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CONTROLLABILITY OF ISOTROPIC VISCOELASTIC BODIES OF MAXWELL–BOLTZMANN TYPE*

L. Pandolfi[1]

Abstract. In this paper we consider a viscoelastic three dimensional body (of Maxwell–Boltzmann type) controlled on (part of) the boundary. We assume that the material is isotropic and homogeneous. If the body is elastic (*i.e.* no dissipation due to past memory), controllability has been studied by several authors. We prove that the viscoelastic body inherits the controllability properties of the corresponding purely elastic system. The proof relays on cosine operator methods combined with moment theory.

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1. Introduction

We consider a linear viscoelastic body occupying a bounded region $\Omega \subseteq \mathbb{R}^3$. We assume that the body is isotropic and homogeneous so that the dynamics of the body is described by the operator $\mathcal{L}$:

$$\mathcal{L}\mathbf{u} = \mu\Delta\mathbf{u} + (\lambda+\mu)\nabla\left(\nabla\cdot\mathbf{u}\right), \quad \mathbf{u} = \mathbf{u}(x) \quad x\in\Omega.$$

Here $\mathbf{u}\in\mathbb{R}^3$ and $\lambda>0$, $\mu>0$ are the Lamé coefficients. We assume that λ and μ are constant (*i.e.* that the body is homogeneous).

Boldface denotes vectors. Note that also the space variable (*i.e.* x) is a vector, with $\dim x = \dim\mathbf{u}$, but we don't use boldface for the space variable (and not for the vector 0).

If the body is elastic, the evolution in time of the displacement is described by the Navier equation

$$\mathbf{u}'' = \mathcal{L}\mathbf{u} + \mathbf{F}, \qquad \begin{cases} \mathbf{u}(0) = \mathbf{u}_0 \\ \mathbf{u}'(0) = \mathbf{u}_1 \end{cases} \tag{1.1}$$

($\mathbf{u} = \mathbf{u}(x,t)$, $\mathbf{F} = \mathbf{F}(x,t)$, $x\in\Omega\subseteq\mathbb{R}^3$ and the prime denotes time derivative) while if the body is viscoelastic (of the Maxwell–Boltzmann type) the evolution in time of the displacement (denoted $\mathbf{w} = \mathbf{w}(x,t)\in\mathbb{R}^3$) is described by

$$\mathbf{w}'' = \mathcal{L}\mathbf{w} + \int_0^t M(t-s)\mathcal{L}\mathbf{w}(s)\,\mathrm{d}s + \mathbf{F}, \qquad \begin{cases} \mathbf{w}(0) = \mathbf{w}_0 \\ \mathbf{w}'(0) = \mathbf{w}_1 \end{cases} \tag{1.2}$$

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[1] Dipartimento di Scienze Matematiche "Giuseppe Luigi Lagrange", Politecnico di Torino, Corso Duca degli Abruzzi 24, 10129 Torino, Italy. luciano.pandolfi@polito.it

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(see [13], p. 112 and 113). In fact, the integral should extend from $-\infty$ but in the study of reachability for linear systems, the control being applied after a time t_0, say $t_0 = 0$, it is not restrictive to assume $\mathbf{w} = 0$ for $t < t_0 = 0$.

We apply a displacement on a part Γ of the boundary of Ω ($\Gamma = \partial\Omega$ is not excluded), both to the elastic and the viscoelastic body, and we are going to study whether is it possible to control the pair ((*velocity of displacement*)/(*displacement*)) to hit a prescribed target at a certain time T. *I.e.* we impose the following boundary condition for $t > 0$:

$$\mathbf{u}(x,t) = \mathbf{f}(x,t) \ x \in \Gamma \ \ \mathbf{u}(x,t) = 0 \ x \in \partial\Omega \setminus \Gamma \tag{1.3}$$

$$\mathbf{w}(x,t) = \mathbf{f}(x,t) \ x \in \Gamma \ \ \mathbf{w}(x,t) = 0 \ x \in \partial\Omega \setminus \Gamma \tag{1.4}$$

(if $\Gamma = \partial\Omega$ then disregard the condition on $\partial\Omega \setminus \Gamma$).

When the initial condition and the affine term $\mathbf{F}$ are zero and we want to stress dependence on the control $\mathbf{f}$, we write $\mathbf{u}^{\mathbf{f}}$, $\mathbf{w}^{\mathbf{f}}$.

Under the assumptions we shall state below, it turns out that $(\mathbf{u}'(t), \mathbf{u}(t))$ and $(\mathbf{w}'(t), \mathbf{w}(t))$ are $\left[\mathrm{H}^{-1}(\Omega)\right]^3 \times \left[\mathrm{L}^2(\Omega)\right]^3$-valued continuous functions of time so that their values at a fixed time T make sense in this space.

If the initial conditions are zero and also $\mathbf{F} = 0$, the *reachable sets at time* T are the sets

$$R_E(T) = \left\{ \left(\left(\mathbf{u}^{\mathbf{f}}\right)'(T), \mathbf{u}^{\mathbf{f}}(T) \right), \ \ \mathbf{f} \in \mathrm{L}^2\left(0, T; \left[\mathrm{L}^2(\Gamma)\right]^3\right) \right\},$$
$$R_V(T) = \left\{ \left(\left(\mathbf{w}^{\mathbf{f}}\right)'(T), \mathbf{w}^{\mathbf{f}}(T) \right), \ \mathbf{f} \in \mathrm{L}^2\left(0, T; \left[\mathrm{L}^2(\Gamma)\right]^3\right) \right\}.$$

Note that (both in the purely elastic and in the viscoelastic case) the reachable set increases with time.

Controllability at time T is the property $R_V(T) = \left[\mathrm{H}^{-1}(\Omega)\right]^3 \times \left[\mathrm{L}^2(\Omega)\right]^3$ for the viscoelastic system, and $R_E(T) = \left[\mathrm{H}^{-1}(\Omega)\right]^3 \times \left[\mathrm{L}^2(\Omega)\right]^3$ in the elastic case.

Under the assumptions we describe below, controllability in the purely elastic case, *i.e.* for the system (1.1)–(1.3), has been studied by several authors (see the references in Sect. 1.2). We are going to prove that the controllability property which holds in the purely elastic case is inherited by the viscoelastic system.

1.1. Notations, assumptions and the main results of this paper

In this section we state the assumptions, we describe preliminary results on the controllability of the elastic system and we state our main results, which will be proved in the next sections.

Assumption A) We assume:

- The kernel $M(t)$ is of class $\mathrm{H}^2(0,T)$ for every $T > 0$.
- The region $\Omega \subseteq \mathbb{R}^3$ is bounded and $\partial\Omega$ is of class C^2.
- We assume that both the Lamé constants λ and μ are positive and constant (*i.e.* the body is homogeneous).
- The subset Γ of $\partial\Omega$ will be called the *active part* of the boundary. The first assumption on Γ is that it is relatively open in $\partial\Omega$.

Known facts for the elastic system (1.1) with boundary conditions (1.3), proved in the references cited in Sect. 1.2:

- Let $T > 0$ and $\mathbf{f} = 0$. Let $\mathbf{F} \in \mathrm{L}^1\left(0, T; \left[\mathrm{L}^2(\Omega)\right]^3\right)$, $\mathbf{u}_0 \in [\mathrm{H}^1_0(\Omega)]^3$, $\mathbf{u}_1 \in [\mathrm{L}^2(\Omega)]^3$. Then

$$t \mapsto \mathbf{u}(t) \in \mathrm{C}\left([0,T]; [\mathrm{H}^1_0(\Omega)]^3\right) \cap \mathrm{C}^1\left([0,T]; [\mathrm{L}^2(\Omega)]^3\right).$$

- Let $T > 0$. Let $\mathbf{F} \in \mathrm{L}^1\left(0, T; [\mathrm{L}^2(\Omega)]^3\right)$, $f \in \mathrm{L}^2\left(0, T; [\mathrm{L}^2(\Gamma)]^3\right)$ and $\mathbf{u}_0 \in [\mathrm{L}^2(\Omega)]^3$, $\mathbf{u}_1 \in [\mathrm{H}^{-1}(\Omega)]^3$. Then

$$t \mapsto \mathbf{u}(t) \in \mathrm{C}\left([0,T]; [\mathrm{L}^2(\Omega)]^3\right) \cap \mathrm{C}^1\left([0,T]; [\mathrm{H}^{-1}(\Omega)]^3\right).$$

- *There exist relatively open subsets Γ of $\partial\Omega$ and times T such that the purely elastic system is controllable at time T, i.e.*

$$R_E(T) = \left[\mathrm{H}^{-1}(\Omega)\right]^3 \times \left[\mathrm{L}^2(\Omega)\right]^3. \tag{1.5}$$

<u>**Assumption B)**</u> The active part Γ of $\partial\Omega$ and $T>0$ are chosen in such a way that the purely elastic system (1.1) with boundary control (1.3) is controllable in time T, *i.e.* equality (1.5) holds.

Remark 1.1 (On the notations)**.** From now on, for the sake of readability, we drop the exponent 3 and we write simply $\mathrm{L}^2(\Omega)$, $\mathrm{H}^1_0(\Omega)$, $\mathrm{H}^{-1}(\Omega)$ instead of $\left[\mathrm{L}^2(\Omega)\right]^3$, $\left[\mathrm{H}^1_0(\Omega)\right]^3$, $\left[\mathrm{H}^{-1}(\Omega)\right]^3$.

We recall the following integration by parts formula, which we shall repeatedly use:

$$\int_\Omega (\mathcal{L}\mathbf{u})\cdot\boldsymbol{\phi}\,\mathrm{d}x = \int_\Gamma (\mathcal{T}\mathbf{u})\cdot\boldsymbol{\phi}\,\mathrm{d}\Gamma - \int_\Gamma (\mathcal{T}\boldsymbol{\phi})\cdot\mathbf{u}\,\mathrm{d}\Gamma + \int_\Omega \mathbf{u}\cdot(\mathcal{L}\boldsymbol{\phi})\,\mathrm{d}x. \tag{1.6}$$

The boundary operator $\mathcal{T}$, the *boundary traction,* is:

$$\mathcal{T}\boldsymbol{\phi} = \mu\,(\mathbf{n}\cdot\nabla)\,\boldsymbol{\phi} + (\lambda+\mu)\,(\nabla\cdot\mathbf{u})\,\mathbf{n}$$

where $\mathbf{n}$ is the exterior normal to $\partial\Omega$.

