VISCOSITY METHODS FOR LARGE DEVIATIONS ESTIMATES OF MULTISCALE STOCHASTIC PROCESSES

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Abstract. We study singular perturbation problems for second order HJB equations in an unbounded setting. The main applications are large deviations estimates for the short maturity asymptotics of stochastic systems affected by a stochastic volatility, where the volatility is modelled by a process evolving at a faster time scale and satisfying some condition implying ergodicity.

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1. Introduction

We study the asymptotic behaviour as $\varepsilon, \delta \to 0$ of stochastic systems in the form

$$\begin{cases} \mathrm{d}X_t = \varepsilon\phi(X_t, Y_t)\mathrm{d}t + \sqrt{2\varepsilon}\sigma(X_t, Y_t)\mathrm{d}W_t & X_0 = x \in \mathbb{R}^n, \\ \mathrm{d}Y_t = \dfrac{\varepsilon}{\delta}b(Y_t)\mathrm{d}t + \sqrt{\dfrac{2\varepsilon}{\delta}}\tau(Y_t)\mathrm{d}W_t & Y_0 = y \in \mathbb{R}^m, \end{cases} \tag{1.1}$$

where $\varepsilon, \delta > 0$, W_t is a standard m-dimensional Brownian motion and the matrix τ is non-degenerate. This is a model of system where the variables Y_t evolve at a much faster time scale $s = \frac{t}{\delta}$ than the other variables X_t. The parameter ε has been added to study the small time behaviour of the system, so time has been rescaled in (1.1) as $t \mapsto \varepsilon t$. Since we expect different limit behaviours depending on the rate $\frac{\varepsilon}{\delta}$, we choose $\delta = \varepsilon^\alpha$ with $\alpha \geq 2$. Motivated by the applications to large deviations that we want to give, we study the limit as $\varepsilon \to 0$ of the following logarithmic functional of the trajectories of (1.1)

$$v^\varepsilon(t, x, y) := \varepsilon \log E\left[\mathrm{e}^{h(X_t)/\varepsilon}|(X_\cdot, Y_\cdot) \quad \text{satisfy} \quad (1.1)\right],$$

where h is a bounded continuous function and we characterize v^ε as the solution of the Cauchy problem with initial data $v^\varepsilon(0, x, y) = h(x)$ for a fully nonlinear parabolic equation in $n + m$ variables.

Our first interest is to characterize the limit v of the functions v^ε as the solution of the Cauchy problem for a first order Hamilton−Jacobi equation in n space dimensions

$$v_t - \bar{H}(x, Dv) = 0 \quad \text{in} \quad]0, T[\times\mathbb{R}^n, \quad v(0, x) = h(x), \tag{1.2}$$

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for a suitable effective Hamiltonian $\bar{H}$. This is a singular perturbation problem for the system (1.1) which we treat by viscosity solutions methods for HJ equations.

The primary motivation for the study of systems of the form (1.1) comes from mathematical models of financial markets with stochastic volatility, where the asset prices are affected by correlated economic factors and the volatility σ is modelled as a function of an Ito process driven by another Brownian motion. This approach seems to have the advantage of taking into account the so called smile effect (caused by the discrepancy between the predicted and market traded option prices) and reproducing more realistic-with fatter and asymmetric tails-returns distributions. Recently an extension to this approach was proposed by Fouque, Papanicolaou, and Sircar in [27], with the aim to model the bursty behaviour of volatility observed in financial markets (that is, the empirically observed tendence of fluctuating to high level and then to low level for small time periods). These phenomena are also related to another feature of volatility, which is mean reversion. The mathematical framework of multiple time scales systems and singular perturbations takes into account both these phenomena.

Following this approach, we model the log-prices of the asset with the slow variable X_t and the volatility σ with a process Y_t evolving on a faster time scale $s = \frac{t}{\delta}$ and which is ergodic, that is, it has a unique invariant long-run distribution and asymptotically decorrelates. Here δ is the fast mean reversion time and it is smaller than the order of maturity ε. The asymptotic analysis as $\varepsilon, \delta \to 0$ gives a simple pricing and hedging theory taking into account the effect of a changing volatility and then providing a correction to the classical Black-Scholes formulas.

The existing techniques to treat singular perturbations problems as (1.2) have been developed so far mainly under assumptions implying some kind of compactness of the fast variable. We refer mainly to the methods of [4], stemming from Evans' perturbed test function method for homogenization [23] and its extensions to singular perturbations [1–3]. A standard hypothesis is for example the periodicity of the coefficients of the stochastic system with respect Y_t, which in particular implies the periodicity in y of the solutions v^ε (see *e.g.* [7] and the end of the introduction for more details).

Fundamental aim of this paper is to consider unbounded fast processes by replacing the compactness with some condition implying ergodicity. A quite natural condition is the following

$$b(y) \cdot y \leq -B|y|^2, \qquad \text{if} \quad |y| > R, \qquad \text{for some} \quad B > 0, R > 0, \tag{1.3}$$

which is reminiscent of other similar conditions about recurrence of diffusion processes in the whole space (see for example [6, 39–41]). The main example is the Ornstein−Uhlenbeck process. The interest in the analysis of such kind of systems is in part related to the financial applications we have in mind; in particular, the standard compactness/periodicity assumptions appear as too restrictive in order to model volatility in financial markets, see the empirical data and the discussion presented in [27] and the references therein.

We identify and study two different regimes depending on how fast the volatility oscillates relative to the horizon length, namely the ultra-fast mean reverting regime $\delta = \varepsilon^\alpha$ (supercritical case), with $\alpha > 2$ and fast mean reverting regime $\delta = \varepsilon^2$ (critical case). The subcritical case $\alpha < 2$ is not analysed in the present paper, since the ergodic problem arising in this case can not be solved (see Rem. 2.6 for more details). This seems to be linked to the fact that, in the slow mean reversion regime ($\alpha < 2$), the long-time behaviour of the Y_t process does not come into play.

The first step is the identification of the effective Hamiltonians in both cases through the resolution of two different ergodic problems. For $\alpha > 2$ the ergodic problem is finding, for any $(\bar{x}, \bar{p}) \in \mathbb{R}^n \times \mathbb{R}^n$, a couple $\lambda \in \mathbb{R}$ and w-viscosity solution of the following uniformly elliptic linear equation

$$\lambda - \operatorname{tr}(\tau\tau^T(y)D^2 w(y)) - b(y) \cdot Dw(y) - |\sigma(\bar{x}, y)^T \bar{p}|^2 = 0. \tag{1.4}$$

Note that $\lambda = \bar{H}(\bar{x}, \bar{p})$ is the effective Hamiltonian and we call w the corrector by analogy with the theory of homogenization. In order to prove the existence of the effective Hamiltonian and of the corrector we approximate the ergodic problem by the so-called approximate δ-ergodic problem, namely

$$\delta w_\delta(y) - \operatorname{tr}(\tau\tau^T(y)D^2 w_\delta(y)) - b(y) \cdot Dw_\delta(y) - |\sigma(\bar{x}, y)^T \bar{p}|^2 = 0.$$

The main result which allows us to conclude the existence is a δ-uniform local Lipschitz bound for w_δ (see Lem. 4.5). For the uniqueness of the effective Hamiltonian, we rely on the ergodicity of the process Y_t (encoded by the Ass. (1.3)) and on the results of Bardi, Cesaroni, Manca [6], where the effective Hamiltonian is uniquely determined by the explicit formula

$$H(\bar{x}, \bar{p}) = \int_{\mathbb{R}^m} |\sigma(\bar{x}, y)^T \bar{p}|^2 \mathrm{d}\mu(y),$$

where μ is the invariant probability measure on $\mathbb{R}^m$ of the stochastic process

$$\mathrm{d}Y_t = b(Y_t)\mathrm{d}t + \sqrt{2}\tau(Y_t)\mathrm{d}W_t.$$

In addition, we prove the existence of the corrector, which is not investigated in [6].

For $\alpha = 2$ the ergodic problem is finding, for any $(\bar{x}, \bar{p}) \in \mathbb{R}^n \times \mathbb{R}^n$, a couple $\lambda \in \mathbb{R}$ and w-viscosity solution of the following uniformly elliptic nonlinear equation

$$\lambda - \mathrm{tr}(\tau\tau^T(y)D^2w(y)) - |\tau(y)^T Dw(y)|^2 - (b(y) + \tau(y)\sigma(\bar{x}, y)^T\bar{p}) \cdot Dw(y) - |\sigma(\bar{x}, y)^T\bar{p}|^2 = 0. \qquad (1.5)$$

Differently from (1.4), equation (1.5) is nonlinear with quadratic growth in the gradient. We remark that the correlation term $\tau(y)\sigma(\bar{x}, y)^T\bar{p} \cdot Dw(y)$ arises only in the fast mean reverting regime, whereas it does not play any role in the ultra-fast mean reverting case (see Rem. 4.1). Note that in the cell problems we use the notation D without subscripts for the gradient of the solution with respect to the y variable.

For the existence of the effective Hamiltonian and the corrector, we proceed analogously to the supercritical case, in particular we rely on an analogous δ-uniform local Lipschitz bound for the solution of the approximate δ-ergodic problem (see Lem. 3.5). We prove the uniqueness of the effective Hamiltonian relying on the results by Ichihara [30], where ergodic problems for Bellman equations (in the case of a nonlinear quadratic term) are solved. For a representation formula we refer to [33], where $\bar{H}$ is written as the convex conjugate of a suitable operator over a space of measures.

Then, the crucial step is the convergence of the functions v^ε to the solution of the limit problem (1.2). The difficulties stem primarily from the unboundedness of the fast variable, and the standard methods relying on the compactness/periodicity of the fast variable have to be modified. Our strategy rely on the following assumption on the fast process

$$b(y) = b - y, \quad \tau(y) = \tau \quad \text{for } |y| \geq R_1 \qquad (1.6)$$

for some $R_1 > 0$, where $b \in \mathbb{R}^m$ is a vector, and τ is bounded and uniformly non-degenerate. In particular, note that (1.6) is stronger as the usual condition implying ergodicity as (1.3) and it is satisfied *e.g.* by the Ornstein−Uhlenbeck process.

Our techniques are based on the perturbed test function method of [4, 23], with some relevant adaptations to the unbounded setting. We mainly rely on the ergodicity of the fast process through the use of a *Liapounov function* (see Sects. 2, 2.3) into the perturbed test function. Further difficulties in the proof of the convergence come from the nonlinearity of the equation satisfied by the v^ε. A fundamental result on which we rely in the proof of the convergence is given in Proposition 5.1, that is, a global δ-uniform Lipschitz bound for the solution of the approximate δ-ergodic problems.

A significant part of the paper is devoted to the proof of Proposition 5.1, which could be considered our main mathematical contribution. Indeed, not only it is a crucial step to prove the convergence, as already mentioned, but also it is a non standard result, at least to our knowledge, for the type of equations we consider, namely uniformly elliptic equations either with linear Hamiltonians in the gradient (supercritical case), or with superlinear quadratic Hamiltonians (critical case). The proof is in some part inspired by a method due to Ishii and Lions [32] (see also [12, 20] and the references therein), which essentially allows to take profit of the uniform ellipticity of the equation to control the Hamiltonian terms. However, we remark that usually the Ishii−Lions method allows to achieve bounds which depend on the L^∞-norm of the solution (at least if we do not assume any

periodicity), whereas our result is a global estimate in all the space independent of such norm. The fundamental hypothesis which enables us to achieve our result is the Ornstein−Uhlenbeck nature of the fast process at infinity encoded by Assumption (1.6).

