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A CLASS OF INFINITE-HORIZON STOCHASTIC DELAY OPTIMAL CONTROL PROBLEMS AND A VISCOSITY SOLUTION TO THE ASSOCIATED HJB EQUATION *

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Abstract. In this paper, we investigate a class of infinite-horizon optimal control problems for stochastic differential equations with delays for which the associated second order Hamilton−Jacobi−Bellman (HJB) equation is a nonlinear partial differential equation with delays. We propose a new concept for the viscosity solution including time t and identify the value function of the optimal control problems as a unique viscosity solution to the associated second order HJB equation.

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1. Introduction

In this article, we consider the following controlled differential equations with delays:

$$\begin{cases} \mathrm{d}X^u(t) = F(X^u(t),(a,X^u_t)_H,u(t))\mathrm{d}t + F_1(X^u(t-\tau))\mathrm{d}t \\ \qquad\qquad +G(X^u(t),(c,X^u_t)_H,u(t))\mathrm{d}W(t),\ t\in[0,+\infty), \\ \quad X^u_0 = x\in H, X^u(0)=x_0\in R^d, \end{cases} \tag{1.1}$$

where

$$X^u_t(\theta)=X^u(t+\theta),\ \theta\in[-\tau,0],\ u(\cdot)\in\mathcal{U}[0,+\infty).$$

In the above equations, H denotes the Hilbert space $L^2([-\tau,0];R^d)$, and $\{W(t),t\geq 0\}$ is an n-dimensional standard Wiener process; the unknown $X^u(s)$, representing the state of the system, is an R^d-valued process; the control process u is left-continuous predictable process with respect to the Wiener filtration and takes its values in a compact subset U of R^{d_1}. The terms $a(\cdot)$ and $c(\cdot)$ are two given functions that satisfy the appropriate smoothness properties. The coefficients F, F_1 and G are assumed to satisfy Lipschitz conditions with respect to appropriate norms. Thus, there exists a unique adapted process $X^u(s,x,x_0),\ s\geq 0$, solution to (1.1).

Keywords and phrases. Second order Hamilton−Jacobi−Bellman equation, viscosity solution, infinite-horizon optimal control, stochastic differential equations with delays, existence and uniqueness.

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We wish to minimize a cost functional of the form

$$J(x, x_0, u) = E \int_0^{+\infty} \mathrm{e}^{-\lambda\sigma} q(X^u(\sigma, x, x_0), u(\sigma))\mathrm{d}\sigma \tag{1.2}$$

over all admissible controls belong to $\mathcal{U}[0, +\infty)$. Here, the term q is a given real function on $R^d \times U$, and the constant λ is sufficiently large. The definition of $\mathcal{U}[0, +\infty)$ will be given in Section 2. We can then introduce the value function

$$V(x, x_0) := \inf_{u\in\mathcal{U}[0,+\infty)} J(x, x_0, u). \tag{1.3}$$

The goal of this article is to characterize this value function V when $x \in \mathcal{D}$ and $x(0) = x_0$. Here $\mathcal{D}$ denotes the set of the right-continuous, R^d-valued functions on $[-\tau, 0]$ such that it belongs to H. We consider the following second order Hamilton−Jacobi−Bellman (HJB) equation:

$$-\lambda V(x) + \mathcal{S}(V)(x) + \mathbf{H}(x, \nabla_{x_0} V(x, x(0)), \nabla^2_{x_0} V(x, x(0))) = 0, \quad x \in D(\mathcal{S}(V)), \tag{1.4}$$

where

$$\begin{aligned}\mathbf{H}(x, p, l) = \inf_{u\in U} [(p, F(x(0), (a, x)_H, u) + F_1(x(-\tau)))_{R^d} + \frac{1}{2}\mathrm{tr}[lG(x(0), (c, x)_H, u) \\ G^\top(x(0), (c, x)_H, u)] + q(x(0), u)], (x, p, l) \in \mathcal{D} \times R^d \times \mathit{\Gamma}(R^d).\end{aligned}$$

Here, $G^\top$ denotes the transpose of the matrix G, $\mathit{\Gamma}(R^d)$ denotes the set of all $(d \times d)$ symmetric matrices and $(\cdot, \cdot)_{R^d}$ denotes the scalar product of R^d. $\nabla_{x_0} V$ and $\nabla^2_{x_0} V$ are first and second Fréchet derivatives of V with respect to the second variable. The definition of a weak infinitesimal generator $\mathcal{S}$ will be given in Section 3.

We will develop a viscosity solution to the second order HJB equation given in (1.4) (see Def. 4.2 for details) and then show that the value function V defined in (1.3) is a unique viscosity solution to the HJB equation given in (1.4).

The type of problem above occurs in many different fields, including finance and economics (see [19] for an overview on their applications). We refer to [9, 25] for mathematical finance and to [6, 24, 27–30] for portfolio optimizations. We also refer to [15–17] for advertising models with delayed effects and to [11, 14] for Pension funds.

These optimal control problems for stochastic systems with delays have been thoroughly investigated in recent years (see [1, 7, 13, 25, 26, 34, 35]). However, to the best of our knowledge, none of these results include our case. References [1, 7, 25, 26] established the stochastic maximum principles. Reference [1] showed a maximum principle of infinite horizon optimal control for stochastic delay equations. References [13, 34, 35] derived the existence and uniqueness of the mild solution for the second order HJB equations associated with stochastic systems with delays. Reference [12] studied the mild solution of the elliptic equations in Hilbert spaces associated with infinite horizon optimal control problems without delays. However, the continuous differentiability of the diffusion term and the drift term in the state equation is required to obtain the results in these references.

It is well known that the approach to the treatment of the optimal control problems given by (1.1) and (1.2) is to reformulate them as optimal control problems of the evolution equation in a Hilbert space (see, *e.g.*, [5, 10] for deterministic cases, and [15] for stochastic cases). However, in such cases, the initial condition of the state equations have the form $X^u_t = x \in H$ and $X^u(t) = x_0 \in R^d$, and the state equations do not contain the term $X^u(\cdot - \tau)$. In [2, 3], the term $X^u(\cdot - \tau)$ in state equations is considered; in these problems, however, $X^u(\cdot - \tau)$ is a linear term. In our case, because the coefficient F_1 is a genuinely nonlinear function about $X^u(\cdot - \tau)$, this form of the initial condition $X^u_t = x \in H$ and $X^u(t) = x_0 \in R^d$ does not guarantee that the value function can be considered as a viscosity solution to the associated HJB equation. To overcome this difficulty, we consider state equations with the initial condition $X^u_0 = x \in \mathcal{D}$ and study the viscosity solution to the associated HJB equation in the set $\mathcal{D}$.

The theory of viscosity solutions for second order HJB equations has been well developed. We refer to [8] for the basic theories of viscosity solutions. References [18, 20–22, 32] extended the notion of viscosity solutions to Hilbert spaces by using a limiting argument based on the existence of a countable basis. To the best of our knowledge, the viscosity solution to the second order HJB equation (1.4) has not been previously considered. The primary difficulty stems from the lack of a local compactness of the set $\mathcal{D}$. Thus, the standard techniques for the proof of the comparison theorem in references [18, 20–22, 32], which rely heavily on compactness arguments, are not applicable in our case.

We continue our previous work [36], in which a parabolic equation was studied. However, the approach of [36] is not directly applicable in our case, because equation (1.4) is time independent. We know that the classical definition for the viscosity solution to the elliptic HJB equation is time independent (see [32]); however, to use the left maximization principle (see Lem. 4.1), we need to introduce a new viscosity solution (see Def. 4.2) that includes the time t. This definition is entirely different from the definition of the viscosity solution in [36]. Hence, one difficulty in defining the viscosity solution in our case is that we must ensure that the value function is a viscosity solution to (1.4). Secondly, to do this, we also need to relax the conditions of viscosity solution. Specifically, we replace the "local maximum point" condition by "global maximum point" condition in the definition of viscosity solution. This relaxation brings a lot of difficulties in the proof of uniqueness for viscosity solution. In fact, the superior properties of the functions in the set $\mathcal{D}$ are needed in the proof of uniqueness for viscosity solution. Thanks to Lemma 4.1, which provides the "global maximum point" condition, the uniqueness of the viscosity solution can be proved under our definition. Thirdly, a dynamic programming principle (DPP) for infinite-horizon stochastic delay optimal control problems (see Thm. 3.3 for details) is also needed, which is difficult because the value function is independent of the times t. Finally, the Lemma 5.3 (see also Thm. 8.3 in [8]) and a limiting process are used to obtain the uniqueness result. Because the "global maximum point" in the set $\mathcal{D}$ is not the classic global maximum point, this makes it very difficult to find the limiting process. To the aforementioned challenges, we use another approximating process to converge to the limiting process.

The plan of this article is as follows. In the next section, some notations are fixed, and stochastic differential equations with delays are examined. The dynamic programming principle (DPP), which will be used in the following sections is proved in Section 3. In Section 4, we define the viscosity solution of the HJB equation (1.4) and show that the value function V defined by (1.3) is a viscosity solution. Section 5 is devoted to proving the uniqueness of the viscosity solution to (1.4), and in Section 6, we study the deterministic cases.

2. Preliminaries

Here, we define the notations used in this paper. We use the symbol $|\cdot|$ to denote the norm in a Banach space $\mathcal{G}$. The norm symbol is subscripted when necessary. For the vectors $x, y \in R^d$, the scalar product is denoted by $(x, y)_{R^d}$, and the Euclidean norm $(x, x)^{\frac{1}{2}}_{R^d}$ is denoted by $|x|$. For every $T > 0$, let $C([0, T], R^d)$ denote the space of the continuous functions from $[0, T]$ to R^d, which is associated with the usual norm $|f|_C = \sup_{\theta \in [0,T]} |f(\theta)|$. Let $\tau > 0$ be fixed. H denotes the real, separable Hilbert space $L^2([-\tau, 0]; R^d)$ for the scalar product $(\cdot, \cdot)_H$. Let $\mathcal{D}$ denote a subset of H:

$$\mathcal{D} := \{x \in H : x \text{ is a right-continuous}, R^d\text{-valued function on } [-\tau, 0]\}.$$

Let $\mathcal{D}_B$ denote a subset of $\mathcal{D}$:

$$\mathcal{D}_B := \{x \in \mathcal{D} : x \text{ is bounded}\}.$$

We define a norm on $\mathcal{D}_B$ as follows:

$$|x|_{\mathcal{D}_B} = \sup_{\theta \in [-\tau, 0]} |x(\theta)|, \quad x \in \mathcal{D}_B.$$

Then, $(\mathcal{D}_B, |\cdot|_{\mathcal{D}_B})$ is a Banach space. For every $N>0$, we define $\mathcal{D}_N$ by

$$\mathcal{D}_N := \left\{ x \in \mathcal{D} : |x|_H \le \tau^{\frac{1}{2}} N, |x(0)| \le N \right\}.$$

We define the $|\cdot|_B$-norm on H by

$$|x|_B^4 := \int_{-\tau}^0 (Bx)^4(s)\mathrm{d}s,$$

where

$$(Bx)(s) = \int_s^0 x(\theta)\mathrm{d}\theta, \quad s \in [-\tau, 0].$$

The notations $\mathcal{D}_N$ and $|\cdot|_B$ will be used to prove the uniqueness of the viscosity solution in Section 5.

Let $0 \le t < +\infty$, $\omega, \bar{\omega} \in \mathcal{D}$ be given. We can then define $\omega \otimes_t \bar{\omega} \in \mathcal{D}$ by

$$\omega \otimes_t \bar{\omega} := \tilde{\omega},$$

where

$$\tilde{\omega}(\theta) = \begin{cases} \bar{\omega}(\theta), & (-t) \vee (-\tau) \le \theta \le 0, \\ \omega(t+\theta), & -\tau \le \theta < (-t) \vee (-\tau). \end{cases}$$

For every $0 \le T < +\infty$, we define $\omega \otimes_{[0,T]} \bar{\omega}$ by

$$\omega \otimes_{[0,T]} \bar{\omega} := \{\omega \otimes_t \bar{\omega} | t \in [0,T]\}.$$

Let $\Omega := \{\omega \in C([0,+\infty), R^n) : \omega(0) = 0\}$, the set of continuous paths with initial value 0, W the canonical process, P the Wiener measure, $\mathcal{F}_\infty$ the complete σ-field generated by $\{W(t), t \ge 0\}$, $\{\hat{\mathcal{F}}_t\}_{t\ge 0}$ the filtration generated by $\{W(t), t \ge 0\}$, $\{\mathcal{F}_t\}_{t \ge 0}$ the filtration generated by $\{W(t), t \ge 0\}$, augmented with the family $\mathcal{N}$ of P-null of $\mathcal{F}_\infty$. The filtration $\{\mathcal{F}_t\}_{t\ge 0}$ satisfies the usual conditions. For every $[a,b] \subset [0,+\infty)$, we also use the notations:

$$\hat{\mathcal{F}}_a^b = \sigma(W(s) - W(a) : s \in [a,b]) \quad \text{and} \quad \mathcal{F}_a^b = \hat{\mathcal{F}}_a^b \vee \mathcal{N}.$$

Let us consider the following controlled state equations:

$$\begin{cases} \mathrm{d}X^u(t) = F(X^u(t), (a, X_t^u)_H, u(t))\mathrm{d}t + F_1(X^u(t-\tau))\mathrm{d}t \\ \qquad\qquad +G(X^u(t), (c, X_t^u)_H, u(t))\mathrm{d}W(t), \ t \in [0,+\infty), \\ \quad X_0^u = x \in H, X^u(0) = x_0 \in R^d, \end{cases} \tag{2.1}$$

where

$$X_t^u(\theta) = X^u(t+\theta), \ \theta \in [-\tau, 0], \ u(\cdot) \in \mathcal{U}[0,+\infty).$$

Here, for every $t \ge 0$, we let $\mathcal{U}[t,+\infty)$ denote

$\mathcal{U}[t,+\infty) := \{u(\cdot) : [t,+\infty) \to U | u(\cdot)$ is an $\{\hat{\mathcal{F}}_t^s\}_{s\ge t}$-left-continuous predictable process with values in a compact subset U of $R^{d_1}\}$;

and

$\mathcal{U}[0,t] := \{u(\cdot) : [0,t] \to U | u(\cdot)$ is an $\{\hat{\mathcal{F}}_s\}_{s\ge 0}$-left-continuous predictable process with values in values in a compact subset U of $R^{d_1}\}$.

We make the following assumptions:

Hypothesis 2.1.

(i) *The mappings* $F\colon R^d \times R \times U \to R^d$ *and* $F_1\colon R^d \to R^d$ *are measurable, and there exists a constant* $L > 0$ *such that for every* $(x_1, y_1), (x_2, y_2) \in R^d \times R, u, u' \in U$,

$$|F(x_1, y_1, u)| \vee |F_1(x_1)| \le L(1 + |x_1| + |y_1|),$$
$$|F(x_1, y_1, u) - F(x_2, y_2, u')| \vee |F_1(x_1) - F_1(x_2)| \le L(|x_1 - x_2| + |y_1 - y_2| + |u - u'|).$$

(ii) *The mapping* $G\colon R^d \times R \times U \to R^{d\times n}$ *is measurable, and there exists a constant* $L > 0$ *such that for every* $(x_1, y_1), (x_2, y_2) \in R^d \times R, u, u' \in U$,

$$|G(x_1, y_1, u)| \le L(1 + |x_1| + |y_1|),$$
$$|G(x_1, y_1, u) - G(x_2, y_2, u')| \le L(|x_1 - x_2| + |y_1 - y_2| + |u - u'|).$$

(iii) $a(\cdot), c(\cdot) \in W^{1,2}([-\tau, 0]; R^d)$ *and* $a(-\tau) = c(-\tau) = 0$.

Remark 2.2. Here $W^{1,2}([-\tau, 0]; R^d) := \{x = (x_1, x_2, \dots, x_d) | x_i \in W^{1,2}([-\tau, 0]; R), i = 1, 2, \dots, d\}$. To obtain the properties in Theorem 2.4, we need to assume $a(\cdot), c(\cdot) \in W^{1,2}([-\tau, 0]; R^d)$. Under the assumption $a(-\tau) = c(-\tau) = 0$, through a simple calculation, we obtain $(a, x)_H \le \tau^{\frac{1}{4}} |a|_{W^{1,2}} |x|_B$ and $(c, x)_H \le \tau^{\frac{1}{4}} |c|_{W^{1,2}} |x|_B$ for every $x \in H$, which will be used in the proof of Theorem 2.4. However, we can choose the function $a(\cdot)$ and $c(\cdot)$ from a wide class, which allows us to consider various economic phenomena.

We say that X^u is a solution to equation (2.1) if it is a continuous, $\{\mathcal{F}_t\}_{t\ge 0}$-predictable process with values in R^d, and it satisfies: P-$a.s.$,

$$X^u(t) = x_0 + \int_0^t F(X^u(\sigma), (a, X^u_\sigma)_H, u(\sigma)) \mathrm{d}\sigma + \int_0^t F_1(X^u(\sigma - \tau)) \mathrm{d}\sigma,$$
$$+ \int_0^t G(X^u(\sigma), (c, X^u_\sigma)_H, u(\sigma)) \mathrm{d}W(\sigma), \quad t \in [0, +\infty), \tag{2.2}$$

where $X^u_0 = x \in H$. To emphasize dependence on initial data, we denote the solution by $X^u(\cdot, x, x_0)$ or $X^u(\cdot, 0, x, x_0)$.

For every $T > 0$ and $p \ge 1$, let $L^p_{\mathcal{P}}(\Omega; C([0, T]; R^d))$ denote the space of predictable processes $\{Y(s), s \in [0, T]\}$ with continuous paths in R^d, such that the norm

$$|Y|^p = \mathbf{E} \sup_{s\in[0,T]} |Y(s)|^p$$

is finite. Elements of $L^p_{\mathcal{P}}(\Omega; C([0, T]; R^d))$ are identified up to indistinguishability.

We first recall a result on the solvability of (2.1) on a bounded interval that is shown in [13, 23].

Theorem 2.3. *Assume that Hypothesis* 2.1 *holds. Then, for all* $T > 0$ *and* $p \ge 1$, *a unique process* $X \in L^p_{\mathcal{P}}(\Omega, C([0, T]; R^d))$ *exists that is a solution to* (2.1) *on* $[0, T]$.

Let $g \in C(R)$, $x \in \mathcal{D}$ be given. We can then define $g(x) \in \mathcal{D}$ by

$$g(x)(\theta) := g(x(\theta)), \quad \theta \in [-\tau, 0].$$

Let us now study equation (2.1) and consider certain continuities for the solution $X^u(\cdot)$ of equation (2.1). These properties will be used in the proof of Theorem 3.2.

Theorem 2.4. *Assume that Hypothesis 2.1 holds. If we let* $\Lambda = 48[3^5+2^63^2+1+2^83^2\tau^2|a|^4_{W^{1,2}}+4\tau^2|c|^4_{W^{1,2}}]L^4+\frac{1}{4}$*, then a unique continuous process* X *exists that is a solution to (2.1). Moreover,*

$$\begin{aligned}
&\mathbf{E}|X^u(t,x,x_0)| \\
&\leq C_1\bigg(1+|x_0|+|x|_B+|F_1(x)|_B+\bigg|\int_{-\tau}^{0}F_1(x(\sigma))\mathrm{d}\sigma\bigg|+\bigg|\int_{-\tau}^{(t-\tau)\wedge 0}F_1(x(\sigma))\mathrm{d}\sigma\bigg|\bigg)\mathrm{e}^{\Lambda t} \\
&\leq C_2(1+|x_0|+|x|)\mathrm{e}^{\Lambda t},\ t\geq 0,
\end{aligned}\tag{2.3}$$

$$\begin{aligned}
&\mathbf{E}|X^u(t,x,x_0)-X^u(t,y,y_0)| \\
&\leq C_3\bigg(|x_0-y_0|+|x-y|_B+|F_1(x)-F_1(y)|_B+\bigg|\int_{-\tau}^{0}F_1(x(\sigma))-F_1(y(\sigma))\mathrm{d}\sigma\bigg| \\
&\quad+\bigg|\int_{-\tau}^{(t-\tau)\wedge 0}F_1(x(\sigma))-F_1(y(\sigma))\mathrm{d}\sigma\bigg|\bigg)\mathrm{e}^{\Lambda t} \\
&\leq C_4(|x_0-y_0|+|x-y|_H)\mathrm{e}^{\Lambda t},\ t\geq 0,
\end{aligned}\tag{2.4}$$

and

$$\mathbf{E}|X^u(t,x,x_0)-X^{u'}(t,x,x_0)|^2\leq C_4\mathbf{E}\int_0^t|u(\sigma)-u'(\sigma)|^2\mathrm{d}\sigma,\ t\geq 0,\tag{2.5}$$

for some constants $C_1,C_2,C_3,C_4>0$ *depending only on* L*,* τ*,* $a(\cdot)$ *and* $c(\cdot)$*.*

Proof. Existence and uniqueness are satisfied by Theorem 2.3. We only need to prove that (2.3), (2.4) and (2.5) hold true. For every $t\geq 0$, we have that

$$\begin{aligned}
&\mathbf{E}\mathrm{e}^{-t}|X^u(t,x,x_0)|^4 \\
&\leq 2^6\mathrm{e}^{-t}|x_0|^4+2^6L^4\mathbf{E}\bigg(\int_0^t\mathrm{e}^{\frac{-t}{4}}(1+|X^u(\sigma,x,x_0)|+\tau^{\frac14}|a|_{W^{1,2}}|X^u_\sigma(x,x_0)|_B)\mathrm{d}\sigma\bigg)^4 \\
&\quad+2^9L^4\mathbf{E}\bigg(\int_0^t\mathrm{e}^{\frac{-t}{4}}(1+|X^u(\sigma,x,x_0)|)\mathrm{d}\sigma\bigg)^4+2^9\mathrm{e}^{-t}\bigg|\int_{-\tau}^{(t-\tau)\wedge 0}F_1(x(\sigma))\mathrm{d}\sigma\bigg|^4 \\
&\quad+2^6L^4\mathrm{e}^{-t}\mathbf{E}\bigg(\int_0^t(1+|X^u(\sigma,x,x_0)|^2+\tau^{\frac12}|c|^2_{W^{1,2}}|X^u_\sigma(x,x_0)|^2_B)\mathrm{d}\sigma\bigg)^2 \\
&\leq 2^6|x_0|^4+2^9\bigg|\int_{-\tau}^{(t-\tau)\wedge 0}F_1(x(\sigma))\mathrm{d}\sigma\bigg|^4+3^12^8(2^63^2|a|^4_{W^{1,2}}+|c|^4_{W^{1,2}})\tau^2L^4|x|^4_B+2^6L^4(2^{10}+3+2^43^3) \\
&\quad+2^63^1[3^5L^4+2^63^2L^4+L^4+2^83^2\tau^2L^4|a|^4_{W^{1,2}}+2^2\tau^2L^4|c|^4_{W^{1,2}}]\bigg(\int_0^t\mathbf{E}\mathrm{e}^{-\sigma}|X^u(\sigma,x,x_0)|^4\mathrm{d}\sigma\bigg).
\end{aligned}$$

The Gronwall Lemma implies that

$$\begin{aligned}
\mathbf{E}\mathrm{e}^{-t}|X^u(t,x,x_0)|^4&\leq 2^9\bigg|\int_{-\tau}^{(t-\tau)\wedge 0}F_1(x(\sigma))\mathrm{d}\sigma\bigg|^4+M_0\mathrm{e}^{K_0t} \\
&\quad+(2^9+2^{12})\bigg(\int_{-\tau}^{0}F_1(x(\sigma))\mathrm{d}\sigma\bigg)^4\mathrm{e}^{K_0t}+2^{12}K_0|F_1(x)|^4_B\mathrm{e}^{K_0t},
\end{aligned}$$

where

$$M_0=2^6|x_0|^4+3^12^8(2^63^2|a|^4_{W^{1,2}}+|c|^4_{W^{1,2}})\tau^2L^4|x|^4_B+2^6L^4(2^{10}+3+2^43^3),$$

and

$$K_0 = 2^6 3^1 [3^5 + 2^6 3^2 + 1 + 2^8 3^2 \tau^2 |a|^4_{W^{1,2}} + 4\tau^2 |c|^4_{W^{1,2}}] L^4.$$

Therefore, there exist some constants $C_1, C_2 > 0$ depending only on L, τ, $a(\cdot)$ and $c(\cdot)$ such that (2.3) holds true.

