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A NOTE ON THE BOUNDARY REGULARITY OF SOLUTIONS TO QUASILINEAR ELLIPTIC EQUATIONS

Giuseppe Riey[1] and Berardino Sciunzi[1,a]

Abstract. We study the summability up to the boundary of the second derivatives of solutions to a class of Dirichlet boundary value problems involving the p-Laplace operator. Our results are meaningful for the cases when the Hopf's Lemma cannot be applied to ensure that there are no critical points of the solution on the boundary of the domain.

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1. Introduction

Let Ω be a bounded smooth domain of $\mathbb{R}^n$, let $p > 1$ be fixed and let $f : \bar{\Omega} \times \mathbb{R} \to \mathbb{R}$ be such that f is Lipshitz continuous on each compact subset of $\bar{\Omega} \times \mathbb{R}$. Let $u \in W_0^{1,p}(\Omega)$ be a weak solution to the problem:

$$\begin{cases} -\Delta_p u = f(x,u) & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega\,, \end{cases} \tag{1.1}$$

namely

$$\int_\Omega |\nabla u|^{p-2} \langle \nabla u, \nabla \varphi \rangle \mathrm{d}x = \int_\Omega f(x,u)\varphi \mathrm{d}x, \qquad \text{for all} \qquad \varphi \in C_c^\infty(\Omega). \tag{1.2}$$

Having in mind [3, 7, 19] it is natural to assume that $u \in C^{1,\alpha}(\overline{\Omega})$, for some $\alpha < 1$. In fact such a regularity results holds under very general assumptions. Results regarding the study of the optimal $C^{1,\alpha}$ regularity, namely estimates on the value of α, can be found in [20], that is based on previous results in [6].

Here we address once again the study of the summability of the second derivatives. From [1,17,18] it is known that under the above assumptions $u \in W^{2,2}_{\mathrm{loc}}(\Omega)$ if $1 < p < 3$, and that, if $p \geq 3$ and the source term f is strictly positive, then $u \in W^{2,q}_{\mathrm{loc}}(\Omega)$ for $q < \frac{p-1}{p-2}$. We remark that, if $p \geq 3$, such a regularity is optimal (see *e.g.* [18]).

When Hopf's boundary Lemma applies there is nothing to be proved up to the boundary since the solutions have no critical points there and therefore the equation is no more degenerate. We therefore refer to cases when Hopf's Lemma cannot be applied.

Our main result is the following.

Keywords and phrases. p-Laplace equations, regularity of the solutions.

[1] Dipartimento di Matematica e Informatica, Università della Calabria, Ponte Pietro Bucci 31B, 87036 Arcavacata di Rende, Cosenza, Italy. riey@mat.unical.it
[a] Corresponding author: sciunzi@mat.unical.it

Theorem 1.1. *Let $u \in C^{1,\alpha}(\overline{\Omega})$ be a weak solution to* (1.1)*. We have that*:

(i) *if $p \leq 2$, then $u \in W^{2,2}(\Omega)$,*
(ii) *if $p > 2$, then $|\nabla u|^{p-1} \in W^{1,2}(\Omega)$.*

This kind of results are somehow close to the Calderón-Zygmund theory. In the case of quasilinear elliptic equation such theory in completely not trivial and has been developed by Mingione and his collaborators. We only refer here to [6,14,15] and the references therein.

Our proofs are based on weighted estimates for the second derivatives of the solution, achieved by means of the linearized operator of a transformed (by a flattening) equation. To get these estimates, we use the techniques developed in [1,2,10,17,18].

Regularity results in our setting, in cases when the assumptions to apply Hopf's Lemma are not satisfied, can be found also in [10], where the study of the summability of the second derivatives of the solutions on the whole Ω is performed for p close to two. An example of application of the above techniques to cases where the Hopf's Lemma fails, can be found also in [8,12].

2. Preliminary results

2.1. The flattening

Given a matrix A, we denote by A^T the transposed of A. For $x \in \mathbb{R}^n$, $|x|$ denotes the euclidian norm of x and, given a $n \times n$ matrix A, $|A|$ is the euclidian norm: $|A| = \sqrt{\sum_{i,j=1}^{n} |a_{ij}|^2}$. For $r > 0$ and $\overline{x} \in \mathbb{R}^n$ we set $B_r(\overline{x}) = \{x \in \mathbb{R}^n : |x - \overline{x}| < r\}$. For $\overline{x} \in \partial\Omega$ and $r > 0$ small enough, consider $B^+ := B_r(\overline{x}) \cap \Omega$ and a flattening operator, that means a diffeomorphism $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ such that $\Phi^{-1}(B_r(\overline{x}) \cap \partial\Omega) \subset \{y_n = 0\}$. We set: $\mathcal{B}^+ = \Phi^{-1}(B^+)$ and we denote by $\mathcal{B}^-$ the reflection of $\mathcal{B}^+$ with respect to the hyperplane $\{y_n = 0\}$. We set $\mathcal{B} = \mathcal{B}^+ \cup \mathcal{B}^- \cup \Phi^{-1}(\partial\Omega \cap B_r(\overline{x}))$.

