}|}, \nabla u^{\tau,T}_{x_i} \right\rangle & \text{for } \tau \le |\nabla u^{\tau,T}(x)| \le T \\ \left(\dfrac{T^{p-1}(p-2)}{|\nabla u^{\tau,T}(x)|^2} \right) \left\langle \dfrac{\nabla u^{\tau,T}}{|\nabla u^{\tau,T}|}, \nabla u^{\tau,T}_{x_i} \right\rangle & \text{otherwise.} \end{cases} \end{aligned} \tag{2.4}$$

Then

$$\frac{\mathrm{d}}{\mathrm{d}x_i} \left[\frac{l'_{\tau,T}(|\nabla u^{\tau,T}(x)|)}{|\nabla u^{\tau,T}(x)|} \cdot \nabla u^{\tau,T}(x) \right] = \frac{l'_{\tau,T}(|\nabla u^{\tau,T}(x)|)}{|\nabla u^{\tau,T}(x)|} \nabla u^{\tau,T}_{x_i}(x) + \left(\frac{\mathrm{d}}{\mathrm{d}x_i} \frac{l'_{\tau,T}(|\nabla u^{\tau,T}(x)|)}{|\nabla u^{\tau,T}(x)|} \right) \nabla u^{\tau,T}(x). \tag{2.5}$$

Both terms above are in $L^2_{\text{loc}}(\Omega)$: in fact, $\frac{l'_{\tau,T}(t)}{t}$ is bounded and, from (2.4), the absolute value of the second term is at most $|\nabla u_{x_i}|$. Hence, $\nabla L_{\tau,T}(\nabla u^{\tau,T})$ is in $W^{1,2}_{\text{loc}}(\Omega)$.

b) Under the assumptions of the Lemma, the Euler–Lagrange equation holds for $u^{\tau,T}$ in the sense that, for $\psi \in W^{1,2}(\Omega)$ with support compactly contained in Ω, we have

$$\int_\Omega [\langle \nabla L_r(\nabla u^{\tau,T}), \nabla\psi \rangle + g_u(x, u^{\tau,T})\psi] \mathrm{d}x = 0.$$

For h sufficiently small, and $s = 1, \ldots, N$, consider the variation $\psi = \delta_{-he_s}\phi$ to obtain

$$\begin{aligned} &\int_\Omega \left\langle \frac{\nabla L_{\tau,T}(\nabla u^{\tau,T}(x + he_s)) - \nabla L_{\tau,T}(\nabla u^{\tau,T}(x))}{h}, \nabla\phi(x) \right\rangle \mathrm{d}x \\ &\qquad = \int_\Omega g_u(x, u^{\tau,T}(x)) \frac{\phi(x - he_s) - \phi(x)}{-h} \mathrm{d}x = \int_\Omega (\delta_{h,e_s} g_u(x, u^{\tau,T}))\phi \mathrm{d}x. \end{aligned} \tag{2.6}$$

Since $\nabla L_{\tau,T}(\nabla u^{\tau,T})$ is in $W^{1,2}(B(\operatorname{supp}\psi, |h|))$, we obtain that the family $(\frac{\nabla L_{\tau,T}(\nabla u^{\tau,T}(x+he_i)) - \nabla L_{\tau,T}(\nabla u^{\tau,T}(x))}{h})_h$ is bounded in $L^2(B(\operatorname{supp}\psi, |h|))$ and we can assume the existence of a sequence (h_n) such that

$$\frac{\nabla L_{\tau,T}(\nabla u^{\tau,T}(x + h_n e_i)) - \nabla L_{\tau,T}(\nabla u^{\tau,T}(x))}{h_n} \rightharpoonup \frac{\mathrm{d}}{\mathrm{d}x_i} \nabla L_{\tau,T}(\nabla u^{\tau,T})$$

so that the left hand side of (2.6) converges to $\int_\Omega \langle \frac{\mathrm{d}}{\mathrm{d}x_s} \nabla L_{\tau,T}(\nabla u^{\tau,T}), \nabla\phi \rangle$.

We also have, from Assumption 2.1, that $g_u(x, u^{\tau,T})$ is in $W^{1,2}(\Omega)$ and hence that the family $\left(\delta_{h,e_s} g_u(x, u^{\tau,T})\right)_h$ is bounded; then we can assume that along the sequence (h_n), we have that $\delta_{h_n,e_s} g_u(x, u^{\tau,T}) \rightharpoonup \frac{\mathrm{d}}{\mathrm{d}x_s} g_u(x, u^{\tau,T})$, thus proving the Lemma. $\square$

Lemma 2.4. *There exists K (independent of τ, T), such that $\|\nabla u^{\tau,T}\|_{L^2(\Omega)} \leq K$ and $\|u^{\tau,T}\|_{L^2(\Omega)} \leq K$.*