Moreover, for every $t > 0$ we have that

$$\begin{aligned}
&\mathbf{E}\mathrm{e}^{-t}|X^u(t,x,x_0) - X^u(t,y,y_0)|^4 \\
&\leq 2^6 L^4 \mathbf{E}\bigg(\int_0^t \mathrm{e}^{\frac{-t}{4}} (|X^u(\sigma,x,x_0) - X^u(\sigma,y,y_0)| + \tau^{\frac{1}{4}} |a|_{W^{1,2}} |X^u_\sigma(x,,x_0) - X^u_\sigma(y,y_0)|_B)\mathrm{d}\sigma \bigg)^4 \\
&\quad + 2^9 L^4 \mathbf{E}\bigg(\int_0^t \mathrm{e}^{\frac{-t}{4}} |X^u(\sigma,x,x_0) - X^u(\sigma,y,y_0)|\mathrm{d}\sigma \bigg)^4 + 2^9 \mathrm{e}^{-t} \bigg| \int_{-\tau}^{(t-\tau)\wedge 0} F_1(x(\sigma)) - F_1(y(\sigma))\mathrm{d}\sigma \bigg|^4 \\
&\quad + 2^6 L^4 \mathrm{e}^{-t} \mathbf{E}\bigg(\int_0^t (|X^u(\sigma,x,x_0) - X^u(\sigma,y,y_0)|^2 + \tau^{\frac{1}{2}} |c|^2_{W^{1,2}} |X^u_\sigma(x,x_0) - X^u_\sigma(y,y_0)|^2_B)\mathrm{d}\sigma \bigg)^2 \\
&\quad + 2^6 \mathrm{e}^{-t} |x_0 - y_0|^4 \\
&\leq 2^6 |x_0 - y_0|^4 + 2^9 \bigg| \int_{-\tau}^{(t-\tau)\wedge 0} F_1(x(\sigma)) - F_1(y(\sigma))\mathrm{d}\sigma \bigg|^4 + 2^9 (2^8 |a|^4_{W^{1,2}} + |c|^4_{W^{1,2}}) \tau^2 L^4 |x-y|^4_B \\
&\quad + 2^7 [2^2 3^3 + 2^2 3^3 + 1 + 2^{10} \tau^2 |a|^4_{W^{1,2}} + 2^2 \tau^2 |c|^4_{W^{1,2}}] L^4 \\
&\quad \times \bigg(\int_0^t \mathbf{E}\mathrm{e}^{-\sigma} |X^u(\sigma,x,x_0) - X^u(\sigma,y,y_0)|^4 \mathrm{d}\sigma \bigg).
\end{aligned}$$

The Gronwall Lemma implies that

$$\begin{aligned}
&\mathbf{E}\mathrm{e}^{-t}|X^u(t,x,x_0) - X^u(t,y,y_0)|^4 \\
&\leq 2^9 \bigg| \int_{-\tau}^{(t-\tau)\wedge 0} F_1(x(\sigma)) - F_1(y(\sigma))\mathrm{d}\sigma \bigg|^4 + M_1 \mathrm{e}^{K_1 t} \\
&\quad + (2^9 + 2^{12}) \bigg(\int_{-\tau}^0 F_1(x(\sigma)) - F_1(y(\sigma))\mathrm{d}\sigma \bigg)^4 \mathrm{e}^{K_1 t} + 2^{12} K_1 |F_1(x) - F_1(y)|^4_B \mathrm{e}^{K_1 t},
\end{aligned}$$

where

$$M_1 = 2^6 |x_0 - y_0|^4 + 2^9 (2^8 |a|^4_{W^{1,2}} + |c|^4_{W^{1,2}}) \tau^2 L^4 |x-y|^4_B,$$

and

$$K_1 = 2^7 [2^2 3^3 + 2^2 3^3 + 1 + 2^{10} \tau^2 |a|^4_{W^{1,2}} + 2^2 \tau^2 |c|^4_{W^{1,2}}] L^4.$$

Therefore, there exist some constants $C_3, C_4 > 0$ depending only on L, τ, $a(\cdot)$ and $c(\cdot)$ such that (2.4) holds true. By the similar (even simpler) process, we can show that (2.5) holds true.

The proof is now complete. □

Remark 2.5. We note that the solution $X^u(\cdot)$ to equation (2.1) is continuous with respect to $t \in [0, +\infty)$ even if the initial value (x, x_0) belongs to the space $H \times R^d$.

3. A DPP for optimal control problems

In this section, we consider the controlled state equation

$$\begin{aligned}
X^u(t) = x_0 &+ \int_0^t F(X^u(\sigma), (a, X^u_\sigma)_H, u(\sigma))\mathrm{d}\sigma + \int_0^t F_1(X^u(\sigma - \tau))\mathrm{d}\sigma, \\
&+ \int_0^t G(X^u(\sigma), (c, X^u_\sigma)_H, u(\sigma))\mathrm{d}W(\sigma), \quad t \in [0, +\infty),
\end{aligned} \tag{3.1}$$

where $X_0^u = x \in H$, $X^u(0) = x_0 \in R^d$, and we consider a cost function of the form

$$J(x, x_0, u) = \mathbf{E} \int_0^{+\infty} \mathrm{e}^{-\lambda\sigma} q(X^u(\sigma, x, x_0), u(\sigma)) \mathrm{d}\sigma. \tag{3.2}$$

Our aim is to minimize the function J over all controls $u \in \mathcal{U}[0, +\infty)$. We define the function $V : H \times R^d \to R$ by

$$V(x, x_0) := \inf_{u \in \mathcal{U}[0,+\infty)} J(x, x_0, u). \tag{3.3}$$

The function V is called the *value function* of the optimal control problems (3.1) and (3.2). The goal of this paper is to characterize this value function V when $x \in \mathcal{D}$ and $x(0) = x_0$.

We make the following assumptions:

Hypothesis 3.1. *The mapping of $q : R^d \times U \to R$ is measurable, and there exists a constant $L > 0$, such that for every $x, y \in R^d, u, u' \in U$,*

$$|q(x, u)| \leq L(1 + |x|), \quad |q(x, u) - q(y, u')| \leq L(|x - y| + |u - u'|).$$

Our first result includes the local boundedness and the continuity of the value function.

Theorem 3.2. *Suppose that Hypotheses 2.1 and 3.1 hold. Then, there exists a constant $C_5 > 0$ such that for every $x, y \in H$, $x_0, y_0 \in R^d$ and $\lambda > \Lambda$,*

$$|V(x, x_0)| \leq C_5(1 + |x_0| + |x|_H), \tag{3.4}$$

and

$$|V(x, x_0) - V(y, y_0)| \leq C_5\bigg(|x_0 - y_0| + |x - y|_B + |F_1(x) - F_1(y)|_B + \bigg|\int_{-\tau}^0 F_1(x(\sigma)) - F_1(y(\sigma))\mathrm{d}\sigma\bigg|\bigg). \tag{3.5}$$

Proof. From the definition of J and from Hypothesis 3.1, we know that

$$\begin{aligned} |J(x, x_0, u) - J(y, y_0, u)| &\leq \int_0^{+\infty} \mathrm{e}^{-\lambda\sigma} \mathbf{E}|q(X^u(\sigma, x, x_0), u(\sigma)) - q(X^u(\sigma, y, y_0), u(\sigma))|\mathrm{d}\sigma \\ &\leq L \int_0^{+\infty} \mathrm{e}^{-\lambda\sigma} \mathbf{E}|X^u(\sigma, x, x_0) - X^u(\sigma, y, y_0)|\mathrm{d}\sigma. \end{aligned}$$

According to Theorem 2.4, we obtain that for a constant $C_5 > 0$,

$$\begin{aligned} &|V(x, x_0) - V(y, y_0)| \leq \sup_{u \in \mathcal{U}[0,+\infty)} |J(x, x_0, u) - J(y, y_0, u)| \\ &\leq C_5\bigg(|x_0 - y_0| + |x - y|_B + |F_1(x) - F_1(y)|_B + \bigg|\int_{-\tau}^0 F_1(x(\sigma)) - F_1(y(\sigma))\mathrm{d}\sigma\bigg|\bigg). \end{aligned}$$

By a similar procedure, we can show that (3.4) holds true, which completes the proof. □

Second, we present the following result, which is called the dynamic programming principle (DPP) for the optimal control problems (3.1) and (3.2).

Theorem 3.3. *Assume that Hypotheses 2.1 and 3.1 hold true. Then, for every $(x, x_0) \in H \times R^d$, $t \in [0, +\infty)$ and $\lambda > \Lambda$, we know that*

$$V(x, x_0) = \inf_{u \in \mathcal{U}[0,+\infty)} \bigg[\int_0^t \mathrm{e}^{-\lambda\sigma} \mathbf{E}q(X^u(\sigma, x, x_0), u(\sigma))\mathrm{d}\sigma + \mathrm{e}^{-\lambda t} \mathbf{E}V(X_t^u(x, x_0), X^u(t, x, x_0))\bigg]. \tag{3.6}$$

To prove the above theorem, we need the following Lemmas 3.5, 3.6 and 3.7. To this end, we need to consider a controlled state equation on an arbitrary interval $[t,+\infty)\subset[0,\infty)$:

$$X^u(s) = x_0 + \int_t^s F(X^u(\sigma),(a,X^u_\sigma)_H,u(\sigma))\mathrm{d}\sigma + \int_t^s F_1(X^u(\sigma-\tau))\mathrm{d}\sigma, \\ + \int_t^s G(X^u(\sigma),(c,X^u_\sigma)_H,u(\sigma))\mathrm{d}W(\sigma), \quad s\in[t,+\infty), \tag{3.7}$$

where $X^u_t = x \in H$, $X^u(t) = x_0 \in R^d$, and we denote by $X^u(\cdot,t,x,x_0,u)$, the solution. Moreover, we consider a cost function of the form

$$J(t,x,x_0,u) = \mathbf{E}\left[\int_t^{+\infty} \mathrm{e}^{-\lambda\sigma} q(X^u(\sigma,t,x,x_0),u(\sigma))\mathrm{d}\sigma \middle| \mathcal{F}_t\right], \quad u\in\mathcal{U}[0,+\infty). \tag{3.8}$$

We define the value function $V:[0,+\infty)\times H\times R^d\to R$ by

$$V(t,x,x_0) := \inf_{u\in\mathcal{U}[t,+\infty)} J(t,x,x_0,u). \tag{3.9}$$

We start from the definition of the cylindrical σ-field $\mathcal{B}^{\mathcal{I}}$.

Definition 3.4. For an interval $\mathcal{I}\subseteq R$, let $R^{\mathcal{I}}$ denote the set of all functions $x:\mathcal{I}\to R$. A finite dimensional rectangle in $R^{\mathcal{I}}$ is any set of the form $\{x : x(t_i)\in J_i, i=1,2,\dots,n\}$ for a non-negative n, intervals $J_i\subseteq R$ and times $t_i\in\mathcal{I}$, $i=1,2,\dots,n$. The cylindrical σ-field $\mathcal{B}^{\mathcal{I}}$ is the σ-field generated by the collection of finite dimensional rectangles.

With the help of the above definition, we can state the following lemma, which is the key to prove Theorem 3.3.

Lemma 3.5. *Assume that Hypotheses 2.1 and 3.1 hold true. Then, for every $(x,x_0)\in H\times R^d$, $t\in[0,+\infty)$ and $\lambda>\Lambda$, we know that*

$$V(x,x_0) = \mathrm{e}^{\lambda t} V(t,x,x_0), \quad t\in[0,+\infty). \tag{3.10}$$

Proof. We note that $\hat{\mathcal{F}}_s = \sigma(W_{[0,s]})$. In fact, by Corollary 12.9 in [4], we obtain that $\sigma(W_{[0,s]})\subseteq\hat{\mathcal{F}}_s$. On the other hand, for every $l\in[0,s]$ and $Q\in\mathcal{B}$, we have that $\{\omega : W(l)\in Q\} = \{\omega : W_{[0,s]}\in D\}\in\sigma(W_{[0,s]})$, where $D=\{x(\cdot)\in R^{[0,s]} : x(l)\in Q\}$. Then we obtain that $\hat{\mathcal{F}}_s\subseteq\sigma(W_{[0,s]})$. Therefore, we can consider $W_{[0,s]}$ as a measurable mapping from $(\Omega,\hat{\mathcal{F}}_s)$ to $(R^{[0,s]},\mathcal{B}^{[0,s]})$.

For every $u\in\mathcal{U}[0,+\infty)$ and $s\geq 0$, by the Theorem 2.1.11 in [33], there exists a measurable function ϕ_s from $(R^{[0,s]},\mathcal{B}^{[0,s]})$ to $(U,\mathcal{B}(U))$ such that

$$u(s) = \phi_s(W_{[0,s]}).$$

We define u' by

$$u'(t+s,\omega) = \phi_s(V_{[0,s]}(\omega)), \quad s\geq 0, \tag{3.11}$$

where $V(l) = W(t+l) - W(t)$, $l\in[0,s]$. Then, $u'(t+\cdot)$ and $u(\cdot)$ have the same distribution.

Now let us show that $u'(\cdot)\in\mathcal{U}[t,+\infty)$. By the definition of u', it is clear that $u'(s)$ is $\hat{\mathcal{F}}^s_t$-measurable for every $s\geq t$. So we only need to prove that $u'(t+\cdot)$ is left-continuous. In fact, for every $\omega\in\Omega$, there exists an $\omega_1\in\Omega$ such that $V(s,\omega) = W(s,\omega_1)$ for all $s\geq 0$. Then,

$$u'(t+s,\omega) = \phi_s(V_{[0,s]}(\omega)) = \phi_s(W_{[0,s]}(\omega_1)) = u(s,\omega_1).$$

Because u is left-continuous, we obtain that $u'(t+\cdot)$ is also left-continuous.

Since the coefficients of (3.7) do not depend on time, by (3.11) we have

$$X^u(\cdot,0,x,x_0) \doteq X^{u'}(t+\cdot,t,x,x_0), \quad t \geq 0,$$

where $\doteq$ denotes equality in distribution. Therefore,

$$\begin{aligned} J(t,x,x_0,u') &= \mathbf{E}\int_t^{+\infty} \mathrm{e}^{-\lambda\sigma} q(X^{u'}(\sigma,t,x,x_0),u'(\sigma))\mathrm{d}\sigma \\ &= \mathrm{e}^{-\lambda t}\mathbf{E}\int_0^{+\infty} \mathrm{e}^{-\lambda\sigma} q(X^{u'}(t+\sigma,t,x,x_0),u'(t+\sigma))\mathrm{d}\sigma = \mathrm{e}^{-\lambda t} J(x,x_0,u). \end{aligned}$$

As a consequence, we obtain that

$$V(x,x_0) \geq \mathrm{e}^{\lambda t} V(t,x,x_0).$$

By the similar procedure, we can show that

$$V(x,x_0) \leq \mathrm{e}^{\lambda t} V(t,x,x_0).$$

This concludes the proof of the Lemma. □

Lemma 3.6. *Assume that Hypotheses 2.1 and 3.1 hold true. Then, for every $u \in \mathcal{U}[0,+\infty)$ and $\lambda > \Lambda$, we know that*

$$J(t,x,x_0,u) \geq \mathrm{e}^{-\lambda t} V(x,x_0), \ (t,x,x_0) \in [0,+\infty) \times H \times R^d. \tag{3.12}$$

Proof. For every $u, u' \in \mathcal{U}[0,+\infty)$, we know that

$$\begin{aligned} \mathbf{E}|J(t,x,x_0,u) - J(t,x,x_0,u')| &\leq \int_t^{+\infty} \mathrm{e}^{-\lambda\sigma}\mathbf{E}|q(X^u(\sigma,t,x,x_0),u(\sigma)) - q(X^{u'}(\sigma,t,x,x_0),u'(\sigma))|\mathrm{d}\sigma \\ &\leq L\int_t^{+\infty} \mathrm{e}^{-\lambda\sigma}\mathbf{E}|X^u(\sigma,t,x,x_0) - X^{u'}(\sigma,t,x,x_0)|\mathrm{d}\sigma. \end{aligned}$$

According to Theorem 2.4, we know that for a constant $C > 0$,

$$\mathbf{E}|J(t,x,x_0,u) - J(t,x,x_0,u')| \leq C\bigg(\int_t^{+\infty} \mathbf{E}\mathrm{e}^{-2\lambda\sigma}|u(\sigma) - u'(\sigma)|^2\mathrm{d}\sigma\bigg)^{\frac{1}{2}}. \tag{3.13}$$

On the other hand, for every $u \in \mathcal{U}[0,+\infty)$, there exists

$$u^n(s) = \sum_{i=1}^n 1_{A_i^n} u^{n,j}(s), \ s > t,$$

where $\{A_i^n\}_{1\leq i\leq n}$ is a partition of $(\Omega, \hat{\mathcal{F}}_t)$, $u^{n,j}$ is an $\{\hat{\mathcal{F}}_t^s\}_{s\geq t}$-left-continuous predictable process, such that, for all $s > t$,

$$u^n(s) \to u(s) \text{ as } n \to +\infty.$$

Then by the uniqueness of solution for state equation (3.1), we obtain that

$$J(t,x,x_0,u^n) = \sum_{i=1}^n 1_{A_i^n} J(t,x,x_0,u^{n,i}).$$

By the definitions of J and V we know $J(t,x,x_0,u^{n,i}) \in R$ and $J(t,x,x_0,u^{n,i}) \geq V(t,x,x_0)$. Then

$$J(t,x,x_0,u^n) \geq V(t,x,x_0).$$

Letting $n \to +\infty$, by (3.10) and (3.13), we obtain (3.12). □

In order to prove the Theorem 3.3, the following lemma is also needed.

Lemma 3.7. *Assume that Hypotheses 2.1 and 3.1 hold true. Then, for every $(x,x_0)\in H\times R^d$, $t\in[0,+\infty)$ and $\lambda>\Lambda$, we know that*

$$J(t,X^u_t(0,x,x_0),X^u(t,0,x,x_0),u)=\mathbf{E}\bigg[\int_t^{+\infty}\mathrm{e}^{-\lambda\sigma}q(X^u(\sigma,0,x,x_0),u(\sigma))\mathrm{d}\sigma\bigg|\mathcal{F}_t\bigg],\quad u\in\mathcal{U}[0,+\infty).$$

Proof. By the definition of J, we obtain that

$$J(t,x,x_0,u(\cdot))=\mathbf{E}\bigg[\int_t^{+\infty}\mathrm{e}^{-\lambda\sigma}q(X^u(\sigma,t,x,x_0),u)\mathrm{d}\sigma\bigg|\mathcal{F}_t\bigg]:=\mathbf{E}[\eta(t,x,x_0,u)|\mathcal{F}_t].$$

By Lemma 1.1 in [31] there exists a sequence of $H\times R^d$-valued $\mathcal{F}_t$-measurable simple functions

$$f_m:\Omega\to H\times R^d,\ \ f_m=\sum_{k=1}^{N_m}h_k^{(m)}I_{\{f_m=h_k^{(m)}\}},\quad N_m\in N,$$

where $h_1^{(m)},\dots,h_m^{(m)}$ are pairwise distinct and $\Omega=\bigcup_{k=1}^{N_m}\{f_m=h_k^{(m)}\}$, such that

$$|f_m(\omega)-(X^u_t(0,x,x_0),X^u(t,0,x,x_0))(\omega)|\downarrow 0\ \text{ for all } \omega\in\Omega \text{ as } n\to\infty.$$

Then we obtain that, for any $B\in\mathcal{F}_t$,

$$\begin{aligned}&\mathbf{E}[I_BJ(t,X^u_t(0,x,x_0),X^u(t,0,x,x_0),u)]\\&=\lim_{m\to\infty}\sum_{k=1}^{N_m}\mathbf{E}\Big(I_BI_{\{f_m=h_k^{(m)}\}}J(t,h_k^{(m)},u)\Big)=\lim_{m\to\infty}\sum_{k=1}^{N_m}\mathbf{E}\Big(I_BI_{\{f_m=h_k^{(m)}\}}\mathbf{E}[\eta(t,h_k^{(m)},u)|\mathcal{F}_t]\Big)\\&=\lim_{m\to\infty}\mathbf{E}\bigg(\sum_{k=1}^{N_m}\Big[I_BI_{\{f_m=h_k^{(m)}\}}\eta(t,h_k^{(m)},u)\Big]\bigg)=\lim_{m\to\infty}\mathbf{E}(I_B\eta(t,y,u)|_{y=f_m})\\&=\mathbf{E}(I_B\eta(t,y,u)|_{y=(X^u_t(0,x,x_0),X^u(t,0,x,x_0))})=\mathbf{E}[I_B\eta(t,X^u_t(0,x,x_0),X^u(t,0,x,x_0),u)].\end{aligned}$$

Using the identities $X^u(\sigma,t,X^u_t(0,x,x_0),X^u(t,0,x,x_0),u)=X^u(\sigma,0,x,x_0,u)$ and $X^u_\sigma(t,X^u_t(0,x,x_0),X^u(t,0,x,x_0),u)=X^u_\sigma(0,x,x_0,u)$, we obtain

$$\begin{aligned}&J(t,X^u_t(0,x,x_0),X^u(t,0,x,x_0),u)=\mathbf{E}[\eta(t,X^u_t(0,x,x_0),X^u(t,0,x,x_0),u)|\mathcal{F}_t]\\&=\mathbf{E}\bigg[\int_t^{+\infty}\mathrm{e}^{-\lambda\sigma}q(X^u(\sigma,0,x,x_0),u(\sigma))\mathrm{d}\sigma\bigg|\mathcal{F}_t\bigg].\end{aligned}$$

The proof is now complete. □

Proof of Theorem 3.3. For any $u\in\mathcal{U}[0,+\infty)$, $t\geq 0$ and $\varepsilon>0$, by Theorem 3.2 and its proof, there is a constant $\delta>0$ such that whenever $|x-y|_H+|x_0-y_0|\leq\delta$,

$$|J(t,x,x_0,u')-J(t,y,y_0,u')|+|V(t,x,x_0)-V(t,y,y_0)|\leq\varepsilon,\ \forall\ u'\in\mathcal{U}[t,+\infty).$$

Let $\{D_i\}_{i\geq 1}$ be a Borel partition of $H\times R^d$, with diameter $\mathrm{diam}(D_i)<\delta$. Choose $(x^i,x_0^i)\in D_i$, then for each i, there is $u_i\in\mathcal{U}[t,+\infty)$ such that

$$J(t,x^i,x_0^i,u_i(\cdot))\leq V(t,x^i,x_0^i)+\varepsilon.$$

Therefore, for any $(x,x_0)\in D_i$, we have that

$$J(t,x,x_0,u_i)\leq J(t,x^i,x_0^i,u_i)+\varepsilon\leq V(t,x^i,x_0^i)+2\varepsilon\leq V(t,x,x_0)+3\varepsilon. \tag{3.14}$$

Now let us define a new control

$$\tilde{u}(\sigma)=\begin{cases}u(\sigma),\ \ \sigma\in[0,t],\\ u_i(\sigma),\ \ \sigma\in(t,+\infty),\ \ (X_t^u(x,x_0),X^u(t,x,x_0))\in D_i.\end{cases}$$

By Lemma 3.7, we obtain

$$J(t,X_t^u(0,x,x_0),X^u(t,0,x,x_0),\tilde{u})=\mathbf{E}\bigg[\int_t^{+\infty}\mathrm{e}^{-\lambda\sigma}q(X^{\tilde{u}}(\sigma,0,x,x_0),\tilde{u}(\sigma))\mathrm{d}\sigma\bigg|\mathcal{F}_t\bigg].$$