We set $\Psi = \Phi^{-1}$ and we use the change of variable:

$$x = \Phi(y), \qquad y = \Psi(x)\,,$$

so that we have:

$$w(y) = u(\Phi(y)) \quad \text{in} \quad \mathcal{B}^+.$$

Denoting by $J\Phi(y)$ the Jacobian matrix of Φ, equation (1.2) becomes:

$$\int_{\mathcal{B}^+} |\nabla_x u(\Phi(y))|^{p-2} \langle \nabla_x u(\Phi(y)), \nabla_x \varphi(\Phi(y))\rangle |\mathrm{det} J\Phi(y)| \mathrm{d}y = \int_{\mathcal{B}^+} g(y, w(y))\psi(y) |\mathrm{det} J\Phi(y)| \mathrm{d}y \tag{2.1}$$

where $\mathrm{det}J\Phi(y)$ denotes the determinant of $J\Phi(y)$ and we have set:

$$g(y, w) = f(\Phi(y), w), \quad \psi(y) = \varphi(\Phi(y)).$$

Hence we have:

$$\nabla_y w(y) = J\Phi(y)^T \nabla_x u(\Phi(y))\,, \qquad \nabla_y \psi(y) = J\Phi(y)^T \nabla_x \varphi(\Phi(y))$$

and (2.1) gives:

$$\begin{aligned} \int_{\mathcal{B}^+} \left| \left[J\Phi(y)^T\right]^{-1} \nabla w(y) \right|^{p-2} \left\langle \left(\left[J\Phi(y)^T\right]^{-1} \right)^T \left[J\Phi(y)^T\right]^{-1} \nabla w(y), \nabla\psi(y) \right\rangle |\mathrm{det} J\Phi(y)| \mathrm{d}y \\ = \int_{\mathcal{B}^+} g(y, w(y))\psi(y) |\mathrm{det} J(\Phi(y))| \mathrm{d}y. \end{aligned} \tag{2.2}$$

Setting:

$$A(y) = \left[J(\Phi(y))^T\right]^{-1}, \quad K(y) = A(y)^T A(y), \quad \rho(y) = |\det J(\Phi(y))| \tag{2.3}$$

it follows that $w(y)$ weakly satisfies

$$-\mathrm{div}\left(\rho(y)|A(y)\nabla w(y)|^{p-2}K(y)\nabla w(y)\right) = g(y, w(y))\rho(y) \tag{2.4}$$

in $\mathcal{B}^+$.

We define the odd extension of $w(y)$ and the even extension of $\rho(y)$ and $A(y)$ (and hence of K) as follows:

$$\bar{w}(y) = \begin{cases} w(y), & \text{if} \quad y_n \geq 0, \\ -w(y_1, \dots, y_{n-1}, -y_n), & \text{if} \quad y_n < 0 \end{cases} \tag{2.5}$$

$$\bar{\rho}(y) = \begin{cases} \rho(y), & \text{if} \quad y_n \geq 0, \\ \rho(y_1, \dots, y_{n-1}, -y_n), & \text{if} \quad y_n < 0 \end{cases} \tag{2.6}$$

$$\bar{A}(y) = \begin{cases} A(y), & \text{if} \quad y_n \geq 0, \\ A(y_1, \dots, y_{n-1}, -y_n), & \text{if} \quad y_n < 0. \end{cases} \tag{2.7}$$

For the function $g(y,t)$ we consider the mixed extension (odd with respect to t and even with respect to y_n):

$$\bar{g}(y,t) = \begin{cases} g(y,t), & \text{if} \quad y_n \geq 0,\, t \geq 0, \\ -g(y,-t), & \text{if} \quad y_n \geq 0,\, t < 0, \\ g(y_1, \dots, y_{n-1}, -y_n, t), & \text{if} \quad y_n < 0,\, t \geq 0, \\ -g(y_1, \dots, y_{n-1}, -y_n, -t), & \text{if} \quad y_n < 0,\, t < 0. \end{cases} \tag{2.8}$$

In this way, $\bar{w}(y)$ satisfies the following equation

$$-\mathrm{div}(\bar{\rho}(y)|\bar{A}(y)\nabla \bar{w}(y)|^{p-2}\bar{K}(y)\nabla \bar{w}(y)) = \bar{g}(y, \bar{w})\bar{\rho}(y) \tag{2.9}$$

in $\mathcal{B}$.

It is easy to verify that (2.9) fulfills the structural assumptions needed to apply standard $C^{1,\alpha}$ regularity results (in $\mathcal{B}$), see for instance [3,7,19].

