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THE CONVERGENCE OF NONNEGATIVE SOLUTIONS FOR THE FAMILY OF PROBLEMS $-\Delta_p u = \lambda e^u$ AS $p \to \infty$ [*,**]

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Abstract. Let $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ be a bounded domain with smooth boundary. We show the existence of a positive real number $\lambda^\star$ such that for each $\lambda \in (0, \lambda^\star)$ and each real number $p > N$ the equation $-\Delta_p u = \lambda e^u$ in Ω subject to the homogeneous Dirichlet boundary condition possesses a nonnegative solution u_p. Next, we analyze the asymptotic behavior of u_p as $p \to \infty$ and we show that it converges uniformly to the distance function to the boundary of the domain.

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1. Introduction

Let $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ be a bounded domain with smooth boundary $\partial\Omega$. For each real number $p > N$ consider the problem

$$\begin{cases} -\Delta_p u = \lambda e^u, & \text{for} \quad x \in \Omega \\ u = 0, & \text{for} \quad x \in \partial\Omega, \end{cases} \tag{1.1}$$

where $\Delta_p \cdot := \operatorname{div}(|\nabla \cdot|^{p-2}\nabla \cdot)$ stands for the p-Laplace operator. We say that $u \in W_0^{1,p}(\Omega)$ is a *weak solution* of problem (1.1) if

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla \phi \, dx = \lambda \int_\Omega e^u \phi \, dx, \quad \forall\ \phi \in W_0^{1,p}(\Omega). \tag{1.2}$$

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Note that the integral from the right-hand side of the above relation is well-defined since for $p > N$ we have $W_0^{1,p}(\Omega) \subset L^\infty(\Omega)$. Moreover, recall that Morrey's inequality holds true, *i.e.* there exists a positive constant C_p such that

$$\|u\|_{L^\infty(\Omega)} \leq C_p \|\nabla u\|_{L^p(\Omega)}, \quad \forall\, u \in W_0^{1,p}(\Omega). \tag{1.3}$$

Furthermore, it is known that $\lim_{p\to\infty} C_p = \|\delta\|_{L^\infty(\Omega)}$, where $\delta(x) := \inf_{y\in\partial\Omega} |x-y|$, $\forall\, x \in \Omega$, is the distance function to the boundary of Ω (see, *e.g.* [7], Prop. 3.1).

Problem (1.1) has been extensively studied in the literature (see, *e.g.* [1, 2, 6, 8, 9, 11, 12] and the reference therein). We recall that Aguilar Crespo and Peral Alonso showed in [2] that for each given $p > N$ problem (1.1) has at least a solution if $\lambda > 0$ is small (see Thm. 1.3 in [2]) while for $\lambda > 0$ large enough problem (1.1) does not have solutions (see Thm. 5.8 in [2]). Note that the existence result from [2] is obtained by using a fixed-point argument. Our first goal in this paper is to obtain a similar result as the one obtained in ([2], Thm. 1.3) but using variational techniques. Actually, we will show that there exists $\lambda^\star > 0$ (which does not depend on p) such that for each $p > N$ and each $\lambda \in (0, \lambda^\star)$ problem (1.1) possesses a nonnegative solution $u_p \in W_0^{1,p}(\Omega)$. Next, we intend to study the convergence of the family of solutions $\{u_p\}$ as $p \to \infty$. More precisely, we will show that u_p converges uniformly in $\overline{\Omega}$ to δ, as $p \to \infty$. This result is not totally unexpected since it is known that for each given positive function $f \in L^\infty(\Omega) \setminus \{0\}$ the family of unique solutions of the family of problems

$$\begin{cases} -\Delta_p v = f, & \text{for} \quad x \in \Omega \\ v = 0, & \text{for} \quad x \in \partial\Omega, \end{cases} \tag{1.4}$$

converges uniformly in $\overline{\Omega}$ to δ (see, *e.g.* Bhattacharya, DiBenedetto and Manfredi [3], or Kawohl [19], or Perez−Llanos and Rossi [23], or Bocea and Mihăilescu [4]). However, even if the right-hand side from (1.1) is a positive function from $L^\infty(\Omega)$ it is obvious that we can not apply directly the result on the family of problems (1.4) in order to derive the convergence of the family of solutions of problem (1.1).