From the definition of V and (3.14), it follows that

$$\begin{aligned}V(x,x_0)\le J(x,x_0,\tilde{u})&=\mathbf{E}\int_0^t\mathrm{e}^{-\lambda\sigma}q(X^{\tilde{u}}(\sigma,0,x,x_0),\tilde{u}(\sigma))\mathrm{d}\sigma+\mathbf{E}\int_t^{+\infty}\mathrm{e}^{-\lambda\sigma}q(X^{\tilde{u}}(\sigma,0,x,x_0),\tilde{u}(\sigma))\mathrm{d}\sigma\\&=\mathbf{E}\int_0^t\mathrm{e}^{-\lambda\sigma}q(X^{\tilde{u}}(\sigma,0,x,x_0),\tilde{u}(\sigma))\mathrm{d}\sigma+\mathbf{E}J(t,X_t^u(0,x,x_0),X^u(t,0,x,x_0),\tilde{u})\\&\le\mathbf{E}\int_0^t\mathrm{e}^{-\lambda\sigma}q(X^{\tilde{u}}(\sigma,0,x,x_0),\tilde{u}(\sigma))\mathrm{d}\sigma+\mathbf{E}V(t,X_t^u(0,x,x_0),X^u(t,0,x,x_0))+3\varepsilon\\&=\mathbf{E}\int_0^t\mathrm{e}^{-\lambda\sigma}q(X^{u}(\sigma,0,x,x_0),u(\sigma))\mathrm{d}\sigma+\mathrm{e}^{-\lambda t}\mathbf{E}V(X_t^u(0,x,x_0),X^u(t,0,x,x_0))+3\varepsilon.\end{aligned}$$

Thus,

$$V(x,x_0)\le\inf_{u\in\mathcal{U}[0,+\infty)}\bigg[\int_0^t\mathrm{e}^{-\lambda\sigma}\mathbf{E}q(X^u(\sigma,0,x,x_0),u(\sigma))\mathrm{d}\sigma+\mathrm{e}^{-\lambda t}\mathbf{E}V(X_t^u(0,x,x_0),X^u(t,0,x,x_0))\bigg].$$

On the other hand, for any $\varepsilon>0$, there exists a $u^\varepsilon\in\mathcal{U}[0,+\infty)$ such that

$$\begin{aligned}V(x,x_0)+\varepsilon&\ge\int_0^t\mathrm{e}^{-\lambda\sigma}\mathbf{E}q(X^{u^\varepsilon}(\sigma,0,x,x_0),u^\varepsilon(\sigma))\mathrm{d}\sigma+\int_t^{+\infty}\mathrm{e}^{-\lambda\sigma}\mathbf{E}q(X^{u^\varepsilon}(\sigma,0,x,x_0),u^\varepsilon(\sigma))\mathrm{d}\sigma\\&=\int_0^t\mathrm{e}^{-\lambda\sigma}\mathbf{E}q(X^{u^\varepsilon}(\sigma,0,x,x_0),u^\varepsilon(\sigma))\mathrm{d}\sigma+\mathbf{E}J(t,X_t^{u^\varepsilon}(0,x,x_0),X^{u^\varepsilon}(t,0,x,x_0),u^\varepsilon).\end{aligned}$$

Then by Lemma 3.6, we have that

$$\begin{aligned}V(x,x_0)+\varepsilon&\ge\int_0^t\mathrm{e}^{-\lambda\sigma}\mathbf{E}q(X^{u^\varepsilon}(\sigma,0,x,x_0),u^\varepsilon(\sigma))\mathrm{d}\sigma+\mathrm{e}^{-\lambda t}\mathbf{E}V(X_t^{u^\varepsilon}(0,x,x_0),X^{u^\varepsilon}(t,0,x,x_0))\\&\ge\inf_{u\in\mathcal{U}[0,+\infty)}\bigg[\int_0^t\mathrm{e}^{-\lambda\sigma}\mathbf{E}q(X^u(\sigma,0,x,x_0),u(\sigma))\mathrm{d}\sigma+\mathrm{e}^{-\lambda t}\mathbf{E}V(X_t^u(0,x,x_0),X^u(t,0,x,x_0))\bigg].\end{aligned}$$

Hence, (3.6) follows. □

Our next goal is to derive the Hamilton−Jacobi−Bellman equation for the value function V. To begin, let us introduce the weak infinitesimal generator $\mathcal{S}$. For a Borel measurable function $f:H\times R^d\to R$, we define

$$\mathcal{S}(f)(x)=\lim_{h\to0^+}\frac{1}{h}[f(\widehat{x}_h,\widehat{x}_h(0))-f(x,x(0))],\quad x\in\mathcal{D},$$

where $\widehat{x}:[-\tau,T]\to R^d$ is an extension of x defined by

$$\widehat{x}(s)=\begin{cases}x(s),&s\in[-\tau,0),\\x(0),&s\ge0,\end{cases}$$

in addition, for every $t \geq 0$, $\widehat{x}_t \in \mathcal{D}$ is defined as

$$\widehat{x}_t(\theta) = \widehat{x}(t+\theta), \quad \theta \in [-\tau, 0].$$

For every measurable function $f : H \times R^d \to R$, we denote $D(\mathcal{S}(f))$ as the domain of the function $\mathcal{S}(f)$ for the set $x \in \mathcal{D}$ such that the above limit exists for this set.

We denote by $C^2_{lip}(H \times R^d)$ the space of functions $\Phi : H \times R^d \to R$ such that $\nabla^2_x \Phi : (H \times R^d) \to (H \times R^d) \times (H \times R^d)$ are continuous and satisfy, for a suitable constant $K > 0$,

$$|\nabla^2_x \Phi(t,x) - \nabla^2_x \Phi(t,y)|_{(H\times R^d)\times(H\times R^d)} \leq K|x-y|_{H\times R^d}, \quad t \in [0,T], \ x, y \in H \times R^d.$$

We also denote by $C^{1,2}_{lip}([0,+\infty) \times (H \times R^d))$ the space of functions $\Phi : [0,+\infty) \times (H \times R^d) \to R$ such that $\Phi_t : [0,+\infty) \times (H \times R^d) \to R$ and $\nabla^2_x \Phi : [0,+\infty) \times (H \times R^d) \to (H \times R^d) \times (H \times R^d)$ are continuous and satisfy, for a suitable constant $K > 0$,

$$|\nabla^2_x \Phi(t,x) - \nabla^2_x \Phi(t,y)|_{(H\times R^d)\times(H\times R^d)} \leq K|x-y|_{H\times R^d}, \quad t \in [0,+\infty), \ x, y \in H \times R^d.$$

Here, and throughout this article, $\nabla_x \Phi$ and $\nabla^2_x \Phi$ are first and second Fréchet derivatives of Φ.

Theorem 3.8. *Let V denote the value function defined by (3.3). If $V \in C^2_{lip}(H \times R^d)$, then V satisfies the HJB equation*

$$-\lambda V(x) + \mathcal{S}(V)(x) + \mathbf{H}(x, \nabla_{x_0} V(x), \nabla^2_{x_0} V(x)) = 0, \quad x \in D(\mathcal{S}(V)), \tag{3.15}$$

where

$$\mathbf{H}(x,p,l) = \inf_{u \in U}[(p, F(x,u) + F_1(x(-\tau)))_{R^d} + \frac{1}{2} tr[lG(x,u)G^\top(x,u)] \\ +q(x(0),u)], \quad (x,p,l) \in \mathcal{D} \times R^d \times \Gamma(R^d),$$

$\Gamma(R^d)$ denotes the set of all $(d \times d)$ symmetric matrices, and $\nabla_{x_0} V$ and $\nabla^2_{x_0} V$ are first and second Fréchet derivatives of V with respect to the second variable. Here, and throughout this article, for simplicity, we let $V(x)$, $F(x,u)$ and $G(x,u)$ denote $V(x, x(0))$, $F(x(0), (a,x)_H, u)$ and $G(x(0), (c,x)_H, u)$, respectively, if $x \in \mathcal{D}$.

To prove the above theorem, we need the following lemma.

Lemma 3.9. *Suppose that Hypothesis 2.1 holds. If $g \in C^{1,2}_{lip}([0,+\infty) \times (H \times R^d))$, then for each $(t,x) \in [0,+\infty) \times D(\mathcal{S}(g(t,\cdot)))$, the following convergence holds uniformly on $u(\cdot) \in \mathcal{U}[0,+\infty)$:*

$$\lim_{\epsilon \to 0^+} \Bigg[\frac{\mathbf{E} g(t+\epsilon, X^u_\epsilon) - g(t,x)}{\epsilon} - \frac{1}{\epsilon} \int_t^{t+\epsilon} \mathbf{E}(\nabla_{x_0} g(t,x), F(x,u(\sigma)) + F_1(x(-\tau)))_{R^d} \mathrm{d}\sigma \\ -\frac{1}{2\epsilon} \int_0^\epsilon \mathbf{E} tr[\nabla^2_{x_0} g(t,x) G(x,u(\sigma)) G^\top(x,u(\sigma))] \mathrm{d}\sigma - g_t(t,x) - \mathcal{S}(g(t,\cdot))(x) \Bigg] = 0. \tag{3.16}$$

Here, and throughout this article, for simplicity, if function $g : [0,+\infty) \times (H \times R^d) \to R$ is measurable, we let $g(t,x)$ denote $g(t,x,x(0))$ when $t \in [0,+\infty)$ and $x \in \mathcal{D}$.

Proof. The proof is very similar to Theorem 5.1.3 in [35]. Here, we omit it. □

The following three lemmas hold are about the properties of the weak infinitesimal generator $\mathcal{S}$, which will be used in the proof of uniqueness for the viscosity solution.

Lemma 3.10 (See Lem. 3.6 in [36]).
If $g(x) = g_0(|x|_H^2)$, $x \in \mathcal{D}$, where $g_0 \in C^1(R)$, then

$$\mathcal{S}(g)(x) = g_0'(|x|_H^2)(x^2(0) - x^2(-\tau)). \tag{3.17}$$

Lemma 3.11 (See Lem. 3.7 in [36]). *If $\psi(x) = \psi_0(|g(x) - \hat{a}|_B^4), \hat{a}, x \in \mathcal{D}$, where $\psi_0 \in C^1(R)$ and $g \in C(R)$, then*

$$\mathcal{S}(\psi)(x) = 4\psi_0'(|(g(x) - \hat{a})|_B^4)(|B(g(x) - \hat{a})|_H^2 B(g(x) - \hat{a}), g(x(0))1_{[-\tau,0]} - g(x))_H. \tag{3.18}$$

Here, the function $1_{[-\tau,0]}$ is the characteristic function of $[-\tau, 0]$.

Lemma 3.12. *If $\psi(x) = \psi_0(\int_{-\tau}^0 x(\theta)\mathrm{d}\theta - \hat{a}), \hat{a} \in R, x \in \mathcal{D}$, where $\psi_0 \in C^1(R)$ and $g \in C(R)$, then*

$$\mathcal{S}(\psi)(g(x)) = \psi_0'\bigg(\int_{-\tau}^0 g(x(\theta))\mathrm{d}\theta - \hat{a}\bigg)(g(x(0)) - g(x(-\tau))). \tag{3.19}$$

Proof. By the definition of ψ, we have that, for some $l \in (0,1)$,

$$\begin{aligned} &\frac{1}{\epsilon}(\psi(g(\widehat{x}_\epsilon)) - \psi(g(x))) = \frac{1}{\epsilon}[\psi'(g(x) + l(g(\widehat{x}_\epsilon) - g(x)))(g(\widehat{x}_\epsilon) - g(x))] \\ &= \frac{1}{\epsilon}\psi_0'\bigg(\int_{-\tau}^0 g(x(\theta)) + l(g(\widehat{x}_\epsilon(\theta)) - g(x(\theta)))\mathrm{d}\theta - \hat{a}\bigg)\int_{-\tau}^0 g(\widehat{x}_\epsilon(\theta)) - g(x(\theta))\mathrm{d}\theta. \end{aligned} \tag{3.20}$$

On the other hand, we have that

$$\begin{aligned} &\frac{1}{\epsilon}\int_{-\tau}^0 g(\widehat{x}_\epsilon(\theta)) - g(x(\theta))\mathrm{d}\theta = \frac{1}{\epsilon}\int_{-\tau+\epsilon}^0 g(x(\theta))\mathrm{d}\theta + g(x(0)) - \frac{1}{\epsilon}\int_{-\tau}^0 g(x(\theta))\mathrm{d}\theta \\ &\to g(x(0)) - g(x(-\tau)) \text{ as } \epsilon \to 0, \end{aligned}$$

Then, letting $\epsilon \to 0$ in (3.20), we get (3.19). □

We conclude this section with the proof for the Theorem 3.8.

Proof of Theorem 3.8. We fix $u \in U$, and then, for every $x \in D(\mathcal{S}(V))$, it follows from (3.6) that

$$0 \le \int_0^s \mathrm{e}^{-\lambda\sigma}\mathbf{E}q(X^u(\sigma,x),u)\mathrm{d}\sigma + \mathrm{e}^{-\lambda s}\mathbf{E}V(X_s^u(x)) - V(x).$$

According to Lemma 3.9, the above inequality implies that

$$\begin{aligned} 0 &\le \lim_{s\to 0^+}\frac{1}{s}\bigg[\int_0^s \mathrm{e}^{-\lambda\sigma}\mathbf{E}q(X^u(\sigma,x),u)\mathrm{d}\sigma + \mathrm{e}^{-\lambda s}\mathbf{E}V(X_s^u(x)) - V(x)\bigg] \\ &= -\lambda V(x) + \mathcal{S}(V)(x) + (\nabla_{x_0}V(x), F(x,u) + F_1(x(-\tau)))_{R^d} \\ &\quad + \frac{1}{2}\mathrm{tr}[\nabla_{x_0}^2 V(x)G(x,u)G^\top(x,u)] + q(x(0),u). \end{aligned}$$

Thus, we know that

$$0 \le -\lambda V(x) + \mathcal{S}(V)(x) + \mathbf{H}(x, \nabla_{x_0}V(x), \nabla_{x_0}^2 V(x)). \tag{3.21}$$

On the other hand, let $x \in D(\mathcal{S}(V))$ be fixed. For any $\varepsilon > 0$ and $s > 0$, according to (3.6), there exists a $\tilde{u} \equiv u^{\varepsilon,s} \in \mathcal{U}[0,+\infty)$ such that

$$\begin{aligned}
\varepsilon s &\geq \int_0^s \mathrm{e}^{-\lambda\sigma}\mathbf{E}q(X^{\tilde{u}}(\sigma,x),\tilde{u}(\sigma))\mathrm{d}\sigma + \mathrm{e}^{-\lambda s}\mathbf{E}V(X_s^{\tilde{u}}(x)) - V(x) \\
&= -\lambda V(x)s + \mathcal{S}(V)(x)s + \left(\nabla_{x_0}V(x), \int_0^s \mathbf{E}[F(x,\tilde{u}(\sigma)) + F_1(x(-\tau))]\mathrm{d}\sigma\right)_{R^d} \\
&\quad + \int_0^s \mathbf{E}q(x(0),\tilde{u}(\sigma))\mathrm{d}\sigma + \int_0^s \frac{1}{2}\mathrm{tr}[\nabla_{x_0}^2 V(x)G(x,\tilde{u}(\sigma))G^\top(x,\tilde{u}(\sigma))]\mathrm{d}\sigma + o(|s|) \\
&\geq -\lambda V(x)s + \mathcal{S}(V)(x)s + \mathbf{H}(x,\nabla_{x_0}V(x),\nabla_{x_0}^2 V(x))s + o(|s|).
\end{aligned}$$

Then, dividing through by s and letting $s \to 0$, we obtain

$$\varepsilon \geq -\lambda V(x) + \mathcal{S}(V)(x) + \mathbf{H}(x,\nabla_{x_0}V(x),\nabla_{x_0}^2 V(x)).$$

Combining this equation with (3.21), we obtain the desired result. □

4. Viscosity solution to the HJB equation: existence theorem

In this section, we introduce a new concept for a time-dependent viscosity solution. In our previous work [36], a new concept for the viscosity solution to the first order HJB equation was introduced. Its uniqueness was proven by Lemma 4.1 in [36]. However, it is not applicable in our case. We need the superior properties of the function in the set $\mathcal{D}$. Before giving the definition of the viscosity solution, let us introduce the following key lemma for the proof of the uniqueness of the viscosity solution.

Lemma 4.1 (Left-maximization principle). *Let $v : [0,+\infty) \times (H \times R^d) \times (H \times R^d) \to R$ be continuous and let there be an integer $k > 0$ such that for every $t \in [0,+\infty), x, x_1, x_2, y, y_1, y_2 \in R^d \times H$,*

$$|v(t,x,y)| \leq L(1 + |x| + |y|)^k,$$

$$|v(t,x_1,y_1) - v(t,x_2,y_2)| \leq L(|x_1 - x_2| + |y_1 - y_2|)^k. \tag{4.1}$$

Finally, let $\overline{\lim}_{t\to+\infty}v(t,x,y) \leq A$ for a constant A uniformly on $x, y \in \mathcal{D}_N$. For simplicity, we denote $v(t,x,x(0),y,y(0))$ by $v(t,x,y)$ when $x, y \in \mathcal{D}$. Then, for each $(\tilde{t},\tilde{x},\tilde{y}) \in [0,+\infty) \times \mathcal{D}_N \times \mathcal{D}_N$ and $M > 0$, if $v(\tilde{t},\tilde{x},\tilde{y}) > A$, there exist $(\bar{t},\bar{x},\bar{y}) \in [0,+\infty) \times \mathcal{D}_N \times \mathcal{D}_N$ and $\bar{k},\bar{l} \in [0, M(\bar{t}-\tilde{t})]$, such that $(\bar{x},\bar{y}) = (\tilde{x}\otimes_{\bar{k}}\bar{x}, \tilde{y}\otimes_{\bar{l}}\bar{y})$, $v(\bar{t},\bar{x},\bar{y}) \geq v(\tilde{t},\tilde{x},\tilde{y})$ and

$$v(\bar{t},\bar{x},\bar{y}) = \sup_{(t,x,y)\in[\bar{t},\infty)\times\mathcal{D}_N\times\mathcal{D}_N} v(t,\bar{x}\otimes_{[0,M(t-\bar{t})]}x, \bar{y}\otimes_{[0,M(t-\bar{t})]}y). \tag{4.2}$$

Here, and throughout this article, for simplicity, we let $\sup_{(t,x,y)\in[\bar{t},\infty)\times\mathcal{D}_N\times\mathcal{D}_N} v(t,\bar{x}\otimes_{[0,M(t-\bar{t})]}x, \bar{y}\otimes_{[0,M(t-\bar{t})]}y)$ denote

$$\sup_{t\in[\bar{t},+\infty)} \sup_{(l,r,x,y)\in[0,M(t-\bar{t})]\times[0,M(t-\bar{t})]\times\mathcal{D}_N\times\mathcal{D}_N} v(t,\bar{x}\otimes_l x, \bar{y}\otimes_r y).$$

Proof. Without loss of generality, we can assume that $v(\tilde{t},\tilde{x},\tilde{y}) \geq v(\tilde{t},\tilde{x}+e1_0,\tilde{y}+f1_0)$ for all $e, f \in R^d$ satisfying $|e + \tilde{x}(0)| \vee |f + \tilde{y}(0)| \leq N$. Because $v(\tilde{t},\tilde{x},\tilde{y}) > A$ and $\overline{\lim}_{t\to+\infty}v(t,x,y) \leq A$ uniformly on $x, y \in \mathcal{D}_N$, there exists a constant $T > 0$ such that

$$v(t,x,y) < v(\tilde{t},\tilde{x},\tilde{y}), \quad t \geq T, \ x, y \in \mathcal{D}_N. \tag{4.3}$$

We set $m_0 = v(\tilde{t}, \tilde{x}, \tilde{y})$ and

$$\bar{m}_0 := \sup_{(t,x,y)\in[\tilde{t},T]\times\mathcal{D}_N\times\mathcal{D}_N} v(t, \tilde{x} \otimes_{[0,M(t-\tilde{t})]} x, \tilde{y} \otimes_{[0,M(t-\tilde{t})]} y) \geq m_0.$$

If $\bar{m}_0 = m_0$, we let $(\bar{t}, \bar{x}, \bar{y}) = (\tilde{t}, \tilde{x}, \tilde{y})$ and finish the procedure. Otherwise, there exist $(t_1, x_1, y_1) \in (\tilde{t}, T] \times \mathcal{D}_N \times \mathcal{D}_N$ and $s_1, l_1 \in [0, M(t_1 - \tilde{t})]$ such that $x_1 = \tilde{x} \otimes_{s_1} x_1, y_1 = \tilde{y} \otimes_{l_1} y_1$ and

$$m_1 := v(t_1, x_1, y_1) \geq \frac{m_0 + \bar{m}_0}{2}.$$

We set

$$\bar{m}_1 := \sup_{(t,x,y)\in[t_1,T]\times\mathcal{D}_N\times\mathcal{D}_N} v(t, x_1 \otimes_{[0,M(t-t_1)]} x, y_1 \otimes_{[0,M(t-t_1)]} y) \geq m_1.$$

If $\bar{m}_1 = m_1$, then we let $(\bar{t}, \bar{x}, \bar{y}) = (t_1, x_1, y_1)$ and finish the procedure. Otherwise, for $i = 2, 3, \ldots$, we can find, $(t_i, x_i, y_i) \in (t_{i-1}, T] \times \mathcal{D}_N \times \mathcal{D}_N$ and $s_i, l_i \in [0, M(t_i - t_{i-1})]$ such that the following hold: $x_i = x_{i-1} \otimes_{s_i} x_i$ and $y_i = y_{i-1} \otimes_{l_i} y_i$; $v(t_i, x_i, y_i) \geq v(t_i, x_i + e1_0, y_i + f1_0)$ for all e, f satisfying $|e + x_i(0)| \vee |f + y_i(0)| \leq N$;

$$m_i := v(t_i, x_i, y_i) \geq \frac{m_{i-1} + \bar{m}_{i-1}}{2};$$

and

$$\bar{m}_i := \sup_{(t,x,y)\in[t_i,T]\times\mathcal{D}_N\times\mathcal{D}_N} v(t, x_i \otimes_{[0,M(t-t_i)]} x, y_i \otimes_{[0,M(t-t_i)]} y) \geq m_i.$$

The procedure continues till the first time $\bar{m}_i = m_i$. The proof is completed by setting $(\bar{t}, \bar{x}, \bar{y}) = (t_i, x_i, y_i)$. For the last case, in which $\bar{m}_i > m_i$ for all $i = 1, 2, \ldots$, we know $t_i \uparrow \bar{t} \in [0, T]$, we can then find $\bar{x}, \bar{y} \in \mathcal{D}$ such that $\bar{x} = x_i \otimes_{\sum_{j=i+1}^{\infty} s_j} \bar{x}, \bar{y} = y_i \otimes_{\sum_{j=i+1}^{\infty} l_j} \bar{y}$. We can choose $|\bar{x}(0)| \vee |\bar{y}(0)| \leq N$ such that $v(\bar{t}, \bar{x}, \bar{y}) \geq v(\bar{t}, \bar{x} + e1_0, \bar{y} + f1_0)$, for all e, f such that $|e + \bar{x}(0)| \vee |f + \bar{y}(0)| \leq N$.