2.2. Properties of the flattening operator Φ

We assume that $\partial\Omega$ is smooth enough and, without loss of generality, we can assume that $\Psi(\bar{x}) = 0$. Therefore we can construct Φ in such a way that:

$$\Phi(y) = y + F(y) \tag{2.10}$$

with F such that $F(0) = \bar{x}$ and $\sup\limits_{|y|<\tau_1} |JF(y)| < \tau_2$ for suitable τ_1 and τ_2 small enough. An explicit representation of Φ can be found for instance in [5]. By (2.10) we have:

$$J\Phi(y) = I + JF(y) \tag{2.11}$$

where I is the identity matrix and JF is the Jacobian matrix of F.

By classical results of linear algebra and the regularity properties of Φ it follows that, there exist $\delta = \delta(\tau_1)$ and $c_1 > 0, c_2 > 0, c_3 > 0, c_4 > 0$ such that:

$$c_1|v| \leq |A(y)v| \leq c_2|v| \quad \forall v \in \mathbb{R}^n, \forall y \in \mathcal{B}_\delta(0) \tag{2.12}$$

and

$$c_3|v| \leq |K(y)v| \leq c_4|v| \quad \forall v \in \mathbb{R}^n, \forall y \in \mathcal{B}_\delta(0). \tag{2.13}$$

3. Main results

3.1. Linearized operator

With a little abuse of notation, even if we are considering the extended functions defined on the whole $\mathcal{B}$, we will omit the bar over the functions defined in (2.5), (2.6),(2.7), (2.8). For a function $v = v(y)$, in the sequel we denote by a subscript j the derivative with respect to y_j and for a function $g = g(y,t)$ we denote by g_j the derivative with respect to y_j and by $g'(y,t)$ the derivative with respect to t.

To compute a linearized equation associated to equation (2.9), we need some basic definition and properties about the weighted Sobolev spaces (for more details about them see for instance [11, 16, 21]).

Let $U \subset \mathbb{R}^n$ be a bounded smooth domain. For $m \geq 1$ and $\mu \in L^1(U)$ the weighted Sobolev space $W^{1,m}_\mu(U)$ (with respect to the weight μ) is defined as the completion of $C^\infty(\overline{U})$ with respect to the norm:

$$\|v\| = \left(\int_U |v|^m\right)^{\frac{1}{m}} + \left(\int_U |\nabla v|^m \mu\right)^{\frac{1}{m}}, \tag{3.1}$$

where ∇v is the distributional derivative. As for the usual Sobolev spaces, the space $W^{1,m}_{0,\mu}(U)$ is defined as the closure of $C^\infty_c(U)$ in $W^{1,m}_\mu(U)$. We set $H^1_\mu(U) = W^{1,2}_\mu(U)$ and $H^1_{0,\mu}(U) = W^{1,2}_{0,\mu}(U)$, which are the Hilbert spaces where the linearized operator associated to equation (2.9) is defined.

We set $Z = \{x \in \mathcal{B} : \nabla w(x) = 0\}$ and we consider $\psi \in C^\infty_c(\mathcal{B} \setminus Z)$. For any $j = 1, \ldots, n$, we take ψ_j as test function in the weak formulation of (2.9) and, integrating by parts, since $w \in C^2(\mathcal{B}\backslash Z)$, we obtain:

$$\begin{aligned}
&\int_{\mathcal{B}} \rho_j(y)|A(y)\nabla w(y)|^{p-2}\langle K(y)\nabla w(y), \nabla\psi(y)\rangle \mathrm{d}y \\
&+ (p-2)\int_{\mathcal{B}} \rho(y)|A(y)\nabla w(y)|^{p-4}\langle A_j(y)^T A(y)\nabla w(y), \nabla w(y)\rangle \cdot \langle K(y)\nabla w(y), \nabla\psi(y)\rangle \mathrm{d}y \\
&+ (p-2)\int_{\mathcal{B}} \rho(y)|A(y)\nabla w(y)|^{p-4}\langle K(y)\nabla w(y), \nabla w_j(y)\rangle \cdot \langle K(y)\nabla w(y), \nabla\psi(y)\rangle \mathrm{d}y \\
&+ \int_{\mathcal{B}} \rho(y)|A(y)\nabla w(y)|^{p-2}\langle K_j(y)\nabla w(y), \nabla\psi(y)\rangle \mathrm{d}y \\
&+ \int_{\mathcal{B}} \rho(y)|A(y)\nabla w(y)|^{p-2}\langle K(y)\nabla w_j(y), \nabla\psi(y)\rangle \mathrm{d}y \\
= &\int_{\mathcal{B}} \left[g_j(y,w(y))\rho(y) + g'(y,w(y))w_j(y)\rho(y) + g(y,w(y))\rho_j(y)\right]\psi \mathrm{d}y.
\end{aligned} \tag{3.2}$$

By a density argument, (3.2) holds for any $\psi \in H^1_\mu(\mathcal{B}) \cap L^\infty(\mathcal{B})$ with compact support in $\mathcal{B} \setminus Z$.