Proof. Choose β such that $2\beta P^2 = \frac{p-1}{8}$ and call k_0 the resulting constant $V + \frac{1}{2\beta}\int |f|^2 + 2\beta P^2 \int |\nabla w^0|^2$. We obtain

$$\frac{3}{8}(p-1)\int |\nabla u^{\tau,T}|^2 \leq 2\frac{(p-2)^2}{p-1}|\Omega| + \frac{1}{8}(p-1)\int |\nabla u^{\tau,T}|^2 + k_0$$

so that $\frac{1}{4}(p-1)\int |\nabla u^{\tau,T}|^2 \leq 2\frac{(p-2)^2}{p-1}|\Omega| + k_0$; hence, there exists k_1 such that

$$\int |\nabla u^{\tau,T}|^2 \leq k_1.$$

From this, making use of $w^0 \in W^{1,2}$ and of Poincaré's inequality, we infer that for some k_2, we also have $\int_\Omega |u^{\tau,T}|^2 \leq k_2$. The constants k_1, k_2 are independent of τ, T.

a) Call P the Poincaré constant in $W_0^{1,2}(\Omega)$; we have $\int |u^{\tau,T}|^2 = \int |w^0 - (u^{\tau,T} - w^0)|^2 \leq 2\int |w^0|^2 + 2P^2 \int |\nabla(u^{\tau,T} - w^0)|^2 \leq 2\int |w^0|^2 + 4P^2 \int |\nabla u^{\tau,T}|^2 + 4P^2 \int |\nabla w^0|^2$. From Assumption 2.1 and from $\lambda_g |u^{\tau,T}| \leq \frac{1}{2\beta}\lambda_g^2 + \frac{1}{2}\beta |u^{\tau,T}|^2$, for a constant β to be fixed, we obtain

$$\int_\Omega g(x, u^{\tau,T}) \geq \int \tau - \frac{1}{2\beta}\int (\lambda_g)^2 - \frac{1}{2}\beta 4P^2 [\int |\nabla u^{\tau,T}|^2 + \int |\nabla w^0|^2]. \tag{2.7}$$

b) From (2.1) we have

$$\begin{aligned} V = \int_\Omega [L(\nabla u(x)) + g(x, u(x))]\mathrm{d}x &\geq \int_\Omega [L_{\tau,T}(\nabla u(x)) + g(x, u(x))]\mathrm{d}x \\ &\geq \int_\Omega [L_{\tau,T}(\nabla u^{\tau,T}(x)) + g(x, u^{\tau,T}(x))]\mathrm{d}x. \end{aligned}$$

Hence, again from (2.1) and from Assumption 2.1,

$$\begin{aligned} \int_\Omega \left[\frac{1}{2}(p-1)|\nabla u^{\tau,T}|^2 - (p-2)|\nabla u^{\tau,T}| - \frac{3-p}{2} + \frac{1}{p}\right] &\leq V - \int_\Omega g(x, u^{\tau,T}) \leq V - \int \tau + \int \lambda_g |u^{\tau,T}| \\ &\leq V + \int -\tau + \frac{1}{2\beta}\int (\lambda_g)^2 + 2\beta P^2 \int |\nabla u^{\tau,T}|^2 + 2\beta P^2 \int |\nabla w^0|^2 \end{aligned}$$

that gives, since $-\frac{3-p}{2} + \frac{1}{p} > 0$ for $p > 2$,

$$\left[\frac{p-1}{2} - 2\beta P^2\right]\int |\nabla u^{\tau,T}|^2 \leq \int (p-2)|\nabla u^{\tau,T}| + V - \int \tau + \frac{1}{2\beta}\int (\lambda_g)^2 + 2\beta P^2 \int |\nabla w^0|^2$$

Choose β such that $2\beta P^2 = \frac{p-1}{8}$ and call k_0 the resulting constant $V - \int \tau + \frac{1}{2\beta}\int (\lambda_g)^2 + 2\beta P^2 \int |\nabla w^0|^2$. We obtain

$$\frac{3}{8}(p-1)\int |\nabla u^{\tau,T}|^2 \leq 2\frac{(p-2)^2}{p-1}|\Omega| + \frac{1}{8}(p-1)\int |\nabla u^{\tau,T}|^2 + k$$

so that $\frac{1}{4}(p-1)\int |\nabla u^{\tau,T}|^2 \leq 2\frac{(p-2)^2}{p-1}|\Omega| + k$; hence, there exists k_1 such that

$$\int |\nabla u^{\tau,T}|^2 \leq k_1.$$

From this, making use of $w^0 \in W^{1,2}$ and of Poincaré's inequality, we infer that for some k_2, we also have $\int_\Omega |u^{\tau,T}|^2 \leq k_2$. The constants k_1, k_2 are independent of τ, T. $\square$

3. Proof of Theorem 1

Proof. Since the case $p = 2$ is well known, we shall assume that $2 < p < 3$.