Now let us show that $\bar{x}, \bar{y} \in \mathcal{D}_N$. Because $|\bar{x}(0)| \leq N$, we only need to prove $|\bar{x}|_H \leq \tau^{\frac{1}{2}} N$. According to the definition of $\bar{x}$, if $\bar{t} - \tilde{t} \leq \tau$, we obtain

$$\begin{aligned}
|\bar{x}|_H^2 &= \int_{-\tau}^{0} |\bar{x}(\theta)|^2 \mathrm{d}\theta = \int_{\bar{t}-\tilde{t}-\tau}^{0} |\tilde{x}(\theta|^2 \mathrm{d}\theta + \sum_{i=1}^{+\infty} \int_{-s_i}^{0} |x_i(\theta)|^2 \mathrm{d}\theta \\
&= \int_{\bar{t}-\tilde{t}-\tau}^{0} |\tilde{x}(\theta|^2 \mathrm{d}\theta + \lim_{m\to+\infty} \sum_{i=1}^{m} \int_{-s_i}^{0} |x_i(\theta)|^2 \mathrm{d}\theta \\
&\leq \overline{\lim}_{m\to+\infty} |x_m|_H^2 \leq \tau^{\frac{1}{2}} N.
\end{aligned} \tag{4.4}$$

If $\bar{t} - \tilde{t} > \tau$, there exists an integer j such that

$$\begin{aligned}
|\bar{x}|_H^2 &= \int_{-\tau}^{0} |\bar{x}(\theta)|^2 \mathrm{d}\theta = \int_{\bar{t}-t_j-\tau}^{0} |x_j(\theta|^2 \mathrm{d}\theta + \sum_{i=j+1}^{+\infty} \int_{-s_i}^{0} |x_i(\theta)|^2 \mathrm{d}\theta \\
&= \int_{\bar{t}-t_j-\tau}^{0} |x_j(\theta|^2 \mathrm{d}\theta + \lim_{m\to+\infty} \sum_{i=j+1}^{m} \int_{-s_i}^{0} |x_i(\theta)|^2 \mathrm{d}\theta \\
&\leq \overline{\lim}_{m\to+\infty} |x_m|_H^2 \leq \tau^{\frac{1}{2}} N.
\end{aligned} \tag{4.5}$$

By (4.4) and (4.5), we obtain $\bar{x} \in \mathcal{D}_N$. By the similar way, we can prove $\bar{y} \in \mathcal{D}_N$.

On the other hand, according to

$$\bar{m}_{i+1} - m_{i+1} \leq \bar{m}_i - \frac{\bar{m}_i + m_i}{2} = \frac{\bar{m}_i - m_i}{2},$$

there exists $\bar{m} \in (m_0, \bar{m}_0)$, such that $\bar{m}_i \downarrow \bar{m}$ and $m_i \uparrow \bar{m}$. From the definitions of $\bar{x}$ and $\bar{y}$, we obtain $x_i(s) \to \bar{x}(s)$ and $y_i(s) \to \bar{y}(s)$ for almost all $s \in [-\tau, 0]$. There exist two subsequences of $x_i(0)$ and $y_i(0)$ still denoted by themselves such that $x_i(0) \to \bar{a}$, $y_i(0) \to \bar{b}$ and $|\bar{a}| \vee |\bar{b}| \le N$, respectively. Thus, according to (4.1), we obtain

$$\bar{m} = \lim_{i\to\infty} m_i = \lim_{i\to\infty} v(t_i, x_i, y_i) \le v(\bar{t}, \bar{x}, \bar{y}).$$

Now let us show that $v(t_i, x_i, y_i) \to v(\bar{t}, \bar{x}', \bar{y}')$, where

$$\bar{x}'(\theta) = \begin{cases} \bar{x}(\theta), & \theta \in [-\tau, 0), \\ \bar{a}, & \theta = 0; \end{cases} \quad \bar{y}'(\theta) = \begin{cases} \bar{y}(\theta), & \theta \in [-\tau, 0), \\ \bar{b}, & \theta = 0. \end{cases}$$

Because if $\bar{t} - \tilde{t} \le \tau$,

$$\begin{aligned} |x_i|^2 &= \int_{t_i - \tilde{t} - \tau}^{0} |\tilde{x}(\theta|^2 \mathrm{d}\theta + \sum_{k=1}^{i} \int_{-s_k}^{0} |x_k(\theta)|^2 \mathrm{d}\theta \\ &\to \int_{\bar{t} - \tilde{t} - \tau}^{0} |\tilde{x}(\theta|^2 \mathrm{d}\theta + \sum_{k=1}^{+\infty} \int_{-s_k}^{0} |x_k(\theta)|^2 \mathrm{d}\theta = |\bar{x}'|^2 \text{ as } i \to \infty, \end{aligned}$$

and if $\bar{t} - \tilde{t} > \tau$, for sufficiently large i, there exists an integer j such that

$$\begin{aligned} |x_i|^2 &= \int_{t_i - t_j - \tau}^{0} |\tilde{x}(\theta|^2 \mathrm{d}\theta + \sum_{k=j+1}^{i} \int_{-s_k}^{0} |x_k(\theta)|^2 \mathrm{d}\theta \\ &\to \int_{\bar{t} - t_j - \tau}^{0} |\tilde{x}(\theta|^2 \mathrm{d}\theta + \sum_{k=j+1}^{+\infty} \int_{-s_k}^{0} |x_k(\theta)|^2 \mathrm{d}\theta = |\bar{x}'|^2 \text{ as } i \to \infty. \end{aligned}$$

By the similar way, we can prove $|y_i|^2 \to |\bar{y}'|^2$ as $i \to \infty$. Then by the dominated convergence theorem, according to (4.1), we obtain

$$\begin{aligned} &\lim_{i\to\infty} |v(t_i, x_i, y_i) - v(\bar{t}, \bar{x}', \bar{y}')| \le \lim_{i\to\infty} [|v(t_i, x_i, y_i) - v(t_i, \bar{x}', \bar{x}')| + |v(t_i, \bar{x}', \bar{y}') - v(\bar{t}, \bar{x}', \bar{y}')|] \\ &\le \lim_{i\to\infty} L(|x_i - \bar{x}'|_H + |x_i(0) - \bar{a}| + |y_i - \bar{y}'|_H + |y_i(0) - \bar{b}|) = 0. \end{aligned}$$

Thus

$$\bar{m} = \lim_{i\to\infty} m_i = \lim_{i\to\infty} v(t_i, x_i, y_i) = v(\bar{t}, \bar{x}', \bar{y}') \le v(\bar{t}, \bar{x}, \bar{y}).$$

We claim that (4.2) holds for this $(\bar{t}, \bar{x}, \bar{y})$. Otherwise, according to (4.3), there exist $(t, x, y) \in (\bar{t}, T] \times \mathcal{D}_N \times \mathcal{D}_N$, $\delta > 0$ and $s, l \in [0, M(t - \bar{t})]$ with $x = \bar{x} \otimes_s x$ and $y = \bar{y} \otimes_l y$, such that

$$v(t, \bar{x} \otimes_s x, \bar{y} \otimes_l y) \ge v(\bar{t}, \bar{x}, \bar{y}) + \delta \ge \bar{m} + \delta,$$

and the following contradiction occurs:

$$v(t, \bar{x} \otimes_s x, \bar{y} \otimes_l y) = v(t, x_i \otimes_{s + \sum_{j=i+1}^{\infty} s_i} (\bar{x} \otimes_s x), y_i \otimes_{l + \sum_{j=i+1}^{\infty} l_i} (\bar{y} \otimes_l y)) \le \bar{m}_i \to \bar{m}.$$

The proof is now finished. □

From the above lemma, we can now give the following definition for the viscosity solution.

Definition 4.2. $w \in C(H \times R^d)$ is called a viscosity subsolution (supersolution) of (3.15) if for every $\varphi \in C^{1,2}_{lip}([0,+\infty) \times (H \times R^d))$, whenever the constants $\gamma, \tilde{\lambda}, \lambda > 0$ and the function $w^\gamma - \varphi$ (resp. $w^\gamma + \varphi$) satisfy $\tilde{\lambda} < \lambda$, $\lambda - \tilde{\lambda} = \gamma$, and

$$(w^\gamma - \varphi)(s,z) = \sup_{(t,x)\in[s,+\infty)\times\mathcal{D}} (w^\gamma - \varphi)(t, z \otimes_{[0,(t-s)]} x),$$

$$(\text{respectively,} \quad (w^\gamma + \varphi)(s,z) = \inf_{(t,x)\in[s,+\infty)\times\mathcal{D}} (w^\gamma + \varphi)(t, z \otimes_{[0,(t-s)]} x),)$$

where $w^\gamma(s,z) = \mathrm{e}^{-\gamma s} w(z)$, $(s,z) \in [0,+\infty) \times \mathcal{D}$ and $z \in D(\mathcal{S}(\varphi(s,\cdot)))$, we have

$$-\tilde{\lambda} w(z) + \mathrm{e}^{\gamma s}\varphi_s(s,z) + \mathrm{e}^{\gamma s}\mathcal{S}(\varphi(s,\cdot))(z) + \mathbf{H}(z, \mathrm{e}^{\gamma s}\nabla_{x_0}\varphi(s,z), \mathrm{e}^{\gamma s}\nabla^2_{x_0}\varphi(s,z)) \geq 0,$$

$$(\text{respectively,} -\tilde{\lambda} w(z) - \mathrm{e}^{\gamma s}\varphi_s(s,z) - \mathrm{e}^{\gamma s}\mathcal{S}(\varphi(s,\cdot))(z) + \mathbf{H}(z, -\mathrm{e}^{\gamma s}\nabla_{x_0}\varphi(s,z), -\mathrm{e}^{\gamma s}\nabla^2_{x_0}\varphi(s,z)) \leq 0).$$

$w \in C(H \times R^d)$ is said to be a viscosity solution to (3.15) if it is both a viscosity subsolution and a viscosity supersolution.

Remark 4.3.

(i) A viscosity solution V of the HJB equation (3.15) is a classical solution if it further lies in $C^2_{lip}(H \times R^d)$ and $\mathcal{D} \subseteq D(\mathcal{S}(V))$.

(ii) The classical definitions of the viscosity solution to the elliptic HJB equation in infinite dimensions are time independent and the uniqueness is proven by using the weak compactness of separable Hilbert spaces (see Exp. [32]). In our case, the elliptic HJB equation is defined on set $\mathcal{D}$, which does not have weak compactness. For the sake of the uniqueness proof, our new concept of the viscosity solution is enhanced to include t. At the same time, our modification leads to a slight additional difficulty in the existence proof.

(iii) Assume that the coefficients $F(x,y,u) = \overline{F}(x,u)$, $G(x,y,u) = \overline{G}(x,u)$, $(x,y,u) \in R^d \times R \times U$ and $F_1 = 0$. Let the function $V(x) : R^d \to R$ be a viscosity solution to (3.15) as a functional of $V(x) : R^d \times H \to R$. Then, V is also a classical viscosity solution as a function of the state.

We conclude this section with the existence proof for the viscosity solution.

Theorem 4.4. *Suppose that Hypotheses 2.1 and 3.1 hold. Then, for $\lambda > \Lambda$, the value function $V(x)$ defined by (3.3) is a viscosity solution to (3.15).*

Proof. First, for every $0 < \tilde{\lambda} < \lambda$, we let $\gamma > 0$ satisfies $\lambda - \tilde{\lambda} = \gamma$ and let $\varphi \in C^{1,2}_{lip}([0,+\infty) \times (H \times R^d))$ such that

$$a_1 = (V^\gamma - \varphi)(t,x) = \sup_{(l,y)\in[t,+\infty)\times\mathcal{D}} (V^\gamma - \varphi)(l, x \otimes_{[0,l-t]} y),$$

where $(t,x) \in [0,+\infty)\times\mathcal{D}$ and $x \in D(\mathcal{S}(\varphi(t,\cdot)))$. Then, for a fixed $u \in U$, according to the dynamic programming principle (Thm. 3.3), we obtain

$$\begin{aligned}\varphi(t,x) = \mathrm{e}^{-\gamma t}V(x) - a_1 &\leq \mathrm{e}^{-\gamma t}\int_0^s \mathrm{e}^{-\lambda\sigma}\mathbf{E}q(X^u(\sigma,x),u(\sigma))\mathrm{d}\sigma + \mathrm{e}^{-\gamma t - \lambda s}\mathbf{E}V(X^u_s(x)) - a_1 \\ &\leq \mathrm{e}^{-\gamma t}\int_0^s \mathrm{e}^{-\lambda\sigma}\mathbf{E}q(X^u(\sigma,x),u(\sigma))\mathrm{d}\sigma + \mathrm{e}^{-\tilde{\lambda}s}\mathbf{E}\varphi(t+s, X^u_s(x)) + \mathrm{e}^{-\tilde{\lambda}s}a_1 - a_1.\end{aligned}$$

Thus,

$$0 \leq \frac{1}{s}\int_0^s \mathrm{e}^{-\lambda\sigma}\mathbf{E}q(X^u(\sigma,x),u(\sigma))\mathrm{d}\sigma + \frac{1}{s}\mathrm{e}^{\gamma t}[\mathrm{e}^{-\tilde{\lambda}s}\mathbf{E}\varphi(t+s, X^u_s(x)) - \varphi(t,x)] + \frac{1}{s}\mathrm{e}^{\gamma t}(\mathrm{e}^{-\tilde{\lambda}s}a_1 - a_1).$$

Now, applying Lemma 3.9, we show that

$$0 \le q(x(0),u) - \tilde{\lambda}V(x) + \mathrm{e}^{\gamma t}[\varphi_t(t,x) + \mathcal{S}(\varphi(t,\cdot))(x) + (\nabla_{x_0}\varphi(t,x), F(x,u) + F_1(x(-\tau)))_{R^d} + \frac{1}{2}\mathrm{tr}[\nabla^2_{x_0}\varphi(t,x)G(x,u)G^\top(x,u)]].$$

Taking the minimum in $u \in U$, we know that V is a viscosity subsolution to (3.15).

Next, for every $0 < \tilde{\lambda} < \lambda$, we let $\gamma > 0$ satisfies $\lambda - \tilde{\lambda} = \gamma$ and let $\varphi \in C^{1,2}_{lip}([0,+\infty) \times (H \times R^d))$ such that

$$a_2 = (V^\gamma + \varphi)(t,x) = \inf_{(l,y)\in[t,+\infty)\times\mathcal{D}} (V^\gamma + \varphi)(l, x \otimes_{[0,(l-t)]} y),$$

where $(t,x) \in [0,+\infty) \times \mathcal{D}$ and $x \in D(\mathcal{S}(\varphi(t,\cdot)))$. For any $\varepsilon > 0$, according to (3.6), one can find a control $u^\varepsilon(\cdot) \equiv u^{\varepsilon,s}(\cdot) \in \mathcal{U}[0,+\infty)$ such that

$$\begin{aligned}\varepsilon s - \varphi(t,x) &= \mathrm{e}^{-\gamma t}V(x) - a_2 + \varepsilon s \\ &\ge \mathrm{e}^{-\gamma t}\int_0^s \mathrm{e}^{-\lambda\sigma}\mathbf{E}q(X^{u^\varepsilon}(\sigma), u^\varepsilon(\sigma))\mathrm{d}\sigma + \mathrm{e}^{-\gamma t - \lambda s}\mathbf{E}V(X^{u^\varepsilon}_s) - a_2 \\ &\ge \mathrm{e}^{-\gamma t}\int_0^s \mathrm{e}^{-\lambda\sigma}\mathbf{E}q(X^{u^\varepsilon}(\sigma), u^\varepsilon(\sigma))\mathrm{d}\sigma - \mathrm{e}^{-\tilde{\lambda}s}\mathbf{E}\varphi(t+s, X^{u^\varepsilon}_s) + \mathrm{e}^{-\tilde{\lambda}s}a_2 - a_2.\end{aligned}$$

Then, by Lemma 3.9, we obtain

$$\begin{aligned}\mathrm{e}^{\gamma t}\varepsilon &\ge \frac{1}{s}\int_0^s \mathrm{e}^{-\lambda\sigma}\mathbf{E}q(X^{u^\varepsilon}(\sigma), u^\varepsilon(\sigma))\mathrm{d}\sigma - \mathrm{e}^{\gamma t}\frac{\mathrm{e}^{-\lambda s}\mathbf{E}\varphi(t+s, X^{u^\varepsilon}_s) - \varphi(t,x)}{s} + \mathrm{e}^{\gamma t}\frac{\mathrm{e}^{-\tilde{\lambda}s}a_2 - a_2}{s} \\ &\ge \tilde{\lambda}\mathrm{e}^{\gamma t}\varphi(t,x) - \mathrm{e}^{\gamma t}\varphi_t(t,x) - \mathrm{e}^{\gamma t}\mathcal{S}(\varphi(t,\cdot))(x) + \frac{1}{s}\mathbf{E}\int_0^s q(x(0), u^\varepsilon(\sigma)) - \mathrm{e}^{\gamma t}(\nabla_{x_0}\varphi(t,x), \\ &\quad F(x,u^\varepsilon(\sigma)) + F_1(x(-\tau)))_{R^d} - \frac{1}{2}\mathrm{e}^{\gamma t}\mathrm{tr}[\nabla^2_{x_0}\varphi(t,x)G(x,u^\varepsilon(\sigma))G^\top(x,u^\varepsilon(\sigma))]\mathrm{d}\sigma - \tilde{\lambda}\mathrm{e}^{\gamma t}a_2 + o(1) \\ &\ge -\tilde{\lambda}V(x) - \mathrm{e}^{\gamma t}\varphi_t(t,x) - \mathrm{e}^{\gamma t}\mathcal{S}(\varphi(t,\cdot))(x) + \inf_{u\in U}[q(x(0),u) - \mathrm{e}^{\gamma t}(\nabla_{x_0}\varphi(t,x), F(x,u) \\ &\quad + F_1(x(-\tau)))_{R^d} - \frac{1}{2}\mathrm{e}^{\gamma t}\mathrm{tr}[\nabla^2_{x_0}\varphi(t,x)G(x,u)G^\top(x,u)]] + o(1).\end{aligned}$$

Letting $\varepsilon \to 0$, we show that

$$\begin{aligned}0 \ge -\tilde{\lambda}V(x) - \mathrm{e}^{\gamma t}\varphi_t(t,x) - \mathrm{e}^{\gamma t}\mathcal{S}(\varphi(t,\cdot))(x) + \inf_{u\in U}[q(x(0),u) - \mathrm{e}^{\gamma t}(\nabla_{x_0}\varphi(t,x), F(x,u) \\ + F_1(x(-\tau)))_{R^d} - \frac{1}{2}\mathrm{e}^{\gamma t}\mathrm{tr}[\nabla^2_{x_0}\varphi(t,x)G(x,u)G^\top(x,u)]].\end{aligned}$$

Therefore, V is also a viscosity supsolution to (3.15). This step completes the proof of Theorem 4.4. □

5. Viscosity solution to the HJB equation: Uniqueness theorem

This section is devoted to a proof of the uniqueness of the viscosity solution to (3.15). This result, together with the results in the previous section, will give a characterization of the value function of the optimal control problems (3.1) and (3.2).

We can now state the main result of this section.

Theorem 5.1. *Suppose that Hypotheses 2.1 and 3.1 hold, and assume* $\lambda > (5L+3L^2+1)\vee(L|a|_H^2+3L^2|c|_H^2)\vee\Lambda$. *Let* W *(resp.* V*) be a viscosity subsolution (resp. supsolution) to (3.15). In addition, let there exist a constant* $\Delta > 0$ *such that for* $(x,x_0),(y,y_0)\in H\times R^d$,

$$|W(x,x_0)|\vee|V(x,x_0)|\leq \Delta(1+|x_0|+|x|_H), \tag{5.1}$$

$$\begin{aligned}&|W(x,x_0)-W(y,y_0)|\vee|V(x,x_0)-V(y,y_0)|\\ \leq{}& \Delta\bigg(|x_0-y_0|+|x-y|_B+|F_1(x)-F_1(y)|_B+\bigg|\int_{-\tau}^0 F_1(x(\sigma))-F_1(y(\sigma))\mathrm{d}\sigma\bigg|\bigg).\end{aligned} \tag{5.2}$$

Then, $W\leq V$.

From this theorem, the viscosity solution to the HJB equation (3.15) can characterize the value function $V(x)$ of our optimal control problems (3.1) and (3.2) as follows:

Theorem 5.2. *Let Hypotheses 2.1 and 3.1 hold, and assume* $\lambda > (5L+3L^2+1)\vee(L|a|_H^2+3L^2|c|_H^2)\vee\Lambda$. *Then, the value function* V *defined by (3.3) is a unique viscosity solution to (3.15) in the class of functions satisfying (3.4) and (3.5).*

Proof. According to Theorem 4.4, we know that V is a viscosity solution to (3.15). Thus, our conclusion follows from Theorems 3.2 and 5.1. □

We are now in a position to prove Theorem 5.1.

Proof of Theorem 5.1. The proof of this theorem is rather long. Thus, we split it into several steps.

Step 1. Definitions of the auxiliary functions and sets.

To prove the theorem, we assume on the contrary that there exists $\varepsilon > 0$ a small number such that $\tilde{m} := \sup_{(x,x_0)\in H\times R^d}[W(x,x_0)-V(x,x_0)-2\varepsilon(|x|_H^2+|x_0|^2)] > 0$. Because simple functions are dense in H, according to (5.2) there exist a simple function $\tilde{y}=\sum_{i=1}^m a_i 1_{[t_i,t_{i+1})}$, $t_i\in[-\tau,0]$, $i=1,2,\ldots,m+1$ and a constant $\tilde{a}\in R^d$ such that $W(\tilde{x})-V(\tilde{x})-2\varepsilon(|\tilde{x}|_H^2+|\tilde{x}(0)|^2) > (\frac{1}{2}\vee \mathrm{e}^{-\frac{\tau}{8}\lambda})\tilde{m}$, where $\tilde{x}=\tilde{y}+\tilde{a}1_0(\cdot)$. First, we can let $\varepsilon>0$ be small enough such that

$$\varepsilon L\sup_{\theta\in[-\tau,0]}|\tilde{x}(\theta)|^2+2\varepsilon L+3\varepsilon L^2 < \frac{\lambda\tilde{m}}{16}\cdot$$

Next, for every $\alpha>0$, we define, for any $(x,y)\in\mathcal{D}\times\mathcal{D}$,

$$\Psi(x,y)=W(x)-V(y)-d_\alpha(x,y)-\varepsilon(|x|_H^2+|x(0)|^2+|y|_H^2+|y(0)|^2),$$

and

$$\Psi^\gamma(t,x,y)=\mathrm{e}^{-\gamma t}\Psi(x,y),$$

where

$$\begin{aligned}d_\alpha(x,y)={}&\frac{\alpha}{2}|x(0)-y(0)|^2+\frac{\alpha}{2}|x(-\tau)-y(-\tau)|^2+\frac{\alpha^2}{4}|x-y|_B^4+\frac{\alpha^2}{4}|F_1(x)-F_1(y)|_B^4\\ &+\frac{\alpha}{2}\bigg(\int_{-\tau}^0 F_1(x(\theta))-F_1(y(\theta))\mathrm{d}\theta\bigg)^2,\end{aligned} \tag{5.3}$$

and

$$\gamma=\frac{\lambda}{2}\cdot$$

Finally, for every $M>0$, we define

$$M_\alpha := \sup_{t\geq 0; x,y\in\mathcal{D}_M} \Psi^\gamma(t, \tilde{x}\otimes_{[0,t]} x, \tilde{x}\otimes_{[0,t]} y),$$

where

$$M_\alpha \geq M_* := \sup_{t\geq 0}\sup_{x\in\mathcal{D}_M; l\in[0,t]} \Psi^\gamma(t,\tilde{x}\otimes_l x, \tilde{x}\otimes_l x) \geq \frac{\tilde{m}}{2}.$$

Step 2. Properties of $\Psi^\gamma(t,x,y)$.