Another result which can be related to our study concerns the family of eigenvalue problems for the p-Laplacian

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2} u & \text{if} \quad x \in \Omega \\ u = 0 & \text{if} \quad x \in \partial\Omega. \end{cases} \tag{1.5}$$

It is known (see *e.g.* Lindqvist [20]) that for each real number $p \in (1, \infty)$ the minimum of the Rayleigh quotient associated to problem (1.5), *i.e.*

$$\lambda_1(p) := \inf_{u \in C_0^\infty(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega |\nabla u|^p \,\mathrm{d}x}{\displaystyle\int_\Omega |u|^p \,\mathrm{d}x}, \tag{1.6}$$

stands for the lowest eigenvalue of problem (1.5) whose corresponding eigenfunctions are minimizers of $\lambda_1(p)$ that do not change sign in Ω. Next, defining

$$\Lambda_\infty := \left\{ \frac{\|\nabla u\|_{L^\infty(\Omega)}}{\|u\|_{L^\infty(\Omega)}} :\; u \in W^{1,\infty}(\Omega) \cap \left(\cap_{q>1} W_0^{1,q}(\Omega) \right) \right\},$$

we know (see Lem. 1.5 and Sect. 2 in [18]) that this infimum is always achieved at δ, that is

$$\Lambda_\infty = \frac{\|\nabla \delta\|_{L^\infty(\Omega)}}{\|\delta\|_{L^\infty(\Omega)}} = \frac{1}{\|\delta\|_{L^\infty(\Omega)}},$$

and

$$\lim_{p\to\infty} \sqrt[p]{\lambda_1(p)} = \Lambda_\infty.$$

The asymptotic behavior as $p \to \infty$ of problems (1.5) has been extensively studied in the literature (see Fukagai, Ito and Narukawa [10], Juutinen, Lindqvist and Manfredi [18], Juutinen and Lindqvist [17]). For instance, in the case when $\lambda = \lambda_1(p)$ and $u_p > 0$ is a solution of (1.5) Juutinen, Lindqvist and Manfredi showed in [18] that there exists a subsequence of $\{u_p\}$ which converges uniformly in Ω to a nontrivial and nonnegative viscosity solution of the limiting problem

$$\begin{cases} \min\{|\nabla u| - \Lambda_\infty u, -\Delta_\infty u\} = 0 & \text{if} \quad x \in \Omega \\ u = 0 & \text{if} \quad x \in \partial\Omega, \end{cases} \tag{1.7}$$

where Δ_∞ is the ∞-Laplace operator, which on sufficiently smooth functions $u : \Omega \to \mathbb{R}$ is given by $\Delta_\infty u := \langle D^2 u \nabla u, \nabla u\rangle = \sum_{i,j=1}^{N} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \frac{\partial^2 u}{\partial x_i \partial x_j}$. Note that δ is not always a viscosity solution of (1.7), but, in the particular case where Ω is a ball, it turns out that δ is the only viscosity solution of (1.7). Thus, in this case the result obtained by Juutinen, Lindqvist and Manfredi in [18] is of the same type as the result we propose to analyze in this paper. However, again, we can not apply directly the result on the family of problems (1.5) in order to derive the convergence of the family of solutions of problem (1.1).

In order to resume the above ideas we point out, once more, the fact that it was already known that for each $p > N$ there exists a positive constant $\lambda_p^\star > 0$ such that for each $\lambda \in (0, \lambda_p^\star)$ problem (1.1) has a weak solution. The main contributions of this paper consist in two aspects. First, we show that $\lambda_p^\star$ does not tend to zero, as $p \to \infty$, and consequently there exists a constant $\lambda^\star > 0$, independent of p, such that for each $p > N$ and each $\lambda \in (0, \lambda^\star)$ problem (1.1) has a nonnegative weak solution u_p. Note also the fact that the existence of u_p is obtained by using a different method from those already used in the literature. Second, we analyze the asymptotic behavior of the family of solutions u_p, as $p \to \infty$, and we prove that it converges uniformly to the unique viscosity solution of the limit problem of problem (1.1), *i.e.*

$$\begin{cases} \min\{|\nabla u| - 1, -\Delta_\infty u\} = 0, & \text{for} \quad x \in \Omega \\ u = 0, & \text{for} \quad x \in \partial\Omega, \end{cases}$$

which is exactly the distance function to the boundary of the domain. The analysis of this convergence is facilitated by the method used in obtaining the solutions u_p and requires a technique which has independent interest and may have other potential applications.