a) From (2.5) we have

$$\frac{\mathrm{d}}{\mathrm{d}x_s}\left(\frac{l'_{\tau,T}(|\nabla u^{\tau,T}|)}{|\nabla u^{\tau,T}|}u^{\tau,T}_{x_i}\right) = \begin{cases} \tau^{p-2}u^{\tau,T}_{x_i x_s} \\ (p-2)|\nabla u^{\tau,T}|^{p-3}\left\langle \dfrac{\nabla u^{\tau,T}}{|\nabla u^{\tau,T}|}, \nabla u^{\tau,T}_{x_s}\right\rangle u^{\tau,T}_{x_i} + |\nabla u^{\tau,T}|^{p-2}u^{\tau,T}_{x_i x_s} \\ \dfrac{T^{p-1}(p-2)}{|\nabla u^{\tau,T}|^2}\left\langle \dfrac{\nabla u^{\tau,T}}{|\nabla u^{\tau,T}|}, \nabla u^{\tau,T}_{x_s}\right\rangle u^{\tau,T}_{x_i} + \left(\dfrac{T^{p-1}(2-p)}{|\nabla u^{\tau,T}|} + (p-1)T^{p-2}\right)u^{\tau,T}_{x_i x_s} \end{cases}$$
$$= \begin{cases} \tau^{p-2}u^{\tau,T}_{x_i x_s}, \\ |\nabla u^{\tau,T}|^{p-2}\left[(p-2)\left\langle \dfrac{\nabla u^{\tau,T}}{|\nabla u^{\tau,T}|}, \nabla u^{\tau,T}_{x_s}\right\rangle \dfrac{u^{\tau,T}_{x_i}}{|\nabla u^{\tau,T}|} + u^{\tau,T}_{x_i x_s}\right], \\ \dfrac{T^{p-2}(p-2)}{|\nabla u^{\tau,T}|}\left[T\left\langle \dfrac{\nabla u^{\tau,T}}{|\nabla u^{\tau,T}|}, \nabla u^{\tau,T}_{x_s}\right\rangle \dfrac{u^{\tau,T}_{x_i}}{|\nabla u^{\tau,T}|} + \left(-T + \dfrac{p-1}{p-2}|\nabla u^{\tau,T}|\right)u^{\tau,T}_{x_i x_s}\right], \end{cases} \tag{3.1}$$

for $|\nabla u^{\tau,T}| < \tau$, for $\tau \le |\nabla u^{\tau,T}| < T$ and for $|\nabla u^{\tau,T}| > T$ respectively.

b) Set

$$\lambda_{\tau,T}(t) = \begin{cases} \tau^{2-p} & \text{for} \quad |t| < \tau \\ t^{2-p} & \text{for} \quad \tau \le |t| \le T \\ \dfrac{t}{T^{p-2}(p-2)[-T + \frac{p-1}{p-2}t]} & \text{for} \quad t \ge T; \end{cases}$$

$\lambda_{\tau,T}$ is globally Lipschitzian and differentiable except at $t = \tau$; then, since $x \to |\nabla u^{\tau,T}(x)|$ is in $W^{1,2}_{\mathrm{loc}}(\Omega)$, the function $x \to \lambda_{\tau,T}(|\nabla u^{\tau,T}(x)|)$ is in $W^{1,2}_{\mathrm{loc}}(\Omega)$ and

$$\frac{\mathrm{d}}{\mathrm{d}x_i}\lambda_{\tau,T}(|\nabla u^{\tau,T}|) = \begin{cases} 0 & \text{for} \quad |\nabla u^{\tau,T}(x)| < \tau \\ (2-p)|\nabla u^{\tau,T}|^{1-p}\left\langle \dfrac{\nabla u^{\tau,T}}{|\nabla u^{\tau,T}|}, \nabla u^{\tau,T}_{x_i}\right\rangle & \text{for} \quad \tau < |\nabla u^{\tau,T}| < T \\ \dfrac{-1}{T^{p-3}(p-2)\left(-T + \dfrac{p-1}{p-2}|\nabla u^{\tau,T}|\right)^2}\left\langle \dfrac{\nabla u^{\tau,T}}{|\nabla u^{\tau,T}|}, \nabla u^{\tau,T}_{x_i}\right\rangle & \text{for} \quad |\nabla u^{\tau,T}| \ge T. \end{cases}$$

Then, the map

$$x \to \gamma^s_{\tau,T}(x) = \lambda_{\tau,T}(|\nabla u^{\tau,T}(x)|)u^{\tau,T}_{x_s}(x)$$

is in $W^{1,2}_{\mathrm{loc}}(\Omega)$ and

$$|\gamma^s_{\tau,T}(x)| \le \begin{cases} \tau^{3-p} & \text{for} \quad |\nabla u^{\tau,T}(x)| < \tau \\ |\nabla u^{\tau,T}(x)|^{3-p} & \text{for} \quad \tau < |\nabla u^{\tau,T}(x)| < T \\ |\nabla u^{\tau,T}(x)| & \text{for} \quad |\nabla u^{\tau,T}(x)| \ge T. \end{cases} \tag{3.2}$$