For every $M,\alpha>0$, from the definition of M_α, we can fix $\bar{t}\geq 0, \bar{k},\bar{l}\in[0,\bar{t}]$ and $\bar{x},\bar{y}\in\mathcal{D}_M$ satisfying

$$\bar{x}=\tilde{x}\otimes_{\bar{k}}\bar{x},\quad \bar{y}=\tilde{x}\otimes_{\bar{l}}\bar{y},\quad \Psi^\gamma(0,\tilde{x},\tilde{x})\leq \Psi^\gamma(\bar{t},\bar{x},\bar{y})\ \text{ and }\ \Psi^\gamma(\bar{t},\bar{x},\bar{y})+\frac{1}{\alpha}>M_\alpha.$$

By the definition of Ψ, we obtain that

$$2\Psi(x,y)=\Psi(x,x)+\Psi(y,y)+V(x)-V(y)+W(x)-W(y)-2d_\alpha(x,y).$$

Therefore,

$$\begin{aligned}\Psi(x,y)&\leq \frac{1}{2}(|\Psi(x,x)|+|\Psi(y,y)|+|V(x)-V(y)|+|W(x)-W(y)|)-d_\alpha(x,y)\\&\leq \tilde{m}+\Delta\bigg(|x(0)-y(0)|+\bigg|\int_{-\tau}^0 F_1(x(\theta))-F_1(y(\theta))\mathrm{d}\theta\bigg|+|x-y|_B+|F_1(x)-F_1(y)|_B\bigg)\\&\quad -d_\alpha(x,y)\\&\leq \tilde{m}+\frac{\Delta^2}{\alpha}+\frac{2}{\alpha}+\frac{\Delta^2}{2\alpha^{\frac{1}{2}}}.\end{aligned}$$

Letting $\alpha\geq (1+\frac{4(2+\Delta)^2}{(\mathrm{e}^{-\frac{\tau}{8}\lambda}-\mathrm{e}^{-\frac{\tau}{4}\lambda})\tilde{m}})^2$, we obtain that

$$\Psi(x,y)\leq \tilde{m}+\frac{1}{2}(\mathrm{e}^{-\frac{\tau}{8}\lambda}-\mathrm{e}^{-\frac{\tau}{4}\lambda})\tilde{m}.$$

As $W(\tilde{x})-V(\tilde{x})-2\varepsilon(|\tilde{x}|_H^2+|\tilde{x}(0)|^2)>(\frac{1}{2}\vee \mathrm{e}^{-\frac{\tau}{8}\lambda})\tilde{m}$, there exists a constant $T\leq\frac{\tau}{2}$ such that for all $M>0$ and $\alpha>N_{\tilde{m},\Delta}:=(1+\frac{4(2+\Delta)^2}{(\mathrm{e}^{-\frac{\tau}{8}\lambda}-\mathrm{e}^{-\frac{\tau}{4}\lambda})\tilde{m}})^2$,

$$\Psi^\gamma(t,x,y)+\frac{1}{\alpha}<M_\alpha,\quad t\geq T,\quad x,y\in\mathcal{D}_M.$$

Now, we can apply Lemma 4.1 to find $\hat{t}\in[0,T), \hat{k},\hat{l}\in[0,(\hat{t}-\bar{t})], \hat{x},\hat{y}\in\mathcal{D}_M$, which satisfies $\hat{x}=\bar{x}\otimes_{\hat{k}}\hat{x},\ \hat{y}=\bar{y}\otimes_{\hat{l}}\hat{y}$ with $\Psi^\gamma(\hat{t},\hat{x},\hat{y})\geq\Psi^\gamma(\bar{t},\bar{x},\bar{y})\geq\Psi^\gamma(0,\tilde{x},\tilde{x})$ such that

$$\Psi^\gamma(\hat{t},\hat{x},\hat{y})\geq \Psi^\gamma(t,\hat{x}\otimes_{[0,(t-\hat{t})]} x,\hat{y}\otimes_{[0,(t-\hat{t})]} y),\ t\geq\hat{t}, x,y\in\mathcal{D}_M. \tag{5.4}$$

In particular, we know that

$$\Psi^\gamma(\hat{t},\hat{x},\hat{y})\geq\Psi^\gamma(t,\hat{x}\otimes_{[0,(t-\bar{t})]}x,\hat{y}),\ \Psi^\gamma(\hat{t},\hat{x},\hat{y})\geq\Psi^\gamma(t,\hat{x},\hat{y}\otimes_{[0,(t-\bar{t})]}y), t\geq\hat{t}, x,y\in\mathcal{D}_M.$$

We should note that $(\hat{t},\hat{x},\hat{y})$ depends on $\bar{t},\bar{k},\bar{l},\bar{x},\bar{y},\alpha,M$.

Step 3. For every $M>0$, we have

$$d_\alpha(\hat{x},\hat{y})\leq\frac{\mathrm{e}^{\gamma T}}{\alpha}+|W(\hat{x})-W(\hat{y})|+|V(\hat{x})-V(\hat{y})|\to 0 \text{ as } \alpha\to+\infty. \tag{5.5}$$

Let us show the above. We can confirm that

$$\begin{aligned}
&\mathrm{e}^{-\gamma\hat{t}}d_\alpha(\hat{x},\hat{y})+\varepsilon\mathrm{e}^{-\gamma\hat{t}}(|\hat{x}|_H^2+|\hat{x}(0)|^2+|\hat{y}|_H^2+|\hat{y}(0)|^2)\\
&\le\frac{1}{\alpha}+\mathrm{e}^{-\gamma\hat{t}}(W(\hat{x})-V(\hat{y}))-M_\alpha\le\frac{1}{\alpha}+\mathrm{e}^{-\gamma\hat{t}}(W(\hat{x})-V(\hat{y}))-M_*\\
&\le\frac{1}{\alpha}+C-M_*,
\end{aligned}\tag{5.6}$$

where $C:=2\Delta(1+M+\tau^{\frac{1}{2}}M)$. We also know that

$$\begin{aligned}
2M_*\le{}&\frac{2}{\alpha}+\mathrm{e}^{-\gamma\hat{t}}(W(\hat{x})-W(\hat{y})+W(\hat{y})-V(\hat{y}))+\mathrm{e}^{-\gamma\hat{t}}(W(\hat{x})-V(\hat{x})+V(\hat{x})-V(\hat{y}))\\
&-2\mathrm{e}^{-\gamma\hat{t}}d_\alpha(\hat{x},\hat{y})-2\varepsilon\mathrm{e}^{-\gamma\hat{t}}(|\hat{x}|_H^2+|\hat{x}(0)|^2+|\hat{y}|_H^2+|\hat{y}(0)|^2)\\
\le{}&\frac{2}{\alpha}+\mathrm{e}^{-\gamma\hat{t}}(|W(\hat{x})-W(\hat{y})|+|V(\hat{x})-V(\hat{y})|)+2M_*-2\mathrm{e}^{-\gamma\hat{t}}d_\alpha(\hat{x},\hat{y}).
\end{aligned}$$

Thus,

$$\mathrm{e}^{-\gamma\hat{t}}d_\alpha(\hat{x},\hat{y})\le\frac{1}{\alpha}+\mathrm{e}^{-\gamma\hat{t}}(|W(\hat{x})-W(\hat{y})|+|V(\hat{x})-V(\hat{y})|).$$

Therefore,

$$d_\alpha(\hat{x},\hat{y})\le\frac{\mathrm{e}^{\gamma T}}{\alpha}+|W(\hat{x})-W(\hat{y})|+|V(\hat{x})-V(\hat{y})|.\tag{5.7}$$

According to (5.6), we obtain

$$\frac{1}{\alpha}d_\alpha(\hat{x},\hat{y})\to 0 \text{ as } \alpha\to+\infty.$$

Then combining (5.2) and (5.7), we see that (5.5) holds.

Step 4. There exists $M>0$ such that (5.4) holds true for all $(t,x,y)\in[\hat{t},+\infty)\times\mathcal{D}\times\mathcal{D}$ and $\alpha>N_{\tilde{m},\Delta}$.

We note that there exists an $M>(1+\tau^{\frac{-1}{2}})(1+\frac{2\Delta}{\varepsilon})$, independent of α, that is sufficiently large that

$$\Psi^\gamma(0,\tilde{x},\tilde{x})=W(\tilde{x})-V(\tilde{x})-2\varepsilon(|\tilde{x}|_H^2+|\tilde{x}(0)|^2)>0>\Psi^\gamma(t,x,y),$$

where $t\in[0,+\infty)$ and $x\in\mathcal{D}\setminus\mathcal{D}_M$ or $y\in\mathcal{D}\setminus\mathcal{D}_M$. Therefore, for this $M>0$, we know that (5.4) holds true for all $(t,x,y)\in[\hat{t},+\infty)\times\mathcal{D}\times\mathcal{D}$ and $\alpha>N_{\tilde{m},\Delta}$.

Step 5. Completion of the proof.

For the fixed $M>0$ in step 4, we find $\hat{t}\in[0,T),\hat{k},\hat{l}\in[0,(\hat{t}-\bar{t})],\hat{x},\hat{y}\in\mathcal{D}_M$ and $\hat{x}=\bar{x}\otimes_{\hat{k}}\hat{x},\hat{y}=\bar{y}\otimes_{\hat{l}}\hat{y}$ with $\Psi^\gamma(\hat{t},\hat{x},\hat{y})\ge\Psi^\gamma(\bar{t},\bar{x},\bar{y})\ge\Psi^\gamma(0,\tilde{x},\tilde{x})$ such that

$$\Psi^\gamma(\hat{t},\hat{x},\hat{y})\ge\Psi^\gamma(t,\hat{x}\otimes_{[0,(t-\hat{t})]}x,\hat{y}\otimes_{[0,(t-\hat{t})]}y),\ t\ge\hat{t},x,y\in\mathcal{D}.\tag{5.8}$$

We put, for $(t,x,x_0),(t,y,y_0)\in[0,+\infty)\times H\times R^d$,

$$\begin{aligned}
W_1(t,x,x_0)&=\mathrm{e}^{-\gamma t}[W(x,x_0)-\varepsilon(|x_0|^2+|x|^2)-\varepsilon|t-\hat{t}|^2-\varepsilon|x-\hat{x}|_B^4-\varepsilon(|x|^2-|\hat{x}|^2)^2],\\
V_1(t,y,y_0)&=\mathrm{e}^{-\gamma t}[V(y,y_0)+\varepsilon(|y_0|^2+|y|^2)+\varepsilon|t-\hat{t}|^2+\varepsilon|y-\hat{y}|_B^4+\varepsilon(|y|^2-|\hat{y}|^2)^2].
\end{aligned}$$

For simplicity, we denote $W_1(t,x,x_0)$ and $V_1(t,y,y_0)$ by $W_1(t,x)$ and $V_1(t,y)$, respectively, when $x,y\in\mathcal{D}$ and $x(0)=x_0$, $y(0)=y_0$. Moreover, we define, for $(t,x,x_0),(t,y,y_0)\in[0,+\infty)\times\mathcal{D}\times R^d$,

$$\tilde{W}_1(t,x,x_0)=\sup_{z=\hat{x}\otimes_{[0,(t-\hat{t})]}z,z(0)=x_0}\Big[W_1(t,z,x_0)-\mathrm{e}^{-\gamma t}p_\alpha(x,z)-\mathrm{e}^{-\gamma t}h_\alpha(z)\Big],\ t\ge\hat{t},$$

$$\tilde{V}_1(t,y,y_0)=\inf_{z=\hat{y}\otimes_{[0,(t-\hat{t})]}z,z(0)=y_0}\Big[V_1(t,z,y_0)+\mathrm{e}^{-\gamma t}p_\alpha(y,z)+\mathrm{e}^{-\gamma t}h_\alpha(z)\Big],\ t\ge\hat{t},$$

and

$$\begin{aligned}\tilde{W}_1(t,x,x_0)&=\tilde{W}_1(\hat{t},x,x_0),\ t<\hat{t},\\ \tilde{V}_1(t,y,y_0)&=\tilde{V}_1(\hat{t},y,y_0),\ t<\hat{t},\end{aligned}$$

where

$$p_\alpha(x,y)=\alpha|x(-\tau)-y(-\tau)|^2+2\alpha^2|x-y|_B^4,$$

and

$$h_\alpha(x)=\alpha\bigg(\int_{-\tau}^0 F_1(x(\theta))-\frac{F_1(\hat{x}(\theta))+F_1(\hat{y}(\theta))}{2}\mathrm{d}\theta\bigg)^2+2\alpha^2\bigg|F_1(x)-\frac{F_1(\hat{x})+F_1(\hat{y})}{2}\bigg|_B^4.$$

Then we obtain that

$$\begin{aligned}&\tilde{W}_1(t,z,x(0))-\tilde{V}_1(t,z,y(0))-\mathrm{e}^{-\gamma t}\frac{\alpha|x(0)-y(0)|^2}{2}\\ &=\sup_{x=\hat{x}\otimes_{[0,(t-\hat{t})]}x,y=\hat{y}\otimes_{[0,(t-\hat{t})]}y}\bigg[W_1(t,x,x(0))-\mathrm{e}^{-\gamma t}p_\alpha(x,z)-\mathrm{e}^{-\gamma t}h_\alpha(x)\\ &\qquad-V_1(t,y,y(0))-\mathrm{e}^{-\gamma t}p_\alpha(y,z)-\mathrm{e}^{-\gamma t}h_\alpha(y)-\mathrm{e}^{-\gamma t}\frac{\alpha|x(0)-y(0)|^2}{2}\bigg]\\ &\le\sup_{x=\hat{x}\otimes_{[0,(t-\hat{t})]}x,y=\hat{y}\otimes_{[0,(t-\hat{t})]}y}\bigg[W_1(t,x,x(0))-\mathrm{e}^{-\gamma t}h_\alpha(x)-V_1(t,y,y(0))-\mathrm{e}^{-\gamma t}h_\alpha(y)\\ &\qquad-\mathrm{e}^{-\gamma t}\frac{\alpha}{2}[|x(-\tau)-y(-\tau)|^2+|x(0)-y(0)|^2+\frac{\alpha}{2}|x-y|_B^4]\bigg]\\ &\le\sup_{x=\hat{x}\otimes_{[0,(t-\hat{t})]}x,y=\hat{y}\otimes_{[0,(t-\hat{t})]}y}\bigg[W_1(t,x,x(0))-V_1(t,y,y(0))-\mathrm{e}^{-\gamma t}d_\alpha(x,y)\bigg]\\ &\le W_1(\hat{t},\hat{x},\hat{x}(0))-V_1(\hat{t},\hat{y},\hat{y}(0))-\mathrm{e}^{-\gamma\hat{t}}d_\alpha(\hat{x},\hat{y}),\end{aligned}\tag{5.9}$$

where the last inequality becomes equality if and only if $t=\hat{t}$ and $x=\hat{x},y=\hat{y}$, the first inequality becomes equality if and only if $z=\frac{x+y}{2}$, and the second inequality becomes equality if $x=\hat{x},y=\hat{y}$. Then we obtain that

$$\begin{aligned}&\tilde{W}_1(t,z,x(0))-\tilde{V}_1(t,z,y(0))-\mathrm{e}^{-\gamma t}\frac{\alpha|x(0)-y(0)|^2}{2}\\ &\le W_1(\hat{t},\hat{x},\hat{x}(0))-V_1(\hat{t},\hat{y},\hat{y}(0))-\mathrm{e}^{-\gamma\hat{t}}d_\alpha(\hat{x},\hat{y}),\quad t\in[\hat{t},T],\end{aligned}$$

and the equality only holds at $\hat{t},\hat{x}(0),\hat{y}(0),\hat{z}=\frac{\hat{x}+\hat{y}}{2}$. Set $r=\frac{1}{2}\hat{t}$, for a given $L>0$, let $\varphi\in C^{1,2}((0,T)\times R^d)$ be a function such that $\tilde{W}_1(t,\hat{z},x_0)-\varphi(t,x_0)$ has a maximum at $(\bar{s},\bar{x}_0)\in(0,T)\times R^d$, moreover, the following inequalities hold true:

$$\begin{aligned}&|\bar{s}-\hat{t}|+|\bar{x}_0-\hat{x}_0|<r,\\ &|\tilde{W}_1(\bar{s},\hat{z},\bar{x}_0)|+|\nabla_x\varphi(\bar{s},\bar{x}_0)|+|\nabla_x^2\varphi(\bar{s},\bar{x}_0)|\le L.\end{aligned}$$

Noting that if $\bar{s}<\hat{t}$, we have $\tilde{W}_1(t,\hat{z},x_0)-\varphi(t+\bar{s}-\hat{t},x_0)$ has a maximum at $(\hat{t},\bar{x}_0)\in(0,T)\times R^d$. Without loss of generality, we can assume $\bar{s}\ge\hat{t}$. We can modify φ and extend it to $(0,+\infty)\times R^d$ and such that $\varphi\in C^{1,2}((0,+\infty)\times R^d)$, $\tilde{W}_1(t,\hat{z},x_0)-\varphi(t,x_0)$ has a strict positive maximum at $(\bar{s},\bar{x}_0)\in[\hat{t},T)\times R^d$ and the above two inequalities hold true. Now we consider the function

$$\Upsilon(t,x,x_0)=W_1(t,x,x_0)-\mathrm{e}^{-\gamma t}p_\alpha(x,\hat{z})-\mathrm{e}^{-\gamma t}h_\alpha(x)-\varphi(t,x_0),\ (t,x)\in[\hat{t},+\infty)\times\mathcal{D}\times R^d.$$

We may assume that φ grow quadratically at ∞. For every $\delta > 0$, by the definition of $\tilde{W}_1$, there exists x^δ such that

$$x^\delta(0) = \bar{x}_0,\ \ x^\delta = \hat{x} \otimes_p x^\delta,\quad W_1(\bar{s}, x^\delta, \bar{x}_0) - \mathrm{e}^{-\gamma\bar{s}} p_\alpha(\hat{z}, x^\delta) - \mathrm{e}^{-\gamma\bar{s}} h_\alpha(x^\delta) \geq \tilde{W}_1(\bar{s}, \hat{z}, \bar{x}_0) - \delta, \tag{5.10}$$

where $p \in [0, (\bar{s} - \hat{t})]$. By Lemma 4.1 and *Step* 4, there exist $M > 0$ independent of δ, $(\check{t}, \check{x}) \in [\bar{s}, \infty) \times \mathcal{D}_M$ and $\check{p} \in [0, (\check{t} - \bar{s})]$ such that

$$\check{x} = x^\delta \otimes_{\check{p}} \check{x}, \qquad \Upsilon(\check{t}, \check{x}, \check{x}(0)) \geq \Upsilon(\bar{s}, x^\delta, \bar{x}_0),$$

and

$$\Upsilon(\check{t}, \check{x}, \check{x}(0)) \geq \Upsilon(t, x, x(0)),$$

where $t \in [\check{t}, +\infty), x \in \mathcal{D}, x = \check{x} \otimes_{[0,(t-\check{t})]} x$. Then

$$\begin{aligned}\tilde{W}_1(\check{t}, \hat{z}, \check{x}(0)) - \varphi(\check{t}, \check{x}(0)) &\geq W_1(\check{t}, \check{x}, \check{x}(0)) - \mathrm{e}^{-\gamma\check{t}} p_\alpha(\check{x}, \hat{z}) - \mathrm{e}^{-\gamma\check{t}} h_\alpha(\check{x}) - \varphi(\check{t}, \check{x}(0))\\ \geq W_1(\bar{s}, x^\delta, \bar{x}_0) - \mathrm{e}^{-\gamma\bar{s}} p_\alpha(x^\delta, \hat{z}) &- \mathrm{e}^{-\gamma\bar{s}} h_\alpha(x^\delta) - \varphi(\bar{s}, \bar{x}_0) \geq \tilde{W}_1(\bar{s}, \hat{z}, \bar{x}_0) - \varphi(\bar{s}, \bar{x}_0) - \delta.\end{aligned}$$

Letting $\delta \to 0$, we obtain

$$\check{t} \to \bar{s},\ \ \check{x}(0) \to \bar{x}_0,\ \text{ as } \delta \to 0.$$

Thus, by the definition of the viscosity subsolution, noting $\gamma = \frac{\lambda}{2}$, we know that

$$\begin{aligned}&-\gamma \mathrm{e}^{-\gamma\check{t}} W(\check{x}) - \gamma \mathrm{e}^{-\gamma\check{t}}[\varepsilon(|\check{x}(0)|^2 + |\check{x}|^2) + \varepsilon|\check{t} - \hat{t}|^2 + \varepsilon|\check{x} - \hat{x}|_B^4 + \varepsilon(|\check{x}|^2 - |\hat{x}|^2)^2 + p_\alpha(\check{x}, \hat{z})\\ &+h_\alpha(\check{x})] + 2\mathrm{e}^{-\gamma\check{t}}\varepsilon(\check{t} - \hat{t}) + \varphi_t(\check{t}, \check{x}(0)) + \mathrm{e}^{-\gamma\check{t}}\varepsilon[4(|B(\check{x} - \hat{x})|_H^2 B(\check{x} - \hat{x}), \check{x}(0)1_{[-\tau,0]} - \check{x})_H\\ &+(2|\check{x}|^2 - 2|\hat{x}|^2 + 1)(\check{x}^2(0) - \check{x}^2(-\tau))] + 8\mathrm{e}^{-\gamma\check{t}}\alpha^2(|B(\check{x} - \hat{z})|_H^2 B(\check{x} - \hat{z}), \check{x}(0)1_{[-\tau,0]} - \check{x})_H\\ &+2\mathrm{e}^{-\gamma\check{t}}\alpha\int_{-\tau}^0 (F_1(\check{x}) - \frac{1}{2}(F_1(\hat{x}) + F_1(\hat{y})))(\theta)\mathrm{d}\theta(F_1(\check{x}(0)) - F_1(\check{x}(-\tau)))\\ &+8\mathrm{e}^{-\gamma\check{t}}\alpha^2(|B(F_1(\check{x}) - \frac{1}{2}(F_1(\hat{x}) + F_1(\hat{y})))|_H^2 B(F_1(\check{x}) - \frac{1}{2}(F_1(\hat{x}) + F_1(\hat{y}))),\\ &F_1(\check{x}(0))1_{[-\tau,0]} - F_1(\check{x}))_H + \mathrm{e}^{-\gamma\check{t}}\mathbf{H}(\check{x}, \mathrm{e}^{\gamma\check{t}}\nabla_x\varphi(\check{t}, \check{x}(0)) + 2\varepsilon\check{x}(0), \mathrm{e}^{\gamma\check{t}}\nabla_x^2\varphi(\check{t}, \check{x}(0)) + 2\varepsilon I) \geq 0,\end{aligned}$$

where I will stand for the identity matrix in any dimension. Letting $\delta \to 0$, by the definition of $\mathbf{H}$, it follows that there exists a constant C such that $b = \varphi_t(\bar{s}, \bar{x}_0) \geq C$. Then by the following Lemma 5.3 to obtain sequences $t_k, s_k \in (0, T)$, $x_0^k, y_0^k \in R^d$ such that $(t_k, x_0^k) \to (\hat{t}, \hat{x}(0))$, $(s_k, y_0^k) \to (\hat{t}, \hat{y}(0))$ as $k \to +\infty$ and the sequences of functions $\varphi_k, \psi_k \in C^{1,2}((0, T) \times R^d)$ such that

$$\tilde{W}_1(t, \hat{z}, x_0) - \varphi_k(t, x_0) \leq 0, \quad \tilde{V}_1(t, \hat{z}, x_0) - \psi_k(t, x_0) \geq 0,$$

equalities hold true at (t_k, x_0^k), (s_k, y_0^k), respectively,

$$\begin{aligned}&(\varphi_k)_t(t_k, x_0^k) \to b_1, \quad (\psi_k)_t(s_k, y_0^k) \to b_2,\\ &\nabla_x\varphi_k(t_k, x_0^k) \to \alpha\mathrm{e}^{-\gamma\hat{t}}(\hat{x}(0) - \hat{y}(0)), \quad \nabla_x\psi_k(s_k, y_0^k) \to \alpha\mathrm{e}^{-\gamma\hat{t}}(\hat{x}(0) - \hat{y}(0)),\\ &\nabla_x^2\varphi_k(t_k, x_0^k) \to \mathrm{e}^{-\gamma\hat{t}}X, \quad \nabla_x^2\psi_k(s_k, y_0^k) \to \mathrm{e}^{-\gamma\hat{t}}Y,\end{aligned}$$

where $b_1 + b_2 = -\gamma\mathrm{e}^{-\gamma\hat{t}}\frac{\alpha}{2}|\hat{x}(0) - \hat{y}(0)|^2$ and X, Y satisfy the following inequality:

$$-4\alpha\begin{pmatrix} I & 0\\ 0 & I\end{pmatrix} \leq \begin{pmatrix} X & 0\\ 0 & Y\end{pmatrix} \leq 2\alpha\begin{pmatrix} I & -I\\ -I & I\end{pmatrix}. \tag{5.11}$$

Noting that if $t_k, s_k < \hat{t}$, we have

$$\tilde{W}_1(t_k, \hat{z}, x_0) - \varphi_k(t_k, x_0) = \tilde{W}_1(\hat{t}, \hat{z}, x_0) - \varphi_k(\hat{t} + t_k - \hat{t}, x_0),$$
$$\tilde{V}_1(s_k, \hat{z}, x_0) - \psi_k(s_k, x_0) = \tilde{V}_1(\hat{t}, \hat{z}, x_0) - \psi_k(\hat{t} + s_k - \hat{t}, x_0).$$

We may assume that $t_k, s_k \geq \hat{t}$. We can modify φ_k, ψ_k and extend them to $(0, +\infty) \times R^d$ such that $\varphi_k, \psi_k \in C^{1,2}((0, +\infty) \times R^d)$ and the above formulae hold true. We may without loss of generality assume that $\varphi_k, -\psi_k$ grow quadratically at ∞.