We recall some results in the literature related to gradient bounds for similar kinds of equations. Gradient bounds for superlinear-type Hamiltonians can be found in Lions [35] and Barles [9], see also Lions and Souganidis [37], Barles and Souganidis [14] and more, recently, Cardaliaguet and Silvestre [19] for nonlinear degenerate parabolic equations. However, we remark that in the previous works the bounds depend usually on the L^∞-norm of the solution. In [14] some results independent of the L^∞-norm of the solutions are established but in periodic environments. We recall also the result of [17] by Capuzzo−Dolcetta, Leoni, Porretta for coercive superlinear Hamiltonians, where a uniform gradient bound is proved, but in some Hölder norm and only in bounded domains. Recently, uniform Lipschitz bounds on the torus for analogous equations as ours (and more general) have been established by Ley and Duc Nguyen in [34].

Our second aim is to derive a large deviation principle for the process X_t^ε as a first application of our convergence result. More specifically, following the approach of [7, 25], we prove that the measures associated to the process X_t in (1.1) satisfy such a principle with good rate function

$$I(x; x_0, t) := \inf \left[\int_0^t \bar{L}\left(\xi(s), \dot{\xi}(s)\right) \, \mathrm{d}s \ \Big| \ \xi \in AC(0, t), \ \xi(0) = x_0, \xi(t) = x \right],$$

where $\bar{L}$ is the *effective Lagrangian* associated to $\bar{H}$ via convex duality. In particular we get that

$$P(X_t^\varepsilon \in B) = \mathrm{e}^{-\inf_{x \in B} \frac{I(x; x_0, t)}{\varepsilon} + o\left(\frac{1}{\varepsilon}\right)}, \ \text{ as } \varepsilon \to 0$$

for any open set $B \subseteq \mathbb{R}^n$. Then, we apply this result to find estimates of option prices near maturity and an asymptotic formula for the implied volatility. We refer to Section 8 for further details.

An important reference to us is the paper [7], where the author together with Bardi and Cesaroni studied small time behaviour for the system defined above under the main assumption of periodicity on the coefficients. We remark that in [7] the resolution of the ergodic problems and the convergence rely deeply on the assumption of periodicity, which allows to take profit of more standard techniques. On the contrary and as already observed, the unboundedness of the fast process assumed in the present paper entails much more difficulties in the proofs, both in the resolution of the ergodic problems (that, is in the local gradient estimates and in the existence and uniqueness of $\bar{H}$) and mainly in the convergence result. However, note that in [7], a rather complete analysis is carried out, considering also the case $1 < \alpha < 2$, and several representation formulas for the effective Hamiltonian are given. Finally, similar financial applications to the ones above mentioned are also given in [7] (see Thm. 7.1, Cors. 8.1 and 8.2). However, we underline that these financial applications take on particular relevance in the present paper due to the unbounded setting, which provides a more realistic model for volatility as appearing in financial markets.

We finally recall the paper [25], where Feng, Fouque, and Kumar study analogous problems for system of the form we consider, only for $\alpha = 2$ and $\alpha = 4$, in the one-dimensional case $n = m = 1$, assuming that Y_t is the Ornstein−Uhlenbeck process and the coefficients in the equation for X_t do not depend on X_t. Their methods are based on the approach to large deviations developed in [26]. Comparing to [25], we remark that we consider more general fast processes satisfying (1.6), we treat vector-valued processes with ϕ and σ depending on X_t in a rather general way and we study all the range $\alpha \geq 2$. Also, our methods are different, mostly from the theory of viscosity solutions for fully nonlinear PDEs and from the theory of homogenization and singular perturbations for such equations.

Organization of the paper

In Section 2 we give the assumptions on the stochastic volatility model and we recall some preliminaries. In Sections 3 and 4 we analyse the ergodic problem and the properties of the effective Hamiltonian in the critical

($\alpha = 2$) and supercritical case ($\alpha > 2$), respectively. Section 5 is devoted to the proof of the Lipschitz bounds for the solution of the ergodic problems for each regimes. In Section 6 we prove the comparison principle for the limit equation (1.2) and in Section 7 we prove the convergence result for each regime of the functions v^ε to the unique viscosity solution of the limit problem (1.2) with $\bar{H}$ identified in the previous sections. Finally in Section 8 we derive a large deviation principle for the stochastic system and we give some financial applications to estimate option prices and asymptotic volatility.

2. ASSUMPTIONS AND PRELIMINARIES

2.1. The stochastic volatility model

We consider fast mean-reverting processes of the following type

$$\begin{cases} \mathrm{d}X_t = \phi(X_t, Y_t)\mathrm{d}t + \sqrt{2}\sigma(X_t, Y_t)\mathrm{d}W_t, & X_0 = x \in \mathbb{R}^n \\ \mathrm{d}Y_t = \frac{1}{\varepsilon^\alpha}b(Y_t)\mathrm{d}t + \sqrt{\frac{2}{\varepsilon^\alpha}}\tau(Y_t)\mathrm{d}W_t, & Y_0 = y \in \mathbb{R}^m, \end{cases} \tag{2.1}$$

where $\varepsilon > 0, \alpha \geq 2, \phi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n, \sigma : \mathbb{R}^n \times \mathbb{R}^m \to \mathbf{M}^{n,m}$ are bounded functions, Lipschitz continuous in (x, y), $b : \mathbb{R}^m \to \mathbb{R}^m$ is Lipschitz continuous, $\tau : \mathbb{R}^m \to \mathbf{M}^{m,m}$ is bounded, Lipschitz continuous and uniformly non degenerate, *i.e.* satisfies for some $\theta > 0$

$$\xi^T \tau(y)\tau(y)^T \xi = |\tau^T(y)\xi|^2 > \theta|\xi|^2 \quad \text{for every } y \in \mathbb{R}, \xi \in \mathbb{R}^m. \tag{2.2}$$

These assumptions will hold throughout the paper.

In order to study small time behaviour of the system (2.1), we rescale time $t \to \varepsilon t$, for $0 < \varepsilon \ll 1$, so that the typical maturity will be of order ε. Denoting the rescaled process by $X_t^\varepsilon, Y_t^\varepsilon$, we get

$$\begin{cases} \mathrm{d}X_t^\varepsilon = \varepsilon\phi(X_t^\varepsilon, Y_t^\varepsilon)\mathrm{d}t + \sqrt{2\varepsilon}\sigma(X_t^\varepsilon, Y_t^\varepsilon)\mathrm{d}W_t, & X_0^\varepsilon = x \in \mathbb{R}^n \\ \mathrm{d}Y_t^\varepsilon = \varepsilon^{1-\alpha}b(Y_t^\varepsilon)\mathrm{d}t + \sqrt{2\varepsilon^{1-\alpha}}\tau(Y_t^\varepsilon)\mathrm{d}W_t, & Y_0^\varepsilon = y \in \mathbb{R}^m. \end{cases} \tag{2.3}$$

The basic assumption on the drift b is the following.

(E) There exist $B > 0$ and R such that

$$b(y) \cdot y \leq -B|y|^2, \quad \text{if} \quad |y| > R;$$

Remark 2.1. Assumption (E) implies the ergodicity of the fast process and, in particular, the existence of a Liapounov function (see Sect. 2.3). This assumption is fundamental for the resolution of the ergodic problems and the identification of the effective Hamiltonians and it will be used in Sections $2, 3, 4$. On the other hand, we remark that (E) is not enough to prove the convergence result, hence it will be be strengthened in Sections $5, 7$ by the following Assumption (U).

The main assumptions on the drift b and the volatility σ are the following.

(U) There exist $b \in \mathbb{R}^m, \tau \in \mathbf{M}^{m,m}$ and R_1 such that

$$b(y) = b - y, \quad \tau(y) = \tau \quad \text{if} \quad |y| > R_1.$$

(S1) Let $\alpha > 2$. Then, for each $x \in \mathbb{R}^n$, there exists $\nu > 0$ such that

$$|\sigma(x, y)^T \xi|^2 \geq \nu|\xi|^2, \quad \text{for all} \quad y \in \mathbb{R}^n, \xi \in \mathbb{R}^m.$$

(S2) Let $\alpha = 2$. Then, for all $x \in \mathbb{R}^n$, there exists $g : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^+$ such that

$$||\sigma(x,y) - \sigma(x,z)||_\infty \leq g(y,z)|y-z| \qquad \text{for all} \quad y, z \in \mathbb{R}^m,$$

and $\forall \eta > 0$, there exists $R_\eta > 0$ such that $g(y,z) \leq \eta$ as $|z|, |y| \geq R_\eta$.

Remark 2.2. As already remarked, (U) is stronger than (E) (and therefore implies the ergodicity of the fast process) and it will be fundamental in Sections 5, 7 in order to prove the global Lipschitz bound for the corrector (Prop. 5.2) and the convergence result. For example, (U) is satisfied by Ornstein−Uhlenbeck type processes, *i.e.* processes Y_t as in (2.3) such that

$$b(y) = b - y, \quad \tau(y) = \tau \qquad \text{for any} \qquad y \in \mathbb{R}^m, \tag{2.4}$$

for some $b \in \mathbb{R}^m$ and $\tau \in \mathbf{M}^{m,m}$ non-degenerate. The Ornstein−Uhlenbeck process is a classical example of a Gaussian process that admits a stationary probability distribution and in particular is a mean-reverting process, namely there is a long-term value towards the process "tends to revert". However, we remark that (U) is more general than (2.4) since it is required only outside some compact which can be arbitrarily big.

Remark 2.3. Assumption (S2) says, roughly speaking, that the Lipschitz constant of $\sigma(x, \cdot)$, considered as a function on $\mathbb{R}^m$ for $x \in \mathbb{R}^n$ fixed, vanishes at infinity. At least to our point of view, (S2) seems not restrictive in the context of financial models, since it influences the behaviour of σ only at infinity, which in general is not "seen" in the financial applications we are interested in. Examples of sufficient conditions for (S2) are

$$\lim_{|y| \to +\infty} g(y,z) = 0 \quad \text{uniformly in } z,$$

$$\lim_{|z| \to +\infty} g(y,z) = 0 \quad \text{uniformly in } y.$$

For example, the above conditions are satisfied by $\sigma(x,y) = \frac{1}{(1+|y|^2)^\alpha}$, for $\alpha > 0$. Then we have (S2) with $g(y,z) = \frac{C}{1+|y|+|z|}$. Without loss of generality we suppose $n = 1$ and $z \geq y \geq 0$. Then

$$\sigma(y) - \sigma(z) = \frac{1}{(1+y^2)^\alpha} \left(1 - \left(1 + \frac{y^2 - z^2}{1+z^2} \right)^\alpha \right).$$

From the inequality $1 - (1+x)^\alpha \leq -x$ for $-1 \leq x \leq 0$, we get

$$\sigma(y) - \sigma(z) \leq \frac{1}{(1+y^2)^\alpha} \frac{(z-y)(z+y)}{1+z^2} \leq \frac{2z}{1+z^2}(y-z).$$

Since we assumed $z \geq y \geq 0$, we can find a constant C independent of y, z such that $\frac{2z}{1+z^2} \leq \frac{C}{1+z+y}$, concluding the proof.