Moreover,

$$\frac{\mathrm{d}}{\mathrm{d}x_i}\gamma^s_{\tau,T}=\begin{cases}\tau^{2-p}u^{\tau,T}_{x_sx_i}(2-p)|\nabla u^{\tau,T}|^{2-p}\left\langle\frac{\nabla u^{\tau,T}}{|\nabla u^{\tau,T}|},\nabla u^{\tau,T}_{x_i}\right\rangle\frac{u^{\tau,T}_{x_s}}{|\nabla u^{\tau,T}|}+|\nabla u^{\tau,T}|^{2-p}u^{\tau,T}_{x_sx_i}\\ \frac{1}{T^{p-2}(p-2)}\left[\frac{-T}{\left(-T+\frac{p-1}{p-2}|\nabla u^{\tau,T}|\right)^2}\langle\frac{\nabla u^{\tau,T}}{|\nabla u^{\tau,T}|},\nabla u^{\tau,T}_{x_i}\rangle u^{\tau,T}_{x_s}+\left(\frac{|\nabla u^{\tau,T}|}{-T+\frac{p-1}{p-2}|\nabla u^{\tau,T}|}\right)u^{\tau,T}_{x_sx_i}\right]\end{cases}$$

$$=\begin{cases}\tau^{2-p}u^{\tau,T}_{x_sx_i},\\ |\nabla u^{\tau,T}|^{2-p}\left[u^{\tau,T}_{x_sx_i}-(p-2)\left\langle\frac{\nabla u^{\tau,T}}{|\nabla u^{\tau,T}|},\nabla u^{\tau,T}_{x_i}\right\rangle\frac{u^{\tau,T}_{x_s}}{|\nabla u^{\tau,T}|}\right],\\ \frac{|\nabla u^{\tau,T}|}{T^{p-2}(p-2)\left(-T+\frac{p-1}{p-2}|\nabla u^{\tau,T}|\right)}\left[u^{\tau,T}_{x_sx_i}-\frac{T}{\left(-T+\frac{p-1}{p-2}|\nabla u^{\tau,T}|\right)}\left\langle\frac{\nabla u^{\tau,T}}{|\nabla u^{\tau,T}|},\nabla u^{\tau,T}_{x_i}\right\rangle\frac{u^{\tau,T}_{x_s}}{|\nabla u^{\tau,T}|}\right],\end{cases}\tag{3.3}$$

for $|\nabla u^{\tau,T}|<\tau$, for $\tau\le|\nabla u^{\tau,T}|<T$ and for $|\nabla u^{\tau,T}|>T$ respectively.

c) From (3.1) and (3.3) we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}x_s}\left(\frac{l'_{\tau,T}(|\nabla u^{\tau,T}|)}{|\nabla u^{\tau,T}|}u^{\tau,T}_{x_i}\right)\cdot\frac{\mathrm{d}}{\mathrm{d}x_i}\gamma^s_{\tau,T}=u^2_{x_ix_s},\ \text{for } |\nabla u^{\tau,T}|\le\tau;\\ &=\left(u_{x_sx_i}-(p-2)\left\langle\frac{\nabla u^{\tau,T}}{|\nabla u^{\tau,T}|},\nabla u^{\tau,T}_{x_i}\right\rangle\frac{u^{\tau,T}_{x_s}}{|\nabla u^{\tau,T}|}\right)\\ &\quad\times\left((p-2)\left\langle\frac{\nabla u^{\tau,T}}{|\nabla u^{\tau,T}|},\nabla u^{\tau,T}_{x_s}\right\rangle\frac{u^{\tau,T}_{x_i}}{|\nabla u^{\tau,T}|}+u_{x_ix_s}\right),\ \text{for } \tau\le|\nabla u^{\tau,T}|\le T;\\ &=\frac{1}{(-T+\frac{p-1}{p-2}|\nabla u^{\tau,T}|)}\left[u^{\tau,T}_{x_sx_i}-\frac{T}{(-T+\frac{p-1}{p-2}|\nabla u^{\tau,T}|)}\left\langle\frac{\nabla u^{\tau,T}}{|\nabla u^{\tau,T}|},\nabla u^{\tau,T}_{x_i}\right\rangle\frac{u^{\tau,T}_{x_s}}{|\nabla u^{\tau,T}|}\right]\\ &\quad\times\left[T\left\langle\frac{\nabla u^{\tau,T}}{|\nabla u^{\tau,T}|},\nabla u^{\tau,T}_{x_s}\right\rangle\frac{u^{\tau,T}_{x_i}}{|\nabla u^{\tau,T}|}+\left(-T+\frac{p-1}{p-2}|\nabla u^{\tau,T}|\right)u^{\tau,T}_{x_ix_s}\right],\ \text{for } |\nabla u^{\tau,T}|>T.\end{aligned}$$