The following property, known as (output) *admissibility* or *direct inequality* is proved for example in ([11], Chap. IV):

Lemma 1.2. *Let $\mathbf{u}$ solve (1.1) and (1.3) with $\mathbf{f}=0$. For every $T>0$ there exists a number M such that*

$$|\mathcal{T}\mathbf{u}|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2(\Gamma))} \le M\left(|\mathbf{u}_0|^2_{\mathrm{H}^1_0(\Omega)} + |\mathbf{u}_1|^2_{\mathrm{L}^2(\Omega)} + |\mathbf{F}|^2_{\mathrm{L}^1(0,T;\mathrm{L}^2(\Omega))}\right). \tag{1.7}$$

Note that the inequality usually proved is

$$|\gamma_1\mathbf{u}|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2(\Gamma))} \le M\left(|\mathbf{u}_0|^2_{\mathrm{H}^1_0(\Omega)} + |\mathbf{u}_1|^2_{\mathrm{L}^2(\Omega)} + |\mathbf{F}|^2_{\mathrm{L}^1(0,T;\mathrm{L}^2(\Omega))}\right)$$

(γ_1 is the normal derivative on $\partial\Omega$) from which (1.7) follows because $\mathbf{u}=0$ on $\partial\Omega$ implies $\nabla u_k = \gamma_1 u_k$ for every component u_k of $\mathbf{u}$.

Our first, ancillary, result is as follows.

Theorem 1.3. *Let Assumption* A) *hold. Then:*

(1) *Let $\mathbf{F}\in\mathrm{L}^1(0,T;\mathrm{L}^2(\Omega))$ and $\mathbf{f}=0$. System (1.2) with initial conditions $\mathbf{w}(0)=\mathbf{w}_0\in\mathrm{H}^1_0(\Omega)$, $\mathbf{w}'(0)=\mathbf{w}_1\in\mathrm{L}^2(\Omega)$ admits a unique solution $\mathbf{w}\in\mathrm{C}\left([0,T];\mathrm{H}^1_0(\Omega)\right)\cap\mathrm{C}^1\left([0,T];\mathrm{L}^2(\Omega)\right)$ for every $T>0$.*
(2) *Let $\mathbf{F}\in\mathrm{L}^1(0,T;\mathrm{L}^2(\Omega))$. System (1.2) with initial conditions $\mathbf{w}(0)=\mathbf{w}_0\in\mathrm{L}^2(\Omega)$, $\mathbf{w}'(0)=\mathbf{w}_1\in\mathrm{H}^{-1}(\Omega)$ and boundary control $\mathbf{f}\in\mathrm{L}^2(0,T;\mathrm{L}^2(\Gamma))$ admits a unique solution $\mathbf{w}\in\mathrm{C}\left([0,T];\mathrm{L}^2(\Omega)\right)\cap\mathrm{C}^1\left([0,T];\mathrm{H}^{-1}(\Omega)\right)$ for every $T>0$.*
(3) *Let $\mathbf{f}=0$, $\mathbf{w}_0\in\mathrm{H}^1_0(\Omega)$, $\mathbf{w}_1\in\mathrm{L}^2(\Omega)$ and let $\mathbf{w}$ solve (1.2) and (1.3). Then, for every $T>0$ and every $\Gamma\subseteq\partial\Omega$ there exists M such that*

$$|\mathcal{T}\mathbf{w}|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2(\Gamma))} \le M\left(|\mathbf{w}_0|^2_{\mathrm{H}^1_0(\Omega)} + |\mathbf{w}_1|^2_{\mathrm{L}^2(\Omega)} + |\mathbf{F}|^2_{\mathrm{L}^1(0,T;\mathrm{L}^2(\Omega))}\right). \tag{1.8}$$

This inequality is the direct inequalities *of equation (1.2).*

We noted already that the statement of Theorem 1.3 holds when $M=0$.

The control result which we intend to prove is:

Theorem 1.4. *Let Assumptions* A) *and* B) *hold so that $R_E(T)=\mathrm{H}^{-1}(\Omega)\times\mathrm{L}^2(\Omega)$ and let $T_1>T$. Then, system (1.2), (1.4) is controllable at time T_1,* i.e. $R_V(T_1)=\mathrm{H}^{-1}(\Omega)\times\mathrm{L}^2(\Omega)$.

The proof is based on two main steps. **Step 1)** proves that if the purely elastic system (1.1) is controllable at time T then the reachable set $R_V(T)$ of the viscoelastic system is closed with finite codimension in $\mathrm{H}^{-1}(\Omega) \times \mathrm{L}^2(\Omega)$. Thanks to the fact that the reachable set is increasing, this holds at every $T_1 \geq T$. **Step 2)** proves that if the purely elastic system is controllable at time T and $T_1 > T$ then we have

$$R_V(T_1) = \mathrm{H}^{-1}(\Omega) \times \mathrm{L}^2(\Omega).$$

The organization of the paper is as follows: the next subsection recalls previous results while Section 2 presents preliminaries on the cosine operator theory and the proof of Theorem 1.3.

The two steps of the proof of Theorem 1.4 are in Section 1.4.

1.2. References to previous work and preliminaries

Controllability of the Navier equation (1.1) has been studied in several papers, of which we cite [7, 8, 11, 22]. The proofs in these papers are based on the *inverse inequality,* obtained using multiplier methods, *i.e.* it is proved that if Γ and T are suitably chosen then there exists $m > 0$ such that

$$m\left(|\mathbf{u}_0|^2_{\mathrm{H}^1_0(\Omega)} + |\mathbf{u}_1|^2_{\mathrm{L}^2(\Omega)}\right) \leq \int_0^T \int_\Gamma \left[\mu|\gamma_1\mathbf{u}|^2 + (\lambda+\mu)|\nabla\cdot\mathbf{u}|^2\right] \,\mathrm{d}\Gamma\,\mathrm{d}t. \tag{1.9}$$

Here it is assumed $\mathbf{f} = 0$, $\mathbf{F} = 0$ and of course $\mathbf{u}_0 \in \mathrm{H}^1_0(\Omega)$, $\mathbf{u}_1 \in \mathrm{L}^2(\Omega)$. As noted in ([11], p. 228), this inequality implies

$$m\left(|\mathbf{u}_0|^2_{\mathrm{H}^1_0(\Omega)} + |\mathbf{u}_1|^2_{\mathrm{L}^2(\Omega)}\right) \leq \int_0^T \int_\Gamma |\mathcal{T}\mathbf{u}|^2 \,\mathrm{d}\Gamma\,\mathrm{d}t \tag{1.10}$$

(for a different $m > 0$) and this is the "dual" formulation of the definition of controllability.

We mention that the inverse inequality in the paper [8] is proved also if the Lamé coefficients are (slowly) space varying and that isotropy is not assumed in [22].

In this paper, systems (1.1) and (1.2) are studied using a cosine operator approach and also moment methods. Let us define the following operators in $\mathrm{L}^2(\Omega)$:

$$\mathrm{dom}A = \mathrm{H}^2(\Omega) \cap \mathrm{H}^1_0(\Omega)\,, \qquad A\boldsymbol{\phi} = \mathcal{L}\boldsymbol{\phi}, \quad \mathcal{A} = i(-A)^{1/2}\,. \tag{1.11}$$

The operator A is selfadjoint with compact resolvent. Hence, $\mathrm{L}^2(\Omega)$ admits an orthonormal basis of eigenvectors of A. Let it be denoted $\{\boldsymbol{\phi}_n\}$ and let $-\lambda_n^2$ be the eigenvalue of $\boldsymbol{\phi}_n$ (it is known that the eigenvalues are negative and we assume that they are ordered in such a way that $\lambda_n \leq \lambda_{n+1}$). The operator $\mathcal{A}$ generates a C_0-group of operators on $\mathrm{L}^2(\Omega)$ so that we can define the operators

$$R_+(t) = \frac{1}{2}\left[e^{\mathcal{A}t} + e^{-\mathcal{A}t}\right], \qquad R_-(t) = \frac{1}{2}\left[e^{\mathcal{A}t} - e^{-\mathcal{A}t}\right]$$

(the operator $R_+(t)$ is called the *cosine operator generated by* A*). Note that* $t \mapsto R_+(t)\mathbf{u}$ *and* $t \mapsto R_-(t)\mathbf{u}$ *belong to* $\mathcal{L}\left(\mathrm{L}^2(\Omega), \mathrm{C}\left([0,T], \mathrm{L}^2(\Omega)\right)\right)$ *and also to* $\mathcal{L}\left(\mathrm{H}^1_0(\Omega), \mathrm{C}\left([0,T], \mathrm{H}^1_0(\Omega)\right)\right)$.

The operators $R_+(t)$ and $R_-(t)$ have the following expansions in series of $\{\boldsymbol{\phi}_n\}$

$$R_+(t)\left(\sum_{n=1}^{+\infty} \alpha_n \boldsymbol{\phi}_n\right) = \sum_{n=1}^{+\infty} \left(\alpha_n \cos\lambda_n t\right) \boldsymbol{\phi}_n(x),$$

$$R_-(t)\left(\sum_{n=1}^{+\infty} \alpha_n \boldsymbol{\phi}_n\right) = i\left(\sum_{n=1}^{+\infty} \left(\alpha_n \sin\lambda_n t\right) \boldsymbol{\phi}_n(x)\right). \tag{1.12}$$

The solutions of problem (1.1)–(1.3) are given by (see [4])

$$\mathbf{u}(t) = R_+(t)\mathbf{u}_0 + \mathcal{A}^{-1}R_-(t)\mathbf{u}_1 + \mathcal{A}^{-1}\int_0^t R_-(t-s)\mathbf{F}(s)\,\mathrm{d}s - \mathcal{A}\int_0^t R_-(t-s)D\mathbf{f}(s)\,\mathrm{d}s. \tag{1.13}$$

The operator D in this formula is the operator $\mathbf{f} \mapsto \mathbf{u} = D\mathbf{f}$ where $\mathbf{u}$ solves

$$\mathcal{L}\mathbf{u} = 0, \quad \begin{cases} \mathbf{u}(x,t) = \mathbf{f}(x,t) \ x \in \varGamma, \\ \mathbf{u}(x,t) = 0 \ x \in \partial\varOmega \setminus \varGamma. \end{cases}$$

It is known (see [6], Thm. 3.6) that $D \in \mathcal{L}\left(\mathrm{L}^2(\partial\varOmega), \mathrm{L}^2(\varOmega)\right)$ and, as noted in ([6], p. 796), it takes values in $\mathrm{H}^{1/2}(\varOmega) = \mathrm{dom}(-A)^{(1/4)-\epsilon}$ ($\epsilon > 0$), hence it is a compact operator.