Assumption (S2) is used to prove the Lipschitz bound for the corrector in the critical case (Prop. 5.1). In particular, we need to assume (S2) to treat the correlation term $\tau(y)\sigma^T(\bar{x},y)\bar{p} \cdot Dw_\delta$, which appears in the ergodic problem for $\alpha = 2$. On the contrary, for $\alpha > 2$ the correlation term do not appear in the ergodic problem (see (4.3) in the following) and then we do not need Assumption (S2).

Assumption (S1) of uniform non-degeneracy of the volatility is due to technical issues arising in the proof of the local gradient bound for the solution of the ergodic problem for $\alpha > 2$. We refer in particular to the proof of Lemma 4.5.

Remark 2.4. Note that in the system (2.3) the processes X_t and Y_t are driven by the same Brownian motion W_t, allowing strong correlation. Also, note that the case of a system as in (2.3) with two different Brownian motions for X_t and Y_t respectively, can be written in the form of (2.3) by suitably redefining the coefficients, in particular, by including the correlation term inside the matrix of the diffusion of the fast process (see also [6], Section 6 for more details).

2.2. The logarithmic transformation method and the HJB equation

We consider the following functional

$$v^\varepsilon(t, x, y) := \varepsilon \log E\left[e^{h(X_t)/\varepsilon} | (X_\cdot, Y_\cdot) \text{ satisfy } (2.3)\right], \tag{2.5}$$

where $h \in BC(\mathbb{R}^n)$ and (X_s, Y_s) satisfies (2.3). Note that the logarithmic form of the functional in (2.5) is motivated by the applications to large deviations that we want to give.

A standard result is that v^ε can be characterized as the unique continuous viscosity solution of the following parabolic problem. We refer to Da Lio and Ley in [21] for a proof.

Proposition 2.5. *Let $\alpha \geq 2$ and define*

$$H^\varepsilon(x, y, p, q, X, Y, Z) := |\sigma^T p|^2 + b \cdot q + \text{tr}(\tau \tau^T Y) + \varepsilon \left(\text{tr}(\sigma \sigma^T X) + \phi \cdot p\right)$$
$$+ 2\varepsilon^{\frac{\alpha}{2}-1}(\tau \sigma^T p) \cdot q + 2\varepsilon^{\frac{1}{2}}\text{tr}(\sigma \tau^T Z) + \varepsilon^{\alpha-2}|\tau^T q|^2.$$

Then v^ε is the unique bounded continuous viscosity solution of the Cauchy problem

$$\begin{cases} \partial_t v^\varepsilon - H^\varepsilon\left(x, y, D_x v^\varepsilon, \dfrac{D_y v^\varepsilon}{\varepsilon^{\alpha-1}}, D_{xx}^2 v^\varepsilon, \dfrac{D_{yy}^2 v^\varepsilon}{\varepsilon^{\alpha-1}}, \dfrac{D_{xy}^2 v^\varepsilon}{\varepsilon^{\frac{\alpha-1}{2}}}\right) = 0 & in \ [0, T] \times \mathbb{R}^n \times \mathbb{R}^m, \\ v^\varepsilon(0, x, y) = h(x) & in \ \mathbb{R}^n \times \mathbb{R}^m. \end{cases} \tag{2.6}$$

Remark 2.6. We treat the range $\alpha \geq 2$ and we do not deal with the case $\alpha < 2$. Indeed, for $\alpha < 2$, the ergodic problem is finding (and characterizing it uniquely) $\lambda \in \mathbb{R}$ and a w-viscosity solution of the following equation:

$$\lambda - 2(\tau(y)\sigma(\bar{x}, y)^T \bar{p}) \cdot D_y w(y) - |\tau(y)^T D_y w(y)|^2 - |\sigma(\bar{x}, y)^T \bar{p}|^2 = 0, \tag{2.7}$$

which is not solvable in general. This seems to be linked to the fact that in the slow mean reversion regime ($\alpha < 2$), the long-time behaviour of the Y_t process does not come into play. Indeed, the ergodicity of the fast process plays no role in (2.7), since the cost ($|\sigma^T \bar{p}|^2$) and the drift $2\tau \sigma^T \bar{p}$ are both bounded and the drift b has disappeared. On the contrary, in the case $\alpha \geq 2$, this role is played by the term $-b \cdot Dw$ where b satisfies Assumption (E). Finally we recall that in [7] the case $\alpha < 2$ is solved thanks to the periodicity assumption.

2.3. A Liapounov-like condition

In this section we prove the existence of a Liapounov function for the following operator

$$\mathcal{G}_{\bar{x}, \bar{p}}(y, q, Y) = -(b(y) + 2\tau(y)\sigma^T(\bar{x}, y)\bar{p}) \cdot q - |\tau^T(y)q|^2 - \text{tr}(\tau \tau^T(y)Y),$$

i.e. we prove that for each $(\bar{x}, \bar{p}) \in \mathbb{R}^n \times \mathbb{R}^n$ there exists a continuous function $\chi_{\bar{x}, \bar{p}} := \chi$, such that $\chi(y) \to +\infty$ as $|y| \to +\infty$ and if $\mathcal{G}[\chi] := \mathcal{G}_{\bar{x}, \bar{p}}(y, D\chi(y), D^2\chi(y))$ then

$$\mathcal{G}[\chi] \to +\infty \text{ as } |y| \to +\infty \text{ in the viscosity sense.} \tag{2.8}$$

The existence of a Liapounov function is reminiscent of other similar conditions about ergodicity of diffusion processes in the whole space; see, for example [15, 16, 29, 36, 38].

Remark 2.7. We observe that

$$\mathcal{G}_{\bar{x}, \bar{p}}(y, q, Y) = -\mathcal{L}_{\bar{x}, \bar{p}}(y, q, Y) - |\tau^T(y)q|^2 \tag{2.9}$$

where, for any $(\bar{x}, \bar{p}) \in \mathbb{R}^n \times \mathbb{R}^n$, $\mathcal{L}_{\bar{x}, \bar{p}}$ is the linear operator

$$\mathcal{L}_{\bar{x}, \bar{p}}(y, q, Y) = (b(y) + 2\tau(y)\sigma^T(\bar{x}, y)\bar{p}) \cdot q + \text{tr}(\tau \tau^T(y)Y),$$

which is the infinitesimal generator of the stochastic process

$$dY_t = (b(Y_t) + 2\tau(Y_t)\sigma^T(\bar{x}, Y_t)\bar{p})dt + \tau(Y_t)dW_t.$$

Note that we consider the additional term $-|\tau^T q|^2$ in (2.9) and this is due to the logarithmic form of the value function v_ε defined in (2.5), which is in turn motivated by the applications to large deviations we are interested in.

Now we prove the following lemma.

Lemma 2.8. *Let* (E) *hold. Then for any* $(\bar{x}, \bar{p}) \in \mathbb{R}^n \times \mathbb{R}^n$ *there exists a Liapounov-like function for the operator* $\mathcal{G}_{\bar{x}, \bar{p}}$.

Proof. Note that a key role in the following proof is played by the behaviour of the drift b at infinity, which is encoded by Assumption (E).
   We take

$$\chi = a|y|^2, \tag{2.10}$$

and by (E) and the boundedness of τ, we have for $|y| \geq R$

$$-b(y) \cdot D\chi(y) - |\tau^T(y)D\chi(y)|^2 \geq 2aB|y|^2 - 4a^2T|y|^2 - 2a|b||y|, \tag{2.11}$$

where $T > 0$ depends on $||\tau||_\infty$. Then by taking

$$a < \frac{B}{2T}, \tag{2.12}$$

the other terms in $\mathcal{G}$ being negligible because of the boundedness of τ and σ, we finally get (2.8). $\qquad\square$

Remark 2.9. We observe that condition (E) reminds classical conditions for ergodicity, see for example [6]. In particular we recall the so-called *recurrence* condition used by Pardoux and Veretennikov [39–41] namely

$$b(y) \cdot y \to -\infty \quad \text{as } |y| \to +\infty. \tag{2.13}$$

Note that (E) is stronger than (2.13). The main reason is that in our context we need to have some additional information on the rate of decay of $b \cdot y$, in particular we need it to be at least quadratic in order to compete with the quadratic growth (in the gradient term) of $\mathcal{G}$ (see also Rem. 2.7).

3. THE CRITICAL CASE: $\alpha = 2$

3.1. Key preliminary results

For any $(\bar{x}, \bar{p}) \in \mathbb{R}^n \times \mathbb{R}^n$, the ergodic problem is finding a constant $\lambda \in \mathbb{R}$ such that the following equation

$$\lambda - \text{tr}(\tau\tau^T(y)D^2w(y)) - |\tau^T(y)Dw|^2 - (b(y) + 2(\tau(y)\sigma^T(\bar{x}, y)\bar{p}) \cdot Dw(y) - |\sigma(\bar{x}, y)^T\bar{p}|^2 = 0. \tag{3.1}$$

has a viscosity solution w. This kind of ergodic problems have been studied by Ichihara [30] and Ichihara and Sheu [31]. We refer in particular to Theorem 2.4 of [30], which we recall in the following proposition.
   Denote

$$\Phi = \{w \in C^2(\mathbb{R}^m) : \text{ there exists } C < 0 \text{ such that } w(y) \leq C(1 + |y|)\}. \tag{3.2}$$

Proposition 3.1. *Let Assumption* (E) *hold. There exists a constant* $\lambda^* \in \mathbb{R}$ *such that* (3.1) *admits a classical solution* $w \in C^2(\mathbb{R}^m)$ *if and only if* $\lambda \leq \lambda^*$. *Moreover, if* (λ, w) *is a solution of* (3.1) *and* $w \in \Phi$, *then* $\lambda = \lambda^*$.

Remark 3.2. We remark that Theorem 2.4 is proved for Hamiltonians which are convex in the gradient variable, whereas in our case the Hamiltonian is concave. The two cases are equivalent, since if we have a solution w of (3.1), then $-w$ is a solution of

$$-\lambda - \operatorname{tr}(\tau\tau^T(y)D^2 w(y)) + H(y, Dw(y)) = 0, \tag{3.3}$$

where

$$H(y, q) = -b(y) \cdot q + |\tau(y)^T q|^2 - 2\tau(y)\sigma(\bar{x}, y)^T \bar{p} \cdot q + |\sigma(\bar{x}, y)\bar{p}|^2. \tag{3.4}$$

which is now convex in the gradient and satisfies the assumptions of [30].

3.2. The ergodic problem and the effective Hamiltonian

For $\delta > 0$, we consider the approximate ergodic problem

$$\delta w_\delta + F(\bar{x}, y, \bar{p}, Dw_\delta, D^2 w_\delta) - |\sigma(\bar{x}, y)\bar{p}|^2 = 0, \tag{3.5}$$

where

$$F(\bar{x}, y, \bar{p}, q, Y) := -\operatorname{tr}(\tau\tau^T(y)Y) - |\tau^T(y)q|^2 - b(y) \cdot q - 2(\tau(y)\sigma^T(\bar{x}, y)\bar{p}) \cdot q. \tag{3.6}$$

Under our standing assumptions we have the following results.