For future use, notice that, summing over s, we have

$$\sum_{i,s}\frac{\mathrm{d}}{\mathrm{d}x_s}\left(\frac{l'_{\tau,T}(|\nabla u^{\tau,T}|)}{|\nabla u^{\tau,T}|}u^{\tau,T}_{x_i}\right)\cdot\frac{\mathrm{d}}{\mathrm{d}x_i}\gamma^s_{\tau,T}$$

$$=\begin{cases}|H_u|^2 & \text{for}\quad |\nabla u^{\tau,T}|\le\tau\\ |H_u|^2-(p-2)^2\left(\frac{\nabla u^{\tau,T}}{|\nabla u^{\tau,T}|}H_u\frac{\nabla u^{\tau,T}}{|\nabla u^{\tau,T}|}\right)^2 & \text{for}\quad \tau\le|\nabla u^{\tau,T}|\le T\\ |H_u|^2-\left(\frac{T}{-T+\frac{p-1}{p-2}|\nabla u^{\tau,T}|}\right)^2\left(\frac{\nabla u^{\tau,T}}{|\nabla u^{\tau,T}|}H_u\frac{\nabla u^{\tau,T}}{|\nabla u^{\tau,T}|}\right)^2 & \text{for}\quad |\nabla u^{\tau,T}|\ge T\end{cases}$$

so that

$$\sum_{i,s} \frac{\mathrm{d}}{\mathrm{d}x_s} \frac{l'_{\tau,T}(|\nabla u^{\tau,T}|)}{|\nabla u^{\tau,T}|} u^{\tau,T}_{x_i} \cdot \frac{\mathrm{d}}{\mathrm{d}x_i} \gamma^s_{\tau,T} \geq \begin{cases} |H_u|^2, \text{ for } |\nabla u^{\tau,T}| \leq \tau \\ (1-(p-2)^2)|H_u|^2, \text{ for } \tau \leq |\nabla u^{\tau,T}| \leq T \\ (p-2)|H_u|^2, \text{ for } |\nabla u^{\tau,T}| \geq T; \end{cases} \tag{3.4}$$

then, setting $\mu = \min\{p-2, (1-(p-2)^2)\}$, so that $\mu > 0$ since $p < 3$, we have

$$\sum_{i,s} \left(\frac{\mathrm{d}}{\mathrm{d}x_s} \frac{l'_{\tau,T}(|\nabla u^{\tau,T}|)}{|\nabla u^{\tau,T}|} u^{\tau,T}_{x_i} \cdot \frac{\mathrm{d}}{\mathrm{d}x_i} \gamma^s_{\tau,T} \right) \geq \mu |H_u|^2. \tag{3.5}$$

d) Let x^0 and δ^0 be such that $B(x^0, 4\delta^0) \subset\subset \Omega$. Let $\eta \in C^\infty_0(B(x^0, 2\delta^0))$ be such that $0 \leq \eta \leq 1$ and that $\eta(x) = 1$ for $x \in B(x^0, \delta^0)$; then the map $\phi = \eta^2 \cdot \gamma^s_{\tau,T}$ is in $W^{1,2}(\Omega)$ with support compactly contained in Ω and $\nabla\phi = 2\eta\nabla\eta\gamma^s_{\tau,T} + \eta^2\nabla\gamma^s_{\tau,T}$.

From Lemma 2.3, we infer that, for every s,

$$\int_\Omega \sum_i \left(\frac{\mathrm{d}}{\mathrm{d}x_s} \left(\frac{l'_{\tau,T}(|\nabla u^{\tau,T}|)}{|\nabla u^{\tau,T}|} u^{\tau,T}_{x_i} \right) \right) \left(\eta^2 \frac{\mathrm{d}}{\mathrm{d}x_i} \gamma^s_{\tau,T} + 2\eta\eta_{x_i}\gamma^s_{\tau,T} \right) \mathrm{d}x = G_s, \tag{3.6}$$

where $G_s = \int_\Omega \eta^2 (\frac{\mathrm{d}}{\mathrm{d}x_s} g_u(\cdot, u^{\tau,T}))\gamma^s_{\tau,T})\mathrm{d}x$; summing over s the previous equations, from (3.4) we obtain

$$\begin{aligned} \int_\Omega \eta^2 \mu |H_{u^{\tau,T}}|^2 \mathrm{d}x &\leq \int_\Omega \eta^2 \sum_{i,s} \left(\left(\frac{\mathrm{d}}{\mathrm{d}x_s} \frac{l'_{\tau,T}(|\nabla u^{\tau,T}|)}{|\nabla u^{\tau,T}|} u^{\tau,T}_{x_i} \right) \cdot \frac{\mathrm{d}}{\mathrm{d}x_i} \gamma^s_{\tau,T} \right) \mathrm{d}x \\ &= -\sum_{i,s} \int_\Omega 2\eta\eta_{x_i}\gamma^s_{\tau,T} \frac{\mathrm{d}}{\mathrm{d}x_s} \left(\frac{l'_{\tau,T}(|\nabla u^{\tau,T}|)}{|\nabla u^{\tau,T}|} u^{\tau,T}_{x_i} \right) \mathrm{d}x + \sum_s G_s. \end{aligned} \tag{3.7}$$