3.2. Hessian estimate

Proposition 3.1. Let $w \in W^{1,\infty}_{\mathrm{loc}}(\mathcal{B})$ be a weak solution of (2.9) and let $p \in (1,\infty)$ be fixed. For $y_0 \in \mathcal{B}$, let $r > 0$ be such that $B_{2r}(y_0) \subset \mathcal{B}$. For $\gamma < n-2$ ($\gamma = 0$ if $n = 2$), there holds:

$$\sup_{z\in\mathcal{B}} \int_{B_r(y_0)} \frac{|\nabla w|^{p-2}|D^2 w|^2}{|y-z|^\gamma} \mathrm{d}y \leq C\,, \tag{3.3}$$

where $C = C(y_0, r, \gamma, p, n, \|w\|_{W^{1,\infty}}, f)$.

Proof. Let $G_\alpha : \mathbb{R} \to \mathbb{R}$ be defined as:

$$G_\alpha(s) = \begin{cases} s & \text{if} \quad |s| \geq 2\alpha, \\ 2\left[s - \alpha\dfrac{s}{|s|}\right] & \text{if} \quad \alpha < |s| < 2\alpha, \\ 0 & \text{if} \quad |s| \leq \alpha, \end{cases}$$

and let φ be a cut-off function such that

$$\varphi \in C_c^\infty(B_{2r}(y_0)) \quad \varphi \equiv 1 \qquad \text{in} \qquad B_r(y_0) \qquad \text{and} \qquad |D\varphi| \leq \frac{2}{r}, \tag{3.4}$$

with $2r < \operatorname{dist}(y_0, \partial\Omega)$. Fix $\gamma < n-2$ (or $\gamma = 0$ if $n = 2$) and set:

$$H_\delta(t) = \frac{G_\delta(t)}{|t|^{\gamma+1}}\,, \quad H_{\delta,z}(y) = H_\delta(|y-z|)$$

and

$$\psi(y) = G_\varepsilon(w_j(y)) H_{\delta,z}(y)\varphi^2(y). \tag{3.5}$$

In the sequel, when there is no possibility of confusion, we omit the dependence on y. Using ψ as test function in (3.2), we have:

$$\begin{aligned}
&\int_{\mathcal{B}} \rho_j |A\nabla w|^{p-2} 2\varphi G_\varepsilon(w_j) H_{\delta,z} \langle K\nabla w, \nabla\varphi\rangle \,\mathrm{d}y \\
&+ \int_{\mathcal{B}} \rho_j |A\nabla w|^{p-2} \varphi^2 G'_\varepsilon(w_j) H_{\delta,z} \langle K\nabla w, \nabla w_j\rangle \,\mathrm{d}y \\
&+ \int_{\mathcal{B}} \rho_j |A\nabla w|^{p-2} \varphi^2 G_\varepsilon(w_j) \langle K\nabla w, \nabla H_{\delta,z}\rangle \,\mathrm{d}y \\
&+ (p-2)\int_{\mathcal{B}} \rho |A\nabla w|^{p-4} 2\varphi G_\varepsilon(w_j) H_{\delta,z} \langle A_j^T A\nabla w, \nabla w\rangle \cdot \langle K\nabla w, \nabla\varphi\rangle \,\mathrm{d}y \\
&+ (p-2)\int_{\mathcal{B}} \rho |A\nabla w|^{p-4} \varphi^2 G'_\varepsilon(w_j) H_{\delta,z} \langle A_j^T A\nabla w, \nabla w\rangle \cdot \langle K\nabla w, \nabla w_j\rangle \,\mathrm{d}y \\
&+ (p-2)\int_{\mathcal{B}} \rho |A\nabla w|^{p-4} \varphi^2 G_\varepsilon(w_j) \langle A_j^T A\nabla w, \nabla w\rangle \cdot \langle K\nabla w, \nabla H_{\delta,z}\rangle \,\mathrm{d}y \\
&+ (p-2)\int_{\mathcal{B}} \rho |A\nabla w|^{p-4} 2\varphi G_\varepsilon(w_j) H_{\delta,z} \langle K\nabla w, \nabla w_j\rangle \cdot \langle K\nabla w, \nabla\varphi\rangle \,\mathrm{d}y \\
&+ (p-2)\int_{\mathcal{B}} \rho |A\nabla w|^{p-4} \varphi^2 G'_\varepsilon(w_j) H_{\delta,z} \langle K\nabla w, \nabla w_j\rangle^2 \,\mathrm{d}y \\
&+ (p-2)\int_{\mathcal{B}} \rho |A\nabla w|^{p-4} \varphi^2 G_\varepsilon(w_j) \langle K\nabla w, \nabla w_j\rangle \cdot \langle K\nabla w, \nabla H_{\delta,z}\rangle \,\mathrm{d}y \\
&+ \int_{\mathcal{B}} \rho |A\nabla w|^{p-2} 2\varphi G_\varepsilon(w_j) H_{\delta,z} \langle K_j\nabla w, \nabla\varphi\rangle \,\mathrm{d}y \\
&+ \int_{\mathcal{B}} \rho |A\nabla w|^{p-2} \varphi^2 G'_\varepsilon(w_j) H_{\delta,z} \langle K_j\nabla w, \nabla w_j\rangle \,\mathrm{d}y \\
&+ \int_{\mathcal{B}} \rho |A\nabla w|^{p-2} \varphi^2 G_\varepsilon(w_j) \langle K_j\nabla w, \nabla H_{\delta,z}\rangle \,\mathrm{d}y \\
&+ \int_{\mathcal{B}} \rho |A\nabla w|^{p-2} 2\varphi G_\varepsilon(w_j) \langle K\nabla w_j, \nabla\varphi\rangle \,\mathrm{d}y \\
&+ \int_{\mathcal{B}} \rho |A\nabla w|^{p-2} \varphi^2 G'_\varepsilon(w_j) H_{\delta,z} \langle K\nabla w_j, \nabla w_j\rangle \,\mathrm{d}y \\
&+ \int_{\mathcal{B}} \rho |A\nabla w|^{p-2} \varphi^2 G_\varepsilon(w_j) \langle K\nabla w_j, \nabla H_{\delta,z}\rangle \,\mathrm{d}y \\
=&\int_{\mathcal{B}} \left[g_j\rho + g' w_j \rho + g\rho_j\right] \varphi^2 G_\varepsilon(w_j) H_{\delta,z} \,\mathrm{d}y.
\end{aligned} \tag{3.6}$$