Our paper is organized as follows. In Section 2 we show the existence of a positive real number $\lambda^\star$ such that for each $\lambda \in (0, \lambda^\star)$ and each real number $p > N$ problem (1.1) has a nonnegative weak solution, say u_p. Next, in Section 3, we analyze the asymptotic behavior of u_p as $p \to \infty$ and we show that it converges uniformly to the distance function to the boundary of the domain.

2. Variational solutions for problem (1.1)

In this section our goal is to show that for $\lambda > 0$ sufficiently small problem (1.1) possesses a nonnegative variational solution. Define the Euler–Lagrange functional associated to problem (1.1), *i.e.* $J_{p,\lambda} : W_0^{1,p}(\Omega) \to \mathbb{R}$ given by

$$J_{p,\lambda}(u) := \frac{1}{p}\int_\Omega |\nabla u|^p \, \mathrm{d}x - \lambda \int_\Omega \mathrm{e}^u \, \mathrm{d}x, \quad \forall\, u \in W_0^{1,p}(\Omega).$$

It is standard to show that $J_{p,\lambda} \in C^1(W_0^{1,p}(\Omega), \mathbb{R})$ and

$$\left\langle J'_{p,\lambda}(u), \phi \right\rangle = \int_\Omega |\nabla u|^{p-2} \nabla u \nabla \phi \, \mathrm{d}x - \lambda \int_\Omega \mathrm{e}^u \phi \, \mathrm{d}x, \quad \forall\, u,\ \phi \in W_0^{1,p}(\Omega).$$

Thus, it is obvious that u is a solution of (1.1) if and only if it is a critical point of $J_{p,\lambda}$.

Note that the Direct Method in the Calculus of Variations can not be applied in this case since $J_{p,\lambda}$ fails to be coercive. Our idea is to analyze problem (1.1) by using Ekeland's Variational Principle in order to find critical points of $J_{p,\lambda}$.

Define

$$\lambda_p^\star := \frac{1}{2|\Omega|\mathrm{e}^{C_p p^{1/p}}},$$

where C_p is the constant given by Morrey's inequality (1.3).

Lemma 2.1. *For each $\lambda \in (0, \lambda_p^\star)$ we have*

$$J_{p,\lambda}(u) \geq \frac{1}{2}, \quad \forall\, u \in W_0^{1,p}(\Omega) \text{ with } \|\nabla u\|_{L^p(\Omega)} = p^{1/p}.$$

Proof. Using the fact that $p > N$ and Morrey's inequality we have

$$\begin{aligned} J_{p,\lambda}(u) &= \frac{1}{p}\int_\Omega |\nabla u|^p \,\mathrm{d}x - \lambda \int_\Omega \mathrm{e}^u \,\mathrm{d}x \\ &\geq \frac{1}{p}\int_\Omega |\nabla u|^p \,\mathrm{d}x - \lambda|\Omega|\mathrm{e}^{\|u\|_{L^\infty(\Omega)}} \\ &\geq \frac{1}{p}\int_\Omega |\nabla u|^p \,\mathrm{d}x - \lambda|\Omega|\mathrm{e}^{C_p\|\nabla u\|_{L^p(\Omega)}}, \quad \forall\, u \in W_0^{1,p}(\Omega). \end{aligned}$$

Then for each $u \in W_0^{1,p}(\Omega)$ with $\|\nabla u\|_{L^p(\Omega)} = p^{1/p}$ and each $\lambda \in (0, \lambda_p^\star)$ we have

$$J_{p,\lambda}(u) \geq 1 - \lambda|\Omega|\mathrm{e}^{C_p p^{1/p}} \geq 1 - \lambda_p^\star|\Omega|\mathrm{e}^{C_p p^{1/p}} = \frac{1}{2}\cdot$$