We use repeatedly the fact that when $\mathbf{u} \in \mathrm{dom}A$ we have

$$\mathcal{T}\mathbf{u} = -D^* A\mathbf{u} \tag{1.14}$$

(a fact that can be proved using the integration by parts (1.6) as in [10], p. 181).

The cosine operator approach to controllability of systems with persistent memory, when $\mathbf{u} = u \in \mathbb{R}$, $\mathbf{w} = w \in \mathbb{R}$ and $\mathcal{L} = \varDelta$, was first used in [14] where the *existence* of the control time was proved, but the control time itself was not identified. The control time was identified in dimension $d = 1$ in subsequent papers and using moment methods, see for example [2, 12, 15, 16] and for the scalar valued wave equation in $\varOmega \subseteq \mathbb{R}^3$ in [17]. In this paper we combine the cosine operator method and the moment method in order to get a proof of controllability for equation (1.2). This is based on the use of the following known estimates for the eigenvalues of the operator A (see [5, 21]): there exist $m > 0$ and M such that

$$m\left(n^{2/3}\right) \le \lambda_n^2 \le M\left(n^{2/3}\right). \tag{1.15}$$

2. Cosine operators and the proof of Theorem 1.3

Now we prove Theorem 1.3 in the case $M \neq 0$. The proof of the items (1) and (2) is similar to that in ([18], Chap. 2) and it is only sketched. The first step is a definition of the solutions of problem (1.2)–(1.4). We first apply formally the MacCamy trick. Let $R(t)$ be the resolvent kernel of $M(t)$, given by

$$R(t) + \int_0^t M(t-s)R(s)\,\mathrm{d}s = M(t). \tag{2.1}$$

We "solve" the Volterra integral equation (1.2) in the "unknown" $\mathcal{L}\mathbf{w}$. We get

$$\mathcal{L}\mathbf{w}(t) = \mathbf{w}''(t) - \mathbf{F}(t) - \int_0^t R(t-s)[\mathbf{w}''(s) - \mathbf{F}(s)]\,\mathrm{d}s.$$

We integrate by parts and we get (we recall $M \in \mathrm{H}^2$)

$$\begin{cases} \mathbf{w}'' = \mathcal{L}\mathbf{w} + a\mathbf{w}'(t) + b\mathbf{w}(t) + \displaystyle\int_0^t K(t-s)\mathbf{w}(s)\,\mathrm{d}s + \mathbf{F}_1(t), \\ \mathbf{F}_1(t) = \mathbf{F}(t) + \displaystyle\int_0^t R(t-s)\mathbf{F}(s)\,\mathrm{d}s - R(t)\mathbf{w}_1 - R'(t)\mathbf{w}_0, \\ a = R(0), \quad b = R'(0), \quad K(t) = R''(t) \end{cases} \tag{2.2}$$

(no unbounded operator in the memory kernel) and the initial and boundary conditions (1.4). *I.e.* MacCamy trick removes the differential operator from the memory term.

Remark 2.1. The term $a\mathbf{w}'(t)$ can be removed from the right hand side if we perform the transformation $\mathbf{v}(t) = e^{-(a/2)t}\mathbf{w}(t)$. The effect on the initial conditions is that $\mathbf{v}(0) = \mathbf{v}_0 = \mathbf{w}_0$ while $\mathbf{v}'(0) = \mathbf{v}_1 = \mathbf{w}_1 - (a/2)\mathbf{w}_0$ and the control is replaced by $e^{-(a/2)t}\mathbf{f}(t)$. Of course this has no influence on controllability and so with obvious modifications of the definitions of $\mathbf{f}(t)$, $\mathbf{F}_1(t)$, b and $K(t)$ we shall work with (2.2) but with $a = 0$ (we shall use the expression $\mathbf{v}'(0) = \mathbf{w}_1 - (a/2)\mathbf{w}_0$ when needed for clarity). Moreover note that the kernel $K(t)$ is square integrable.

We explicitly note that if $(\mathbf{w}_0, \mathbf{w}_1) \in \mathrm{H}^1_0(\Omega) \times \mathrm{L}^2(\Omega)$ (respectively $(\mathbf{w}_0, \mathbf{w}_1) \in \mathrm{L}^2(\Omega) \times \mathrm{H}^{-1}(\Omega)$) then $\mathbf{F}_1(t)$ belongs to $\mathrm{L}^1(0, T; \mathrm{L}^2(\Omega))$ (respectively to $\mathrm{L}^1(0, T; \mathrm{H}^{-1}(\Omega))$) and it depends continuously on $(\mathbf{w}_0, \mathbf{w}_1)$ in the specified spaces. So, the direct inequality hold for (2.2) when $\mathbf{f} = 0$.

Using formula (1.13) we get the following Volterra integral equation of the second kind for $\mathbf{w}(t)$:

$$\mathbf{w}(t) = \mathbf{u}(t) + \mathcal{A}^{-1}\int_0^t R_-(t-s)\mathbf{F}_1(s)\,\mathrm{d}s + \mathcal{A}^{-1}L * \mathbf{w} \tag{2.3}$$

where

$$L(t)\mathbf{w} = bR_-(t)\mathbf{w} + \int_0^t K(t-r)R_-(r)\mathbf{w}\,\mathrm{d}r \tag{2.4}$$

and $*$ denotes the convolution:

$$(L * w)(t) = \int_0^t L(t-s)w(s)\,\mathrm{d}s.$$

The function $\mathbf{u}(t)$ in (2.3) is the solution of (1.1) when $\mathbf{u}_0 = \mathbf{w}_0$ and $\mathbf{u}_1 = \mathbf{w}_1 - (a/2)\mathbf{w}_0$ and $\mathbf{F}(t) = 0$ (recall $a = 0$, see Rem. 2.1):

$$\mathbf{u}(t) = -\mathcal{A}\int_0^t R_-(t-s)D\mathbf{f}(s)\,\mathrm{d}s + R_+(t)\mathbf{u}_0 + \mathcal{A}^{-1}R_-(t)\mathbf{u}_1.$$

Now we observe that the function

$$t \mapsto \mathcal{A}^{-1}\int_0^t R_-(t-s)\mathbf{F}_1(s)\,\mathrm{d}s$$

belongs to $\mathrm{C}\left([0,T];\mathrm{H}^1_0(\Omega)\right) \cap \mathrm{C}^1\left([0,T];\mathrm{L}^2(\Omega)\right)$ if $(\mathbf{w}_0, \mathbf{w}_1) \in \mathrm{H}^1_0(\Omega) \times \mathrm{L}^2(\Omega)$ and it belongs to $\mathrm{C}\left([0,T];\mathrm{L}^2(\Omega)\right) \cap \mathrm{C}^1\left([0,T];\mathrm{H}^{-1}(\Omega)\right)$ if $(\mathbf{w}_0, \mathbf{w}_1) \in \mathrm{L}^2(\Omega) \times \mathrm{H}^{-1}(\Omega)$ and this is precisely the regularity of $\mathbf{u}(t)$. The fact that the solution of a Volterra integral equation retains the regularity of the affine term proves items (1) and (2) in Theorem 1.3. Now we prove the direct inequality, *i.e.* the statement (3). In this proof, $T > 0$ is fixed and we use M to denote constants (possibly depending on T) which are not the same at every occurrence.

Remark 2.2. We shall use the following observation: the initial conditions enters into the affine term (and $\mathbf{w}_1$ is transformed to $\mathbf{w}_1 - (a/2)\mathbf{w}_0$) when we used MacCamy trick. If the original equation to be studied is

$$\boldsymbol{\psi}'' = \mathcal{L}\boldsymbol{\psi} + b\boldsymbol{\psi}(t) + \int_0^t K(t-s)\boldsymbol{\psi}(s)\,\mathrm{d}s, \qquad \begin{cases} \boldsymbol{\psi}(0) = \boldsymbol{\xi} \\ \boldsymbol{\psi}'(0) = \boldsymbol{\eta} \\ \boldsymbol{\psi} = 0 \text{ on } \partial\Omega \end{cases}$$

Then $\boldsymbol{\psi}$ solves the Volterra integral equation (2.3) with $\mathbf{F}_1 = 0$ and $\mathbf{u}(t) = R_+(t)\boldsymbol{\xi} + \mathcal{A}^{-1}R_-(t)\eta$.

In the proof of the statement (3) of Theorem 1.3 we assume $\mathbf{f} = 0$, $\mathbf{w}_0 \in \mathrm{H}^1_0(\Omega)$, $\mathbf{w}_1 \in \mathrm{L}^2(\Omega)$ and $\mathbf{F} \in \mathrm{L}^1(0,T;\mathrm{L}^2(\Omega))$. Let

$$\begin{aligned} \tilde{\mathbf{u}}(t) &= \mathbf{u}(t) + \mathcal{A}^{-1}\int_0^t R_-(t-s)\mathbf{F}_1(s)\,\mathrm{d}s \\ &= R_+(t)\mathbf{u}_0 + \mathcal{A}^{-1}R_-(t)\mathbf{u}_1 + \mathcal{A}^{-1}\int_0^t R_-(t-s)\mathbf{F}_1(s)\,\mathrm{d}s, \\ \tilde{\mathbf{u}}(0) &= \mathbf{u}_0 = \mathbf{w}_0 \in \mathrm{H}^1_0(\Omega), \quad \tilde{\mathbf{u}}'(0) = \mathbf{u}_1 = \mathbf{w}_1 - \frac{a}{2}\mathbf{w}_0 \in \mathrm{L}^2(\Omega). \end{aligned}$$

Picard iteration applied to (2.3) gives

$$\mathbf{w}(t)=\tilde{\mathbf{u}}(t)+\mathcal{A}^{-1}\int_0^t L(t-s)\tilde{\mathbf{u}}(s)\,\mathrm{d}s+A^{-1}\left[\sum_{n=2}^{+\infty}\mathcal{A}^{-n+2}\left(L^{(*n)}\right)*\tilde{\mathbf{u}}\right](t) \tag{2.5}$$

where $L^{(*n)}$ denotes iterated convolution.

We prove the direct inequality in the case $\mathbf{w}_0\in\mathcal{D}(\Omega)$, $\mathbf{w}_1\in\mathcal{D}(\Omega)$ and $\mathbf{F}\in\mathcal{D}(\Omega\times(0,T))$. In this case, $\mathbf{w}\in\mathrm{C}\left([0,T];\mathrm{H}^1_0(\Omega)\cap\mathrm{H}^2(\Omega)\right)=\mathrm{C}\left([0,T];\mathrm{dom}A\right)$. The inequality is then extended to $\mathbf{w}_0\in\mathrm{H}^1_0(\Omega)$, $\mathbf{w}_1\in\mathrm{L}^2(\Omega)$, $\mathbf{F}\in\mathrm{L}^1(0,T;\mathrm{L}^2(\Omega))$ by continuity and density.