In the sequel, c, as well as C, will denote positive constants, possibly depending on $r, y_0, \|w\|_{W^{1,\infty}(B_{2r}(\bar{x}))}$ but not on z, whose value can vary from line to line.

We set:

$$\begin{aligned} I_1^\delta = &\int_{\mathcal{B}} \rho_j |A\nabla w|^{p-2} 2\varphi G_\varepsilon(w_j) H_{\delta,z} \langle K\nabla w, \nabla\varphi\rangle \,\mathrm{d}y \\ &+ (p-2)\int_{\mathcal{B}} \rho |A\nabla w|^{p-4} 2\varphi G_\varepsilon(w_j) H_{\delta,z} \langle A_j^T A\nabla w, \nabla w\rangle \cdot \langle K\nabla w, \nabla\varphi\rangle \,\mathrm{d}y \\ &+ \int_{\mathcal{B}} \rho |A\nabla w|^{p-2} 2\varphi G_\varepsilon(w_j) H_{\delta,z} \langle K_j\nabla w, \nabla\varphi\rangle \,\mathrm{d}y \end{aligned} \tag{3.7}$$

$$\begin{aligned} I_2^\delta = &\int_{\mathcal{B}} \rho_j |A\nabla w|^{p-2} \varphi^2 G_\varepsilon(w_j) \langle K\nabla w, \nabla H_{\delta,z}\rangle \,\mathrm{d}y \\ &+ (p-2)\int_{\mathcal{B}} \rho |A\nabla w|^{p-4} \varphi^2 G_\varepsilon(w_j) \langle A_j^T A\nabla w, \nabla w\rangle \cdot \langle K\nabla w, \nabla H_{\delta,z}\rangle \,\mathrm{d}y \\ &+ \int_{\mathcal{B}} \rho |A\nabla w|^{p-2} \varphi^2 G_\varepsilon(w_j) \langle K_j\nabla w, \nabla H_{\delta,z}\rangle \,\mathrm{d}y \end{aligned} \tag{3.8}$$

$$\begin{aligned} I_3^\delta = (p-2) &\int_{\mathcal{B}} \rho |A\nabla w|^{p-4} 2\varphi G_\varepsilon(w_j) H_{\delta,z} \langle K\nabla w, \nabla w_j\rangle \cdot \langle K\nabla w, \nabla\varphi\rangle \,\mathrm{d}y \\ &+ \int_{\mathcal{B}} \rho |A\nabla w|^{p-2} 2\varphi G_\varepsilon(w_j) \langle K\nabla w_j, \nabla\varphi\rangle \,\mathrm{d}y \end{aligned} \tag{3.9}$$

$$\begin{aligned} I_4^\delta = &\int_{\mathcal{B}} \rho_j |A\nabla w|^{p-2} \varphi^2 G'_\varepsilon(w_j) H_{\delta,z} \langle K\nabla w, \nabla w_j\rangle \,\mathrm{d}y \\ &+ (p-2)\int_{\mathcal{B}} \rho |A\nabla w|^{p-4} \varphi^2 G'_\varepsilon(w_j) H_{\delta,z} \langle A_j^T A\nabla w, \nabla w\rangle \cdot \langle K\nabla w, \nabla w_j\rangle \,\mathrm{d}y \\ &+ \int_{\mathcal{B}} \rho |A\nabla w|^{p-2} \varphi^2 G'_\varepsilon(w_j) H_{\delta,z} \langle K_j\nabla w, \nabla w_j\rangle \,\mathrm{d}y \end{aligned} \tag{3.10}$$