The proof of the lemma is complete. □

Remark 2.2. Since by ([7], Prop. 3.1) we know that

$$C_p = p|B_1(0)|^{-1/p} N^{-N(p+1)/p^2}(p-1)^{N(p-1)/p^2}(p-N)^{(N-p^2)/p^2}\lambda_1(p)^{(N-p)/p^2},$$

and $\lim\limits_{p\to\infty} C_p = \|\delta\|_{L^\infty(\Omega)}$ and $\lim\limits_{p\to\infty} p^{1/p} = 1$ it follows that

$$\lim_{p\to\infty} \lambda_p^\star = \frac{1}{2|\Omega|\mathrm{e}^{\|\delta\|_{L^\infty(\Omega)}}} > 0.$$

Consequently, defining

$$\lambda^\star := \inf_{p>N} \lambda_p^\star, \tag{2.1}$$

and taking into account that function $(1,\infty) \ni p \longrightarrow \lambda_1(p)$ is continuous (see, Lindqvist [21] or Huang [13]) we deduce that $\lambda^\star > 0$ and consequently

$$\lambda_p^\star \geq \lambda^\star > 0, \quad \forall\, p > N.$$

The main result of this section is given by the following theorem.

Theorem 2.3. *Let $\lambda^\star > 0$ be given by (2.1). Then for each $\lambda \in (0, \lambda^\star)$ and each $p > N$ problem (1.1) has a nonnegative solution $u_p \in \overline{B}_{p^{1/p}}(0) \subset W_0^{1,p}(\Omega)$ which is characterized by $J_{p,\lambda}(u_p) = \inf\limits_{\overline{B}_{p^{1/p}}(0)} J_{p,\lambda}$.*

Proof. Fix $\lambda \in (0, \lambda^\star)$ and let $r_p := p^{1/p}$.

For each $u \in B_{r_p}(0) \subset W_0^{1,p}(\Omega)$ we have

$$J_{p,\lambda}(u) \geq \frac{1}{p} \|\nabla u\|_{L^p(\Omega)}^p - \lambda |\Omega| \mathrm{e}^{C_p \|\nabla u\|_{L^p(\Omega)}}.$$

By Lemma 2.1 we have

$$\inf_{\partial B_{r_p}(0)} J_{p,\lambda} \geq \frac{1}{2} > 0,$$

while

$$J_{p,\lambda}(0) = -\lambda |\Omega| < 0.$$

It follows that

$$-\infty < \overline{c_p} := \inf_{B_{r_p}(0)} J_{p,\lambda} < 0.$$

Let $0 < \epsilon < \inf_{\partial B_{r_p}(0)} J_{p,\lambda} - \inf_{B_{r_p}(0)} J_{p,\lambda}$. Applying Ekeland's variational principle to the functional $J_{p,\lambda} : \overline{B_{r_p}(0)} \to \mathbb{R}$, we find $u_\epsilon \in \overline{B_{r_p}(0)}$ such that

$$J_{p,\lambda}(u_\epsilon) < \inf_{B_{r_p}(0)} J_{p,\lambda} + \epsilon$$
$$J_{p,\lambda}(u_\epsilon) < J_{p,\lambda}(u) + \epsilon \, \|\nabla u - \nabla u_\epsilon\|_{L^p(\Omega)}, \quad u \neq u_\epsilon.$$

Since

$$J_{p,\lambda}(u_\epsilon) \leq \inf_{B_{r_p}(0)} J_{p,\lambda} + \epsilon \leq \inf_{B_{r_p}(0)} J_{p,\lambda} + \epsilon < \inf_{\partial B_{r_p}(0)} J_{p,\lambda},$$

we deduce that $u_\epsilon \in B_{r_p}(0)$.

Now, we define $I_{p,\lambda} : \overline{B_{r_p}(0)} \to \mathbb{R}$ by $I_{p,\lambda}(u) = J_{p,\lambda}(u) + \epsilon \, \|\nabla u - \nabla u_\epsilon\|_{L^p(\Omega)}$. It is clear that u_ϵ is a minimum point of $I_{p,\lambda}$ and thus

$$\frac{I_{p,\lambda}(u_\epsilon + tv) - I_{p,\lambda}(u_\epsilon)}{t} \geq 0$$

for small $t > 0$ and any $v \in B_1(0)$. The above relation yields

$$\frac{J_{p,\lambda}(u_\epsilon + tv) - J_{p,\lambda}(u_\epsilon)}{t} + \epsilon \, \|\nabla v\|_{L^p(\Omega)} \geq 0.$$

Letting $t \to 0$ it follows that $\langle J'_{p,\lambda}(u_\epsilon), v \rangle + \epsilon \, \|\nabla v\|_{L^p(\Omega)} > 0$ and we infer that $\|J'_{p,\lambda}(u_\epsilon)\| \leq \epsilon$.