Using (1.14) we see that there exists $M=M(T)$ such that

$$\begin{aligned}|\mathcal{T}\mathbf{w}|^2_{\mathrm{L}^2((0,T);\mathrm{L}^2(\Gamma))}\le M\Bigg(&|\mathcal{T}\tilde{\mathbf{u}}|^2_{\mathrm{L}^2((0,T);\mathrm{L}^2(\Gamma))}\\&+\left|\mathcal{T}\left(\mathcal{A}^{-1}\int_0^t L(t-s)\tilde{\mathbf{u}}(s)\,\mathrm{d}s\right)\right|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2(\Gamma))}\\&+\left|D^*\left(\sum_{n=2}^{+\infty}\mathcal{A}^{-n+2}\left(L^{(*n)}\right)*\tilde{\mathbf{u}}\right)\right|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2(\Gamma))}\Bigg).\end{aligned}$$

The known properties of the solution $\mathbf{u}$ of the elastic system (1.1) implies that (for a possibly different constant $M=M_T$)

$$\begin{aligned}|\mathcal{T}\tilde{\mathbf{u}}|^2_{\mathrm{L}^2((0,T);\mathrm{L}^2(\Gamma))}+\left|D^*\left(\sum_{n=2}^{+\infty}\mathcal{A}^{-n+2}\left(L^{(*n)}\right)*\tilde{\mathbf{u}}\right)\right|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2(\Gamma))}\le&\\ M\left(|\mathbf{w}_0|^2_{\mathrm{H}^1_0(\Omega)}+|\mathbf{w}_1|^2_{\mathrm{L}^2(\Omega)}+|\mathbf{F}|^2_{\mathrm{L}^1(0,T;\mathrm{L}^2(\Omega))}\right)&.\end{aligned}$$

In order to complete the proof we show that we have also

$$\left|\mathcal{T}\left(\mathcal{A}^{-1}\int_0^t L(t-s)\tilde{\mathbf{u}}(s)\,\mathrm{d}s\right)\right|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2(\Gamma))}\le M\left(|\mathbf{w}_0|^2_{\mathrm{H}^1_0(\Omega)}+|\mathbf{w}_1|^2_{\mathrm{L}^2(\Omega)}+|\mathbf{F}|^2_{\mathrm{L}^1(0,T;\mathrm{L}^2(\Omega))}\right).$$

We note that

$$\mathcal{A}^{-1}\int_0^t L(t-s)\tilde{\mathbf{u}}(s)\,\mathrm{d}s=\mathcal{A}^{-1}\int_0^t L(t-s)\mathbf{u}(s)\,\mathrm{d}s+A^{-1}\int_0^t L(t-s)\int_0^s R_-(s-r)\mathbf{F}_1(r)\,\mathrm{d}r\,\mathrm{d}s$$

and the required inequality holds for the second addendum. The first addendum is

$$\begin{aligned}\mathcal{A}^{-1}\int_0^t L(t-s)\mathbf{u}(s)\,\mathrm{d}s=\mathcal{A}^{-1}&\left[\int_0^t bR_-(t-s)R_+(s)\mathbf{u}_0\,\mathrm{d}s+\int_0^t K(r)\int_0^{t-r}R_-(t-r-s)R_+(s)\mathbf{u}_0\,\mathrm{d}r\,\mathrm{d}s\right]\\&+A^{-1}\left[\int_0^t bR_-(t-s)R_-(s)\mathbf{u}_1\,\mathrm{d}s+\int_0^t K(r)\int_0^{t-r}R_-(t-r-s)R_-(s)\mathbf{u}_1\,\mathrm{d}r\,\mathrm{d}s\right].\end{aligned}$$

Using (1.14), we see that:

$$\begin{aligned}\left|\mathcal{T}A^{-1}\left[\int_0^t bR_-(t-s)R_-(s)\mathbf{u}_1\,\mathrm{d}s+\int_0^t K(r)\int_0^{t-r}R_-(t-r-s)R_-(s)\mathbf{u}_1\,\mathrm{d}r\,\mathrm{d}s\right]\right|\le&\\ M|\mathbf{u}_1|_{\mathrm{L}^2(\Omega)}\le M\left(|\mathbf{w}_0|_{\mathrm{L}^2(\Omega)}+|\mathbf{w}_1|_{\mathrm{L}^2(\Omega)}\right)\le M\left(|\mathbf{w}_0|_{\mathrm{H}^1_0(\Omega)}+|\mathbf{w}_1|_{\mathrm{L}^2(\Omega)}\right)&\end{aligned}$$

(recall $\mathbf{u}_1=\mathbf{w}_1-(a/2)\mathbf{w}_0$).

We consider the term containing $\mathbf{w}_0$ using

$$R_-(s)R_+(r) = \frac{1}{2}\left(R_-(s+r) + R_-(s-r)\right).$$

So we have

$$\mathcal{T}\mathcal{A}^{-1}\int_0^t R_-(t-s)R_+(s)\mathbf{w}_0 \,\mathrm{d}s = \frac{t}{2}\mathcal{T}\mathcal{A}^{-1}R_-(t)\mathbf{w}_0 + \frac{1}{2}D^* R_-(t)\mathbf{w}_0.$$

The second addendum, as an element of $\mathrm{C}\left([0,T];\mathrm{L}^2(\Gamma)\right)$, depends continuously on $\mathbf{w}_0 \in \mathrm{H}^1_0(\Omega)$.

The function $\mathcal{A}^{-1}R_-(t)\mathbf{w}_0$ is the solution of (1.1) with $\mathbf{f} = 0$, $\mathbf{F} = 0$ and initial data $\mathbf{u}_0 = 0$, $\mathbf{u}_1 = \mathbf{w}_0 \in \mathrm{H}^1_0(\Omega) \subseteq \mathrm{L}^2(\Omega)$ so that

$$|\mathcal{T}\mathcal{A}^{-1}R_-(t)\mathbf{w}_0|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2(\Gamma))} \leq M|\mathbf{w}_0|^2_{\mathrm{L}^2(\Omega)} \leq M|\mathbf{w}_0|^2_{\mathrm{H}^1_0(\Omega)}.$$

The remaining iterated integral is treated analogously.

See [19,20] for similar arguments and [14] for a different proof (in the case of the wave equation).

3. The proof of controllability

In this section we prove the controllability result in Theorem 1.4. As noted, the proof is in two steps. Let $R_E(T) = \mathrm{H}^{-1}(\Omega) \times L^2(\Omega)$. Then we prove

- $R_V(T)$, at the same time T (and so also at larger times), is closed with finite codimension.
- if $\epsilon > 0$ and $\boldsymbol{\phi} \perp R_V(T+\epsilon)$ then $\boldsymbol{\phi} = 0$ and so $R_V(T+\epsilon) = \mathrm{H}^{-1}(\Omega) \times \mathrm{L}^2(\Omega)$.

3.1. The first step

The first step is simple and it is based on the Picard formula (2.5). Let $\mathbf{w}_0 = 0$, $\mathbf{w}_1 = 0$ and $\mathbf{F} = 0$ so that $\tilde{\mathbf{u}} = \mathbf{u}$. We consider the operator $\mathbf{f} \mapsto \mathbf{w}^{\mathbf{f}}(T)$ which, according to (2.5), is the sum of three terms, which are linear and continuous as functions of $\mathbf{f} \in \mathrm{L}^2(0,T;\mathrm{L}^2(\Gamma))$. The first one is

$$\mathbf{f} \mapsto \mathbf{u}(T) = \mathbf{u}^{\mathbf{f}}(T):$$

this operator is surjective by assumptions (in fact, even more: $\mathbf{f} \mapsto \left(\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{u}^{\mathbf{f}}(T), \mathbf{u}^{\mathbf{f}}(T)\right)$ is surjective). The third operator

$$\mathbf{f} \mapsto A^{-1}\left[\sum_{n=2}^{+\infty} \mathcal{A}^{-n+2}\left(L^{(*n)}\right) * \mathbf{u}\right](T)$$

takes values in $\mathrm{dom}A = \mathrm{H}^1_0(\Omega) \cap \mathrm{H}^2(\Omega)$, and so it is a compact operator from $\mathrm{L}^2(0,T;\mathrm{L}^2(\Gamma))$ to $\mathrm{L}^2(\Omega)$.

The operator D takes values in $\mathrm{H}^{1/2}(\Omega)$ and this subspace is invariant under $e^{\mathcal{A}t}$. So the operators of $\mathbf{f} \in \mathrm{L}^2(0,T;\mathrm{L}^2(\Gamma))$

$$\mathbf{f} \mapsto \mathcal{A}^{-1}\int_0^T L(T-s)\mathbf{u}^{\mathbf{f}}(s)\,\mathrm{d}s = -\int_0^T L(T-s)\int_0^s R_-(s-r)D\mathbf{f}(r)\,\mathrm{d}r\,\mathrm{d}s$$

with values in $\mathrm{L}^2(\Omega)$ is compact.

The function $\mathbf{f} \mapsto \mathbf{w}'(T)$, with values in $\mathrm{H}^{-1}(\Omega)$ is treated analogously. Note that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{A}^{-1}L(t)\mathbf{u} = bR_+(t)\mathbf{u} + \int_0^t K(r)R_+(t-r)\mathbf{u}\,\mathrm{d}r$$

belongs to $\mathrm{C}\left([0,T];\mathrm{L}^2(\Omega)\right)$ and $\mathrm{L}^2(\Omega)$ is compactly embedded in $\mathrm{H}^{-1}(\Omega)$ and so the transformation

$$\mathbf{f}\mapsto\left[\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{A}^{-1}\int_0^t L(t-s)\mathbf{u}(s)\,\mathrm{d}s\right]_{t=T}$$

from $\mathrm{L}^2(0,T;\mathrm{L}^2(\Gamma))$ to $\mathrm{H}^{-1}(\Omega)$ is compact. The same holds for the derivative of the last term in (2.5) computed for $t=T$, which is

$$\mathcal{A}^{-1}\int_0^t R_+(t-r)\left[b\mathbf{g}(r)+\int_0^r K(t-r-s)\mathbf{g}(s)\,\mathrm{d}s\right]\mathrm{d}r,$$
$$\mathbf{g}=\mathbf{u}+\sum_{n=3}^{+\infty}\mathcal{A}^{-n+2}L^{(*(n-1))}*\mathbf{u}.$$

In conclusion, the reachable set $R_V(T)$ is the image of the operator $\mathbf{f}\mapsto\left(\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{u}^{\mathbf{f}}(T),\mathbf{u}^{\mathbf{f}}(T)\right)$, which is surjective in $\mathrm{H}^{-1}(\Omega)\times\mathrm{L}^2(\Omega)$ since we assumed that system (1.1) is controllable at time T, perturbed by the addition of a compact operator. This implies that $R_V(T)$ is closed with finite codimension.