$$\begin{aligned} I_5^\delta = (p-2) &\int_{\mathcal{B}} \rho |A\nabla w|^{p-4} \varphi^2 G_\varepsilon(w_j) \langle K\nabla w, \nabla w_j\rangle \cdot \langle K\nabla w, \nabla H_{\delta,z}\rangle \,\mathrm{d}y \\ &+ \int_{\mathcal{B}} \rho |A\nabla w|^{p-2} \varphi^2 G_\varepsilon(w_j) \langle K\nabla w_j, \nabla H_{\delta,z}\rangle \,\mathrm{d}y \end{aligned} \tag{3.11}$$

$$I_6^\delta = (p-2)\int_{\mathcal{B}} \rho |A\nabla w|^{p-4} \varphi^2 G'_\varepsilon(w_j) H_{\delta,z} \langle K\nabla w, \nabla w_j\rangle^2 \,\mathrm{d}y \tag{3.12}$$

$$I_7^\delta = \int_{\mathcal{B}} \rho |A\nabla w|^{p-2} \varphi^2 G'_\varepsilon(w_j) H_{\delta,z} \langle K\nabla w_j, \nabla w_j\rangle \,\mathrm{d}y \tag{3.13}$$

$$I_8^\delta = \int_{\mathcal{B}} \left[g_j\rho + g' w_j \rho + g\rho_j\right] \varphi^2 G_\varepsilon(w_j) H_{\delta,z} \,\mathrm{d}y. \tag{3.14}$$

If $p \geq 2$, then I_6^δ is positive and hence we trivially have:

$$I_6^\delta + I_7^\delta \geq I_7^\delta. \tag{3.15}$$

If $p < 2$, then by the definition of K we have:

$$|A\nabla w|^{p-4}\langle K\nabla w, \nabla w_j\rangle^2 = |A\nabla w|^{p-4}\langle A\nabla w, A\nabla w_j\rangle^2 \le |A\nabla w|^{p-2}|A\nabla w_j|^2\,, \tag{3.16}$$

which implies:

$$(p-2)|A\nabla w|^{p-4}\langle K\nabla w, \nabla w_j\rangle^2 \ge (p-2)|A\nabla w|^{p-2}|A\nabla w_j|^2 \tag{3.17}$$

and hence:

$$I_6^\delta + I_7^\delta \ge (p-1)I_7^\delta. \tag{3.18}$$

We can summarize (3.15) and (3.18) saying that, for every $p > 1$, it holds:

$$I_6^\delta + I_7^\delta \ge \min\{1, p-1\}I_7^\delta. \tag{3.19}$$

By (3.6) and (3.19) we infer:

$$\min\{1, p-1\}I_7^\delta \le I_6^\delta + I_7^\delta \le \sum_{i=1}^{5}|I_i^\delta| + I_8^\delta. \tag{3.20}$$

By the regularity properties of the diffeomorphism Φ we have that there exist $c, M > 0$ such that:

$$c \le \rho(y) \le M, \qquad |\rho_j(y)| \le M \qquad \forall y \in \mathcal{B}\,, \quad \forall j = 1, \dots, n. \tag{3.21}$$

By (2.12), (2.13), (3.20) and (3.21), estimating the terms in the righthand side of (3.20), we get:

$$\begin{aligned}
\int_{\mathcal{B}} |\nabla w|^{p-2}|\nabla w_j|^2 G_\varepsilon'(w_j)H_{\delta,z}\varphi^2 \mathrm{d}y \le{}& c\int_{\mathcal{B}} |G_\varepsilon(w_j)||\nabla w|^{p-1}|\nabla\varphi|H_{\delta,z}\varphi \mathrm{d}y\\
&+ c\int_{\mathcal{B}} |G_\varepsilon(w_j)||\nabla w|^{p-1}|\nabla H_{\delta,z}|\varphi^2 \mathrm{d}y\\
&+ c\int_{\mathcal{B}} |G_\varepsilon(w_j)||\nabla w|^{p-2}|\nabla w_j||\nabla\varphi|H_{\delta,z}\varphi\\
&+ c\int_{\mathcal{B}} |G_\varepsilon'(w_j)||\nabla w|^{p-1}|\nabla w_j|H_{\delta,z}\varphi^2\\
&+ c\int_{\mathcal{B}} |G_\varepsilon(w_j)||\nabla w|^{p-2}|\nabla w_j||\nabla H_{\delta,z}|\varphi^2\\
&+ c\int_{\mathcal{B}} [|g_j\rho| + |g\rho_j| + |g'\rho||w_j|]|G_\varepsilon(w_j)|H_{\delta,z}\varphi^2 \mathrm{d}y.
\end{aligned} \tag{3.22}$$