We deduce that there exists a sequence $\{w_m\}_m \subset B_{r_p}(0)$ such that

$$J_{p,\lambda}(w_m) \to \overline{c}_p \quad \text{and} \quad J'_{p,\lambda}(w_m) \to 0. \tag{2.2}$$

It is clear that $\{w_m\}_m$ is bounded in $W_0^{1,p}(\Omega)$. Thus, there exists $u_p \in W_0^{1,p}(\Omega)$ such that, up to a subsequence, $\{w_m\}_m$ converges weakly to u_p in $W_0^{1,p}(\Omega)$ and uniformly in Ω. Thus, we deduce that

$$\lim_{m\to\infty} \int_\Omega \mathrm{e}^{w_m}(w_m - u_p) \, \mathrm{d}x = 0,$$

and

$$\lim_{m\to\infty} \langle J'_{p,\lambda}(w_m), w_m - u_p \rangle = 0.$$

Thus, we get

$$\lim_{m\to\infty}\int_\Omega |\nabla w_m|^{p-2}\nabla w_m \nabla(w_m-u_p)\,\mathrm{d}x=0,$$

and consequently, $\{w_m\}_m$ converges strongly to u_p in $W_0^{1,p}(\Omega)$. So, by (2.2),

$$J_{p,\lambda}(u_p)=\overline{c_p}<0 \quad\text{and}\quad J^{'}_{p,\lambda}(u_p)=0. \tag{2.3}$$

Consequently, u_p is a weak solution of (1.1). Finally, note that for each $v\in W_0^{1,p}(\Omega)$ we have $J_{p,\lambda}(v)\geq J_{p,\lambda}(|v|)$. This information implies that a minimizer of $J_{p,\lambda}$ on $B_{r_p}(0)$ is nonnegative in Ω.

The proof of Theorem 2.3 is complete. □

3. The convergence of the sequence of solutions given by Theorem 2.3 as $p\to\infty$

The goal of this section is to prove the following result.

Theorem 3.1. *Let $\lambda^\star>0$ be given by (2.1). For each $\lambda\in(0,\lambda^\star)$ let u_p be the nonnegative solution of problem (1.1) given by Theorem 2.3. Then u_p converges uniformly in Ω to $\delta=\operatorname{dist}(\cdot,\partial\Omega)$.*

In order to prove Theorem 3.1 we first establish the following result.

Lemma 3.2. *Let $\lambda^\star>0$ be given by (2.1). Fix $\lambda\in(0,\lambda^\star)$ and let u_p be the positive solution of problem (1.1) given by Theorem 2.3. Then there is a subsequence $\{u_p\}$ which converges uniformly in Ω, as $p\to\infty$, to some function $u_\infty\in C(\overline{\Omega})$ with $u_\infty\geq 0$ in Ω.*

Proof. Fix $q>N$. For each $p>q$ we have

$$\int_\Omega |\nabla u_p|^q\,\mathrm{d}x\leq\left(\int_\Omega|\nabla u_p|^p\,\mathrm{d}x\right)^{q/p}|\Omega|^{1-q/p}\leq p^{q/p}|\Omega|^{1-q/p}\leq(\mathrm{e}^{1/e})^q|\Omega|^{1-q/p}\leq(\mathrm{e}^{1/e})^q(1+|\Omega|).$$

Thus, $\{|\nabla u_p|\}_p$ is uniformly bounded in $L^q(\Omega)$. The fact that $q>N$ guarantees that the embedding of $W_0^{1,q}(\Omega)$ into $C(\overline{\Omega})$ is compact. Taking into account the reflexivity of the space $W_0^{1,q}(\Omega)$ it follows that there exists a subsequence (not relabeled) of $\{u_p\}$ and a function $u_\infty\in C(\overline{\Omega})$ such that $u_p\rightharpoonup u_\infty$ weakly in $W_0^{1,q}(\Omega)$ and $u_p\to u_\infty$ uniformly in Ω. Moreover, the fact that $u_p\geq 0$ in Ω for each $p>N$ implies that $u_\infty\geq 0$ in Ω.