e) On the other hand we have

$$\begin{aligned} &\left(\frac{\mathrm{d}}{\mathrm{d}x_s} \left(\frac{l'_{\tau,T}(|\nabla u^{\tau,T}|)}{|\nabla u^{\tau,T}|} u^{\tau,T}_{x_i} \right) \right) \gamma^s_{\tau,T} \\ &= \begin{cases} u^{\tau,T}_{x_i x_s} u^{\tau,T}_{x_s} \\ \left((p-2) \left\langle \frac{\nabla u^{\tau,T}}{|\nabla u^{\tau,T}|}, \nabla u^{\tau,T}_{x_s} \right\rangle \frac{u^{\tau,T}_{x_i}}{|\nabla u^{\tau,T}|} + u^{\tau,T}_{x_i x_s} \right) u^{\tau,T}_{x_s} \\ \left(\left\langle \frac{\nabla u^{\tau,T}}{|\nabla u^{\tau,T}|}, \nabla u^{\tau,T}_{x_s} \right\rangle \frac{T u^{\tau,T}_{x_i}}{|\nabla u^{\tau,T}|} + \left(-T + \frac{p-1}{p-2} |\nabla u^{\tau,T}| \right) u^{\tau,T}_{x_i x_s} \right) \left(\frac{1}{-T + \frac{p-1}{p-2}|\nabla u^{\tau,T}|} \right) u^{\tau,T}_{x_s} \end{cases} \end{aligned}$$

respectively for $|\nabla u^{\tau,T}| < \tau$, for $\tau \leq |\nabla u^{\tau,T}| < T$ and for $|\nabla u^{\tau,T}| > T$. Then we obtain

$$\begin{aligned} &\sum_{i,s} 2\eta\eta_{x_i} \left(\frac{\mathrm{d}}{\mathrm{d}x_s} \left(\frac{l'_{\tau,T}(|\nabla u^{\tau,T}|)}{|\nabla u^{\tau,T}|} u^{\tau,T}_{x_i} \right) \right) \gamma^s_{\tau,T} \\ &= \begin{cases} 2\eta(\nabla\eta^T H_{u^{\tau,T}} \nabla u^{\tau,T}) \\ 2\eta \left[(\nabla\eta^T H_{u^{\tau,T}} \nabla u^{\tau,T}) + (p-2) \left(\frac{\nabla u^{\tau,T}}{|\nabla u^{\tau,T}|} H_{u^{\tau,T}} \frac{\nabla u^{\tau,T}}{|\nabla u^{\tau,T}|} \right) \langle \nabla u^{\tau,T}, \nabla\eta \rangle \right] \\ 2\eta \left[(\nabla\eta^T H_{u^{\tau,T}} \nabla u^{\tau,T}) + \frac{T}{-T + \frac{p-1}{p-2}|\nabla u^{\tau,T}|} \left(\frac{\nabla u^{\tau,T}}{|\nabla u^{\tau,T}|} H_{u^{\tau,T}} \frac{\nabla u^{\tau,T}}{|\nabla u^{\tau,T}|} \right) \langle \nabla u^{\tau,T}, \nabla\eta \rangle \right] \end{cases} \end{aligned}$$

and, for a constant α to be fixed, we have

$$\sum_{i,s} 2\eta\eta_{x_i}\left(\frac{\mathrm{d}}{\mathrm{d}x_s}(\frac{l'_{\tau,T}(|\nabla u^{\tau,T}|)}{|\nabla u^{\tau,T}|}u^{\tau,T}_{x_i})\right)\gamma^s_{\tau,T}$$
$$\leq \begin{cases} \alpha\eta^2|H_{u^{\tau,T}}|^2+\frac{1}{\alpha}D^2_\eta|\nabla u^{\tau,T}|^2 \\ \alpha\eta^2|H_{u^{\tau,T}}|^2+\frac{1}{\alpha}D^2_\eta|\nabla u^{\tau,T}|^2+(p-2)[\alpha\eta^2|H_{u^{\tau,T}}|^2+\frac{1}{\alpha}D^2_\eta|\nabla u^{\tau,T}|^2] \\ \alpha\eta^2|H_{u^{\tau,T}}|^2+\frac{1}{\alpha}D^2_\eta|\nabla u^{\tau,T}|^2+(p-2)\left[\alpha\eta^2|H_{u^{\tau,T}}|^2+\frac{1}{\alpha}D^2_\eta|\nabla u^{\tau,T}|^2\right] \end{cases}$$

so that, for some constant $C(p)$, we have

$$\sum_{i,s} 2\eta\eta_{x_i}\left(\frac{\mathrm{d}}{\mathrm{d}x_s}\left(\frac{l'_{\tau,T}(|\nabla u^{\tau,T}|)}{|\nabla u^{\tau,T}|}u^{\tau,T}_{x_i}\right)\right)\gamma^s_{\tau,T}\leq C(p)\left[\alpha\eta^2|H_{u^{\tau,T}}|^2+\frac{1}{\alpha}D^2_\eta|\nabla u^{\tau,T}|^2\right] \tag{3.8}$$

f) It is left to estimate

$$\sum_s|G_s|=\sum_s\left|\int_\Omega\eta^2\left(\frac{\mathrm{d}}{\mathrm{d}x_s}g_u(\cdot,u^{\tau,T})\right)\gamma^s_{\tau,T})\mathrm{d}x\right|=\sum_s\left|\int f(\eta^2\frac{\mathrm{d}}{\mathrm{d}x_s}\gamma^s_{\tau,T}+2\eta\eta_{x_s}\gamma^s_{\tau,T})\right|.$$