Proposition 3.3. *Let Assumption* (E) *hold. For any* $(\bar{x}, \bar{p})$ *fixed, there exists a unique solution* $w_\delta \in C^2(\mathbb{R}^m)$ *of* (3.5) *satisfying*

$$-\frac{1}{\delta} \inf_{y \in \mathbb{R}^m} \left| \sigma(\bar{x}, y)^T \bar{p} \right|^2 \leq w_\delta(y) \leq \frac{1}{\delta} \sup_{y \in \mathbb{R}^m} \left| \sigma(\bar{x}, y)^T \bar{p} \right|^2, \tag{3.7}$$

such that

$$\lim_{\delta \to 0} \delta w_\delta(y) = \operatorname{const} := \bar{H}(\bar{x}, \bar{p}) \text{ locally uniformly.}$$

Moreover $\bar{H}(\bar{x}, \bar{p})$ *is the unique constant such that* (3.1) *has a solution* $w \in C^2(\mathbb{R}^m)$ *satisfying*

$$|w(y)| \leq \bar{C} \left(1 + \log(\sqrt{|y|^2 + 1}) \right) \quad \text{for all } y \in \mathbb{R}^m. \tag{3.8}$$

Finally w *is the unique* (*up to and additive constant*) *solution to* (3.1) *for* $\lambda = \bar{H}(\bar{x}, \bar{p})$.

Remark 3.4. The growth estimate (3.8) implies that w solution of (3.1) belongs to the class Φ defined in (3.2), allowing us to apply Proposition 3.1 and deriving the uniqueness of $\bar{H}$. Note that (3.8) is stronger than the growth required in Φ, in particular it would be enough to prove (3.8) with a linear function of y in the right-hand side.

First, we prove the following local gradient bound for the solution of the δ-ergodic problem.

Lemma 3.5. *Let* $\delta > 0$ *and* $w_\delta \in C^2(\mathbb{R}^m)$ *be the unique bounded solution of* (3.5). *Then for all* $k \in \mathbb{N}$ *and* $\bar{x}, \bar{p} \in \mathbb{R}^n$, *there exists* $C > 0$ *such that it holds*

$$\max_{y \in \bar{B}_k} |D_y w_\delta(y; \bar{x}, \bar{p})| \leq C, \tag{3.9}$$

where B_k *is the ball with radius* k *and center* 0 *and* C *depends on* k *and* $\bar{p}$.

Proof. We refer to [7] where we proved the result by the Bernstein method under the assumption of periodicity; the extension to a local bound follows by cut-off functions arguments, following the derivation of similar estimates in [24]. We refer also to [33], Lemma 2.4 for an analogous result. We only note that a key role is played by the coercivity in the gradient of the ergodic problem, more precisely by the quadratic term in the gradient $|\tau^T Dw_\delta|^2$. □

Now we prove Proposition 3.3.

Proof of Proposition 3.3. We split the proof into two steps. In step 1 we prove the existence of a couple $(w, \lambda) \in C(\mathbb{R}^m) \times \mathbb{R}$ solution to (3.1); in step 2 we prove that $w \in C^2(\mathbb{R}^m)$, (3.8) and the uniqueness of such λ. Note that the uniqueness up to an additive constant of w follows from Theorem 2.2 of [30].

Step. 1-*Existence.*

We use the methods of [5] based on the small discount approximation (3.5). Note that the PDE (3.5) has bounded forcing term $|\sigma^T(\bar{x}, y)\bar{p}|^2$ since σ is bounded. The existence and uniqueness of a viscosity solution with the δ dependent bound (3.7) follows from the Perron-Ishii method and the comparison principle in [21]. Moreover $w_\delta \in C^2(\mathbb{R}^m)$, thanks to the Lipschitz uniform estimate of Lemma 3.5 and by elliptic regularity theory of convex uniformly elliptic equations, see [42, 43].

Now we prove that $\delta w_\delta(y)$ converges along a subsequence of $\delta \to 0$ to the constant $\bar{H}(\bar{x}, \bar{p})$ and $w_\delta(y) - w_\delta(0)$ converges to the corrector w. The hard part is proving equicontinuity estimates for δw_δ. We proceed by a diagonal argument. By the local Lipschitz estimates of Lemma 3.5, we have

$$|w_\delta(y) - w_\delta(z)| \le C_1 |y - z| \quad y, z \in \bar{B}_1, \tag{3.10}$$

where for convenience we denote by C_k the constant of Lemma 3.5 in B_k for $k \in \mathbb{N}$. Then δw_δ is equicontinuous in $\bar{B}_1$. The equiboundedness follows from the comparison principle with constant sub and super solutions, namely $\min_{y \in \mathbb{R}^m} |\sigma(y, \bar{x})^T \bar{p}|^2$ and $\max_{y \in \mathbb{R}^m} |\sigma(y, \bar{x})^T \bar{p}|^2$. Then by Ascoli$-$Arzela theorem, there exists a subsequence $\delta_n^1 w_{\delta_n^1}$ of δw_δ, converging uniformly in $\bar{B}_1$ to a constant λ^1, since by (3.10) we have

$$|\delta w_\delta(y) - \delta w_\delta(z)| \le \delta C_1 |y - z| \quad y, z \in \bar{B}_1$$

and then

$$\delta w_\delta(y) - \delta w_\delta(z) \to 0 \quad \forall y, z \in \bar{B}_1 \text{ as } \delta \to 0.$$

By the same argument, $\delta_n^1 w_{\delta_n^1}$ is equibounded and equicontinuous in $\bar{B}_2$. Then, there exists a subsequence $\delta_n^2 w_{\delta_n^2}$ of $\delta_n^1 w_{\delta_n^1}$, converging uniformly in $\bar{B}_2$ to a constant λ^2, such that

$$\lambda^1 = \lambda^2 =: \lambda.$$

Similarly, we construct for all $k \in \mathbb{N}$, a sequence $\{\delta_n^k w_{\delta_n^k}\}_n$ converging as $n \to \infty$ uniformly in $\bar{B}_k$ to a constant $\lambda^k = \lambda$. Note that the subsequence $\{\delta_n^n w_{\delta_n^n}\}_n$ converges locally uniformly to λ. In fact for any $k \in \mathbb{N}$ we have that $\{\delta_n^n w_{\delta_n^n}\}_n$ is a subsequence of $\{\delta_n^k w_{\delta_n^k}\}_n$ for all $n \ge k$, from which we deduce that $\{\delta_n^n w_{\delta_n^n}\}_n$ converges uniformly in $\bar{B}_k$ for all $k \in \mathbb{N}$.

Now define $v_\delta := w_\delta(y) - w_\delta(0)$. Notice that, for all k, v_δ is equibounded in $\bar{B}_k$, since, by Lemma 3.5, we have

$$|v_\delta(y)| = |w_\delta(y) - w_\delta(0)| \le C_k |y|, \quad y \in \bar{B}_k$$

and, again by Lemma 3.5, v_δ is equicontinuous in B_k since

$$|v_\delta(y) - v_\delta(z)| = |w_\delta(y) - w_\delta(z)| \le C_k |y - z|, \quad y, z \in \bar{B}_k.$$

By an analogous diagonal argument, we find sequences $\{v_{\delta_n^k}\}_n$ such that $v_{\delta_n^{k+1}}$ is a subsequence of $v_{\delta_n^k}$, and converges uniformly in $\bar{B}_k$ to a function v^k. Moreover for all $k \in \mathbb{N}$, we have

$$v^{k+1}(y) = v^k(y) \quad y \in \bar{B}_k.$$

Then, if we define $w : \mathbb{R}^m \to \mathbb{R}$ such that

$$w(y) = v^k(y) \quad y \in \bar{B}_k, \tag{3.11}$$

we conclude that

$$\{v_{\delta_n^n}\}_n \to w \quad \text{locally uniformly.} \tag{3.12}$$

Now we prove that (λ, w) satisfy (3.1). From (3.5) we get

$$\delta v_\delta + \delta w_\delta(0) + F(\bar{x}, y, \bar{p}, D_y v_\delta, D_{yy}^2 v_\delta) - |\sigma^T(\bar{x}, y)\bar{p}|^2 = 0, \quad \text{in } \mathbb{R}^m. \tag{3.13}$$

Since v_δ is locally equibounded, $\delta v_\delta \to 0$ locally uniformly and the claim follows recalling that $\delta w_\delta \to \lambda$ and using the stability property of viscosity solutions.

Finally the corrector inherits the property (3.9) of Lemma 3.5, that is, for all $k \in \mathbb{N}$ and $(barx, \bar{p}) \in \mathbb{R}^n \times \mathbb{R}^n$, there exists $C > 0$ such that

$$\max_{y \in \bar{B}_k} |D_y w(y; \bar{x}, \bar{p})| \le C, \tag{3.14}$$

where C depends on k and $\bar{p}$.

Step. 2-*Uniqueness of λ*

The uniqueness is given by Proposition 3.1, once proved that $w \in \Phi$. The C^2 regularity follows from the uniform Lipschitz estimate (3.14) and the regularity theory of convex uniformly elliptic equations, see [42, 43].

Note that, in order to prove that $w \in \Phi$, we prove the (stronger) growth condition (3.8). We prove the claim for the upper bound, since the proof of the lower bound is analogous.

We take the approximate problem (3.5) and we prove that the function $g = C \log(\sqrt{|y|^2 + 1})$, for some positive constant C large enough, is a supersolution of (3.5), that is, we prove

$$\delta g(y) - (b(y) + 2\tau(y)\sigma^T(\bar{x}, y)\bar{p}) \cdot Dg - |\tau^T(y)Dg(y)|^2 - \text{tr}(\tau\tau^T(y)D^2g) - |\sigma(\bar{x}, y)\bar{p}|^2 \ge 0. \tag{3.15}$$

Take $|y| \ge R$ where R is defined in (E). By (E) and the boundedness of σ, we have

$$\delta g(y) - (b(y) + 2\tau(y)\sigma^T(\bar{x}, y)\bar{p}) \cdot Dg - |\tau^T(y)Dg(y)|^2 - \text{tr}(\tau\tau^T(y)D^2g) - |\sigma(\bar{x}, y)\bar{p}|^2$$
$$\ge 2CB\frac{|y|^2}{|y|^2 + 1} - \frac{KC(1 + |\bar{p}|)|y|}{|y|^2 + 1} - KC^2\frac{|y|^2}{(|y|^2 + 1)^2} - |\sigma(\bar{x}, y)\bar{p}|^2, \tag{3.16}$$

where K depends on $B > 0$ defined in (E) and on $||\sigma(\bar{x}, \cdot)||_\infty$. Then, in order to prove that g is a supersolution of (3.15), we prove that the right-hand side in (3.16) is non negative. We factorise $\frac{|y|^2}{|y|^2+1}$ and we prove that

$$2CB - \frac{KC(1 + |\bar{p}|)}{|y|} - \frac{KC^2}{|y|^2 + 1} - \sup_y |\sigma^T \bar{p}|^2 \frac{|y|^2 + 1}{|y|^2} \ge 0. \tag{3.17}$$

Note that when y goes to infinity in (3.17) the leading order term is $2CB - \sup_y |\sigma^T \bar{p}|^2$. Then the claim follows by taking C such that $2CB = 2 + \frac{3}{2}\sup_y |\sigma^T \bar{p}|^2$ and $y \in \mathbb{R}^m \setminus \bar{B}_{\bar{R}}$ for some $\bar{R} > R$ such that

$$\frac{KC(1 + |\bar{p}|)}{|y|} + \frac{KC^2}{|y|^2 + 1} \le 2, \quad \frac{|y|^2 + 1}{|y|^2} \le \frac{3}{2}.$$