The proof of Lemma 3.2 is complete. □

Assuming that solution u_p of equation (1.1) is smooth enough, we can rewrite equation (1.1) as

$$-|\nabla u_p|^{p-2}\Delta u_p-(p-2)|\nabla u_p|^{p-4}\Delta_\infty u_p=\lambda\mathrm{e}^{u_p}, \tag{3.1}$$

where $\Delta_\infty\cdot:=\sum_{i,j=1}^N\frac{\partial\cdot}{\partial x_i}\frac{\partial\cdot}{\partial x_j}\frac{\partial^2\cdot}{\partial x_i\partial x_j}$ is the so called ∞-Laplace operator. This equation is nonlinear, but elliptic, thus it makes sense to consider its viscosity solutions.

Let $y\in\mathbb{R}$, $z\in\mathbb{R}^N$ and S be a real symmetric matrix in $\mathbb{M}^{N\times N}$. Consider the following continuous function

$$H_p(y,z,S)=-|z|^{p-2}\mathrm{Trace}(S)-(p-2)|z|^{p-4}\langle Sz,z\rangle-\lambda\mathrm{e}^y.$$

We are interested in finding viscosity solutions of the partial differential equation

$$\begin{cases}H_p(u,\nabla u,D^2u)=0, & \text{for}\quad x\in\Omega\\ u=0, & \text{for}\quad x\in\partial\Omega.\end{cases} \tag{3.2}$$

Definition 3.3.

(a) An upper semicontinuous function $u : \Omega \to \mathbb{R}$ is a viscosity subsolution of (3.2) if, $u|_{\partial\Omega} \leq 0$ and, whenever $x_0 \in \Omega$ and $\psi \in C^2(\Omega)$ are such that $u(x_0) = \psi(x_0)$ and $u(x) < \psi(x)$ for all $x \in B_r(x_0)\backslash\{x_0\}$ then

$$H_p(\psi(x_0), \nabla\psi(x_0), D^2(\psi(x_0))) \leq 0.$$

(b) A lower semicontinuous function $u : \Omega \to \mathbb{R}$ is a viscosity supersolution of (3.2) if, $u|_{\partial\Omega} \geq 0$ and, whenever $x_0 \in \Omega$ and $\psi \in C^2(\Omega)$ are such that $u(x_0) = \psi(x_0)$ and $u(x) > \psi(x)$ for all $x \in B_r(x_0)\backslash\{x_0\}$ then

$$H_p(\psi(x_0), \nabla\psi(x_0), D^2(\psi(x_0))) \geq 0.$$

(c) A continuous function $u : \Omega \to \mathbb{R}$ is a viscosity solution of (3.2) if it is both a viscosity subsolution and a viscosity supersolution of (3.2).

The following result can be obtained by using the ideas of Juutinen, Lindqvist and Manfredi from ([18], Lem. 1.8). We include the proof for reader's convenience.

Lemma 3.4. *A continuous weak solution of* (1.1) *is a viscosity solution of* (3.2).

Proof. We start by checking that u_p is a viscosity supersolution of (3.2). Let $x_0 \in \Omega$ and let $\psi \in C^2(\Omega)$ be a test function such that $u_p(x_0) = \psi(x_0)$ and $u_p(x) > \psi(x)$ for all $x \in B_r(x_0)\backslash\{x_0\}$.

We want to show that

$$-\Delta_p\psi(x_0) = -|\nabla\psi(x_0)|^{p-2}\Delta\psi(x_0) - (p-2)|\nabla\psi(x_0)|^{p-4}\Delta_\infty\psi(x_0) \geq \lambda \mathrm{e}^{\psi(x_0)}.$$

Assume by contradiction that this is not the case. By continuity, there exists a radius $r > 0$ such that

$$-|\nabla\psi(x)|^{p-2}\Delta\psi(x) - (p-2)|\nabla\psi(x)|^{p-4}\Delta_\infty\psi(x) < \lambda \mathrm{e}^{\psi(x)},$$

for every $x \in B_r(x_0)\backslash\{x_0\}$. Taking r smaller if necessary, we may assume that $u_p > \psi$ in $B_r(x_0)\backslash\{x_0\}$. Set $m := \inf_{|x-x_0|=r}(u_p - \psi)(x)$ and define $w(x) := \psi(x) + \frac{m}{2}$. The function w verifies $w(x_0) > u_p(x_0)$ and $w(x) < u_p(x)$ for all $x \in \partial B_r(x_0)$. Moreover, it holds that

$$-\Delta_p w(x) < \lambda \mathrm{e}^{\psi(x)}, \text{ in } B_r(x_0). \tag{3.3}$$