Recalling that $\gamma < n-2$ and that, for $s < n$, $\displaystyle\int_{\mathcal{B}} \frac{1}{|y-z|^s}\mathrm{d}y$ is uniformly bounded (because $\mathcal{B}$ is bounded), for fixed $\varepsilon > 0$ we can use dominate convergence to let δ to 0 and we get:

$$\begin{aligned}
\int_{\mathcal{B}} \frac{|\nabla w|^{p-2}|\nabla w_j|^2 G_\varepsilon'(w_j)\varphi^2}{|y-z|^\gamma}\mathrm{d}y \le{}& c\int_{\mathcal{B}} \frac{|G_\varepsilon(w_j)||\nabla w|^{p-1}|\nabla\varphi|\varphi}{|y-z|^\gamma}\mathrm{d}y\\
&+ c\int_{\mathcal{B}} \frac{|G_\varepsilon(w_j)||\nabla w|^{p-1}\varphi^2}{|y-z|^{\gamma+1}}\mathrm{d}y\\
&+ c\int_{\mathcal{B}} \frac{|G_\varepsilon(w_j)||\nabla w|^{p-2}|\nabla w_j||\nabla\varphi|\varphi}{|y-z|^\gamma}\mathrm{d}y\\
&+ c\int_{\mathcal{B}} \frac{|G_\varepsilon'(w_j)||\nabla w|^{p-1}|\nabla w_j|\varphi^2}{|y-z|^\gamma}\mathrm{d}y\\
&+ c\int_{\mathcal{B}} \frac{|G_\varepsilon(w_j)||\nabla w|^{p-2}|\nabla w_j|\varphi^2}{|y-z|^{\gamma+1}}\mathrm{d}y\\
&+ c\int_{\mathcal{B}} \frac{[|g_j\rho| + |g\rho_j| + |g'\rho||w_j|]|G_\varepsilon(w_j)|\varphi^2}{|y-z|^\gamma}\mathrm{d}y.
\end{aligned} \tag{3.23}$$

We set:

$$
\begin{aligned}
J_1 &= \int_{\mathcal{B}} \frac{|G_\varepsilon(w_j)||\nabla w|^{p-1}|\nabla \varphi|\varphi}{|y-z|^\gamma}\mathrm{d}y\\
J_2 &= \int_{\mathcal{B}} \frac{|G_\varepsilon(w_j)||\nabla w|^{p-1}\varphi^2}{|y-z|^{\gamma+1}}\mathrm{d}y\\
J_3 &= \int_{\mathcal{B}} \frac{|G_\varepsilon(w_j)||\nabla w|^{p-2}|\nabla w_j||\nabla \varphi|\varphi}{|y-z|^\gamma}\mathrm{d}y\\
J_4 &= \int_{\mathcal{B}} \frac{|G'_\varepsilon(w_j)||\nabla w|^{p-1}|\nabla w_j|\varphi^2}{|y-z|^\gamma}\mathrm{d}y\\
&= \int_{\mathcal{B}\cap\{w_j>\varepsilon\}} \frac{|G'_\varepsilon(w_j)||\nabla w|^{p-1}|\nabla w_j|\varphi^2}{|y-z|^\gamma}\mathrm{d}y\\
J_5 &= \int_{\mathcal{B}} \frac{|G_\varepsilon(w_j)||\nabla w|^{p-2}|\nabla w_j|\varphi^2}{|y-z|^{\gamma+1}}\mathrm{d}y\\
J_6 &= \int_{\mathcal{B}} \frac{[|g_j\rho|+|g\rho_j|+|g'\rho||w_j|]|G_\varepsilon(w_j)|\varphi^2}{|y-z|^\gamma}\mathrm{d}y.
\end{aligned}
$$

We remark that the second and the fifth term in the sum at right-hand side of (3.22) and (3.23) disappears if $n=2$, because $\nabla H_{\delta,z}=0$ if $\gamma=0$.

By definition of G_ε it follows that:

$$|G_\varepsilon(w_j)| \le 2|w_j| \tag{3.24}$$

and

$$|G'_\varepsilon(w_j)| \le C. \tag{3.25}$$

By properties of g and ρ and the regularity of w, recalling that $\gamma<n-2$, we have:

$$J_6 \le c\int_{\mathcal{B}} \frac{1}{|y-z|^\gamma}\mathrm{d}y \le C. \tag{3.26}$$

We recall that for $a,b\in\mathbb{R}$ and $\theta>0$ there holds the Young inequality:

$$ab \le \theta a^2 + \frac{1}{4\theta}b^2. \tag{3.27}$$

For $n\ge 3$ (as above, for $n=2$ we have $J_5=0$ and $J_2=0$), using (3.27) and recalling that $\gamma<n-2$, we get:

$$J_2 \le C\,, \tag{3.28}$$

$$J_4 \le c\int_{\mathcal{B}} \frac{|\nabla w|^{p-2}|\nabla w_j|\varphi^2\chi_{\{w_j>\varepsilon\}}}{|y-z|^\gamma} \le \theta\int_{\mathcal{B}} \frac{|\nabla w|^{p-2}|\nabla w_j|^2\varphi^2\chi_{\{w_j>\varepsilon\}}}{|x-y|^\gamma}\mathrm{d}y + C \tag{3.29}$$

and

$$J_5 \le \theta\int_{\mathcal{B}} \frac{|\nabla w|^{p-2}|\nabla w_j|^2\varphi^2}{|x-y|^\gamma}\mathrm{d}y + C. \tag{3.30}$$