Up to now we proved that the function $C \log(\sqrt{|y|^2 + 1})$ is a supersolution of (3.5) in $\mathbb{R}^m \setminus B_{\bar{R}}$. If $\max_{\bar{B}_{\bar{R}}} w_\delta \le 0$ then

$$w_\delta(y) \le \max_{\bar{B}_{\bar{R}}} w_\delta \le C \log(\sqrt{|y|^2 + 1}) \quad y \in \partial B_{\bar{R}},$$

and then by the comparison principle we have

$$w_\delta(y) \le C \log(\sqrt{|y|^2 + 1}) \quad y \in \mathbb{R}^m.$$

Now suppose that $\max_{\bar{B}_{\bar{R}}} w_\delta \geq 0$ and notice that in this case $C \log(\sqrt{|y|^2 + 1}) + \max_{\bar{B}_{\bar{R}}} w_\delta$ is still a supersolution of (3.5) in $\mathbb{R}^m \setminus B_{\bar{R}}$. Then, again by the comparison principle, we get

$$w_\delta(y) \leq C \log(\sqrt{|y|^2 + 1}) + \max_{\bar{B}_{\bar{R}}} w_\delta \quad y \in \mathbb{R}^m. \tag{3.18}$$

Since w_δ satisfies (3.18)

$$v_\delta(y) = w_\delta(y) - w_\delta(0) \leq C \log(\sqrt{|y|^2 + 1}) + \max_{\bar{B}_{\bar{R}}} w_\delta(y) - w_\delta(0) \quad y \in \mathbb{R}^m.$$

We estimate the term $\max_{\bar{B}_{\bar{R}}} w_\delta(y) - w_\delta(0)$ by Lemma 3.5 and we get

$$v_\delta(y) \leq C \log(\sqrt{|y|^2 + 1}) + C_{\bar{R}}$$

and thanks to (3.12) we conclude (3.8) by taking $\bar{C} = \max\{C, C_{\bar{R}}\}$. $\qquad\square$

We recall some properties satisfied by $\bar{H}$. For a proof we refer to [7], Proposition 3.3.

Proposition 3.6. *Let Assumption* (E) *hold.*

(a) $\bar{H}$ *is continuous on* $\mathbb{R}^n \times \mathbb{R}^n$;
(b) *the function* $p \to \bar{H}(x, p)$ *is convex;*
(c)

$$\inf_{y \in \mathbb{R}^m} |\sigma^T(\bar{x}, y)\bar{p}|^2 \leq \bar{H}(\bar{x}, \bar{p}) \leq \sup_{y \in \mathbb{R}^m} |\sigma^T(\bar{x}, y)\bar{p}|^2; \tag{3.19}$$

(d) *For all* $0 < \mu < 1$ *and* $x, z, q, p \in \mathbb{R}^n$, *it holds*

$$\mu \bar{H}\left(x, \frac{p}{\mu}\right) - \bar{H}(z, q) \geq \frac{1}{\mu - 1} \sup_{y \in \mathbb{R}^m} |\sigma^T(x, y)p - \sigma^T(z, y)q|^2. \tag{3.20}$$

Finally we observe that equations like (3.1) have been studied in a non compact setting by Khaise and Sheu in [33]. They prove the existence of a constant $\bar{H}$ such that there is a unique (up to an additive constant) smooth solution w of (3.1) with prescribed growth. Moreover they provide a representation formula for $\bar{H}$ as the convex conjugate of a suitable operator over a space of measures.

4. THE SUPERCRITICAL CASE: $\alpha > 2$

The ergodic problem is finding, for any $(\bar{x}, \bar{p}) \in \mathbb{R}^n \times \mathbb{R}^n$ fixed, a unique constant $\lambda \in \mathbb{R}$ such that the following uniformly elliptic linear equation has a viscosity solution w

$$\lambda - \text{tr}(\tau\tau^T(y)D^2 w(y)) - b(y) \cdot Dw(y) - |\sigma(\bar{x}, y)^T \bar{p}|^2 = 0. \tag{4.1}$$

Remark 4.1. Note that in (4.1) the correlation term $\tau(y)\sigma(\bar{x}, y)^T Dw(y)$ does not arise, differently from the critical case see the ergodic problem (3.1). This suggests that the correlation between the asset and the volatility does not play any role when the volatility evolves very fast, that is, in the ultra-fast mean reverting regime.

This kind of ergodic problems has been studied in [6], see in particular Proposition 4.2 and Theorem 4.3.

Proposition 4.2. *Let Assumption* (E) *hold. For any* $(\bar{x}, \bar{p}) \in \mathbb{R}^n \times \mathbb{R}^n$, *there exists a unique invariant probability measure* μ *for the process*

$$dY_t = b(Y_t)dt + \sqrt{2}\tau(Y_t)dW_t. \tag{4.2}$$

Remark 4.3. For the details we refer to [6], Proposition 4.2. We just observe that the proof relies strongly on the existence of a Liapounov function as proved in the paragraph 2.3 for the infinitesimal generator of the process (4.2), that is, the operator $\mathcal{L}(y, q, Y) = \mathrm{tr}(\tau\tau^T(y)Y) - b(y) \cdot q$.

Consider the approximate δ-*ergodic problem* for fixed $(\bar{x}, \bar{p}, \bar{X})$

$$\delta w_\delta(y) - |\sigma(\bar{x}, y)^T \bar{p}|^2 - b(y) \cdot D_y w_\delta(y) - \mathrm{tr}(\tau(y)\tau(y)^T D_{yy}^2 w_\delta(y)) = 0 \text{ in } \mathbb{R}^m. \tag{4.3}$$

We have the following proposition.

Proposition 4.4. *Let Assumption* (E) *and* (S1) *holds. For any fixed* $(\bar{x}, \bar{p})$ *there exists a unique solution* $w_\delta \in C^2(\mathbb{R}^m)$ *of* (4.3) *satisfying*

$$-\frac{1}{\delta} \inf_{y \in \mathbb{R}^m} |\sigma(\bar{x}, y)^T \bar{p}|^2 \leq w_\delta(y) \leq \frac{1}{\delta} \sup_{y \in \mathbb{R}^m} |\sigma(\bar{x}, y)^T \bar{p}|^2 \tag{4.4}$$

such that

$$\lim_{\delta \to 0} \delta w_\delta(y) = \int_{\mathbb{R}^m} |\sigma(\bar{x}, y)^T \bar{p}|^2 \, \mathrm{d}\mu(y) := \bar{H}(\bar{x}, \bar{p}) \text{ locally uniformly }, \tag{4.5}$$

where μ *is the unique invariant probability measure of the process* (4.2). *Moreover there exists a viscosity solution* $w \in C^2(\mathbb{R}^m)$ *of* (4.1) *with* $\lambda = \bar{H}(\bar{x}, \bar{p})$ *satisfying* (3.8).

First we prove the following local gradient bound for the solution of the δ-ergodic problem.

Lemma 4.5. *Let* (S1) *hold. Let* $\delta > 0$ *and* $w_\delta \in C^2(\mathbb{R}^m)$ *be the unique bounded solution of* (4.3). *Then for all* $k \in \mathbb{N}$ *and* $\bar{x}, \bar{p} \in \mathbb{R}^n$, *there exists* $C > 0$ *such that it holds*

$$\max_{y \in \bar{B}_k} |D_y w_\delta(y; \bar{x}, \bar{p})| \leq C, \tag{4.6}$$

where B_k *is the ball of radius* k *and center* 0 *and* C *depends on* k *and on* $\bar{p}$.

Proof. We suppose that $\bar{p} \neq 0$, otherwise $w_\delta = 0$ is the unique solution of (4.3). We observe that, by Assumption (S1), if $\delta \leq \nu|\bar{p}|^2$, 1 is a subsolution of (4.3). Then, for such δ, $w_\delta \geq 1$. Let $\bar{y}$ such that $w_\delta(\bar{y}) = \min_{\bar{B}_k} w_\delta(y)$ and denote $M := w_\delta(\bar{y}) - 1 \geq 0$. Let for $y \in \bar{B}_k$

$$v_\delta(y) = \log(w_\delta(y) - M). \tag{4.7}$$

Then v_δ satisfies on $\bar{B}_k$

$$\delta\left(1 + \mathrm{e}^{-v_\delta(y)} M\right) - \mathrm{tr}\left(\tau\tau^T D^2 v_\delta(y)\right) - b \cdot D v_\delta(y) - |\tau^T D v_\delta(y)|^2 - \mathrm{e}^{-v_\delta(y)}|\sigma^T \bar{p}|^2 = 0 \tag{4.8}$$

and

$$v_\delta \geq 0 \text{ and } v_\delta(\bar{y}) = 0. \tag{4.9}$$

Note that for $y \in \bar{B}_k$, we have $1 + \mathrm{e}^{-v_\delta(y)} M \geq 0$ and $\mathrm{e}^{-v_\delta(y)} \leq 1$. Then, by the coercivity of (4.8) and analogously to the critical case (see the proof of Lem. 3.5), we prove that there exists some positive constant C, depending on k and $\bar{p}$, such that

$$\max_{y \in \bar{B}_k} |D_y v_\delta(y; \bar{x}, \bar{p})| \leq C. \tag{4.10}$$

By (4.7) and (4.9), we have for $y \in \bar{B}_k$

$$D w_\delta(y) = D v_\delta(y) \mathrm{e}^{v_\delta(y)} = D v_\delta(y) \mathrm{e}^{v_\delta(y) - v_\delta(\bar{y})}$$

and, by (4.10), we finally get (4.6). □

Proof of Proposition 4.4. For the identification of $\bar{H}$ and in particular for the proof of (4.5) we refer to [6], Theorem 4.3. For the existence of the corrector we note that the proof can be carried out analogously as in the critical case and we refer to the proof of Proposition 3.3.   □

We observe that $\bar{H}$ satisfies the properties $(a), (b), (c), (d)$ of Proposition 3.6, which can be proved with similar arguments.

5. GRADIENT BOUNDS

In this section, we prove global uniform Lipschitz bounds for the solution of the approximate δ-ergodic problems and of the true cell problems.

We recall that both $||\tau||_\infty$ and $||\sigma||_\infty$ are assumed finite.

The results are stated in the following propositions.

Proposition 5.1. *Let Assumptions* (U), (S1) *and* (S2) *hold. Let* $w_\delta \in C^2(\mathbb{R}^m)$ *be the unique bounded solution of* (3.5) *for* $\alpha = 2$ *and of* (4.3) *for* $\alpha > 2$. *Then for all* $x, y \in \mathbb{R}^m$ *we have*

$$|w_\delta(y; \bar{x}, \bar{p}) - w_\delta(x; \bar{x}, \bar{p})| \le C|x - y|, \tag{5.1}$$

where C *is a positive constant, depending on* $\bar{x}, \bar{p}, ||\tau||_\infty, ||\sigma||_\infty, m$, *the Lipschitz constants of* τ, b, σ *and is independent of* δ.

As a straightforward corollary of Proposition 5.1, we get the following global gradient bound for the correctors.