Multiplying (3.3) by $(w - u_p)^+$, which vanishes on $\partial B_r(x_0)$ we get

$$\int_{B_r(x_0)\cap[w>u_p]} |\nabla w(x)|^{p-2}\nabla w(x)\nabla(w-u_p)(x)\,\mathrm{d}x < \lambda \int_{B_r(x_0)\cap[w>u_p]} \mathrm{e}^{\psi(x)}(w-u_p)(x)\,\mathrm{d}x.$$

Testing in (1.2) with $(w - u_p)^+$ extended by zero outside $B_r(x_0)$ we obtain

$$\int_{B_r(x_0)\cap[w>u_p]} |\nabla u_p(x)|^{p-2}\nabla u_p(x)\nabla(w-u_p)(x)\,\mathrm{d}x = \lambda \int_{B_r(x_0)\cap[w>u_p]} \mathrm{e}^{u_p(x)}(w-u_p)(x)\,\mathrm{d}x.$$

Subtracting the last two relations and using the fact that $u_p > \psi$ on $B_r(x_0) \setminus \{x_0\}$ we get

$$\begin{aligned} 0 &\leq \int_{B_r(x_0)\cap[w>u_p]} (|\nabla w(x)|^{p-1} - |\nabla u_p(x)|^{p-1})(|\nabla w(x)| - |\nabla u_p(x)|)\,\mathrm{d}x \\ &\leq \int_{B_r(x_0)\cap[w>u_p]} (|\nabla w(x)|^{p-2}\nabla w(x) - |\nabla u_p(x)|^{p-2}\nabla u_p(x))\nabla(w(x) - u_p(x))\,\mathrm{d}x \\ &< \lambda \int_{B_r(x_0)\cap[w>u_p]} [\mathrm{e}^{\psi(x)} - \mathrm{e}^{u_p(x)}](w(x) - u_p(x))\,\mathrm{d}x \leq 0, \end{aligned}$$

a contradiction. Therefore, u_p is a viscosity supersolution of problem (3.2). The fact that u_p is a viscosity subsolution runs as above and we omit the details. □

By Lemma 3.2 we may extract a subsequence $u_p \to u_\infty$ uniformly in $\overline{\Omega}$ as $p \to \infty$. Next, our goal is to identify the limit equation verified by u_∞.

Theorem 3.5. *Let u_∞ be the function obtained as a uniform limit of a subsequence of $\{u_p\}$ in Lemma 3.2. Then u_∞ is a viscosity solution of problem*

$$\begin{cases} \min\{|\nabla u| - 1, -\Delta_\infty u\} = 0, & \text{for} \quad x \in \Omega \\ u = 0, & \text{for} \quad x \in \partial\Omega. \end{cases} \tag{3.4}$$

Proof. First, we show that u_∞ is a supersolution of (3.4). Fix $x_0 \in \Omega$ and a function $\psi \in C^2(\overline{\Omega})$ such that $u_\infty(x_0) = \psi(x_0)$ and $u_\infty(x) > \psi(x)$, for any $x \in B_r(x_0)\backslash\{x_0\}$. Since $u_p \to u_\infty$ uniformly, there exists a sequence $\{x_p\}_p \in \Omega$ such that $x_p \to x_0$, $u_p(x_p) = \psi(x_p)$ and $u_p - \psi$ has a local minimum at x_p (see details for example in Bocea, Mihăilescu and Stancu−Dumitru [5], Thm. 3.1).