We have $\frac{\mathrm{d}}{\mathrm{d}x_s}g_u(x,u^{\tau,T})=\frac{\mathrm{d}}{\mathrm{d}x_s}f(x)+G'(u^{\tau,T}(x))\frac{\mathrm{d}}{\mathrm{d}x_s}u^{\tau,T}(x)$; from the estimate (independent of τ,T) for $\|\nabla u^{\tau,T}\|_{L^2(\Omega)}$ of Lemma 2.4 and the estimate for $|\gamma^s_{\tau,T}|$ in (3.2), we obtain that, for some k_1, $\|\gamma^s_{\tau,T}\|_{L^2(\Omega)}\leq k_1$; we also have that

$$\left\|G'(u^{\tau,T}(x))\frac{\mathrm{d}}{\mathrm{d}x_s}u^{\tau,T}(x)\right\|_{L^2(\Omega)}\leq\Lambda_G K$$

and hence, for some K_G, we obtain

$$\sum_s|G_s|\leq K_G. \tag{3.9}$$

g) From (3.6)–(3.8) and (3.9) we obtain

$$\int_\Omega\eta^2\mu|H_{u^{\tau,T}}|^2\mathrm{d}x\leq C(p)\int_\Omega[\alpha\eta^2|H_{u^{\tau,T}}|^2+\frac{1}{\alpha}D^2_\eta|\nabla u^{\tau,T}|^2]\mathrm{d}x+K_G. \tag{3.10}$$

Choose α so that $C(p)\alpha=\dfrac{1}{2}\mu$. Recalling the estimate of Lemma 2.4 for $\|\nabla u^{\tau,T}\|_{L^2(\Omega)}$, we obtain that, for a constant H (independent of τ,T), we have

$$\int_\Omega\eta^2|H_{u^{\tau,T}}|^2\mathrm{d}x\leq H. \tag{3.11}$$

h) Set $\tau=\frac{1}{n}$ and $T=n$; set also $u^{\frac{1}{n},n}=u^n$ and $L^{\frac{1}{n},n}=L^n$. Recalling the estimates of Lemma 2.4, we can assume that

$$\int_{B(x^0,\delta^0)}|\nabla u^n|^2\leq K^2 \tag{3.12}$$

and

$$\int_{B(x^0,\delta^0)}|H_{u^n}|^2\leq K^2. \tag{3.13}$$

Then, the family $(\nabla u^n)_n$ is contained in a compact subset of $L^2(B(x^0,\delta^0))$.

Consider $\left(\frac{l'_n(|\nabla u^n|)}{|\nabla u^n|}u^n_{x_i}\right)_n$. From (3.1) we have

$$\left|\frac{\mathrm{d}}{\mathrm{d}x_s}\left(\frac{l'_n(|\nabla u^n|)}{|\nabla u^n|}u^n_{x_i}\right)\right| \le \begin{cases} |u^n_{x_i x_s}| & \text{for} \quad |\nabla u^n| < \tau \\ |\nabla u^n|^{p-2}\left[(p-2)|\nabla u^n_{x_s}| + |u^n_{x_i x_s}|\right] & \text{for} \quad \tau \le |\nabla u^n| < T \\ n^{p-2}(p-2)\left[|\nabla u^n_{x_s}| + |u^n_{x_i x_s}\right] & \text{for} \quad |\nabla u^n| > T. \end{cases}$$

We can write

$$|\nabla u^n|^{p-2}|\nabla u^n_{x_s}| \le \frac{p-2}{p-1}|\nabla u^n|^{p-1} + \frac{1}{p-1}|\nabla u^n_{x_s}|^{p-1}$$

and

$$|\nabla u^n|^{p-2}|u^n_{x_i x_s}| \le \frac{p-2}{p-1}|\nabla u^n|^{p-1} + \frac{1}{p-1}|u^n_{x_i x_s}|^{p-1}.$$

Since $\int_{B(x^0,\delta^0)}[|\nabla u^n|^{p-1}]^{\frac{2}{p-1}} = \int_\Omega |\nabla u^n|^2$, and, similarly, $\int_{B(x^0,\delta^0)}[|\nabla u^n_{x_s}|^{p-1}]^{\frac{2}{p-1}} = \int_\Omega |\nabla u^n_{x_s}|^2$ and the same for $|u^n_{x_i x_s}|^{p-1}$, from (3.13) and (3.12), we can assume that, for some constant K_p, independent of n, we have that, for $i = 1, \ldots, N$,

$$\int_{B(x^0,\delta^0)}\left|\nabla\left(\frac{l'_n(|\nabla u^n|)}{|\nabla u^n|}u^n_{x_i}\right)\right|^{\frac{2}{p-1}} \le K_p$$

and, since $\frac{2}{p-1} > 1$, that the family $(\frac{l'_n(|\nabla u^n|)}{|\nabla u^n|}\nabla u^n)_n$ is contained in a compact subset of $L^{\frac{2}{p-1}}(B(x^0,\delta^0))$.