Proposition 5.2. *Let Assumptions* (U), (S1) *and* (S2) *hold. When* $\alpha = 2$ *let* $w \in C^2(\mathbb{R}^m)$ *be a solution of* (3.1) *for* $\lambda = \bar{H}(\bar{x}, \bar{p})$ *where* $\bar{H}(\bar{x}, \bar{p})$ *is defined in Proposition* 3.3; *when* $\alpha > 2$ *let* $w \in C^2(\mathbb{R}^m)$ *be the solution* (*defined in Prop.* 4.4) *of* (4.1) *for* $\lambda = \bar{H}(\bar{x}, \bar{p})$. *Then*

$$\sup_{y \in \mathbb{R}^m} |D_y w(y; \bar{x}, \bar{p})| \le C, \tag{5.2}$$

where C *is a positive constant, depending on* $\bar{x}, \bar{p}, ||\tau||_\infty, ||\sigma||_\infty, m$ *and the Lipschitz constants of* τ, b, σ.

The main result is Proposition 5.1. The proof is carried out in in two steps. In Step 1 we prove an Hölder bound not uniform in δ (see Prop. 5.3). The method is essentially based on the Ishii−Lions method and relies mainly on the uniform ellipticity of the equation. Then the result of Proposition 5.3 is used in Step 2 to complete the proof of Proposition 5.1. We remark that the result is non standard mainly because we do not use any compactness or periodicity of the coefficients, namely our result holds in all the space and is independent of δ.

The proof of Propositions 5.3 and 5.1 are carried out only for $\alpha = 2$ since the case $\alpha > 2$ is analogous and even simpler.

Note that, thanks to the uniform local estimate previously proved in Lemma 3.5 for $\alpha = 2$ and Lemma 4.5 for $\alpha > 2$ (see respectively Sects. 3 and 4), the main difficulties come from the behaviour at infinity, which we treat by the assumptions (U) and (S2).

Step. 1-Global Hölder bounds

The proof of Proposition 5.3 is based on the Ishii−Lions method which allows us to take profit of the uniform ellipticity. As usual in the Ishii−Lions method, the estimate that we prove in (5.3) is not uniform in δ. This is the main difference between Proposition 5.3 and Proposition 5.1 and, mainly for this reason, the proof of Proposition 5.3 is more standard.

Note that in the following proof we do not need Assumptions (S1) and (S2), which, on the contrary, are fundamental in the proof of Proposition 5.1.

Proposition 5.3. *Let Assumption* (U) *hold. Let* $w_\delta \in C^2(\mathbb{R}^m)$ *be the unique bounded solution of* (3.5) *for* $\alpha = 2$ *and of* (4.3) *for* $\alpha > 2$. *Then there exists* $C_\delta > 0$ *and* $\alpha \in (0,1)$ *such that*

$$|w_\delta(x; \bar{x}, \bar{p}) - w_\delta(y; \bar{x}, \bar{p})| \le C_\delta |x - y|^\alpha \quad \text{for all } x, y \in \mathbb{R}^m, \tag{5.3}$$

where C_δ *depends on* $\delta, \alpha, ||\tau||_\infty, ||\sigma(\bar{x}, \cdot)||_\infty, \bar{p}$, *the Lipschitz constants of* τ, b, σ *and* θ *of* (2.2).

Proof. We give the proof for $\alpha = 2$ since the case $\alpha > 2$ is analogous and even simpler.

Throughout the following proof we denote either by (a, b) or $a \cdot b$ the scalar product for any $a, b \in \mathbb{R}^m$. For convenience of notation in the following we drop the dependence on $\bar{x}, \bar{p}$ by denoting the solution of (3.5) by w_δ.

Let $\delta > 0$ and $\alpha \in (0, 1)$ be fixed and consider the function

$$w_\delta(x) - w_\delta(y) - C_\delta |x - y|^\alpha, \tag{5.4}$$

for some constant $C_\delta > 0$ large enough. Note that C_δ will be chosen suitably at the end of the proof and will depend on $\delta, \alpha, ||\tau||_\infty, ||\sigma(\bar{x}, \cdot)||_\infty, \bar{p}$, the Lipschitz constants of τ, b, σ and θ of (2.2). For clearness of exposition, we keep track only of the dependence on δ.

We suppose that

$$\sup\{w_\delta(x) - w_\delta(y) - C_\delta |x - y|^\alpha\} = M > 0.$$

Let $R > 0$ and consider the function

$$\Phi(x, y) = w_\delta(x) - w_\delta(y) - C_\delta |x - y|^\alpha - \psi_R(x) - \psi_R(y), \tag{5.5}$$

where

$$\psi_R(z) = \psi\left(\frac{\sqrt{|z|^2 + 1}}{R}\right) \tag{5.6}$$

and $\psi \in C^2([0, +\infty))$ satisfies

$$\begin{cases} \psi(s) = 2||w_\delta||_\infty + 1 & \text{if} \quad s \ge 1 \\ \psi(0) = 0, \ \psi \ge 0, \ \psi' \ge 0, \end{cases} \tag{5.7}$$

where we note that $||w_\delta||_\infty$ depends on δ as in (3.7). We claim that

$$M_R = \sup \Phi(x, y) \to M \text{ as } R \to +\infty.$$

In fact

$$M_R \le M \quad \text{for any } R > 0.$$

On the other hand

$$M_R \ge w_\delta(x) - w_\delta(y) - C_\delta |x - y|^\alpha - \psi_R(x) - \psi_R(y) \text{ for all } x, y \in \mathbb{R}^m, R > 0,$$

then

$$\lim_{R \to +\infty} M_R \ge w_\delta(x) - w_\delta(y) - C_\delta |x - y|^\alpha \text{ for all } x, y \in \mathbb{R}^m$$

and we conclude

$$\lim_{R \to +\infty} M_R \ge \sup\{w_\delta(x) - w_\delta(y) - C_\delta |x - y|^\alpha\} = M.$$

Then we can suppose for R large enough

$$M_R \ge \frac{M}{2} > 0. \tag{5.8}$$

We observe that if $\sqrt{|x|^2 + 1} \geq R$

$$\Phi(x, y) \leq -1 < 0$$

and the same holds when $\sqrt{|y|^2 + 1} \geq R$. Then, there exists (x_R, y_R) point of maximum of Φ such that

$$M_R = w_\delta(x_R) - w_\delta(y_R) - C_\delta|x_R - y_R|^\alpha - \psi_R(x_R) - \psi_R(y_R). \tag{5.9}$$

Note that (x_R, y_R) depends also on δ and that we omit the dependence. Note also that

$$|x_R - y_R| > 0, \tag{5.10}$$

otherwise by (5.9) we have

$$M_R = -\psi_R(x_R) - \psi_R(y_R)$$

and we get a contradiction by (5.8) and the definition of ψ_R.

By (??), (5.8) and the definition of ψ_R, we also have

$$C_\delta|x_R - y_R|^\alpha \leq 2||w_\delta||_\infty := A_\delta.$$

Then

$$|x_R - y_R| \leq \left(\frac{A_\delta}{C_\delta}\right)^{\frac{1}{\alpha}}. \tag{5.11}$$

From now on we omit the dependence on R and we write

$$(x_R, y_R) = (x, y).$$

The main result is the following lemma.

Lemma 5.4. *Under the above notations and Assumption* (U), *there exist positive constants* K, K_1, K_2, K_3, K_4 *such that*

$$0 \leq KC_\delta\alpha(\alpha - 1)|x - y|^{\alpha-2} + KC_\delta\alpha|x - y|^{\alpha+1} + K_1C_\delta\alpha|x - y|^\alpha + K_2\alpha C_\delta^2|x - y|^{2\alpha-1}$$
$$+ K_2o_R(1)C_\delta\alpha|x - y|^{\alpha-1} + K_3\alpha C_\delta|x - y|^{\alpha-1} + K_4|x - y| + o_R(1).$$

where by $o_R(1)$ *we mean that* $o_R(1) \to 0$ *as* $R \to +\infty$. *Moreover* K, K_1, K_2, K_3, K_4 *depends only on* $\bar{p}, ||\sigma||_\infty, ||\tau||_\infty$, *the Lipschitz constants of* τ, b, σ *and* θ *of* (2.2).

Proof. Let

$$r_x = D_x\psi_R = 2R^{-1}\psi'\left(\frac{\sqrt{|x|^2 + 1}}{R}\right)x(\sqrt{|x|^2 + 1})^{-1} \tag{5.12}$$

and

$$r_y = D_y\psi_R = 2R^{-1}\psi'\left(\frac{\sqrt{|y|^2 + 1}}{R}\right)y(\sqrt{|y|^2 + 1})^{-1}, \tag{5.13}$$

then for each δ fixed

$$|r_x|, |r_y| \leq o_R(1), \quad ||D^2\psi_R||_\infty \leq o_R(1), \tag{5.14}$$

where $o_R(1)$ means that $\lim_{R \to +\infty} o_R(1) = 0$.

We remark that in the rest of the proof we denote by $o_R(1)$ any function such that $o_R(1) \to 0$ as $R \to +\infty$. We also denote

$$s = C_\delta\alpha|x - y|^{\alpha-2}(x - y). \tag{5.15}$$

Note that the function in (5.5) is smooth near (x, y) by (5.10). Then, since w_δ is a viscosity solution of (3.5) and since (x, y) is a maximum point of the function in (5.5), we have

$$0 \le \mathrm{tr}(\tau(x)\tau(x)^T D^2 w_\delta(x)) - \mathrm{tr}(\tau(y)\tau(y)^T D^2 w_\delta(y)) + L(x,y) + G(x,y) + E(x,y) + F(x,y) + D(x,y) + o_R(1),$$
$$(5.16)$$

where we used (5.14) to estimate the ψ_R-terms and we denoted

$$D(x,y) = \delta w_\delta(y) - \delta w_\delta(x);$$

$$L(x,y) = s \cdot (b(x) - b(y)) + b(y) \cdot r_y + b(x) \cdot r_x;$$

$$G(x,y) = |\tau(x)^T(s + r_x)|^2 - |\tau(y)^T(s - r_y)|^2;$$

$$E(x,y) = 2\tau(x)\sigma(\bar{x}, x)^T \bar{p} \cdot (s + r_x) - 2\tau(y)\sigma(\bar{x}, y)^T \bar{p} \cdot (s - r_y);$$

$$F(x,y) = |\sigma^T(\bar{x}, x)\bar{p}|^2 - |\sigma^T(\bar{x}, y)\bar{p}|^2.$$

First we estimate the second order terms in (5.16), by proving the following lemma.

Lemma 5.5. *Under the above notations, we have*

$$\mathrm{tr}(\tau(x)\tau(x)^T D^2 w_\delta(x)) - \mathrm{tr}(\tau(y)\tau(y)^T D^2 w_\delta(y)) \le KC_\delta\alpha(\alpha-1)|x-y|^{\alpha-2} + KC_\delta\alpha|x-y|^{\alpha+1} + o_R(1), \quad (5.17)$$

where K is a positive constant (depending on θ of (2.2) and on the Lipschitz constant of τ) and by $o_R(1)$ we mean that $\lim_{R \to +\infty} o_R(1) = 0$.