Recalling that $|\nabla\varphi| \le \dfrac{2}{\rho}$, we also have:

$$J_1 \le C \tag{3.31}$$

and

$$J_3 \le \theta\int_{\mathcal{B}} \frac{|\nabla w|^{p-2}|\nabla w_j|^2\varphi^2}{|x-y|^\gamma}\mathrm{d}y + C. \tag{3.32}$$

After setting $\vartheta = c\theta$, we choose θ such that $\vartheta < 1$. By the above estimates we get:

$$\int_{\mathcal{B}\cap\{w_j>\varepsilon\}} \frac{|\nabla w|^{p-2}|\nabla w_j|^2}{|y-z|^\gamma} \left(G'_\varepsilon(w_j) - \vartheta\right) \varphi^2 \mathrm{d}y \le c. \tag{3.33}$$

By definition of G_ε it follows that, $\forall s > 0$, $G'_\varepsilon(s)$ converges to 1 as ε goes to 0 and hence by Fatou's Lemma we get:

$$\int_{\mathcal{B}\setminus\{w_j=0\}} \frac{|\nabla w|^{p-2}|\nabla w_j|^2}{|y-z|^\gamma} \varphi^2 \mathrm{d}y \le c \tag{3.34}$$

and hence:

$$\int_{\mathcal{B}\setminus Z} \frac{|\nabla w|^{p-2}|\nabla w_j|^2}{|y-z|^\gamma} \varphi^2 \mathrm{d}y \le c\,, \tag{3.35}$$

where c depends on $y_0, r, n, p, \gamma, g, \Phi, \|w\|_{W^{1,\infty}(B_{2r}(\bar{x}))}$ but it does not depend on z. Recalling the properties of φ, we proved that

$$\sup_{z\in\mathcal{B}} \int_{B_r(y_0)\setminus Z} \frac{|\nabla w|^{p-2}|D^2 w|^2}{|y-z|^\gamma} \mathrm{d}y \le c. \qquad \square$$

3.3. Proof of Theorem 1.1

Taking $\gamma = 0$ in (3.3), we have that:

$$\int_{B_r(y_0)} |\nabla w|^{p-2}|D^2 w|^2 \mathrm{d}y \le C$$

and hence by the properties of Φ we immediately get that the same kind of estimate holds for u in $\Phi(\mathcal{B})$ and, recalling that $\overline{\Omega}$ is compact, we get that there exists $C > 0$ such that:

$$\int_\Omega |\nabla u|^{p-2}|D^2 u|^2 \le C. \tag{3.36}$$

If $p \le 2$, we trivially have $\int_\Omega |D^2 u|^2 \le \int_\Omega |\nabla u|^{p-2}|D^2 u|^2$ and hence (i) of Theorem 1.1 follows by (3.36).

If $p > 2$, since for suitable $c > 0$ there holds $|\nabla\left(|\nabla u|^{p-1}\right)| \le c|\nabla u|^{p-2}|D^2 u|$, recalling that ∇u is bounded, we have that there exists $C > 0$:

$$|\nabla\left(|\nabla u|^{p-1}\right)|^2 \le c^2|\nabla u|^{2(p-2)}|D^2 u|^2 \le C|\nabla u|^{p-2}|D^2 u|^2$$

and hence (ii) of Theorem 1.1 follows by (3.36), provided that, for every $i \in \{1, \ldots, n\}$, the i-th generalized derivative of $|\nabla u|^{p-1}$ coincides with the classical one, both denoted with $\left(|\nabla u|^{p-1}\right)_i$ almost everywhere in Ω. In fact, for $\varphi \in C^\infty_c(\Omega)$, we have:

$$\int_\Omega \left(|\nabla u|^{p-1}\right)_i \varphi \frac{G_\varepsilon(|\nabla u|)}{|\nabla u|} = -\int_\Omega |\nabla u|^{p-1}\varphi_i \frac{G_\varepsilon(|\nabla u|)}{|\nabla u|} - \int_{\Omega\cap\{\varepsilon<|\nabla u|<2\varepsilon\}} |\nabla u|^{p-1}\varphi\frac{1}{\varepsilon}. \tag{3.37}$$

Since we are considering the case $p > 2$, we have:

$$\left|\int_{\Omega\cap\{\varepsilon<|\nabla u|<2\varepsilon\}} |\nabla u|^{p-1}\varphi\frac{1}{\varepsilon}\right| \le \int_{\Omega\cap\{\varepsilon<|\nabla u|<2\varepsilon\}} |\nabla u|^{p-2}|\varphi| \le C\varepsilon. \tag{3.38}$$

Moreover by (3.36) we have that $\left(|\nabla u|^{p-1}\right)_i \in L^1(\Omega)$ and, recalling that $\frac{G_\varepsilon(|\nabla u|)}{|\nabla u|} \le 2$, as ε tends to 0, we can use dominated convergence in the other terms in (3.37) and we get the thesis.

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