Now we consider the function, for $t, s \in [\hat{t}, +\infty)$ and $x, y \in \mathcal{D}$,

$$\Upsilon_1(t, s, x, y) = W_1(t, x) - V_1(s, y) - \mathrm{e}^{-\gamma t}\hat{p}_\alpha(x) - \mathrm{e}^{-\gamma s}\hat{p}_\alpha(y) - \varphi_k(t, x(0)) + \psi_k(s, y(0)),$$

where

$$\hat{p}_\alpha(x) = p_\alpha(x, \hat{z}) + h_\alpha(x), \quad x \in \mathcal{D}.$$

For every k and $\delta > 0$, by the definitions of $\tilde{W}_1$ and $\tilde{V}_1$, there exist $x^{k,\delta}$ and $y^{k,\delta}$ such that

$$x^{k,\delta}(0) = x_0^k, \ x^{k,\delta} = \hat{x} \otimes_{p_k} x^{k,\delta}, \quad W_1(t_k, x^{k,\delta}) - \mathrm{e}^{-\gamma t_k}\hat{p}_\alpha(x^{k,\delta}) \geq \tilde{W}_1(t_k, \hat{z}, x_0^k) - \delta, \tag{5.12}$$

and

$$y^{k,\delta}(0) = y_0^k, \ y^{k,\delta} = \hat{y} \otimes_{l_k} y^{k,\delta}, \quad V_1(s_k, y^{k,\delta}) + \mathrm{e}^{-\gamma s_k}\hat{p}_\alpha(y^{k,\delta}) \leq \tilde{V}_1(s_k, \hat{z}, y_0^k) + \delta, \tag{5.13}$$

where $p_k \in [0, (t_k - \hat{t})]$ and $l_k \in [0, (s_k - \hat{t})]$. By Lemma 4.1 and *Step* 4, there exist $M > 0$ independent of δ, $(\check{t}, \check{s}, \check{x}, \check{y}) \in [t_k, +\infty) \times [s_k, +\infty) \times \mathcal{D}_M \times \mathcal{D}_M$ and $\check{p} \in [0, (\check{t} - t_k)], \check{l} \in [0, (\check{s} - s_k)]$ such that

$$\check{x} = x^{k,\delta} \otimes_{\check{p}} \check{x}, \quad \check{y} = y^{k,\delta} \otimes_{\check{l}} \check{y},$$
$$\Upsilon_1(\check{t}, \check{s}, \check{x}, \check{y}) \geq \Upsilon_1(t_k, s_k, x^{k,\delta}, y^{k,\delta}),$$

and

$$\Upsilon_1(\check{t}, \check{s}, \check{x}, \check{y}) \geq \Upsilon_1(t, s, x, y),$$

where $t \in [\check{t}, +\infty), s \in [\check{s}, +\infty), x, y \in \mathcal{D}, x = \check{x}\otimes_{[0,(t-\check{t})]}, y = \check{y} \otimes_{[0,(t-\check{s})]} y$. Then

$$\begin{aligned}
&\tilde{W}_1(\check{t}, \hat{z}, \check{x}(0)) - \tilde{V}_1(\check{s}, \hat{z}, \check{y}(0)) - \varphi_k(\check{t}, \check{x}(0)) + \psi_k(\check{s}, \check{y}(0)) \\
&\geq W_1(\check{t}, \check{x}) - V_1(\check{s}, \check{y}) - \mathrm{e}^{-\gamma\check{t}}\hat{p}_\alpha(\check{x}) - \varphi_k(\check{t}, \check{x}(0)) - \mathrm{e}^{-\gamma\check{s}}\hat{p}_\alpha(\check{y}) + \psi_k(\check{s}, \check{y}(0)) \\
&\geq W_1(t_k, x^{k,\delta}) - V_1(s_k, y^{k,\delta}) - \mathrm{e}^{-\gamma t_k}\hat{p}_\alpha(x^{k,\delta}) - \varphi_k(t_k, x_0^k) - \mathrm{e}^{-\gamma s_k}\hat{p}_\alpha(y^{k,\delta}) + \psi_k(t_k, y_0^k) \\
&\geq \tilde{W}_1(t_k, \hat{z}, x_0^k) - \tilde{V}_1(s_k, \hat{z}, y_0^k) - \varphi_k(t_k, x_0^k) + \psi_k(s_k, y_0^k) - 2\delta.
\end{aligned}$$

Moreover, letting $\delta \to 0$, we obtain

$$\check{t} \to t_k, \ \check{x}(0) \to x_0^k, \ \check{s} \to s_k, \ \check{y}(0) \to y_0^k, \ \text{as } \delta \to 0;$$

and

$$\begin{aligned}
&W_1(\check{t}, \check{x}) - V_1(\check{s}, \check{y}) - \mathrm{e}^{-\gamma\check{t}}\hat{p}_\alpha(\check{x}) - \mathrm{e}^{-\gamma\check{s}}\hat{p}_\alpha(\check{y}) - \varphi_k(\check{t}, \check{x}(0)) + \psi_k(\check{s}, \check{y}(0)) \\
&\to \tilde{W}_1(t_k, \hat{z}, x_0^k) - \tilde{V}_1(s_k, \hat{z}, y_0^k) - \varphi_k(t_k, x_0^k) + \psi_k(s_k, y_0^k) \text{ as } \delta \to 0.
\end{aligned}$$

Letting $\delta \to 0$ and $k \to \infty$, by (5.9) we show that

$$\begin{aligned}
&\lim_{k\to\infty}\lim_{\delta\to 0}\left[W_1(\check{t}, \check{x}, \check{x}(0)) - V_1(\check{s}, \check{y}, \check{y}(0)) - \mathrm{e}^{-\gamma\check{t}}\hat{p}_\alpha(\check{x}) - \mathrm{e}^{-\gamma\check{s}}\hat{p}_\alpha(\check{y})\right] \\
&= \lim_{k\to\infty}\left[\tilde{W}_1(t_k, \hat{z}, x_0^k) - \tilde{V}_1(s_k, \hat{z}, y_0^k)\right] \\
&= \tilde{W}_1(\hat{t}, \hat{z}, \hat{x}(0)) - \tilde{V}_1(\hat{t}, \hat{z}, \hat{y}(0)) \\
&= W_1(\hat{t}, \hat{x}, \hat{x}(0)) - V_1(\hat{t}, \hat{y}, \hat{y}(0)) - \mathrm{e}^{-\gamma\hat{t}}d_\alpha(\hat{x}, \hat{y}) + \mathrm{e}^{-\gamma\hat{t}}\frac{\alpha|\hat{x}(0) - \hat{y}(0)|^2}{2}.
\end{aligned} \tag{5.14}$$

We claim that

$$\lim_{k\to\infty}\lim_{\delta\to 0}[|\check{x}-\hat{x}|_B^4+(|\check{x}|^2-|\hat{x}^2|)^2+|\check{y}-\hat{y}|_B^4+(|\check{y}|^2-|\hat{y}|^2)^2]=0. \tag{5.15}$$

In fact, if not, there exist a $\nu>0$ and two subsequence of k and δ, still denoted by themselves, such that

$$\varepsilon[|\check{x}-\hat{x}|_B^4+(|\check{x}|^2-|\hat{x}^2|)^2+|\check{y}-\hat{y}|_B^4+(|\check{y}|^2-|\hat{y}|^2)^2]\geq\nu.$$

We can assume $\check{s}\leq\check{t}$, then

$$\begin{aligned}
&W_1(\check{t},\check{x},\check{x}(0))-V_1(\check{s},\check{y},\check{x}(0))-\mathrm{e}^{-\gamma\check{t}}\hat{p}_\alpha(\check{x})-\mathrm{e}^{-\gamma\check{s}}\hat{p}_\alpha(\check{y})\\
&=\mathrm{e}^{-\gamma\check{t}}[W(\check{x})-\varepsilon(|\check{x}(0)|^2+|\check{x}|^2)-\varepsilon|\check{x}-\hat{x}|_B^4-\varepsilon|\check{t}-\hat{t}|^2-\varepsilon(|\check{x}|^2-|\hat{x}|^2)^2-\hat{p}_\alpha(\check{x})]\\
&\quad+\mathrm{e}^{-\gamma\check{s}}[-V(\check{y})-\varepsilon(|\check{y}(0)|^2+|\check{y}|^2)-\varepsilon|\check{s}-\hat{s}|^2-\varepsilon|\check{y}-\hat{y}|_B^4-\varepsilon(|\check{y}|^2-|\hat{y}|^2)^2-\hat{p}_\alpha(\check{y})]\\
&\leq\mathrm{e}^{-\gamma\check{t}}[W(\check{x})-\varepsilon(|\check{x}(0)|^2+|\check{x}|^2)-\varepsilon|\check{t}-\hat{t}|^2]-\mathrm{e}^{-\gamma\check{s}}V(\check{y})\\
&\quad-\varepsilon\mathrm{e}^{-\gamma\check{t}}(|\check{y}(0)|^2+|\check{y}|^2+|\check{s}-\hat{s}|^2)-\mathrm{e}^{-\gamma\check{t}}d_\alpha(\check{x},\check{y})+\mathrm{e}^{-\gamma\check{t}}\frac{\alpha|\check{x}(0)-\check{y}(0)|^2}{2}-\mathrm{e}^{-\gamma\check{t}}\nu\\
&=\mathrm{e}^{-\gamma\check{t}}W(\check{x})-\mathrm{e}^{-\gamma\check{t}}V(\check{y})-\varepsilon\mathrm{e}^{-\gamma\check{t}}(|\check{x}(0)|^2+|\check{x}|^2+|\check{y}(0)|^2+|\check{y}|^2)+\mathrm{e}^{-\gamma\check{t}}V(\check{y})\\
&\quad-\mathrm{e}^{-\gamma\check{s}}V(\check{y})-\varepsilon\mathrm{e}^{-\gamma\check{t}}(|\check{s}-\hat{s}|^2+|\check{t}-\hat{t}|^2)-\mathrm{e}^{-\gamma\check{t}}d_\alpha(\check{x},\check{y})+\mathrm{e}^{-\gamma\check{t}}\frac{\alpha|\check{x}(0)-\check{y}(0)|^2}{2}-\mathrm{e}^{-\gamma\check{t}}\nu\\
&\leq W_1(\hat{t},\hat{x},\hat{x}(0))-V_1(\hat{t},\hat{y},\hat{y}(0))-\mathrm{e}^{-\gamma\hat{t}}d_\alpha(\hat{x},\hat{y})+\mathrm{e}^{-\gamma\check{t}}\frac{\alpha|\check{x}(0)-\check{y}(0)|^2}{2}\\
&\quad+\mathrm{e}^{-\gamma\check{t}}V(\check{y})-\mathrm{e}^{-\gamma\check{s}}V(\check{y})-\varepsilon\mathrm{e}^{-\gamma\check{t}}(|\check{s}-\hat{s}|^2+|\check{t}-\hat{t}|^2)-\mathrm{e}^{-\gamma\check{t}}\nu.
\end{aligned}$$

Letting $\delta\to 0$ and $k\to\infty$, we obtain

$$\begin{aligned}
&\overline{\lim}_{k\to\infty}\overline{\lim}_{\delta\to 0}\Big[W_1(\check{t},\check{x},\check{x}(0))-V_1(\check{s},\check{y},\check{x}(0))-\mathrm{e}^{-\gamma\check{t}}\hat{p}_\alpha(\check{x})-\mathrm{e}^{-\gamma\check{s}}\hat{p}_\alpha(\check{y})\Big]\\
&\leq W_1(\hat{t},\hat{x},\hat{x}(0))-V_1(\hat{t},\hat{y},\hat{y}(0))-\mathrm{e}^{-\gamma\hat{t}}d_\alpha(\hat{x},\hat{y})+\mathrm{e}^{-\gamma\hat{t}}\frac{\alpha|\hat{x}(0)-\hat{y}(0)|^2}{2}-\mathrm{e}^{-\gamma\hat{t}}\nu.
\end{aligned}$$

This is in contradiction to (5.14). Then we get (5.15) holds true. Therefore, from the definition of viscosity solution it follows that

$$\begin{aligned}
&-\gamma\mathrm{e}^{-\gamma\check{t}}W(\check{x})-\gamma\mathrm{e}^{-\gamma\check{t}}[\varepsilon(|\check{x}(0)|^2+|\check{x}|^2)+\varepsilon|\check{t}-\hat{t}|^2+\varepsilon|\check{x}-\hat{x}|_B^4+\varepsilon(|\check{x}|^2-|\hat{x}|^2)^2+\hat{p}_\alpha(\check{x})]\\
&+2\mathrm{e}^{-\gamma\check{t}}\varepsilon(\check{t}-\hat{t})+(\varphi_k)_t(\check{t},\check{x}(0))+\mathrm{e}^{-\gamma\check{t}}\varepsilon[4(|B(\check{x}-\hat{x})|_H^2B(\check{x}-\hat{x}),\check{x}(0)1_{[-\tau,0]}-\check{x})_H\\
&+(2|\check{x}|^2-2|\hat{x}|^2+1)(\check{x}^2(0)-\check{x}^2(-\tau))]+8\mathrm{e}^{-\gamma\check{t}}\alpha^2(|B(\check{x}-\check{z})|_H^2B(\check{x}-\check{z}),\check{x}(0)1_{[-\tau,0]}-\check{x})_H\\
&+2\mathrm{e}^{-\gamma\check{t}}\alpha\int_{-\tau}^0(F_1(\check{x})-\frac{1}{2}(F_1(\hat{x})+F_1(\hat{y})))(\theta)\mathrm{d}\theta(F_1(\check{x}(0))-F_1(\check{x}(-\tau)))\\
&+8\mathrm{e}^{-\gamma\check{t}}\alpha^2(|B(F_1(\check{x})-\frac{1}{2}(F_1(\hat{x})+F_1(\hat{y})))|_H^2B(F_1(\check{x})-\frac{1}{2}(F_1(\hat{x})+F_1(\hat{y}))),F_1(\check{x}(0))1_{[-\tau,0]}\\
&-F_1(\check{x}))_H+\mathrm{e}^{-\gamma\check{t}}\mathbf{H}(\check{x},\mathrm{e}^{\gamma\check{t}}\nabla_x\varphi_k(\check{t},\check{x}(0))+2\varepsilon\check{x}(0),\mathrm{e}^{\gamma\check{t}}\nabla_x^2\varphi_k(\check{t},\check{x}(0))+2\varepsilon I)\geq 0,
\end{aligned}\tag{5.16}$$

and

$$
\begin{aligned}
&-\gamma e^{-\gamma\check{s}}V(\check{y})+\gamma e^{-\gamma\check{s}}[\varepsilon(|\check{y}(0)|^2+|\check{y}|^2)+\varepsilon|\check{s}-\hat{t}|^2+\varepsilon|\check{y}-\hat{y}|_B^4+\varepsilon(|\check{y}|^2-|\hat{y}|^2)^2+\hat{p}_\alpha(\check{y})]\\
&-2e^{-\gamma\check{s}}\varepsilon(\check{s}-\hat{t})-(\psi_k)_t(\check{s},\check{y}(0))-e^{-\gamma\check{s}}\varepsilon[4(|B(\check{y}-\hat{y})|_H^2B(\check{y}-\hat{y}),\check{y}(0)1_{[-\tau,0]}-\check{y})_H\\
&-(2|\check{y}|^2-2|\hat{y}|^2+1)(\check{y}^2(0)-\check{y}^2(-\tau))]-8e^{-\gamma\check{s}}\alpha^2(|B(\check{y}-\hat{z})|_H^2B(\check{y}-\hat{z}),\check{y}(0)1_{[-\tau,0]}-\check{y})_H\\
&-2e^{-\gamma\check{t}}\alpha\int_{-\tau}^0(F_1(\check{y})-\frac{1}{2}(F_1(\hat{x})+F_1(\hat{y})))(\theta)d\theta(F_1(\check{y}(0))-F_1(\check{y}(-\tau)))\\
&-8e^{-\gamma\check{t}}\alpha^2(|B(F_1(\check{y})-\frac{1}{2}(F_1(\hat{x})+F_1(\hat{y})))|_H^2B(F_1(\check{y})-\frac{1}{2}(F_1(\hat{x})+F_1(\hat{y}))),F_1(\check{y}(0))1_{[-\tau,0]}\\
&-F_1(\check{y}))_H+e^{-\gamma\check{s}}\mathbf{H}(\check{y},-e^{\gamma\check{s}}\nabla_x\psi_k(\check{s},\check{y}(0))-2\varepsilon\check{y}(0),-e^{\gamma\check{s}}\nabla_x^2\psi_k(\check{s},\check{y}(0))-2\varepsilon I)\le 0,
\end{aligned}
\tag{5.17}
$$

Letting $\delta\to 0$ and $k\to\infty$, by (5.16) and (5.17) we obtain

$$
\begin{aligned}
&-\gamma W(\hat{x})-\gamma\varepsilon(|\hat{x}(0)|^2+|\hat{x}|^2)-\gamma\hat{p}_\alpha(\hat{x})+e^{\gamma\hat{t}}b_1+\varepsilon[\hat{x}^2(0)-\hat{x}^2(-\tau)]\\
&+8\alpha^2(|B(\hat{x}-\hat{z})|_H^2B(\hat{x}-\hat{z}),\hat{x}(0)1_{[-\tau,0]}-\hat{x})_H+2\alpha\int_{-\tau}^0(F_1(\hat{x})-\frac{1}{2}(F_1(\hat{x})+F_1(\hat{y})))(\theta)d\theta\\
&\times(F_1(\hat{x}(0))-F_1(\hat{x}(-\tau)))+8\alpha^2(|B(F_1(\hat{x})-\frac{1}{2}(F_1(\hat{x})+F_1(\hat{y})))|_H^2B(F_1(\hat{x})-\frac{1}{2}(F_1(\hat{x})+F_1(\hat{y}))),\\
&F_1(\hat{x}(0))1_{[-\tau,0]}-F_1(\hat{x}))_H+\mathbf{H}(\hat{x},\alpha(\hat{x}(0)-\hat{y}(0))+2\varepsilon\hat{x}(0),X+2\varepsilon I)\ge 0,
\end{aligned}
\tag{5.18}
$$

and

$$
\begin{aligned}
&-\gamma V(\hat{y})+\gamma\varepsilon(|\hat{y}(0)|^2+|\hat{y}|^2)+\gamma\hat{p}_\alpha(\hat{y})-e^{\gamma\hat{t}}b_2-\varepsilon[\hat{y}^2(0)-\hat{y}^2(-\tau)]\\
&-8\alpha^2(|B(\hat{y}-\hat{z})|_H^2B(\hat{y}-\hat{z}),\hat{y}(0)1_{[-\tau,0]}-\hat{y})_H-2\alpha\int_{-\tau}^0(F_1(\hat{y})-\frac{1}{2}(F_1(\hat{x})+F_1(\hat{y})))(\theta)d\theta\\
&\times(F_1(\hat{y}(0))-F_1(\hat{y}(-\tau)))-8\alpha^2(|B(F_1(\hat{y})-\frac{1}{2}(F_1(\hat{x})+F_1(\hat{y})))|_H^2B(F_1(\hat{y})-\frac{1}{2}(F_1(\hat{x})+F_1(\hat{y}))),\\
&F_1(\hat{y}(0))1_{[-\tau,0]}-F_1(\hat{y}))_H+\mathbf{H}(\hat{y},\alpha(\hat{x}(0)-\hat{y}(0))-2\varepsilon\hat{y}(0),-Y-2\varepsilon I)\le 0.
\end{aligned}
\tag{5.19}
$$

Combining (5.18) and (5.19), we obtain

$$
\begin{aligned}
&\gamma[W(\hat{x})-V(\hat{y})]+\gamma\varepsilon(|\hat{x}(0)|^2+|\hat{x}|^2+|\hat{y}(0)|^2+|\hat{y}|^2)+\gamma(\hat{p}_\alpha(\hat{x})+\hat{p}_\alpha(\hat{y}))\\
\le&-\gamma\frac{\alpha}{2}|\hat{x}(0)-\hat{y}(0)|^2+\varepsilon[\hat{x}^2(0)-\hat{x}^2(-\tau)+\hat{y}^2(0)-\hat{y}^2(-\tau)]\\
&+\alpha^2(|B(\hat{x}-\hat{y})|_H^2B(\hat{x}-\hat{y}),\hat{x}(0)1_{[-\tau,0]}-\hat{x}-\hat{y}(0)1_{[-\tau,0]}+\hat{y})_H\\
&+\alpha\int_{-\tau}^0(F_1(\hat{x})-F_1(\hat{y}))(\theta)d\theta(F_1(\hat{x}(0))-F_1(\hat{x}(-\tau))-F_1(\hat{y}(0))+F_1(\hat{y}(-\tau)))\\
&+\alpha^2(|B(F_1(\hat{x})-F_1(\hat{y}))|_H^2B(F_1(\hat{x})-F_1(\hat{y})),F_1(\hat{x}(0))1_{[-\tau,0]}-F_1(\hat{x})\\
&-F_1(\hat{y}(0))1_{[-\tau,0]}+F_1(\hat{y}))_H+\mathbf{H}(\hat{x},\alpha(\hat{x}(0)-\hat{y}(0))+2\varepsilon\hat{x}(0),X+2\varepsilon I)\\
&-\mathbf{H}(\hat{y},\alpha(\hat{x}(0)-\hat{y}(0))-2\varepsilon\hat{y}(0),-Y-2\varepsilon I).
\end{aligned}
\tag{5.20}
$$