The arbitrariness of x_0 allows us to extend the previous results from $B(x^0,\delta^0)$ to any $\omega \subset\subset \Omega$.

i) We claim that u is a solution to the Euler–Lagrange equation, *i.e.*, that, for every $\eta \in C^1_c(\Omega)$,

$$\int_\Omega [\langle \nabla L(\nabla u(x)), \nabla\eta(x)\rangle + g_u(x, u(x))\eta(x)]\mathrm{d}x = 0.$$

Fix $\eta \in C^1_c(\Omega)$. There exists a subsequence (n_ν) such that $u^{n_\nu} \to u$ in $L^2(\operatorname{supp}\eta)$, that $\nabla u^{n_\nu} \to \nabla u$ in $L^2(\operatorname{supp}\eta)$ and pointwise a.e., and that $\frac{l'_{n_\nu}(|\nabla u^{n_\nu}|)}{|\nabla u^{n_\nu}|}\nabla u^{n_\nu} \to d$ in $L^{\frac{2}{p-1}}(\operatorname{supp}\eta)$. Moreover, we have that

$$\int_\Omega [\langle \nabla L^{n_\nu}(\nabla u^{n_\nu}), \nabla\eta\rangle + g_u(x, u^{n_\nu})\eta]\,\mathrm{d}x = 0$$

and that $g_u(x, u) = f(x) + G(u)$, where G is uniformly Lipschitzian of Lipschitz constant Λ_G. Then, from $\int_\Omega |g_u(x, u^{n_\nu}) - g_u(x, u)|\mathrm{d}x \le \Lambda_G \int_\Omega |u^{n_\nu} - u|\mathrm{d}x$ we obtain

$$\int_\Omega [\langle \mathrm{d}, \nabla\eta\rangle + g_u(x, u)\eta]\,\mathrm{d}x = 0$$

and we have to show that $d = \nabla L(\nabla u)$.

Fix x such that $\nabla u^{n_\nu}(x) \to \nabla u(x)$. When $\nabla u(x) \neq 0$, there exists a ball $B(\nabla u(x), \varepsilon)$ such that for ν sufficiently large, $\nabla L^{n_\nu}(\xi) = \nabla L(\xi)$ for every $\xi \in B(\nabla u(x), \varepsilon)$, so that $d(x) = \lim \nabla L(\nabla u^{n_\nu}) = \nabla L(\nabla u(x))$; when $\nabla u(x) = 0$, we can assume that $|\nabla u^{n_\nu}| \le 1$; fix arbitrarily τ, then $l'_{n_\nu}(|\nabla u^{n_\nu}|) \le |\nabla u^{n_\nu}| \cdot \max\{\tau^{p-2}, |\nabla u^{n_\nu}|^{p-2}\}$, thus proving the claim.

This proves statement i) of the Theorem 2.2.

j) To prove statement ii), first notice that, in this case, a solution u to the Euler–Lagrange equation is a solution to the problem of minimizing

$$\int_\Omega [L(\nabla v(x)) + f(x)v(x)]\mathrm{d}x \quad \text{on} \quad u + W^{1,p}_0(\Omega):$$

in fact, the problem is jointly convex in $(\nabla v, v)$. We claim that, for this problem, uniqueness of solutions holds. Once this claim is proved, part i) will apply to u.

Proof of this claim: let u and w be solutions to the minimization problem; then, for every $\eta \in C_c^1(\Omega)$, we have

$$\int_\Omega \langle |\nabla u|^{p-2}\nabla u - |\nabla w|^{p-2}\nabla w, \nabla\eta\rangle \mathrm{d}x = 0.$$

Since $|\nabla u|^{p-2}\nabla u - |\nabla w|^{p-2}\nabla w \in L^q$ and $u - w \in W_0^{1,p}$, by approximating $u - w$ with a sequence in $C_c^1(\Omega)$, we obtain

$$\int_\Omega \langle |\nabla u|^{p-2}\nabla u - |\nabla w|^{p-2}\nabla w, \nabla u - \nabla w\rangle \mathrm{d}x = 0.$$

On the other hand, from [3, 11], we have, for any ξ_1 and ξ_2, that

$$\langle |\xi_1|^{p-2}\xi_1 - |\xi_2|^{p-2}\xi_2, \xi_1 - \xi_2\rangle \geq 2^{2-p}|\xi_1 - \xi_2|^p$$

and hence that

$$\int_\Omega 2^{2-p}|\nabla u - \nabla w|^p \mathrm{d}x = 0$$

that implies that $u = w$. □

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