3.2. The second step

The second step is more involved and requires several substeps.

3.2.1. Step 2-1: The elements of $[R_V(T)]^\perp$

We elaborate on formula (2.5) which can be written as:

$$\mathbf{w}(t)=\tilde{\mathbf{u}}(t)+\int_0^t H(s)\tilde{\mathbf{u}}(t-s)\,\mathrm{d}s,\qquad H(t)\mathbf{v}=\sum_{n=1}^{+\infty}\mathcal{A}^{-n}L^{(*n)}\mathbf{v}. \tag{3.1}$$

We study the reachable set, hence we assume $\mathbf{F}=0$ and null initial conditions, so that

$$\tilde{\mathbf{u}}(t)=\mathbf{u}(t)=\mathcal{A}\mathbf{v}(t),\qquad \mathbf{v}(t)=\int_0^t R_-(t-s)D\left(-\mathbf{f}(s)\right)\,\mathrm{d}s.$$

Of course, in the study of the reachable set, $-\mathbf{f}$ can be renamed $\mathbf{f}$.

We characterize the elements $(\boldsymbol{\xi},\boldsymbol{\eta})\in\mathrm{H}^1_0(\Omega)\times\mathrm{L}^2(\Omega)$ which annihilates $R_V(T)$ (here $T>0$ is arbitrary). *I.e.* we characterize those elements $(\boldsymbol{\xi},\boldsymbol{\eta})\in\mathrm{H}^1_0(\Omega)\times\mathrm{L}^2(\Omega)$ such that

$$\langle\!\langle\left(\mathbf{w}^{\mathbf{f}}\right)'(T),\boldsymbol{\xi}\rangle\!\rangle+\int_\Omega \mathbf{w}^{\mathbf{f}}(x,T)\boldsymbol{\eta}(x)\,\mathrm{d}x=0$$

for every $\mathbf{f}\in\mathrm{L}^2(0,T;\mathrm{L}^2(\Gamma))$. The crochet denotes the pairing of $\mathrm{H}^{-1}(\Omega)$ and $\mathrm{H}^1_0(\Omega)$. This set of annihilators is shortly denoted $[R_V(T)]^\perp$ and we write $(\boldsymbol{\xi},\boldsymbol{\eta})\perp R_V(T)$.

Note that if it happens that $\mathbf{w}'(T)\in\mathrm{L}^2(\Omega)$ then

$$\langle\!\langle\mathbf{w}'(T),\boldsymbol{\xi}\rangle\!\rangle=\int_\Omega \mathbf{w}'(x,T)\boldsymbol{\xi}(x)\,\mathrm{d}x.$$

Of course we can study $R_V(T)^\perp$ by assuming $f\in\mathcal{D}(\Gamma\times(0,T))$ so that the following computations are justified. First we note that when $f\in\mathcal{D}(\Gamma\times(0,T))$ we have $\mathbf{w}(t)\in\mathrm{L}^2(\Omega)$ since

$$\mathbf{w}'(t)=\mathcal{A}\int_0^t R_-(s)D\mathbf{f}'(t-s)\,\mathrm{d}s+\mathcal{A}\int_0^t H(s)\int_0^{t-s}R_-(r)D\mathbf{f}'(t-s-r)\,\mathrm{d}r\,\mathrm{d}s.$$

Hence (in the last step we integrate by parts in time and we note that $R_-(t)$ and $H(t)$ commute),

$$\begin{aligned}\langle\langle \mathbf{w}'(T),\boldsymbol{\xi}\rangle\rangle &= \int_\Omega \boldsymbol{\xi}(x)\mathcal{A}\int_0^T R_-(T-s)Df'(s)\,\mathrm{d}s\,\mathrm{d}x \\ &\quad+\int_\Omega \boldsymbol{\xi}(x)\mathcal{A}\int_0^T H(s)\int_0^{T-s} R_-(T-s-r)Df'(r)\,\mathrm{d}r\,\mathrm{d}s\,\mathrm{d}x \\ &= \int_\Gamma\int_0^T f'(s)\left[D^*\mathcal{A}R_-(T-s)\boldsymbol{\xi}\right]\mathrm{d}s\,\mathrm{d}\Gamma \\ &\quad+\int_\Gamma\int_0^T f'(r)\left[D^*\mathcal{A}\int_0^{T-r} H(s)R_-(T-r-s)\boldsymbol{\xi}\,\mathrm{d}s\right]\mathrm{d}r\,\mathrm{d}\Gamma \\ &= \int_\Gamma\int_0^T f(r)\left\{D^*A\left[R_+(T-r)\boldsymbol{\xi}+\int_0^{T-r} H(s)R_+(T-r-s)\boldsymbol{\xi}\,\mathrm{d}s\right]\right\}\mathrm{d}r\,\mathrm{d}\Gamma. \end{aligned} \tag{3.2, 3.3}$$

Note that the last integral makes sense thanks to the direct inequality, because (3.1) shows that the bracket is the solution of (2.2) with $\mathbf{f}=0$, $\mathbf{F}=0$ and initial condition $\mathbf{w}(0)=\boldsymbol{\xi}\in \mathrm{H}^1_0(\Omega)$, $\mathbf{w}'(0)=0$.

Analogously,

$$\begin{aligned}&\int_\Omega \boldsymbol{\eta}(x)\left[\mathcal{A}\int_0^T R_-(T-r)D\mathbf{f}(r)\,\mathrm{d}r+\mathcal{A}\int_0^T H(s)\int_0^{T-s} R_-(r)D\mathbf{f}(T-s-r)\,\mathrm{d}r\,\mathrm{d}s\right]\mathrm{d}x \\ &=\int_0^T\int_\Gamma \mathbf{f}(r)D^*\mathcal{A}R_-(T-r)\boldsymbol{\eta}\,\mathrm{d}r\,\mathrm{d}\Gamma+\int_0^T\int_\Gamma \mathbf{f}(r)D^*\mathcal{A}\int_0^{T-r} H(T-r-s)R_-(s)\boldsymbol{\eta}\,\mathrm{d}s\,\mathrm{d}\Gamma\,\mathrm{d}r \\ &=\int_0^T\int_\Gamma \mathbf{f}(r)\left\{D^*A\left[\mathcal{A}^{-1}R_-(T-r)\boldsymbol{\eta}+\int_0^{T-r} H(T-r-s)\mathcal{A}^{-1}R_-(s)\boldsymbol{\eta}\,\mathrm{d}s\right]\right\}\mathrm{d}\Gamma\,\mathrm{d}r. \end{aligned} \tag{3.4}$$

The bracket is the solution of (1.2) with $\mathbf{f}=0$, $\mathbf{F}=0$ and initial condition $\mathbf{w}(0)=0$, $\mathbf{w}'(0)=\boldsymbol{\eta}\in \mathrm{L}^2(\Omega)$ so that the brace belongs to $\mathrm{L}^2(0,T;\mathrm{L}^2(\Gamma))$ thanks to the direct inequality.

We have $(\boldsymbol{\xi},\boldsymbol{\eta})\perp R_V(T)$ when (3.3) and (3.4) sum to zero. Taking into account that the previous computation holds for every $f\in\mathcal{D}(\Gamma\times(0,T))$ (which is dense in $\mathrm{L}^2(0,T;\mathrm{L}^2(\Gamma))$ we get

$$D^*A\left\{\mathbf{u}(t)+\int_0^t H(t-r)\mathbf{u}(r)\,\mathrm{d}r\right\}=0,\qquad \mathbf{u}(t)=R_+(t)\boldsymbol{\xi}+\mathcal{A}^{-1}R_-(t)\boldsymbol{\eta}.$$

Let

$$\boldsymbol{\psi}(t)=\mathbf{u}(t)+\int_0^t H(t-r)\mathbf{u}(r)\,\mathrm{d}r.$$

The function $\boldsymbol{\psi}(t)$ is the solution of the Volterra integral equation

$$\boldsymbol{\psi}(t)=\mathbf{u}(t)+\mathcal{A}^{-1}\int_0^t L(t-s)\boldsymbol{\psi}(s)\,\mathrm{d}s.$$

We compare with Remark 2.2 and we see that $\boldsymbol{\psi}(t)$ solves

$$\boldsymbol{\psi}''=\mathcal{L}\boldsymbol{\psi}+b\boldsymbol{\psi}(t)+\int_0^t K(t-s)\boldsymbol{\psi}(s)\,\mathrm{d}s,\qquad \begin{cases}\boldsymbol{\psi}(0)=\boldsymbol{\xi}\\ \boldsymbol{\psi}'(0)=\boldsymbol{\eta}\\ \boldsymbol{\psi}=0 \text{ on } \partial\Omega.\end{cases} \tag{3.5}$$

Note that the operator $\mathcal{L}$ is not in the memory term and that $K(t)$ is square integrable.

In conclusion, we can state:

Theorem 3.1. *We have* $(\boldsymbol{\xi},\boldsymbol{\eta}) \perp R_V(T)$ *if and only if the solution* $\boldsymbol{\psi}(x,t)$ *of the problem* (3.5) *satisfy the condition*

$$D^* A\boldsymbol{\psi} = 0. \tag{3.6}$$

As we noted, $D^*A = -\mathcal{T}$ when applied to the elements of dom$A$, in particular when $\boldsymbol{\xi} \in \mathcal{D}(\Omega)$ and $\boldsymbol{\eta} \in \mathcal{D}(\Omega)$. The direct inequality implies that $\mathcal{T}\boldsymbol{\psi}(\cdot) \in \mathcal{L}\left(\mathrm{H}^1_0(\Omega)\times \mathrm{L}^2(\Omega), \mathrm{L}^2(0,T;\mathrm{L}^2(\Gamma)\right)$ and so $\mathcal{T}\boldsymbol{\psi}(\cdot)$ is the continuous extension of $D^*A\boldsymbol{\psi}(\cdot)$ to every initial condition in $\mathrm{H}^1_0(\Omega)\times \mathrm{L}^2(\Omega)$. Hence we can also state:

Corollary 3.2. *We have* $(\boldsymbol{\xi},\boldsymbol{\eta}) \perp R_V(T)$ *if and only if the solution* $\boldsymbol{\psi}(x,t)$ *of the problem* (3.5) *satisfies*

$$\mathcal{T}\boldsymbol{\psi} = 0 \text{ in } \mathrm{L}^2(0,T;\mathrm{L}^2(\Gamma)).$$

3.2.2. Step 2-2*: Fourier expansion and regularity of the elements of* $[\mathsf{R_V}(\mathsf{T}+\epsilon)]^\perp$

We recall that A is a selfadjoint operator with compact resolvent so that its spectrum is a sequence of eigenvalues and there exists an orthonormal basis $\{\boldsymbol{\phi}_n\}$ whose elements are eigenvectors of A. We denoted $-\lambda_n^2$ the eigenvalue of $\boldsymbol{\phi}_n$ (eigenvalues can be repeated, each one a finite number of times) and we assumed that the order has been chosen so to have $\lambda_n \le \lambda_{n+1}$. The bases of respectively $\mathrm{H}^1_0(\Omega)$ and $\mathrm{H}^{-1}(\Omega)$ which correspond to the orthonormal basis $\{\boldsymbol{\phi}_n\}$ of the eigenvectors of A in $\mathrm{L}^2(\Omega)$ are respectively $\{\boldsymbol{\phi}_n/\lambda_n\}$ and $\{\lambda_n\boldsymbol{\phi}_n\}$.