On the other hand, by a simple calculation we obtain

$$\mathbf{H}(\hat{x},\alpha(\hat{x}(0)-\hat{y}(0))+2\varepsilon\hat{x}(0),X+2\varepsilon I)-\mathbf{H}(\hat{y},\alpha(\hat{x}(0)-\hat{y}(0))-2\varepsilon\hat{y}(0),-Y-2\varepsilon I)\le\sup_{u\in U}(J_1+J_2+J_3),\quad(5.21)$$

where

$$\begin{aligned}J_1&=(F(\hat{x}(0),(a,\hat{x})_H,u)+F_1(\hat{x}(-\tau)),\alpha(\hat{x}(0)-\hat{y}(0))+2\varepsilon\hat{x}(0))_{R^d}\\&\quad-(F(\hat{y}(0),(a,\hat{y})_H,u)+F_1(\hat{y}(-\tau)),\alpha(\hat{x}(0)-\hat{y}(0))-2\varepsilon\hat{y}(0))_{R^d}\\&\le\alpha L(|\hat{x}(0)-\hat{y}(0)|^2+|\hat{x}(0)-\hat{y}(0)|[\tau^{\frac14}|a|_{W^{1,2}}|\hat{x}-\hat{y}|_B+|\hat{x}(-\tau)-\hat{y}(-\tau)|])\qquad(5.22)\\&\quad+2\varepsilon L|\hat{x}(0)|(2+|\hat{x}(-\tau)|+|\hat{x}(0)|+|a|_H|\hat{x}|)\\&\quad+2\varepsilon L|\hat{y}(0)|(2+|\hat{y}(-\tau)|+|\hat{y}(0)|+|a|_H|\hat{y}|);\end{aligned}$$

$$\begin{aligned}J_2&=\frac12\mathrm{tr}[(X+2\varepsilon I)G(\hat{x}(0),(c,\hat{x})_H,u)G^\top(\hat{x}(0),(c,\hat{x})_H,u)]\\&\quad-\frac12\mathrm{tr}[(-Y-2\varepsilon I)G(\hat{y}(0),(c,\hat{y})_H,u)G^\top(\hat{y}(0),(c,\hat{y})_H,u)]\\&\le\alpha|G(\hat{x}(0),(c,\hat{x})_H,u)-G(\hat{y}(0),(c,\hat{y})_H,u)|^2\\&\quad+\varepsilon|G(\hat{x}(0),(c,\hat{x})_H,u)G^\top(\hat{x}(0),(c,\hat{x})_H,u)|\\&\quad+\varepsilon|G(\hat{y}(0),(c,\hat{y})_H,u)G^\top(\hat{y}(0),(c,\hat{y})_H,u)|\\&\le\alpha L^2(|\hat{x}(0)-\hat{y}(0)|^2+\tau^{\frac12}|c|^2_{W^{1,2}}|\hat{x}-\hat{y}|^2_B)\\&\quad+3\varepsilon L^2(2+|\hat{x}(0)|^2+|\hat{y}(0)|^2+|c|^2_H|\hat{x}|^2+|c|^2_H|\hat{y}|^2);\qquad(5.23)\end{aligned}$$

and

$$J_3=q(\hat{x}(0),u)-q(\hat{y}(0),u)\le L|\hat{x}(0)-\hat{y}(0)|.\qquad(5.24)$$

Combining (5.20)–(5.24), we obtain

$$\begin{aligned}&\gamma[W(\hat{x})-V(\hat{y})]+\gamma\varepsilon(|\hat{x}(0)|^2+|\hat{x}|^2+|\hat{y}(0)|^2+|\hat{y}|^2)+\gamma(\hat{p}_\alpha(\hat{x})+\hat{p}_\alpha(\hat{y}))\\&\le-\gamma\frac{\alpha}{2}|\hat{x}(0)-\hat{y}(0)|^2+\varepsilon[\hat{x}^2(0)-\hat{x}^2(-\tau)+\hat{y}^2(0)-\hat{y}^2(-\tau)]\\&\quad+\frac{\alpha}{2}|\hat{x}(0)-\hat{y}(0)|^2+\frac{\alpha^3}{2}\tau^{\frac52}|\hat{x}-\hat{y}|^6_B+\frac{\alpha L^2}{2}|\hat{x}(0)-\hat{y}(0)|^2+\frac{\alpha^3}{2}\tau^{\frac52}|F_1(\hat{x})-F_1(\hat{y})|^6_B\\&\quad+\alpha\bigg(\int_{-\tau}^0F_1(\hat{x}(\theta))-F_1(\hat{y}(\theta))\mathrm{d}\theta\bigg)^2+\frac{\alpha L^2}{2}(|\hat{x}(0)-\hat{y}(0)|^2+|\hat{x}(-\tau)-\hat{y}(-\tau)|^2)\\&\quad+2\alpha L|\hat{x}(0)-\hat{y}(0)|^2+\alpha L[\tau^{\frac12}|a|^2_{W^{1,2}}|\hat{x}-\hat{y}|^2_B+|\hat{x}(-\tau)-\hat{y}(-\tau)|^2]\\&\quad+(5\varepsilon L+3\varepsilon L^2)(|\hat{x}(0)|^2+|\hat{y}(0)|^2)+\varepsilon L(|\hat{x}(-\tau)|^2+|\hat{y}(-\tau)|^2)\\&\quad+(\varepsilon L|a|^2_H+3\varepsilon L^2|c|^2_H)(|\hat{x}|^2+|\hat{y}|^2)+\alpha L^2(|\hat{x}(0)-\hat{y}(0)|^2+\tau^{\frac12}|c|^2_{W^{1,2}}|\hat{x}-\hat{y}|^2_B)\\&\quad+L|\hat{x}(0)-\hat{y}(0)|+4\varepsilon L+6\varepsilon L^2.\qquad(5.25)\end{aligned}$$

Recalling $\hat{x}, \hat{y} \in \mathcal{D}_M$ and $\Psi^\gamma(\hat{t}, \hat{x}, \hat{y}) \geq \Psi^\gamma(0, \tilde{x}, \tilde{x})$, and combining it with $\varepsilon L \sup_{\theta\in[-\tau,0]} |\tilde{x}(\theta)|^2 + 2\varepsilon L + 3\varepsilon L^2 < \frac{\lambda\tilde{m}}{16}$ and $\lambda > (5L + 3L^2 + 1) \vee (L|a|_H^2 + 3L^2|c|_H^2)$, it follows that

$$
\begin{aligned}
&\frac{\gamma}{2}\tilde{m} < \gamma[W(\tilde{x}) - V(\tilde{x}) - 2\varepsilon(|\tilde{x}|_H^2 + |\tilde{x}(0)|^2)] \leq \gamma\Psi^\gamma(\hat{t}, \hat{x}, \hat{y}) \\
&\leq \gamma[W(\hat{x}) - V(\hat{y}) - d_\alpha(\hat{x}, \hat{y}) - \varepsilon(|\hat{x}|_H^2 + |\hat{x}(0)|^2 + |\hat{y}|_H^2 + |\hat{y}(0)|^2)] \\
&\leq (2 + 8L^2 + 8L - \lambda)\frac{\alpha}{4}|\hat{x}(0) - \hat{y}(0)|^2 + \frac{\alpha^3}{2}\tau^{\frac{5}{2}}(|\hat{x} - \hat{y}|_B^6 + |F_1(\hat{x}) - F_1(\hat{y})|_B^6) \\
&\quad + \alpha\left(\int_{-\tau}^0 F_1(\hat{x}(\theta)) - F_1(\hat{y}(\theta))\mathrm{d}\theta\right)^2 + (L^2 + 2L)\frac{\alpha}{2}|\hat{x}(-\tau) - \hat{y}(-\tau)|^2 \\
&\quad + (L\tau^{\frac{1}{2}}|a|_{W^{1,2}}^2 + L^2\tau^{\frac{1}{2}}|c|_{W^{1,2}}^2)\alpha|\hat{x} - \hat{y}|_B^2 + L|\hat{x}(0) - \hat{y}(0)| \\
&\quad + (5\varepsilon L + 3\varepsilon L^2 + \varepsilon)(|\hat{x}(0)|^2 + |\hat{y}(0)|^2) + \varepsilon L(|\hat{x}(-\tau)|^2 + |\hat{y}(-\tau)|^2) \\
&\quad + (\varepsilon L|a|_H^2 + 3\varepsilon L^2|c|_H^2)(|\hat{x}|^2 + |\hat{y}|^2) \\
&\quad + 4\varepsilon L + 6\varepsilon L^2 - \lambda\varepsilon(|\hat{x}|_H^2 + |\hat{x}(0)|^2 + |\hat{y}|_H^2 + |\hat{y}(0)|^2) \\
&\leq (2 + 8L^2 + 8L - \lambda)\frac{\alpha}{4}|\hat{x}(0) - \hat{y}(0)|^2 + \frac{\alpha^3}{2}\tau^{\frac{5}{2}}(|\hat{x} - \hat{y}|_B^6 + |F_1(\hat{x}) - F_1(\hat{y})|_B^6) \\
&\quad + \alpha\left(\int_{-\tau}^0 F_1(\hat{x}(\theta)) - F_1(\hat{y}(\theta))\mathrm{d}\theta\right)^2 + (L^2 + 2L)\frac{\alpha}{2}|\hat{x}(-\tau) - \hat{y}(-\tau)|^2 \\
&\quad + (L\tau^{\frac{1}{2}}|a|_{W^{1,2}}^2 + L^2\tau^{\frac{1}{2}}|c|_{W^{1,2}}^2)\alpha|\hat{x} - \hat{y}|_B^2 + L|\hat{x}(0) - \hat{y}(0)| + \frac{\lambda\tilde{m}}{8}\cdot
\end{aligned}
$$

Letting $\alpha \to +\infty$, the following contradiction is induced:

$$\frac{\lambda\tilde{m}}{4} \leq \frac{\lambda\tilde{m}}{8}\cdot$$

The proof is now complete. □

The following Lemma is needed in the proof of Theorem 5.1.

Lemma 5.3. *Let $u_i \in C((0,T) \times R^d)$, $i = 1, 2$, $\varphi : (0,T) \times R^{2d} \to R$ be once continuously differentiable in t and twice continuously differentiable in x. Suppose that*

$$u_1(t, x_1) + u_2(t, x_2) - \varphi(t, x_1, x_2)$$

has a maximum at $(\hat{t}, \hat{x}) = (\hat{t}, \hat{x}_1, \hat{x}_2) \in (0,T) \times R^{2d}$. Assume, moreover, that there is an $r > 0$ such that for every $M > 0$ there is an C such that for $i = 1, 2$

$$
\begin{aligned}
&b_i \geq C \text{ whenever } (b_i, q_i, X_i) \in \mathcal{P}^{2,+}u_i(t, x_i), \\
&|x_i - \hat{x}_i| + |t - \hat{t}| \leq r \text{ and } |u_i(t, x_i)| + |q_i| + ||X_i|| \leq M.
\end{aligned} \tag{5.26}
$$

Then for each $\varepsilon > 0$ there are $X_i \in \Gamma(R^d)$ such that

$$
\begin{cases}
(i)\ (b_i, \nabla_{x_i}\varphi(\hat{x}), X_i) \in \bar{\mathcal{P}}^{2,+}u_i(\hat{t}, \hat{x}_i) \text{ for } i = 1, 2, \\
(ii)\ -\left(\dfrac{1}{\varepsilon} + ||D||\right) I \leq \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix} \leq D + \varepsilon D^2, \\
(iii)\ b_1 + b_2 = \varphi_t(\hat{t}, \hat{x}_1, \hat{x}_2),
\end{cases} \tag{5.27}
$$

where $D = \nabla_x^2\varphi(\hat{t}, \hat{x}_1, \hat{x}_2)$. (For the definitions of the parabolic "superjet" $\mathcal{P}^{2,+}u$ and its closure $\bar{\mathcal{P}}^{2,+}u$ see [8]).

Proof. Let $v_i(t,x_i)=u_i(T-t,x_i)$, $\psi(t,x_1,x_2)=\varphi(T-t,x_1,x_2)$, $(t,x_1,x_2)\in(0,T)\times R^d\times R^d$, $i=1,2$, then

$$v_1(t,x_1)+v_2(t,x_2)-\psi(t,x_1,x_2)$$

has a maximum at $(T-\hat{t},\hat{x})=(T-\hat{t},\hat{x}_1,\hat{x}_2)\in(0,T)\times R^{2d}$.

Moreover, for every $M>0$, let

$$(-b_i,q_i,X_i)\in\mathcal{P}^{2,+}v_i(t,x_i),\quad |x_i-\hat{x}_i|+|t-T+\hat{t}|\le r \text{ and } |v_i(t,x_i)|+|q_i|+||X_i||\le M.$$

Then

$$(b_i,q_i,X_i)\in\mathcal{P}^{2,+}u_i(T-t,x_i),\quad |x_i-\hat{x}_i|+|t-T+\hat{t}|\le r \text{ and } |u_i(T-t,x_i)|+|q_i|+||X_i||\le M.$$

Therefore, by (5.26), there is an C such that $-b_i\le C$, $i=1,2$. Thus, v_i, $i=1,2$, and ψ satisfy the condition of Theorem 8.3 in [8], then we obtain that for each $\varepsilon>0$ there are $X_i\in\Gamma(R^d)$ such that

$$\begin{cases}(i)\ (-b_i,\nabla_{x_i}\varphi(\hat{x}),X_i)\in\bar{\mathcal{P}}^{2,+}v_i(T-\hat{t},\hat{x}_i) \text{ for } i=1,2,\\ (ii)\ \ -(\frac{1}{\varepsilon}+||D||)I\le\begin{pmatrix}X_1 & 0\\ 0 & X_2\end{pmatrix}\le D+\varepsilon D^2,\\ (iii)\ \ -b_1-b_2=\psi_t(T-\hat{t},\hat{x}_1,\hat{x}_2).\end{cases}\tag{5.28}$$

Thus, we obtain (5.27) holds true. □

6. Deterministic cases

In this section, we study the deterministic delay optimal control problems and the associated first order HJB equation.

We consider the controlled state equation

$$X^u(t)=x_0+\int_0^t F(X^u(\sigma),(a,X^u_\sigma)_H,u(\sigma))\mathrm{d}\sigma+\int_0^t F_1(X^u(\sigma-\tau))\mathrm{d}\sigma,\quad t\in[0,+\infty),\tag{6.1}$$

where $X^u_0=x\in H$, and a cost function of the form

$$J(x,x_0,u)=\int_0^{+\infty}\mathrm{e}^{-\lambda\sigma}q(X^u(\sigma,x,x_0),u(\sigma))\mathrm{d}\sigma,\tag{6.2}$$

where

$$u(\cdot)\in\mathcal{U}[0,+\infty):=\{u(\cdot):[0,+\infty)\to U|\ u(\cdot)\text{ is measurable}\},$$

with U representing a compact subset U of R^{d_1}. Our aim is to minimize the function J over all controls $u\in\mathcal{U}[0,+\infty)$. We define the function $V:H\times R^d\to R$ by

$$V(x,x_0):=\inf_{u\in\mathcal{U}[0,+\infty)}J(x,x_0,u).\tag{6.3}$$

The function V is called the *value function* of the optimal control problems (6.1) and (6.2).

Theorem 6.1. *Assume that Hypothesis 2.1 holds. If we let $\Lambda = 2L(1+\tau|a|_{W^{1,2}})$, then a unique continuous function $X:[0,+\infty)\to R^d$ exists that is a solution to (6.1). Moreover,*

$$\begin{aligned}|X^u(t,x,x_0)| &\le C_1\bigg(1+|x_0|+\bigg|\int_{-\tau}^{0}F_1(x(\sigma))\mathrm{d}\sigma\bigg|+\bigg|\int_{-\tau}^{(t-\tau)\wedge 0}F_1(x(\sigma))\mathrm{d}\sigma\bigg|+|Bx|_H+|BF_1(x)|_H\bigg)\mathrm{e}^{\Lambda t}\\ &\le C_2(1+|x_0|+|x|_H)\mathrm{e}^{\Lambda t}, \qquad\qquad t\ge 0,\end{aligned} \tag{6.4}$$

for some constants $C_1, C_2>0$ depending only on L, τ and $a(\cdot)$, and

$$\begin{aligned}&|X^u(t,x,x_0)-X^u(t,y,y_0)|\\ &\le C_3\bigg(|x_0-y_0|+\bigg|\int_{-\tau}^{0}F_1(x(\sigma))-F_1(y(\sigma))\mathrm{d}\sigma\bigg|\\ &\quad+\bigg|\int_{-\tau}^{(t-\tau)\wedge 0}F_1(x(\sigma))-F_1(y(\sigma))\mathrm{d}\sigma\bigg|+|B(x-y)|_H+|B(F_1(x)-F_1(y))|_H\bigg)\mathrm{e}^{\Lambda t}\\ &\le C_4(|x_0-y_0|+|x-y|_H)\mathrm{e}^{\Lambda t},\ \ t\ge 0,\end{aligned} \tag{6.5}$$

for some constants $C_3, C_4>0$ depending only on L, τ and $a(\cdot)$.

Proof. Existence and uniqueness are satisfied by Theorem 2.2. We only need to prove that (6.4) and (6.5) hold true.

By the definition of X^u, we know that

$$\begin{aligned}|X^u(t,x,x_0)| &\le |x_0|+\int_0^t |F(X^u(\sigma,x,x_0),(a,X^u_\sigma(x,x_0))_H,u(\sigma))|\mathrm{d}\sigma+\bigg|\int_0^t F_1(X^u(\sigma-\tau,x,x_0))\mathrm{d}\sigma\bigg|\\ &\le |x_0|+\bigg|\int_{-\tau}^{(t-\tau)\wedge 0}F_1(x(\sigma))\mathrm{d}\sigma\bigg|+\tau L|a|_{W^{1,2}}|Bx|_H+Lt\\ &\quad+2L(1+\tau|a|_{W^{1,2}})\int_0^t|X^u(\sigma,x,x_0)|\mathrm{d}\sigma.\end{aligned}$$

The Gronwall Lemma implies that

$$\begin{aligned}|X^u(t,x,x_0)| \le &\bigg(|x_0|+\bigg|\int_{-\tau}^{0}F_1(x(\sigma))\mathrm{d}\sigma\bigg|+\tau L|a|_{W^{1,2}}|Bx|_H+\frac{L}{\Lambda}\bigg)\mathrm{e}^{\Lambda t}\\ &+\bigg|\int_{-\tau}^{(t-\tau)\wedge 0}F_1(x(\sigma))\mathrm{d}\sigma\bigg|+\Lambda^{\frac12}|BF_1(x)|_H\mathrm{e}^{\Lambda\tau},\end{aligned}$$

where $\Lambda = 2L(1+\tau|a|_{W^{1,2}})$. Therefore, there exist some constants $C_1, C_2>0$ depending only on L, τ and $a(\cdot)$ such that (6.4) holds true. Now, let us show that (6.5) holds.

$$\begin{aligned}|X^u(t,x,x_0)-X^u(t,y,y_0)| \le\ &|x_0-y_0|+\bigg|\int_{-\tau}^{(t-\tau)\wedge 0}F_1(x(\sigma))-F_2(y(\sigma))\mathrm{d}\sigma\bigg|+\tau L|a|_{W^{1,2}}|B(x-y)|_H\\ &+2L(1+\tau|a|_{W^{1,2}})\bigg(\int_0^t|X^u(\sigma,x,x_0)-X^u(\sigma,y,y_0)|\mathrm{d}\sigma\bigg).\end{aligned}$$

The Gronwall Lemma implies that

$$\begin{aligned}|X^u(t,x,x_0)-X^u(t,y,y_0)| \le\ &\bigg(|x_0-y_0|+\bigg|\int_{-\tau}^{0}F_1(x(\sigma))-F_1(y(\sigma))\mathrm{d}\sigma\bigg|+\tau L|a|_{W^{1,2}}|B(x-y)|_H\bigg)\mathrm{e}^{\Lambda t}\\ &+\bigg|\int_{-\tau}^{(t-\tau)\wedge 0}F_1(x(\sigma))-F_1(y(\sigma))\mathrm{d}\sigma\bigg|+\Lambda^{\frac12}|B(F_1(x)-F_1(y))|_H\mathrm{e}^{\Lambda\tau},\end{aligned}$$

where $\Lambda = 2L(1 + \tau|a|_{W^{1,2}})$. Therefore, there exist some constants $C_3, C_4 > 0$ depending only on L, τ and $a(\cdot)$ such that (6.5) holds. The proof is now complete. $\square$

By the similar (even simpler) procedure in Section 3, we can show that Theorems 3.2 and 3.3 also hold true for the value function V defined in (6.3) when $\Lambda = 2L(1 + \tau|a|_{W^{1,2}})$, and Lemma 3.9 holds for $g \in C^1(H \times R^d)$ and $G = 0$.

Now we consider the following first order HJB equation

$$-\lambda V(x) + \mathcal{S}(V)(x) + H(x, \nabla_{x_0} V(x)) = 0, \quad x \in D(\mathcal{S}(V)), \tag{6.6}$$

where

$$H(x, p) = \inf_{u \in U} [(p, F(x, u) + F_1(x(-\tau)))_{R^d} + q(x(0), u)], \ (x, p) \in \mathcal{D} \times R^d.$$

Here, and throughout this article, for simplicity, we let $V(x)$ and $F(x, u)$ denote $V(x, x(0))$ and $F(x(0), (a, x)_H, u)$, respectively, if $x \in \mathcal{D}$.

Theorem 6.2. *Let V denote the value function defined by (6.3). If $V \in C^1(H \times R^d)$, then V satisfies the HJB equation (6.6).*

Proof. The proof is very similar to Theorem 3.8. Here, we omit it. $\square$

We now give the following definition for the viscosity solution.

Definition 6.3. $w \in C(H \times R^d)$ is called a viscosity subsolution (supersolution) of (6.6) if for every $\varphi \in C^1([0, +\infty) \times (H \times R^d))$, whenever the constants $\gamma, \tilde{\lambda}, \lambda > 0$ and the function $w^\gamma - \varphi$ (resp. $w^\gamma + \varphi$) satisfy $\tilde{\lambda} < \lambda$, $\lambda - \tilde{\lambda} = \gamma$, and

$$(w^\gamma - \varphi)(s, z) = \sup_{(t,x) \in [s,+\infty) \times \mathcal{D}} (w^\gamma - \varphi)(t, z \otimes_{[0,(t-s)]} x),$$

$$(\text{respectively}, \quad (w^\gamma + \varphi)(s, z) = \inf_{(t,x) \in [s,+\infty) \times \mathcal{D}} (w^\gamma + \varphi)(t, z \otimes_{[0,(t-s)]} x),)$$

where $w^\gamma(s, z) = \mathrm{e}^{-\gamma s} w(z)$, $(s, z) \in [0, +\infty) \times \mathcal{D}$ and $z \in D(\mathcal{S}(\varphi(s, \cdot)))$, we have

$$-\tilde{\lambda} w(z) + \mathrm{e}^{\gamma s} \varphi_s(s, z) + \mathrm{e}^{\gamma s} \mathcal{S}(\varphi(s, \cdot))(z) + \mathbf{H}(z, \mathrm{e}^{\gamma s} \nabla_{x_0} \varphi(s, z)) \geq 0,$$

$$(\text{respectively}, -\tilde{\lambda} w(z) - \mathrm{e}^{\gamma s} \varphi_s(s, z) - \mathrm{e}^{\gamma s} \mathcal{S}(\varphi(s, \cdot))(z) + \mathbf{H}(z, -\mathrm{e}^{\gamma s} \nabla_{x_0} \varphi(s, z)) \leq 0).$$

$w \in C(H \times R^d)$ is said to be a viscosity solution to (6.6) if it is both a viscosity subsolution and a viscosity supersolution.

By the similar (even simpler) procedure of Theorem 4.4, we have the existence result for the viscosity solution.

Theorem 6.4. *Suppose that Hypotheses 2.1 and 3.1 hold. Then, for $\lambda > \Lambda$, the value function V defined by (6.3) is a viscosity solution to (6.6).*

We now state the comparison principle for the viscosity solution.

Theorem 6.5. *Suppose that Hypotheses 2.1 and 3.1 hold, and assume $\lambda > 2L(1+\tau|a|_{W^{1,2}}) \vee (5L+1) \vee L|a|_H^2$. Let W (resp. V) be a viscosity subsolution (resp. supsolution) to (6.6). In addition, let there exist a constant $\Delta > 0$ such that for $(x, x_0), (y, y_0) \in H \times R^d$,*

$$|W(x, x_0)| \vee |V(x, x_0)| \leq \Delta(1 + |x_0| + |x|_H), \tag{6.7}$$

$$|W(x,x_0)-W(y,y_0)|\vee|V(x,x_0)-V(y,y_0)|$$
$$\leq \Delta\Big(|x_0-y_0|+\Big|\int_{-\tau}^0 F_1(x(\sigma))-F_1(y(\sigma))\mathrm{d}\sigma\Big|+|B(x-y)|_H+|B(F_1(x)-F_1(y))|_H\Big). \tag{6.8}$$

Then, $W\leq V$*.*

From this theorem, the viscosity solution to the HJB equation (6.6) can characterize the value function V of our optimal control problems (6.1) and (6.2) as follows:

Theorem 6.6. *Let Hypotheses 2.1 and 3.1 hold, and assume* $\lambda>2L(1+\tau|a|_{W^{1,2}})\vee(5L+1)\vee L|a|_H^2$*. Then, the value function* V *defined by* (6.3) *is a unique viscosity solution to* (6.6) *in the class of functions satisfying* (3.4) *and* (3.5)*.*

Proof. According to Theorem 6.4, we know that V is a viscosity solution to (6.6). Thus, our conclusion follows from Theorem 3.2 and Theorem 6.5. □

We are now in a position to prove Theorem 6.5.