Proof. We observe that, for any orthonormal basis e_i, $i = 1, \ldots m$ of $\mathbb{R}^m$, we can write

$$\mathrm{tr}(\tau(x)\tau(x)^T D^2 w_\delta(x)) = \sum_{i=1}^m (\tau(x)\tau(x)^T D^2 w_\delta(x) e_i, e_i) = \sum_{i=1}^m (D^2 w_\delta(x)\tau(x) e_i, \tau(x) e_i). \quad (5.18)$$

Denote $\phi(t) = C_\delta t^\alpha$, $f(z) = |z|$. By the maximum point property and the second inequality of (5.14), we get

$$(D^2 w_\delta(x) p, p) - (D^2 w_\delta(y) q, q) \le \phi'(f(x-y))(D^2 f(x-y)(p-q), (p-q))$$
$$+ \phi''(f(x-y))(Df(x-y), p-q)^2 + o_R(1) \quad (5.19)$$

for any $p, q \in \mathbb{R}^m$.

Next we remark that $|Df|^2 = 1$ and therefore, by differentiating this identity, we have $D^2 f Df = 0$. By (2.2), we can set

$$e_1 = \frac{\tau(x)^{-1} Df(x-y)}{|\tau(x)^{-1} Df(x-y)|}, \quad \tilde{e}_1 = -\frac{\tau(y)^{-1} Df(x-y)}{|\tau(y)^{-1} Df(x-y)|}.$$

If $e_1, \tilde{e}_1$ are collinear, the we complete the basis with orthogonal unit vectors $e_i = \tilde{e}_i \in e_1^\top$, $2 \le i \le m$. Otherwise, in the plane $\mathrm{span}\{e_1, \tilde{e}_1\}$, we consider a rotation $\mathcal{R}$ of angle $\frac{\pi}{2}$ and we define

$$e_2 = \mathcal{R} e_1, \quad \tilde{e}_2 = -\mathcal{R}\tilde{e}_1.$$

Since $\mathrm{span}\{e_1, e_2\}^\top = \mathrm{span}\{\tilde{e}_1, \tilde{e}_2\}^\top$, we can complete the orthonormal basis with unit vectors $e_i = \tilde{e}_i \in \mathrm{span}\{e_1, e_2\}^\top$, $3 \le i \le m$.

By (2.2), we have

$$\theta \le \frac{1}{|\tau(x)^{-1} Df(x-y)|^2} \le ||\tau||_\infty^2.$$

Define

$$r_1 = \tau(x) e_1 \quad t_1 = \tau(y)\tilde{e}_1.$$

Since $|Df| = 1$ and $D^2fDf = 0$ and by choosing $p = r_1, q = r_1$ in (5.19), we get

$$(D^2w_\delta(x)r_1, r_1) - (D^2w_\delta(y)t_1, t_1) \le \phi''(f(x-y))(Df(x-y), r_1 - t_1)^2 + o_R(1)$$
$$= C_\delta\alpha(\alpha - 1)|x - y|^{\alpha-2}(Df(x-y), r_1 - t_1)^2 + o_R(1).$$

Notice that

$$\alpha(\alpha - 1) < 0. \tag{5.20}$$

By (2.2), we have

$$(Df(x-y), r_1 - t_1)^2 = \left(\frac{1}{|\tau(x)^{-1}Df(x-y)|^2} + \frac{1}{|\tau(y)^{-1}Df(x-y)|}\right)^2 \ge 4\theta.$$

Then

$$(D^2w_\delta(x)r_1, r_1) - (D^2w_\delta(y)t_1, t_1) \le 4\theta C_\delta\alpha(\alpha - 1)|x - y|^{\alpha-2} + o_R(1). \tag{5.21}$$

Therefore in the right hand side we have a very negative term by a double effect, first because we will choose C_δ large but also because, by doing so, $|x - y|$ becomes smaller and smaller and $|x - y|^{\alpha-2}$ larger and larger.

Now we choose in (5.19) for all $i \in \{1, \ldots, m-1\}$

$$p = \tau(x)e_i \quad q = \tau(y)\tilde{e}_i.$$

Since τ is Lipschitz, we get

$$(D^2w_\delta(x)\tau(x)e_i, \tau(x)e_i) - (D^2w_\delta(y)\tau(y)\tilde{e}_i, \tau(y)\tilde{e}_i) \le KC_\delta\alpha|x - y|^{\alpha+1} + o_R(1),$$

where K depends on the Lipschitz constant of τ. Then, by summing the previous equation on i and adding (5.21), we get

$$\sum_{i=1}^{m}(D^2w_\delta(x)\tau(x)e_i, \tau(x)e_i) - \sum_{i=1}^{m}(D^2w_\delta(y)\tau(y)\tilde{e}_i, \tau\tilde{e}_i) \le KC_\delta\alpha(\alpha-1)|x-y|^{\alpha-2} + KC_\delta\alpha|x-y|^{\alpha+1} + o_R(1),$$

when by K we denote a constant depending on the Lipschitz constant of τ and on θ. Then, by (5.18) with e_i defined as above (and $\tilde{e}_i$ for $\text{tr}(\tau(y)\tau(y)^T D^2w_\delta(y)))$, we finally get (5.17). $\qquad\square$

Then (5.16) becomes

$$0 \le KC_\delta\alpha(\alpha-1)|x-y|^{\alpha-2} + KC_\delta\alpha|x-y|^{\alpha+1} + L(x,y) + G(x,y) + E(x,y) + F(x,y) + D(x,y) + o_R(1), \tag{5.22}$$

Finally we estimate the left terms D, L, G, E, F in (5.22). First note that

$$D(x,y) = \delta w_\delta(y) - \delta w_\delta(x) \le 0;$$

First note that, by (5.15) and since b is Lipschitz, we have

$$L(x,y) \le K_1 C_\delta\alpha|x-y|^\alpha + b(y) \cdot r_y + b(x) \cdot r_x.$$

where K_1 depends on the Lipschitz constant of b. Note that

$$b(y) \cdot r_y + b(x) \cdot r_x \le o_R(1).$$

Indeed, the previous inequality holds from the second of (5.14) when x, y are uniformly bounded in R. Now suppose $|x| \to +\infty$ as $R \to +\infty$ (the argument being similar if $|y| \to +\infty$). By assumption (U) we have

$$b(x) \cdot r_x = (b - x) \cdot r_x$$

and by (5.12), we have

$$x \cdot r_x = 2R^{-1}|x|^2 \psi' \left(\frac{\sqrt{|x|^2+1}}{R} \right) (\sqrt{|x|^2+1})^{-1}$$

and since $\psi' \geq 0$ by definition of ψ_R we have

$$x \cdot r_x \geq 0. \tag{5.23}$$

Then by (5.23) and (5.14), we get

$$(b-x) \cdot r_x \leq o_R(1).$$

Then

$$L(x,y) \leq K_1 C_\delta \alpha |x-y|^\alpha + o_R(1).$$

Now we estimate the G-term. By the first of (5.14), (5.15) and since τ is bounded, we have

$$G(x,y) \leq |\tau^T(x)s|^2 - |\tau^T(y)s|^2 + K_2 o_R(1) C_\delta \alpha |x-y|^{\alpha-1} + o_R(1),$$

where K_2 depends on $||\tau||_\infty$. Note that from now on we denote by K_2 a constant depending on $||\tau||_\infty$ and the Lipschitz constant of τ and which may change from line to line. Since τ is bounded by (5.15), we have

$$|\tau^T(x)s| + |\tau^T(y)s| \leq K_2 C_\delta \alpha |x-y|^{\alpha-1}$$

and since τ is Lipschitz and by (5.15), we have

$$|\tau^T(x)s| - |\tau^T(y)s| \leq K_2 C_\delta \alpha |x-y|^\alpha.$$

Then we get

$$|\tau^T(x)s|^2 - |\tau^T(y)s|^2 \leq K_2 \alpha C_\delta^2 \alpha |x-y|^{2\alpha-1},$$

and we conclude

$$G(x,y) \leq K_2 \alpha C_\delta^2 |x-y|^{2\alpha-1} + K_2 o_R(1) C_\delta \alpha |x-y|^{\alpha-1} + o_R(1). \tag{5.24}$$

Next we estimate E using the boundedness of σ and we get

$$E(x,y) \leq K_3 \alpha C_\delta |x-y|^{\alpha-1} + o_R(1),$$

where $K_3 > 0$ depends on $\bar{p}, ||\tau||_\infty, ||\sigma||_\infty$.

Finally, by the Lipschitz continuity and boundedness of σ, we have

$$F(x,y) \leq K_4 |x-y|,$$

where K_4 depends on $||\sigma||_\infty$ and the Lipschitz constant of σ and on $\bar{p}$.

Then, by all the previous estimates, (5.22) becomes

$$0 \leq KC_\delta \alpha(\alpha-1)|x-y|^{\alpha-2} + KC_\delta \alpha|x-y|^{\alpha+1} + K_1 C_\delta \alpha |x-y|^\alpha + K_2 \alpha C_\delta^2 |x-y|^{2\alpha-1}$$
$$+ K_2 o_R(1) C_\delta \alpha |x-y|^{\alpha-1} + K_3 \alpha C_\delta |x-y|^{\alpha-1} + K_4|x-y| + o_R(1). \tag{5.25}$$

This concludes the proof of Lemma 5.4. □

We divide (5.25) by $C_\delta |x-y|^{\alpha-2}$ and we get

$$0 \leq K\alpha(\alpha-1) + K\alpha|x-y|^3 + K_1 \alpha|x-y|^2 + K_2 \alpha C_\delta |x-y|^{\alpha+1} + K_2 o_R(1)\alpha|x-y|$$
$$+ K_3 \alpha|x-y| + K_4 C_\delta^{-1}|x-y|^{3-\alpha} + o_R(1)C_\delta^{-1}|x-y|^{2-\alpha}. \tag{5.26}$$

Note that by (5.11), we have

$$|x-y| \leq A_\delta^{\frac{1}{\alpha}} C_\delta^{-\frac{1}{\alpha}}, \tag{5.27}$$

then

$$C_\delta^{-1}|x-y|^{3-\alpha} \le A_\delta^{\frac{3-\alpha}{\alpha}} C_\delta^{-\frac{3}{\alpha}};$$

$$C_\delta^{-1}|x-y|^{2-\alpha} \le A_\delta^{\frac{2-\alpha}{\alpha}} C_\delta^{-\frac{2}{\alpha}};$$

$$C_\delta|x-y|^{\alpha+1} \le A_\delta^{\frac{\alpha+1}{\alpha}} C_\delta^{-\frac{1}{\alpha}}.$$

By all the previous estimates and by taking R large enough such that $o_R(1) \le 1$, (5.26) becomes

$$0 \le K\alpha(\alpha-1) + K\alpha A_\delta^{\frac{3}{\alpha}} C_\delta^{-\frac{3}{\alpha}} + K_1\alpha A_\delta^{\frac{2}{\alpha}} C_\delta^{-\frac{2}{\alpha}} + K_2\alpha A_\delta^{\frac{\alpha+1}{\alpha}} C_\delta^{-\frac{1}{\alpha}} + K_2 o_R(1)\alpha A_\delta^{\frac{1}{\alpha}} C_\delta^{-\frac{1}{\alpha}}$$
$$+ K_3\alpha A_\delta^{\frac{1}{\alpha}} C_\delta^{-\frac{1}{\alpha}} + K_4 A_\delta^{\frac{3-\alpha}{\alpha}} C_\delta^{-\frac{3}{\alpha}} + o_R(1) A_\delta^{\frac{2-\alpha}{\alpha}} C_\delta^{-\frac{2}{\alpha}}. \quad (5.28)$$