Let $(\boldsymbol{\xi},\boldsymbol{\eta}) \in [R_V(T+\epsilon)]^\perp$, $(\boldsymbol{\xi},\boldsymbol{\eta}) \in \mathrm{H}^1_0(\Omega)\times \mathrm{L}^2(\Omega)$. We can expand in series of the eigenvectors and we get

$$\boldsymbol{\xi}(x) = \sum_{n=1}^{+\infty} \boldsymbol{\phi}_n(x)\xi_n, \quad \boldsymbol{\eta}(x) = \sum_{n=1}^{+\infty} \boldsymbol{\phi}_n(x)\eta_n, \qquad \{\lambda_n\xi_n\}\in l^2,\ \{\eta_n\}\in l^2. \tag{3.7}$$

First we prove the following result (which holds also with $\epsilon = 0$):

Lemma 3.3. *Assume* $R_E(T) = \mathrm{H}^{-1}(\Omega)\times \mathrm{L}^2(\Omega)$ *and* $\epsilon \ge 0$. *Let* $(\boldsymbol{\xi},\boldsymbol{\eta}) \in [R_V(T+\epsilon)]^\perp$ *and* $(\boldsymbol{\xi},\boldsymbol{\eta}) \ne 0$. *Then at least one of the series in* (3.7) *is not a finite sum.*

Proof. Let us denote $\boldsymbol{\psi}(t)$ the solution of equation (3.5) with $\boldsymbol{\psi}(0) = \boldsymbol{\xi}$ and $\boldsymbol{\psi}'(0) = \boldsymbol{\eta}$ and let us assume that both the expansions (3.7) are finite sums, of N terms at most. We expand also the solution $\boldsymbol{\psi}(t)$,

$$\boldsymbol{\psi}(t) = \sum_{n=1}^{N} \boldsymbol{\phi}_n(x)\left[\psi_n^0(t)\xi_n + \psi_n^1(t)\eta_n\right]$$

where both $\psi_n^0(t)$ and $\psi_n^1(t)$ solve the scalar equation

$$\psi_n''(t) = -\lambda_n^2\psi_n(t) + b\psi_n(t) + \int_0^t K(t-s)\psi_n(s)\,\mathrm{d}s$$

with initial conditions respectively

$$\psi_n^0(0) = 1,\ (\psi_n^0)'(0) = 0, \quad \psi_n^1 = 0,\ (\psi_n^1)'(0) = 1.$$

The orthogonality condition is

$$0 = \mathcal{T}\boldsymbol{\psi}(t) = \sum_{n=1}^{N} (\mathcal{T}\boldsymbol{\phi}_n(x))\left[\psi_n^0(t)\xi_n + \psi_n^1(t)\eta_n\right]. \tag{3.8}$$

This sum cannot have only one nonzero addendum. To see this, let $\xi_n = 0$ and $\eta_n = 0$ for $n \ne n_0$ and either $\xi_{n_0} \ne 0$ or $\eta_{n_0} \ne 0$ (or both). Computing either (3.8) or its derivative with $t = 0$ we get

$$\left(\mathcal{T}\boldsymbol{\phi}_{n_0}(x)\right) = 0. \tag{3.9}$$

It is a fact that the equality (3.9) cannot hold if Γ has been chosen in such a way that the elastic system (1.1) is controllable at some time T. The proof, based on the inverse inequality of equation (1.1), is the same as that with $d=1$ and can be found in [9] (see also [18], Lem. 4.3).

Even more: *the nonzero elements of* (3.8) *must correspond to at least two different eigenvalues.* In fact, equation (3.8) computed with $t=0$ gives

$$\mathcal{T}\left(\sum_{n=1}^{N}\xi_n\boldsymbol{\phi}_n(x)\right)=0$$

and if every $\boldsymbol{\phi}_n$ correspond to the same eigenvalue $-\lambda^2$ then $\sum_{n=1}^{N}\xi_n\boldsymbol{\phi}_n(x)$ is an eigenvector of A. As we noted, the equality is impossible unless the sum is zero.

Analogous argument if the derivative of both the sides of (3.8) is computed with $t=0$.

Now we prove that if the sum is finite then we can reduce ourselves to have a sum of terms which correspond to only one eigenvalue, and we proved that this is not possible. In fact, computing the second derivatives of (3.8) we get

$$\sum_{n=1}^{N}\lambda_n^2\left(\mathcal{T}\boldsymbol{\phi}_n(x)\right)\left[\psi_n^0(t)\xi_n+\psi_n^1(t)\eta_n\right]=0 \tag{3.10}$$

since the other terms appearing in the computation of the derivative sum to zero thanks to (3.8).

We subtract (3.8), multiplied with λ_N^2, and (3.10) (as in [19]) so to obtain a new sum like (3.8) but with at most $N-1$ terms:

$$\sum_{n=1}^{N-1}\left(\mathcal{T}\boldsymbol{\phi}_n(x)\right)\left\{\psi_n^0(t)\left[\lambda_n^2-\lambda_N^2\right]\xi_n+\psi_n^1(t)\left[\lambda_n^2-\lambda_N^2\right]\eta_n\right\}=0.$$

After a finite number of iteration we remain with terms which correspond to the same eigenvalue (possibly, only one term) and we have seen that this is impossible. □

Now we have this information, that at least one of the series (3.7) is not a finite sum. So, the orthogonality condition (3.8) has to be replaced with

$$0=\mathcal{T}\boldsymbol{\psi}(t)=\sum_{n=1}^{+\infty}\left(\mathcal{T}\boldsymbol{\phi}_n(x)\right)\left[\psi_n^0(t)\xi_n+\psi_n^1(t)\eta_n\right] \tag{3.11}$$

(the exchange of $\mathcal{T}$ and the series is justified by the direct inequality). We are going to prove that also this case is impossible and so it must be $\boldsymbol{\xi}=0$ and $\boldsymbol{\eta}=0$, but now we need the assumption that the purely elastic system (1.1) is controllable at time T and that $(\boldsymbol{\xi},\boldsymbol{\eta})\in\left[R_V(T+\epsilon)\right]^{\perp}$ with $\epsilon>0$; *i.e.* that the equality (3.11) holds in $\mathrm{L}^2(0,T+\epsilon;\mathrm{L}^2(\Gamma))$. The condition $\epsilon>0$ is used in the proof of the following lemma:

Lemma 3.4. *Let* $R_E(T)=\mathrm{H}^{-1}(\Omega)\times\mathrm{L}^2(\Omega)$ *and let* $\epsilon>0$. *If* $(\boldsymbol{\xi},\boldsymbol{\eta})\in\left[R_V(T+\epsilon)\right]^{\perp}$ *then*

$$\xi_n=\frac{\tilde{\xi}_n}{\lambda_n^3},\quad \eta_n=\frac{\tilde{\eta}_n}{\lambda_n^2},\quad \{\tilde{\xi}_n\}\in l^2,\ \{\tilde{\eta}_n\}\in l^2. \tag{3.12}$$

Proof. We expand in series of the eigenfunctions $\boldsymbol{\phi}_n$ the solution $\mathbf{u}(t)$ of the purely elastic problem (1.1)–(1.3) when $\mathbf{F}=0$, $\mathbf{u}_0=0$ and $\mathbf{u}_1=0$. Using (1.1) and (1.6) (recall that $-\mathbf{f}$ was renamed $\mathbf{f}$) we get:

$$\mathbf{u}(t)=\sum_{n=1}^{+\infty}\boldsymbol{\phi}_n(x)u_n(t),\qquad u_n''=-\lambda_n^2u_n+\int_\Gamma\left(\mathcal{T}\boldsymbol{\phi}_n\right)\cdot\mathbf{f}(t)\,\mathrm{d}\Gamma.$$

Hence,

$$u_n(t) = \frac{1}{\lambda_n} \int_0^t \sin \lambda_n (t-s) \left[\int_\Gamma (\mathcal{T} \boldsymbol{\phi}_n) \cdot \mathbf{f}(s) \, \mathrm{d}\Gamma \right] \mathrm{d}s$$

and we have the following expansions of $\mathbf{u}(t)$ and $\mathbf{u}'(t)$ (we compute with $\mathbf{u}_0 = 0$, $\mathbf{u}_1 = 0$ and $\mathbf{F} = 0$):

$$\mathbf{u}(t) = -\sum_{n=1}^{+\infty} \boldsymbol{\phi}_n(x) \int_0^t \int_\Gamma \left(\frac{\mathcal{T}\boldsymbol{\phi}_n}{\lambda_n} \sin \lambda_n (t-s) \right) \cdot \mathbf{f}(s) \, \mathrm{d}\Gamma \, \mathrm{d}s,$$

$$\mathbf{u}'(t) = -\sum_{n=1}^{+\infty} (\lambda_n \boldsymbol{\phi}_n(x)) \int_0^t \int_\Gamma \left(\frac{\mathcal{T}\boldsymbol{\phi}_n}{\lambda_n} \cos \lambda_n (t-s) \right) \cdot \mathbf{f}(s) \, \mathrm{d}\Gamma \, \mathrm{d}s.$$

So, controllability is equivalent to the surjectivity of the map

$$\mathbf{f} \mapsto \left\{ \left[\int_0^T \int_\Gamma \left(\frac{\mathcal{T}\boldsymbol{\phi}_n}{\lambda_n} \cos \lambda_n s \right) \cdot \mathbf{f}(T-s) \, \mathrm{d}\Gamma \, \mathrm{d}s \right], \left[\int_0^T \int_\Gamma \left(\frac{\mathcal{T}\boldsymbol{\phi}_n}{\lambda_n} \sin \lambda_n s \right) \cdot \mathbf{f}(T-s) \, \mathrm{d}\Gamma \, \mathrm{d}s \right] \right\} \in l^2 \times l^2.$$

Here $l^2 = l^2(\mathbb{N})$, $\mathbb{N} = 1, 2, \ldots$ This transformation is continuous since $\mathbf{f} \mapsto (\mathbf{u}'(T), \mathbf{u}(T))$ is continuous from $\mathrm{L}^2(0, T; \mathrm{L}^2(\Gamma))$ to $\mathrm{H}^{-1}(\Omega) \times \mathrm{L}^2(\Omega)$.