By Lemma 3.4 the function u_p is a viscosity solution of (3.2) and therefore

$$-|\nabla\psi(x_p)|^{p-2}\Delta\psi(x_p) - (p-2)|\nabla\psi(x_p)|^{p-4}\Delta_\infty\psi(x_p) \geq \lambda \mathrm{e}^{\psi(x_p)}. \tag{3.5}$$

By relation (3.5) it is clear that $|\nabla\psi(x_p)| \neq 0$. Multiplying (3.5) by $1/[(p-2)|\nabla\psi(x_p)|^{p-4}]$ we get

$$-\frac{|\nabla\psi(x_p)|^2}{p-2}\Delta\psi(x_p) - \Delta_\infty\psi(x_p) \geq \frac{\lambda \mathrm{e}^{\psi(x_p)}}{(p-2)|\nabla\psi(x_p)|^{p-4}}. \tag{3.6}$$

Letting $p \to \infty$ we obtain

$$-\Delta_\infty\psi(x_0) \geq \limsup_{p\to\infty}\left[\frac{\lambda^{1/p}\mathrm{e}^{\psi(x_p)/p}}{(p-2)^{1/p}|\nabla\psi(x_p)|^{1-4/p}}\right]^p.$$

In particular, we find

$$-\Delta_\infty\psi(x_0) \geq 0. \tag{3.7}$$

Next, we claim that

$$|\nabla\psi(x_0)| - 1 \geq 0. \tag{3.8}$$

Suppose that this is not the case, and then

$$\frac{1}{|\nabla\psi(x_0)|} > 1.$$

Using that fact we deduce

$$\lim_{p\to\infty}\frac{\lambda^{1/p}\mathrm{e}^{\psi(x_p)/p}}{(p-2)^{1/p}|\nabla\psi(x_p)|^{1-4/p}} = \frac{1}{|\nabla\psi(x_0)|} > 1.$$

It follows that there exists $\epsilon_0 > 0$ such that

$$\frac{\lambda^{1/p}\mathrm{e}^{\psi(x_p)/p}}{(p-2)^{1/p}|\nabla\psi(x_p)|^{1-4/p}} \geq 1 + \epsilon_0, \quad \text{for each } p > 1 \text{ sufficiently large.}$$

The above estimates yield

$$\limsup_{p\to\infty}\left[\frac{\lambda^{1/p}\mathrm{e}^{\psi(x_p)/p}}{(p-2)^{1/p}|\nabla\psi(x_p)|^{1-4/p}}\right]^p = +\infty,$$

which contradicts (3.7). Thus, (3.8) holds true.

Therefore (3.7) and (3.8) yield

$$\min\{-\Delta_\infty\psi(x_0), |\nabla\psi(x_0)| - 1\} \geq 0. \tag{3.9}$$

In order to end the proof of this theorem it remains to check that u_∞ is a viscosity subsolution of (3.4). Let $x_0 \in \Omega$ be fixed and let $\psi \in C^2(\Omega)$ be a test function such that $u_\infty(x_0) = \psi(x_0)$ and $u_\infty(x) < \psi(x)$, for any each x in a neighborhood of x_0. We have to check that

$$\min\{-\Delta_\infty \psi(x_0), |\nabla\psi(x_0)| - 1\} \leq 0.$$

If $|\nabla\psi(x_0)| = 0$ then the above inequality obviously holds true. It suffices to show that if $|\nabla\psi(x_0)| > 0$ and

$$|\nabla\psi(x_0)| - 1 > 0, \tag{3.10}$$

then $-\Delta_\infty\psi(x_0) \leq 0$. We follow the arguments considered in the supersolution case and we can construct a sequence $x_p \to x_0$ as $n \to \infty$ such that

$$-\frac{|\nabla\psi(x_p)|^2}{p-2}\Delta\psi(x_p) - \Delta_\infty\psi(x_p) \leq \left[\frac{\lambda^{1/p}\mathrm{e}^{\psi(x_p)/p}}{(p-2)^{1/p}|\nabla\psi(x_p)|^{1-4/p}}\right]^p.$$

Letting $p \to \infty$ from (3.10) we obtain

$$-\Delta_\infty\psi(x_0) \leq \liminf_{p\to\infty}\left[\frac{\lambda^{1/p}\mathrm{e}^{\psi(x_p)/p}}{(p-2)^{1/p}|\nabla\psi(x_p)|^{1-4/p}}\right]^p = 0,$$

which ends the proof. □

Proof of Theorem 3.1 (*concluded*). It is well-known that equation (3.4) has as unique solution δ, namely the distance function to the boundary of Ω (see Jensen [15], or Juutinen [16], or Ishibashi and Koike [14], p. 546). Thus, using the results of Lemma 3.2 and Theorem 3.5, we conclude that the entire sequence u_p converges uniformly to δ in Ω, as $p \to \infty$.

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