Proof of Theorem 6.5. The proof of this theorem is rather long. Thus, we split it into several steps.

Step 1. Definitions of the auxiliary functions and sets.

To prove the theorem, we assume on the contrary that there exists $\varepsilon>0$ a small number such that $\tilde{m}:=\sup_{(x,x_0)\in H\times R^d}[W(x,x_0)-V(x,x_0)-2\varepsilon(|x|_H^2+|x_0|^2)]>0$. Because simple functions are dense in H, according to (6.8) there exist a simple function $\tilde{y}=\sum_{i=1}^m a_i 1_{[t_i,t_{i+1})}$, $t_i\in[-\tau,0]$, $i=1,2,\dots,m+1$ and a constant $\tilde{a}\in R^d$ such that $W(\tilde{x})-V(\tilde{x})-2\varepsilon(|\tilde{x}|_H^2+|\tilde{x}(0)|^2)>(\frac{1}{2}\vee \mathrm{e}^{-\frac{\tau}{8}\lambda})\tilde{m}$, where $\tilde{x}=\tilde{y}+\tilde{a}1_0(\cdot)$. First, we can let $\varepsilon>0$ be small enough such that

$$\varepsilon L\sup_{\theta\in[-\tau,0]}|\tilde{x}(\theta)|^2+2\varepsilon L<\frac{\lambda\tilde{m}}{16}$$

Next, for every $\alpha>0$, we define, for any $(x,y)\in\mathcal{D}\times\mathcal{D}$,

$$\Psi(x,y)=W(x)-V(y)-\frac{\alpha}{2}d(x,y)-\varepsilon(|x|_H^2+|x(0)|^2+|y|_H^2+|y(0)|^2),$$

and

$$\Psi^\gamma(t,x,y)=\mathrm{e}^{-\gamma t}\Psi(x,y),$$

where

$$d(x,y)=|x(0)-y(0)|^2+|x(-\tau)-y(-\tau)|^2+|B(x-y)|_H^2$$
$$+\Big|\int_{-\tau}^0 F_1(x(\sigma))-F_1(y(\sigma))\mathrm{d}\sigma\Big|^2+|B(F_1(x)-F_1(y))|_H^2, \tag{6.9}$$

and

$$\gamma=\frac{\lambda}{3}\cdot$$

Finally, for every $M>0$, we define

$$M_\alpha:=\sup_{t\geq 0;x,y\in\mathcal{D}_M}\Psi^\gamma(t,\tilde{x}\otimes_{[0,t]}x,\tilde{x}\otimes_{[0,t]}y),$$

where

$$M_\alpha\geq M_*:=\sup_{t\geq0}\sup_{x\in\mathcal{D}_M;l\in[0,t]}\Psi^\gamma(t,\tilde{x}\otimes_l x,\tilde{x}\otimes_l x)\geq\frac{\tilde{m}}{2}\cdot$$

Step 2. Properties of $\Psi^\gamma(t,x,y)$.

For every $M,\alpha>0$, from the definition of M_α, we can fix $\bar{t}\geq 0, \bar{k},\bar{l}\in[0,\bar{t}]$ and $\bar{x},\bar{y}\in\mathcal{D}_M$ satisfying

$$\bar{x}=\tilde{x}\otimes_{\bar{k}}\bar{x},\quad \bar{y}=\tilde{x}\otimes_{\bar{l}}\bar{y},\quad \Psi^\gamma(0,\tilde{x},\tilde{x})\leq\Psi^\gamma(\bar{t},\bar{x},\bar{y})\ \text{ and }\ \Psi^\gamma(\bar{t},\bar{x},\bar{y})+\frac{1}{\alpha}>M_\alpha.$$

By the definition of Ψ, we obtain that

$$2\Psi(x,y)=\Psi(x,x)+\Psi(y,y)+V(x)-V(y)+W(x)-W(y)-\alpha d(x,y).$$

Therefore,

$$\begin{aligned}\Psi(x,y)&\leq\frac{1}{2}(|\Psi(x,x)|+|\Psi(y,y)|+|V(x)-V(y)|+|W(x)-W(y)|)-\frac{\alpha}{2}d(x,y)\\&\leq\tilde{m}+4\Delta d^{\frac{1}{2}}(x,y)-\frac{\alpha}{2}d(x,y)\leq\tilde{m}+\frac{8\Delta^2}{\alpha}.\end{aligned}$$

Letting $\alpha\geq 1+\frac{16\Delta^2}{(\mathrm{e}^{-\frac{\tau}{8}\lambda}-\mathrm{e}^{-\frac{\tau}{4}\lambda})\tilde{m}}$, we obtain that

$$\Psi(x,y)\leq\tilde{m}+\frac{1}{2}(\mathrm{e}^{-\frac{\tau}{8}\lambda}-\mathrm{e}^{-\frac{\tau}{4}\lambda})\tilde{m}.$$

As $W(\tilde{x})-V(\tilde{x})-2\varepsilon(|\tilde{x}|_H^2+|\tilde{x}(0)|^2)>(\frac{1}{2}\vee\mathrm{e}^{-\frac{\tau}{8}\lambda})\tilde{m}$, there exists a constant $T\leq\frac{3\tau}{4}$ such that for all $M>0$ and $\alpha>M_{\tilde{m},\Delta}:=\frac{2}{(\mathrm{e}^{-\frac{\tau}{8}\lambda}-\mathrm{e}^{-\frac{\tau}{4}\lambda})\tilde{m}}\vee(1+\frac{16\Delta^2}{(\mathrm{e}^{-\frac{\tau}{8}\lambda}-\mathrm{e}^{-\frac{\tau}{4}\lambda})\tilde{m}})$,

$$\Psi^\gamma(t,x,y)+\frac{1}{\alpha}<M_\alpha,\quad t\geq T,\quad x,y\in\mathcal{D}_M.$$

Now, we can apply Lemma 4.1 to find $\hat{t}\in[0,T),\hat{k},\hat{l}\in[0,(\hat{t}-\bar{t})],\hat{x},\hat{y}\in\mathcal{D}_M$, which satisfies $\hat{x}=\bar{x}\otimes_{\hat{k}}\hat{x},\ \hat{y}=\bar{y}\otimes_{\hat{l}}\hat{y}$ with $\Psi^\gamma(\hat{t},\hat{x},\hat{y})\geq\Psi^\gamma(\bar{t},\bar{x},\bar{y})\geq\Psi^\gamma(0,\tilde{x},\tilde{x})$ such that

$$\Psi^\gamma(\hat{t},\hat{x},\hat{y})\geq\Psi^\gamma(t,\hat{x}\otimes_{[0,(t-\hat{t})]}x,\hat{y}\otimes_{[0,(t-\hat{t})]}y),\ \ t\geq\hat{t},x,y\in\mathcal{D}_M. \tag{6.10}$$

In particular, we know that

$$\Psi^\gamma(\hat{t},\hat{x},\hat{y})\geq\Psi^\gamma(t,\hat{x}\otimes_{[0,(t-\bar{t})]}x,\hat{y}),\ \Psi^\gamma(\hat{t},\hat{x},\hat{y})\geq\Psi^\gamma(t,\hat{x},\hat{y}\otimes_{[0,(t-\bar{t})]}y),t\geq\hat{t},x,y\in\mathcal{D}_M.$$

We should note that $(\hat{t},\hat{x},\hat{y})$ depends on $\bar{t},\bar{k},\bar{l},\bar{x},\bar{y},\alpha,M$.

Step 3. For every $M>0$, we have

$$\frac{\alpha}{2}d(\hat{x},\hat{y})\leq\frac{\mathrm{e}^{\gamma T}}{\alpha}+|W(\hat{x})-W(\hat{y})|+|V(\hat{x})-V(\hat{y})|\to 0\text{ as }\alpha\to+\infty. \tag{6.11}$$

Let us show the above. We can confirm that

$$\begin{aligned}&\frac{\alpha}{2}\mathrm{e}^{-\gamma\hat{t}}d(\hat{x},\hat{y})+\varepsilon\mathrm{e}^{-\gamma\hat{t}}(|\hat{x}|_H^2+|\hat{x}(0)|^2+|\hat{y}|_H^2+|\hat{y}(0)|^2)\\&\leq\frac{1}{\alpha}+\mathrm{e}^{-\gamma\hat{t}}(W(\hat{x})-V(\hat{y}))-M_\alpha\leq\frac{1}{\alpha}+\mathrm{e}^{-\gamma\hat{t}}(W(\hat{x})-V(\hat{y}))-M_*\\&\leq\frac{1}{\alpha}+C-M_*,\end{aligned} \tag{6.12}$$

where $C := 2\Delta(1 + M + \tau^{\frac{1}{2}} M)$. We also know that

$$\begin{aligned}2M_* \leq& \frac{2}{\alpha} + \mathrm{e}^{-\gamma\hat{t}}(W(\hat{x}) - W(\hat{y}) + W(\hat{y}) - V(\hat{y})) + \mathrm{e}^{-\gamma\hat{t}}(W(\hat{x}) - V(\hat{x}) + V(\hat{x}) - V(\hat{y}))\\ &-\alpha \mathrm{e}^{-\gamma\hat{t}} d(\hat{x}, \hat{y}) - 2\varepsilon \mathrm{e}^{-\gamma\hat{t}}(|\hat{x}|_H^2 + |\hat{x}(0)|^2 + |\hat{y}|_H^2 + |\hat{y}(0)|^2)\\ \leq& \frac{2}{\alpha} + \mathrm{e}^{-\gamma\hat{t}}(|W(\hat{x}) - W(\hat{y})| + |V(\hat{x}) - V(\hat{y})|) + 2M_* - 2\alpha \mathrm{e}^{-\gamma\hat{t}} d(\hat{x}, \hat{y}).\end{aligned}$$

Thus,

$$\alpha \mathrm{e}^{-\gamma\hat{t}} d(\hat{x}, \hat{y}) \leq \frac{1}{\alpha} + \mathrm{e}^{-\gamma\hat{t}}(|W(\hat{x}) - W(\hat{y})| + |V(\hat{x}) - V(\hat{y})|).$$

Therefore,

$$\alpha d(\hat{x}, \hat{y}) \leq \frac{\mathrm{e}^{\gamma T}}{\alpha} + |W(\hat{x}) - W(\hat{y})| + |V(\hat{x}) - V(\hat{y})|. \tag{6.13}$$

According to (6.12), we obtain $d(\hat{x}, \hat{y}) \to 0$ as $\alpha \to +\infty$. Then combining (6.8) and (6.13), we see that (6.11) holds.

Step 4. There exists $M > 0$ such that (6.10) holds true for all $(t, x, y) \in [\hat{t}, +\infty) \times \mathcal{D} \times \mathcal{D}$ and $\alpha > N_{\tilde{m},\Delta}$.

We note that there exists an $M > (1 + \tau^{\frac{-1}{2}})(1 + \frac{2\Delta}{\varepsilon})$, independent of α, that is sufficiently large that

$$\Psi^\gamma(0, \tilde{x}, \tilde{x}) = W(\tilde{x}) - V(\tilde{x}) - 2\varepsilon(|\tilde{x}|_H^2 + |\tilde{x}(0)|^2) > 0 > \Psi^\gamma(t, x, y),$$

where $t \in [0, +\infty)$ and $x \in \mathcal{D} \setminus \mathcal{D}_M$ or $y \in \mathcal{D} \setminus \mathcal{D}_M$. Therefore, for this $M > 0$, we know that (6.10) holds true for all $(t, x, y) \in [\hat{t}, +\infty) \times \mathcal{D} \times \mathcal{D}$ and $\alpha > N_{\tilde{m},\Delta}$.

Step 5. Completion of the proof.

For the fixed $M > 0$ in step 4, we find $\hat{t} \in [0, T), \hat{k}, \hat{l} \in [0, (\hat{t} - \bar{t})], \hat{x}, \hat{y} \in \mathcal{D}_M$ and $\hat{x} = \bar{x} \otimes_{\hat{k}} \hat{x}, \hat{y} = \bar{y} \otimes_{\hat{l}} \hat{y}$ with $\Psi^\gamma(\hat{t}, \hat{x}, \hat{y}) \geq \Psi^\gamma(\bar{t}, \bar{x}, \bar{y}) \geq \Psi^\gamma(0, \tilde{x}, \tilde{x})$ such that

$$\Psi^\gamma(\hat{t}, \hat{x}, \hat{y}) \geq \Psi^\gamma(t, \hat{x} \otimes_{[0,(t-\bar{t})]} x, \hat{y} \otimes_{[0,(t-\bar{t})]} y), \ \ t \geq \hat{t}, x, y \in \mathcal{D}. \tag{6.14}$$

Then, we know that

$$\Psi^\gamma(\hat{t}, \hat{x}, \hat{y}) \geq \Psi^\gamma(t, \hat{x} \otimes_{[0,(t-\bar{t})]} x, \hat{y}), t \geq \hat{t}, x \in \mathcal{D}.$$

Thus, by the definition of the viscosity solution, we know that

$$\begin{aligned}&-\tilde{\lambda} W(\hat{x}) - (\lambda - \tilde{\lambda})(V(\hat{y}) + \frac{\alpha}{2} d(\hat{x}, \hat{y}) + \varepsilon(|\hat{x}|_H^2 + |\hat{x}(0)|^2 + |\hat{y}|_H^2 + |\hat{y}(0)|^2))\\ &+\varepsilon(|\hat{x}(0)|^2 - |\hat{x}(-\tau)|^2) + \alpha(B(\hat{x} - \hat{y}), \hat{x}(0)1_{[-\tau,0]}(\cdot) - \hat{x})_H\\ &+\alpha(B(F_1(\hat{x}) - F_1(\hat{y})), F_1(\hat{x}(0))1_{[-\tau,0]}(\cdot) - F_1(\hat{x}))_H\\ &+\alpha(F_1(\hat{x}(0)) - F_1(\hat{x}(-\tau))) \int_{-\tau}^0 F_1(\hat{x}(\theta)) - F_1(\hat{y}(\theta)) \mathrm{d}\theta\\ &+H(\hat{x}, \alpha(\hat{x}(0) - \hat{y}(0)) + 2\varepsilon \hat{x}(0)) \geq 0,\end{aligned} \tag{6.15}$$

and also, according to (6.14), that

$$\Psi^\gamma(\hat{t}, \hat{x}, \hat{y}) \geq \Psi^\gamma(t, \hat{x}, \hat{y} \otimes_{[0,(t-\bar{t})]} y), \quad t \geq \hat{t}, y \in \mathcal{D}.$$

Thus, we obtain

$$\begin{aligned}&-\tilde{\lambda}V(\hat{y})+(\lambda-\tilde{\lambda})(-W(\hat{x})+\frac{\alpha}{2}d(\hat{x},\hat{y})+\varepsilon(|\hat{x}|_H^2+|\hat{x}(0)|^2+|\hat{y}|_H^2+|\hat{y}(0)|^2))\\&-\varepsilon(|\hat{y}(0)|^2-|\hat{y}(-\tau)|^2)+\alpha(B(\hat{x}-\hat{y}),\hat{y}(0)1_{[-\tau,0]}-\hat{y})_H\\&+\alpha(B(F_1(\hat{x})-F_1(\hat{y})),F_1(\hat{y}(0))1_{[-\tau,0]}(\cdot)-F_1(\hat{y}))_H\\&+\alpha(F_1(\hat{y}(0))-F_1(\hat{y}(-\tau)))\int_{-\tau}^0 F_1(\hat{x}(\theta))-F_1(\hat{y}(\theta))\mathrm{d}\theta\\&+H(\hat{y},\alpha(\hat{x}(0)-\hat{y}(0))-2\varepsilon\hat{y}(0))\leq 0.\end{aligned}\tag{6.16}$$

Note that $\tilde{\lambda}=\lambda-\gamma=\frac{2\lambda}{3}$. Combining (6.15) and (6.16), we obtain

$$\begin{aligned}&\frac{\lambda}{3}[W(\hat{x})-V(\hat{y})+\alpha d(\hat{x},\hat{y})+2\varepsilon(|\hat{x}|_H^2+|\hat{x}(0)|^2+|\hat{y}|_H^2+|\hat{y}(0)|^2)]\\&\leq\varepsilon(|\hat{y}(0)|^2-|\hat{y}(-\tau)|^2+|\hat{x}(0)|^2-|\hat{x}(-\tau)|^2)\\&\quad+\alpha(B(\hat{x}-\hat{y}),\hat{x}(0)1_{[-\tau,0]}(\cdot)-\hat{x}-\hat{y}(0)1_{[-\tau,0]}(\cdot)+\hat{y})_H\\&\quad+\alpha(B(F_1(\hat{x})-F_1(\hat{y})),F_1(\hat{x}(0))1_{[-\tau,0]}(\cdot)-F_1(\hat{x})-F_1(\hat{y}(0))1_{[-\tau,0]}(\cdot)+F_1(\hat{y}))_H\\&\quad+\alpha(F_1(\hat{x}(0))-F_1(\hat{x}(-\tau))-F_1(\hat{y}(0))+F_1(\hat{y}(-\tau)))\int_{-\tau}^0 F_1(\hat{x}(\theta))-F_1(\hat{y}(\theta))\mathrm{d}\theta\\&\quad+H(\hat{x},\alpha(\hat{x}(0)-\hat{y}(0))+2\varepsilon\hat{x}(0))-H(\hat{y},\alpha(\hat{x}(0)-\hat{y}(0))-2\varepsilon\hat{y}(0)).\end{aligned}\tag{6.17}$$

On the other hand, by a simple calculation we obtain

$$\begin{aligned}&H(\hat{x},\alpha(\hat{x}(0)-\hat{y}(0))+2\varepsilon\hat{x}(0))-H(\hat{y},\alpha(\hat{x}(0)-\hat{y}(0))-2\varepsilon\hat{y}(0))\\&\leq\sup_{u\in U}(J_1+J_2),\end{aligned}\tag{6.18}$$

where

$$\begin{aligned}J_1&=(F(\hat{x}(0),(a,\hat{x})_H,u)+F_1(\hat{x}(-\tau)),\alpha(\hat{x}(0)-\hat{y}(0))+2\varepsilon\hat{x}(0))_{R^d}\\&\qquad-(F(\hat{y}(0),(a,\hat{y})_H,u)+F_1(\hat{y}(-\tau)),\alpha(\hat{x}(0)-\hat{y}(0))-2\varepsilon\hat{y}(0))_{R^d}\\&\leq\alpha L(|\hat{x}(0)-\hat{y}(0)|^2+|\hat{x}(0)-\hat{y}(0)|[|a|_{W^{1,2}}|B(\hat{x}-\hat{y})|_H+|\hat{x}(-\tau)-\hat{y}(-\tau)|])\\&\quad+2\varepsilon L|\hat{x}(0)|(2+|\hat{x}(-\tau)|+|\hat{x}(0)|+|a|_H|\hat{x}|_H)\\&\quad+2\varepsilon L|\hat{y}(0)|(2+|\hat{y}(-\tau)|+|\hat{y}(0)|+|a|_H|\hat{y}|_H);\end{aligned}\tag{6.19}$$

and

$$J_2=q(\hat{x}(0),u)-q(\hat{y}(0),u)\leq L|\hat{x}(0)-\hat{y}(0)|.\tag{6.20}$$

Combining (6.17)−(6.20), we obtain

$$
\begin{aligned}
&\frac{\lambda}{3}[W(\hat{x}) - V(\hat{y}) + \alpha d(\hat{x},\hat{y}) + 2\varepsilon(|\hat{x}|_H^2 + |\hat{x}(0)|^2 + |\hat{y}|_H^2 + |\hat{y}(0)|^2)] \\
&\le \varepsilon(|\hat{y}(0)|^2 + |\hat{x}(0)|^2) + \frac{\alpha}{2}p(\hat{x},\hat{y}) + 2\varepsilon L|\hat{x}(0)|(2 + |\hat{x}(-\tau)| + |\hat{x}(0)| + |a|_H|\hat{x}|_H) \\
&\quad +2\varepsilon L|\hat{y}(0)|(2 + |\hat{y}(-\tau)| + |\hat{y}(0)| + |a|_H|\hat{y}|_H) + L|\hat{x}(0) - \hat{y}(0)| \\
&\le \varepsilon(|\hat{y}(0)|^2 + |\hat{x}(0)|^2) + \frac{\alpha}{2}p(\hat{x},\hat{y}) + 5\varepsilon L(|\hat{y}(0)|^2 + |\hat{x}(0)|^2) + \varepsilon L(|\hat{x}(-\tau)|^2 + |\hat{y}(-\tau)|^2) \\
&\quad +\varepsilon L|a|_H^2(|\hat{x}|_H^2 + |\hat{y}|_H^2) + 4\varepsilon L + L|\hat{x}(0) - \hat{y}(0)|,
\end{aligned}
$$

where

$$
\begin{aligned}
p(\hat{x},\hat{y}) = {} & |B(\hat{x}-\hat{y})|_H^2 + [\tau(1+L^2) + 2L^2]|\hat{x}(0) - \hat{y}(0)|^2 + 2L^2|x(-\tau) - y(-\tau)|^2 \\
& +|B(F_1(\hat{x}) - F_1(\hat{y}))|_H^2 + \left|\int_{-\tau}^0 F_1(x(\theta)) - F_1(y(\theta))d\theta\right|^2 \\
& +2L(2|\hat{x}(0) - \hat{y}(0)|^2 + |a|_{W^{1,2}}^2|B(\hat{x}-\hat{y})|_H^2 + |\hat{x}(-\tau) - \hat{y}(-\tau)|^2).
\end{aligned}
$$

Recalling $\hat{x},\hat{y} \in \mathcal{D}_M$ and $\Psi^\gamma(\hat{t},\hat{x},\hat{y}) \ge \Psi^\gamma(0,\tilde{x},\tilde{x})$, and combining it with $\varepsilon L \sup_{\theta\in[-\tau,0]} |\tilde{x}(\theta)|^2 + 2\varepsilon L < \frac{\lambda\tilde{m}}{16}$ and $\lambda > 2L(1+\tau|a|_{W^{1,2}}) \vee (5L+1) \vee L|a|_H^2$, it follows that

$$
\begin{aligned}
&\frac{\tilde{\lambda}\tilde{m}}{4} < \frac{\tilde{\lambda}}{2}[W(\tilde{x}) - V(\tilde{x}) - 2\varepsilon(|\tilde{x}|_H^2 + |\tilde{x}(0)|^2)] \le \frac{\tilde{\lambda}}{2}\Psi^\gamma(\hat{t},\hat{x},\hat{y}) \\
&\le \frac{\tilde{\lambda}}{2}[W(\hat{x}) - V(\hat{y}) - \frac{\alpha}{2}d(\hat{x},\hat{y}) - \varepsilon(|\hat{x}|_H^2 + |\hat{x}(0)|^2 + |\hat{y}|_H^2 + |\hat{y}(0)|^2)] \\
&\le \frac{\alpha}{2}p(\hat{x},\hat{y}) + L|\hat{x}(0) - \hat{y}(0)| + (5L+1)\varepsilon(|\hat{y}(0)|^2 + |\hat{x}(0)|^2) + \varepsilon L(|\hat{x}(-\tau)|^2 + |\hat{y}(-\tau)|^2) \\
&\quad +\varepsilon L|a|_H^2(|\hat{x}|_H^2 + |\hat{y}|_H^2) + 4\varepsilon L - \lambda\varepsilon(|\hat{x}|_H^2 + |\hat{x}(0)|^2 + |\hat{y}|_H^2 + |\hat{y}(0)|^2) \\
&\le \frac{\alpha}{2}p(\hat{x},\hat{y}) + L|\hat{x}(0) - \hat{y}(0)| + \frac{\lambda\tilde{m}}{8}\cdot
\end{aligned}
$$

Letting $\alpha \to +\infty$, the following contradiction is induced:

$$\frac{\lambda\tilde{m}}{6} \le \frac{\lambda\tilde{m}}{8}\cdot$$

The proof is now complete. □

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