As usual with Fourier series, it is convenient to introduce

$$\lambda_{-n}^2 = -\lambda_n^2, \ \boldsymbol{\phi}_{-n} = \boldsymbol{\phi}_n$$

and we see that controllability of the purely elastic system is equivalent to surjectivity of the following operator $\mathbb{M}$ (here $\mathbb{Z}' = \mathbb{Z} \setminus \{0\}$ and $l^2 = l^2(\mathbb{Z}') = l^2(\mathbb{Z}'; \mathbb{C})$):

$$\mathbb{M} \in \mathcal{L}\left(\mathrm{L}^2\left(0, T; \mathrm{L}^2(\Gamma)\right), l^2\right): \quad \mathbb{M}\mathbf{f} = \left\{ \int_0^T \int_\Gamma \left(\frac{\mathcal{T}\boldsymbol{\phi}_n}{\lambda_n} e^{i\lambda_n s} \right) \cdot \mathbf{f}(T-s) \, \mathrm{d}\Gamma \, \mathrm{d}s \right\} \tag{3.13}$$

(see Lems. 4.6 and 5.1 in [18]): *controllability of problem* (1.1) *and* (1.3) *is equivalent to the surjectivity of the bounded operator* $\mathbb{M}$ *in* (3.13) *from* $\mathrm{L}^2(0, T; \mathrm{L}^2(\Gamma))$ *to* $l^2(\mathbb{Z}'; \mathbb{C})$. In turn, this is equivalent to the fact that the sequence

$$\left\{ \left(\boldsymbol{\Psi}_n e^{i\lambda_n t} \right) \right\}_{n \in \mathbb{Z}'} \quad \text{where} \quad \boldsymbol{\Psi}_n = \frac{1}{\lambda_n} \mathcal{T} \boldsymbol{\phi}_n$$

is a Riesz sequence in $\mathrm{L}^2(0, T; \mathrm{L}^2(\Gamma))$ *i.e.* it can be transformed to an orthonormal sequence using a linear bounded and boundedly invertible transformation.

We need the following pieces of information (see [18], Chap. 3, for details on the Riesz sequences):

Lemma 3.5. *The following properties hold:*

(1) *if* $\{e_n\}$ *is a Riesz sequence in a Hilbert space* H *then* $\sum \alpha_n e_n$ *converges in* H *if and only if* $\{\alpha_n\} \in l^2$*;*
(2) *if* $\left\{k_n e^{i\lambda_n t}\right\}$ *is a Riesz sequence in* $\mathrm{L}^2(0, T; H)$ *(*H *is a Hilbert space and* $k_n \in H$*) and if* $\sum \alpha_n k_n e^{i\lambda_n t}$ *converges in* $\mathrm{L}^2(0, T+\epsilon; H)$ *to an* H^1 *function then* $\alpha_n = \delta_n / \lambda_n$ *and* $\{\delta_n\} \in l^2$. This result requires $\epsilon > 0$.

We continue the proof of Lemma 3.4: we go back to examine the orthogonality condition $\mathcal{T}\boldsymbol{\psi}(t) = 0$ which can be written as:

$$\sum_{n=1}^{+\infty} \boldsymbol{\Psi}_n Z_n(t) = 0, \quad Z_n(t) = \lambda_n \left[\psi_n^0(t) \xi_n + \psi_n^1(t) \eta_n \right]. \tag{3.14}$$

Hence $Z_n(t)$ solves

$$Z_n'' = -\lambda_n^2 Z_n + bZ_n + \int_0^t K(t-s)Z_n(s)\,\mathrm{d}s, \qquad Z_n(0) = \lambda_n\xi_n, \quad Z_n'(0) = \lambda_n\eta_n. \tag{3.15}$$

So we have

$$\begin{aligned} Z_n(t) &= (\lambda_n\xi_n)\cos\lambda_n t + \eta_n \sin\lambda_n t + \frac{b}{\lambda_n}\int_0^t \sin\lambda_n(t-s)Z_n(s)\,\mathrm{d}s \\ &\quad + \frac{1}{\lambda_n}\int_0^t \sin\lambda_n(t-s)\int_0^s K(s-r)Z_n(r)\,\mathrm{d}r\,\mathrm{d}s \end{aligned} \tag{3.16}$$

(note that ξ_n and η_n are real numbers and that $\{\lambda_n\xi_n\} \in l^2$ because $\xi \in \mathrm{H}_0^1(\Omega)$).

Gronwall inequality shows that the sequence of continuous functions $\{Z_n(t)\}$ is uniformly bounded on compact intervals.

We have

$$(\lambda_n\xi_n)\cos\lambda_n t + \eta_n\sin\lambda_n t = e^{i\lambda_n t}c_n + e^{-i\lambda_n t}\bar{c}_n, \qquad c_n = \frac{1}{2}\left[\lambda_n\xi_n - i\eta_n\right].$$

We introduce the notations

$$U_n = e^{i\lambda_n t}c_n + e^{-i\lambda_n t}\bar{c}_n, \quad S_n = \sin\lambda_n t, \quad C_n = \cos\lambda_n t, \quad H_n = bS_n + S_n * K.$$

Three steps of Picard iteration applied to (3.16) give

$$Z_n = U_n + \frac{1}{\lambda_n}U_n * \left[\sum_{\nu=1}^{4}\frac{1}{\lambda_n^{\nu-1}}H_n^{(*\nu)}\right] + \frac{1}{\lambda_n^5}M_n$$

where $M_n = M_n(t)$ are continuous functions, and the sequences $\{M_n(t)\}$, $\{(1/\lambda_n)M_n'(t)\}$ are bounded on bounded intervals. In fact,

$$M_n = U_n * \left[\sum_{\nu=5}^{7}\frac{1}{\lambda_n^{\nu-5}}H^{(*\nu)}\right] + \frac{1}{\lambda_n^3}H_n^{(*8)} * Z_n.$$

The orthogonality condition (3.14) takes the form

$$-\sum_{n=1}^{+\infty}\boldsymbol{\Psi}_n U_n = \sum_{n=1}^{+\infty}\boldsymbol{\Psi}_n\frac{1}{\lambda_n}U_n * \left[\sum_{\nu=1}^{4}\frac{1}{\lambda_n^{\nu-1}}H_n^{(*\nu)}\right] + \sum_{n=1}^{+\infty}\boldsymbol{\Psi}_n\frac{1}{\lambda_n^5}M_n. \tag{3.17}$$

In fact, we can distribute the series since every one of the obtained series converges since **1)** the last series converges because the estimate (1.15) implies that

$$\sum_{n=1}^{+\infty}\frac{1}{\lambda_n^4} < +\infty.$$

2) the previous series converge since $S_n * U_n$ is a linear combination of exponentials $e^{i\lambda_n t}$ multiplied with a polynomial of degree at most 1 and our assumption on the time T implies that $\boldsymbol{\Psi}_n e^{i\lambda_n t}$ is a Riesz sequence.

The right hand side of (3.17) is an H^1 function on *every interval*. In fact, computing the derivative of the right hand side termwise we get the following sum of L^2-convergent series, where $V_n = U_n' = e^{i\lambda_n t}(i\lambda_n c_n) - e^{-i\lambda_n t}(i\lambda_n\bar{c}_n)$:

$$\sum_{n=1}^{+\infty}\boldsymbol{\Psi}_n\frac{1}{\lambda_n}(c_n+\bar{c}_n)\left[\sum_{\nu=1}^{4}\frac{1}{\lambda_n^{\nu-1}}H_n^{(*\nu)}\right] + \sum_{n=1}^{+\infty}\boldsymbol{\Psi}_n\left(\frac{1}{\lambda_n}V_n\right) * \left[\sum_{\nu=1}^{4}\frac{1}{\lambda_n^{\nu-1}}H^{(*\nu)}\right] + \sum_{n=1}^{+\infty}\boldsymbol{\Psi}_n\frac{1}{\lambda_n^5}M_n'. \tag{3.18}$$

Convergence of the series is seen by an argument similar to the one used above, thanks to (1.15) and to the fact that $\{(1/\lambda_n)M_n'(t)\}$ is a bounded sequence. So, we can apply Lemma 3.5 and we conclude

$$c_n = \frac{1}{\lambda_n}\tilde{c}_n,\ \{\tilde{c}_n\}\in l^2 \ \text{ hence }\ \lambda_n\xi_n = \frac{1}{\lambda_n}\delta_n,\ \eta_n = \frac{1}{\lambda_n}\sigma_n,\ \{\delta_n\}\in l^2,\ \{\sigma_n\}\in l^2. \tag{3.19}$$

We equate the derivative of the left hand side of (3.17) and (3.18) and we find the equality

$$-\sum_{n=1}^{+\infty}\boldsymbol{\Psi}_n V_n = \sum_{n=1}^{+\infty}\boldsymbol{\Psi}_n\frac{1}{\lambda_n}\left(\tilde{c}_n+\bar{\tilde{c}}_n\right)\left[\sum_{\nu=1}^{4}\frac{1}{\lambda_n^{\nu-1}}H_n^{(*\nu)}\right] \tag{3.20}$$

$$+\sum_{n=1}^{+\infty}\boldsymbol{\Psi}_n\left(\frac{1}{\lambda_n}V_n\right)*\left[\sum_{\nu=1}^{4}\frac{1}{\lambda_n^{\nu-1}}H^{(*\nu)}\right]+\sum_{n=1}^{+\infty}\boldsymbol{\Psi}_n\frac{1}{\lambda_n^5}M_n'. \tag{3.21}$$

Using

$$c_n+\bar{c}_n = \frac{1}{\lambda_n}\left(\tilde{c}_n+\bar{\tilde{c}}_n\right),\qquad \frac{1}{\lambda_n}V_n(t) = \frac{i}{\lambda_n}\left[e^{i\lambda_n t}\tilde{c}_n - e^{-i\lambda_n t}\bar{\tilde{c}}_n\right]$$

we see differentiability of the first and second series on the right hand side. Using again the estimate (1.15) we see that also the last series is differentiable, since the explicit expression of $M_n'(t)$ is

$$\frac{1}{\lambda_n^5}M_n'(t) = \frac{1}{\lambda_n^5}\left[(c_n+\bar{c}_n)\sum_{\mu=5}^{7}\frac{1}{\lambda_n^{\mu-5}}H_n^{(*\mu)} + V_n' * \left(\sum_{\mu=5}^{7}\frac{1}{\lambda_n^{\mu-5}}H_n^{(*\mu)}\right)\right]$$
$$+\frac{1}{\lambda_n^7}\left(C_n + C_n * K\right) * H_n^{(*7)} * Z_n.$$

In conclusion, the left hand side of (3.21) is an H^1 function on every interval $[0,T]$. Applying again Lemma 3.5 we get

$$\tilde{c}_n = \frac{1}{\lambda_n}\hat{c}_n,\ \{\hat{c}_n\}\in l^2\ \text{ so that }\ \lambda_n\xi_n = \frac{1}{\lambda_n^2}\left(\hat{c}_n+\bar{\hat{c}}_n\right),\quad \eta_n = \frac{-i}{\lambda_n^2}\left(\bar{\hat{c}}_n - \hat{c}_n,\right)$$

as we wanted. □

Note that we proved also this result:

Theorem 3.6. *The first and the second derivatives of the series in* (3.14) *can be computed termwise.*

3.2.3. Step 2-3*: End of the proof*

We use Theorem 3.6: the second derivative of (3.14) computed termwise gives the equality

$$\sum_{n=1}^{+\infty}\boldsymbol{\Psi}_n Z_n''(t) = -\sum_{n=1}^{+\infty}\boldsymbol{\Psi}_n\left(\lambda_n^2 Z_n(t)\right) - \int_0^t M(t-s)\left[\sum_{n=1}^{+\infty}\boldsymbol{\Psi}_n\left(\lambda_n^2 Z_n(s)\right)\right]\mathrm{d}s = 0$$

so that we have also

$$\sum_{n=1}^{+\infty}\boldsymbol{\Psi}_n\lambda_n^2 Z_n(t) = \sum_{n=1}^{+\infty}\left(\mathcal{T}\boldsymbol{\phi}_n(x)\right)\left[\psi_n^0(t)\frac{\tilde{\xi}_n}{\lambda_n}+\psi_n^1(t)\tilde{\eta}_n\right] = 0. \tag{3.22}$$

Let N_1 be the first index such that $\xi_{N_1}^2+\eta_{N_1}^2\neq 0$. Combining (3.14) and (3.22) we get the new equality

$$\sum_{n=N_1}^{+\infty}\boldsymbol{\Psi}_n\left(\lambda_{N_1}^2-\lambda_n^2\right)Z_n(t) = \sum_{n=N_2}^{+\infty}\boldsymbol{\Psi}_n\left(\lambda_{N_1}^2-\lambda_n^2\right)Z_n(t)$$
$$= \sum_{n=N_2}^{+\infty}\left(\mathcal{T}\boldsymbol{\phi}_n(x)\right)\left[\left(\frac{\lambda_{N_1}^2}{\lambda_n^2}-1\right)\frac{\tilde{\xi}_n}{\lambda_n}\psi_n^0(t)+\left(\frac{\lambda_{N_1}^2}{\lambda_n^2}-1\right)\tilde{\eta}_n\psi_n^1(t)\right] = 0$$

and $N_2 > N_1$. So, the element $(\boldsymbol{\xi}^1, \boldsymbol{\eta}^1) \in \mathrm{H}_0^1(\Omega) \times \mathrm{L}^2(\Omega)$ whose Fourier coefficients are

$$\xi_n^1 = \left(\frac{\lambda_{N_1}^2}{\lambda_n^2} - 1\right)\frac{\tilde{\xi}_n}{\lambda_n}, \quad \eta_n^1 = \left(\frac{\lambda_{N_1}^2}{\lambda_n^2} - 1\right)\tilde{\eta}_n, \qquad \{\lambda_n\tilde{\xi}_n\} \in l^2,\ \{\tilde{\eta}_n\} \in l^2$$

is a second element in $[R_V(T+\epsilon)]^\perp$.

The elements $(\boldsymbol{\xi}, \boldsymbol{\eta})$ and $(\boldsymbol{\xi}^1, \boldsymbol{\eta}^1)$ *are not colinear,* since the Fourier coefficient of index N_1 is nonzero for the first pair, while it is zero for the second one.

Now we use the fact that $(\boldsymbol{\xi}_1, \boldsymbol{\eta}_1) \in \mathrm{H}_0^1(\Omega) \times \mathrm{L}^2(\Omega)$ and we iterate the procedure. We remove the first nonzero Fourier coefficient of $(\boldsymbol{\xi}_1, \boldsymbol{\eta}_1)$ and so we get an element $(\boldsymbol{\xi}_2, \boldsymbol{\eta}_2) \in \mathrm{H}_0^1(\Omega) \times \mathrm{L}^2(\Omega)$ which belongs to $[R_V(T+\epsilon)]^\perp$ and *which is linearly independent from the previous ones.* If the series of $(\boldsymbol{\xi}, \boldsymbol{\eta})$ is not a finite sum then the procedure can be repeated and we get the *contradictory statement* that $[R_V(T+\epsilon)]^\perp$ has infinite codimension.

We combine this fact with Lemma 3.3 and we get $\boldsymbol{\xi} = 0$, $\boldsymbol{\eta} = 0$ as we wanted to prove.

4. An abstract setup

Although every concrete case has its specific features, it makes sense to extract an abstract setup from the previous arguments. So doing, we shall see the role of the assumptions and possible extensions.

The first assumptions are regularity of $M(t)$ and of the boundary of the bounded region Ω. Controllability under weaker regularity assumptions has not been studied in the viscoelastic case (in view of the application of control methods to inverse problems as in [3], also the study of the reachable set on (parts of) unbounded regions Ω would have its interest).

System (1.2)–(1.4) can be written as

$$w'' = A(w - Df) + \int_0^t M(t-s)A\left(w(s) - Df(s)\right)\,\mathrm{d}s + F(t)$$

where w takes values in a Hilbert space H and the control f takes values in a Hilbert space U (in this abstract setup the use of boldface is not needed).

After the MacCamy trick (and multiplication with an the exponential) we get

$$w''(t) = Aw(t) + bw(t) + \int_0^t K(t-s)w(s)\,\mathrm{d}s + Bf(t) + F_1(t), \quad B = -AD. \tag{4.1}$$

When $f = 0$ the solution of this equation will be denoted ψ.

We assumed $-A$ selfadjoint positive with compact resolvent and the existence of $\sigma \in (0,1)$ such that $\mathrm{im}B \subseteq (\mathrm{dom}(-A)^\sigma)$, so that $A^{-1}B$ is compact.

Equation (4.1) makes sense in $(\mathrm{dom}A)'$ and it turned out that $w \in \mathrm{C}([0,T], H) \cap \mathrm{C}^1\left([0,T]\left(\mathrm{dom}(-A)^{1/2}\right)'\right)$ for every $f \in \mathrm{L}^2(0,T;U)$. This is a consequence of the corresponding property when $M = 0$, and it is used to give a sense to the definition of the reachable set at every time $T \geq 0$.

Note that in several applications it will be $A = A_0 + A_1$ where A_0 has the properties above while A_1 will be A_0 bounded (or compact). This case is still to be studied, under suitable assumptions on the perturbation A_1.

The previous assumptions and *admissibility, i.e.* continuity of $B^*\phi(\cdot)$ from $\left(\left(\mathrm{dom}(-A)^{1/2}\right) \times H\right)$ to $\mathrm{L}^2(0,T;U)$ for every $T > 0$ (when f, F are zero) are what we need to characterize $[R_V(T)]^\perp$ (in $H \times \mathrm{dom}\left((-A)^{1/2}\right)'$), as in Theorem 3.1 and to prove that $\dim [R_V(T)]^\perp < +\infty$ if the system with $M(t) \equiv 0$ is controllable at time T.

In order to conclude controllability of the system with memory, we need the analogous of **Step 2-2**, hence of the regularity property in Lemma 3.4. This depends on a new assumption, that the eigenvalues $-\lambda_n^2$ of the operator A verify

$$\sum_{n=1}^{+\infty} \frac{1}{\lambda_n^{2d}} < +\infty. \tag{4.2}$$

In the case studied in this paper, condition (4.2) holds with $d = 2$ and the condition $d = 2$ has been used in the proof of Lemma 3.4 when we used *tree* iteration of Picard method. In general, for elliptic operators, property (4.2) holds with a suitably large value of d, which depends on dimΩ (see [1], Sects. 13–14). If $d > 2$, Lemma 3.4 can still be proved, using more iterations of the Picard method to represent the solution of (3.16). After that, we can proceed with steps similar to those in Section 3.2 to prove controllability, since $B^*\phi(t) \equiv 0$ in $\mathrm{L}^2(0, T; U)$ implies $\phi(t) \equiv 0$ when the system with $M(t) \equiv 0$ is controllable at time T, thanks to the inverse inequality.

Note that in this abstract setting the condition that the Lamé coefficients are constant (*i.e.* the material is homogeneous) has no role and existing controllability results for nonhomogeneous elastic materials (as in [8]) are simply lifted to the corresponding viscoelastic case. Even more, if the material is not homogeneous the operator A is the restriction to $\mathrm{H}^1_0(\Omega) \cap \mathrm{H}^2(\Omega)$ of the operator $\nabla \cdot (a_{ijkl}\,(u_{i,j} + u_{j,i})$ (where $_{,k}$ denotes the kth partial derivative and a_{ijkl} is a tensor with suitable symmetry and positivity properties). Controllability of classes of purely elastic nonisotropic bodies has been studied (the paper [22] studies controllability of homogeneous orthotropic materials) and also these results can be extended to viscoelastic materials, using the abstract setting outlined here.

Finally, we consider again the assumptions on the memory kernel $M(t)$. We assumed that $M(t)$ does not depend on the space variable x. If the system is not homogeneous then M might depend upon x. It is easy to conjecture that this case can be handled without much difficulty. A more delicate point is the assumption that $M(t)$ is real valued. If the material is not isotropic then $M(t)$ might be tensor valued. The extension of the results in [22] when $M(t)$ is a tensor will require annoying computations.

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