Then, the claim of the proposition follows by taking C_δ in (5.4) large enough in order to get a contradiction with (5.28). For example we take $C_\delta > \bar{C}_\delta$ where $\bar{C}_\delta$ satisfies

$$K\alpha(\alpha-1) + K\alpha A_\delta^{\frac{3}{\alpha}} \bar{C}_\delta^{-\frac{3}{\alpha}} + K_1\alpha A_\delta^{\frac{2}{\alpha}} \bar{C}_\delta^{-\frac{2}{\alpha}} + K_2\alpha A_\delta^{\frac{\alpha+1}{\alpha}} \bar{C}_\delta^{-\frac{1}{\alpha}} + K_2 o_R(1)\alpha A_\delta^{\frac{1}{\alpha}} \bar{C}_\delta^{-\frac{1}{\alpha}}$$
$$+ K_3\alpha A_\delta^{\frac{1}{\alpha}} \bar{C}_\delta^{-\frac{1}{\alpha}} + K_4 A_\delta^{\frac{3-\alpha}{\alpha}} \bar{C}_\delta^{-\frac{3}{\alpha}} + o_R(1) A_\delta^{\frac{2-\alpha}{\alpha}} \bar{C}_\delta^{-\frac{2}{\alpha}} < 0.$$

Note that $\bar{C}_\delta$ depends on K_i, $i=1,2,3,4$ and on δ, α, K.   □

Step. 2-Proof of Proposition 5.1

Proof. Note that, under the assumption (U), (3.5) reads for $|y| > R_1$

$$\delta w_\delta + F(\bar{x}, y, \bar{p}, Dw_\delta, D^2 w_\delta) - |\sigma(\bar{x}, y)\bar{p}|^2 = 0, \quad (5.29)$$

where

$$F(\bar{x}, y, \bar{p}, q, Y) := -\mathrm{tr}(\tau\tau^T Y) - |\tau^T q|^2 - (b - y, q) - (2\tau\sigma^T(\bar{x}, y)\bar{p}, q).$$

Note also that throughout the following proof we denote either by (a, b) or $a \cdot b$ the scalar product for any $a, b \in \mathbb{R}^m$.

Let $\bar{R} > R_1$ be large enough (which will be chosen suitably at the end of the proof) and take $C_{\bar{R}}$ the constant of Lemma 3.5 for $k = \bar{R}$. Then we have for all $x, y \in \bar{B}_{\bar{R}}$

$$|w_\delta(x; \bar{x}, \bar{p}) - w_\delta(y; \bar{x}, \bar{p})| \le C_{\bar{R}}|x-y|. \quad (5.30)$$

For convenience of notation in the following we drop the dependence on $\bar{x}, \bar{p}$ by denoting the solution of (5.29) by w_δ.

In this first part of the proof we proceed analogously as in the proof of Proposition 5.3. The new part of the proof starts from Lemma 5.6. We give a sketch and for all the details we refer to the beginning of the proof of Proposition 5.3.

We proceed by contradiction and we suppose that

$$\sup\{w_\delta(x) - w_\delta(y) - C|x-y|\} = M > 0, \quad (5.31)$$

where C is a positive constant large enough, that is $C > \max\{C_{\bar{R}}, C_{\bar{R}+1}\}$.

Let $R > 0$ and consider the function

$$\Phi(x, y) = w_\delta(x) - w_\delta(y) - C|x-y| - \psi_R(x) - \psi_R(y), \quad (5.32)$$

where

$$\psi_R(z) = \psi\left(\frac{\sqrt{|z|^2 + 1}}{R}\right) \tag{5.33}$$

where ψ is defined in (5.7). By standard argument (see also the proof of Prop. 5.3), we prove that

$$M_R = \sup \Phi(x, y) \to M \text{ as } R \to +\infty,$$

then we can suppose for R large enough

$$M_R \geq \frac{M}{2} > 0, \tag{5.34}$$

and by definition of ψ_R we get that, for R large enough, there exist (x_R, y_R) such that

$$M_R = w_\delta(x_R) - w_\delta(y_R) - C|x_R - y_R| - \psi_R(x_R) - \psi_R(y_R). \tag{5.35}$$

Note also that

$$|x_R - y_R| > 0. \tag{5.36}$$

We prove the following lemma, whose result is essential in order to use assumption (U) in the rest of the proof.

Lemma 5.6. *Under the above notations, we have that, for R large enough, there exists a point of maximum (x_R, y_R) of the function Φ such that $(x_R, y_R) \in \left(\mathbb{R}^m \setminus \bar{B}_{\bar{R}}\right) \times \left(\mathbb{R}^m \setminus \bar{B}_{\bar{R}}\right)$. Moreover*

$$\liminf_{R \to +\infty} |x_R - y_R| > 0. \tag{5.37}$$

Proof. Let (x_R, y_R) be a point of maximum of Φ defined in (5.32) (see the above arguments for the existence). If $(x_R, y_R) \in \left(\mathbb{R}^m \setminus \bar{B}_{\bar{R}}\right) \times \left(\mathbb{R}^m \setminus \bar{B}_{\bar{R}}\right)$, the claim is proved. Otherwise, there are three possible cases (up to subsequences):

(i) $(x_R, y_R) \in \bar{B}_{\bar{R}} \times \bar{B}_{\bar{R}}$;
(ii) $(x_R, y_R) \in \bar{B}_{\bar{R}} \times \left(\mathbb{R}^m \setminus \bar{B}_{\bar{R}}\right)$;
(iii) $(x_R, y_R) \in \left(\mathbb{R}^m \setminus \bar{B}_{\bar{R}}\right) \times \bar{B}_{\bar{R}}$.

Suppose we are in case (i). We apply the local estimate on $\bar{B}_{\bar{R}}$ (5.30) and by the choice of C in (5.31), we get a contradiction with (5.34).

Now we deal with case (ii) and we observe that case (iii) can be treated analogously. We prove that there exists $z_R \in \mathbb{R}^m \setminus \bar{B}_{\bar{R}}$ such that (z_R, y_R) is still a maximum point of the function Φ. Note that we can suppose that $y_R \in \mathbb{R}^m \setminus \bar{B}_{\bar{R}+1}$. Indeed, if $y_R \in \bar{B}_{\bar{R}+1}$, we use the local estimate on $\bar{B}_{\bar{R}+1}$ and by the choice of C in (5.31), we get a contradiction with (5.34). Let z_R, z'_R be respectively the points where the segment between x_R and y_R intersects the boundary of $B_{\bar{R}+1}$ and of $B_{\bar{R}}$. Note that

$$|x_R - y_R| = |x_R - z_R| + |z_R - y_R| \tag{5.38}$$

and

$$|x_R - z_R| = |x_R - z'_R| + 1. \tag{5.39}$$

Then, by (5.38), we have

$$\max \Phi = \Phi(x_R, y_R) \leq \Phi(z_R, y_R) + w_\delta(x_R) - w_\delta(z_R) - C|x_R - z_R| - \psi_R(x_R) + \psi_R(z_R),$$

and by the local estimate (5.30) on $\bar{B}_{\bar{R}+1}$ coupled with (5.39), we get

$$\max \Phi \leq \Phi(z_R, y_R) + C_{\bar{R}+1}|x_R - z'_R| + C_{\bar{R}+1} - C|x_R - z'_R| - C - \psi_R(x_R) + \psi_R(z_R).$$

By the choice of C in (5.31) we get

$$\max \Phi \leq C_{\bar{R}+1} - C + \Phi(z_R, y_R) - \psi_R(x_R) + \psi_R(z_R)$$

and, by taking R large enough so that $C_{\bar{R}+1} - C - \psi_R(x_R) + \psi_R(z_R) \leq 0$, we conclude

$$\max \Phi \leq \Phi(z_R, y_R).$$

Then, for R large enough, $(z_R, y_R) \in \left(\mathbb{R}^m \setminus \bar{B}_{\bar{R}}\right) \times \left(\mathbb{R}^m \setminus \bar{B}_{\bar{R}}\right)$ is a point of maximum of the function Φ. This conclude the proof of the first claim.

Now we prove (5.37). By contradiction, we suppose that

$$\liminf_{R \to +\infty} |x_R - y_R| = 0.$$

By (5.35) and the definition of ψ_R, we have

$$M_R \leq w_\delta(x_R) - w_\delta(y_R).$$

Now we use Proposition 5.3 and by (5.3), we get

$$M_R \leq C_\delta |x_R - y_R|^\alpha.$$

Then, since $M_R \to M > 0$, we get the following contradiction

$$0 < \liminf_{R \to +\infty} M_R \leq \liminf_{R \to +\infty} C_\delta |x_R - y_R|^\alpha = 0,$$

concluding the proof. $\square$

From now on we omit the dependence on R and we write

$$(x_R, y_R) = (x, y).$$

We prove the following lemma.

Lemma 5.7. *Under the above notations and assumptions, there exists two positive constants K_1, K_2 such that*

$$C|x - y| \leq CK_1 g(x, y)|x - y| + K_2|x - y| + o_R(1), \tag{5.40}$$

where $g : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^+$ is such that $\forall \varepsilon > 0$ there exists R_ε such that $g(x, y) \leq \varepsilon$ for all $|x|, |y| \geq R_\varepsilon$. Moreover K_1, K_2 depends only on $\bar{p}, ||\sigma||_\infty, ||\tau||_\infty$ and by $o_R(1)$ we mean that $\lim_{R \to +\infty} o_R(1) = 0$.

Remark 5.8. Note that $C|x - y|$, on the left side in (5.40), remains strictly positive for $R \to +\infty$ (by Lemma (5.6)). This term stems from the Ornstein−Uhlenbeck term $-(b-y)\cdot Dw_\delta$ in the ergodic problem (5.29).

Proof. We denote

$$r_x := D\psi_R(x) = R^{-1}\psi'\left(\frac{\sqrt{|x|^2 + 1}}{R}\right) x(\sqrt{|x|^2 + 1})^{-1} \tag{5.41}$$

$$r_y := D\psi_R(y) = R^{-1}\psi'\left(\frac{\sqrt{|y|^2 + 1}}{R}\right) y(\sqrt{|y|^2 + 1})^{-1}. \tag{5.42}$$

We remark that

$$|r_x|, |r_y| \leq R^{-1}||\psi'||_\infty, \tag{5.43}$$

where $||\psi'||_\infty$ depends on δ. Similarly we argue for the second derivatives of ψ_R and we get

$$||D^2\psi_R(z)||_\infty \le o_R(1), \tag{5.44}$$

where $o_R(1)$ means that $\lim_{R\to+\infty} o_R(1) = 0$.

Note that in the rest of the proof we denote by $o_R(1)$ any function respectively such that $o_R(1) \to 0$ as $R \to +\infty$. We also denote

$$s = C\frac{x-y}{|x-y|}. \tag{5.45}$$

Notice that the function in (5.32) is smooth since for R big enough $x \ne y$ by Lemma 5.6. Then, since w_δ is a viscosity solution of (5.29) and since (x, y) is a maximum point of the function in (5.32), we have

$$L(x, y) \le \text{tr}(\tau\tau^T D^2 w_\delta(x)) - \text{tr}(\tau\tau^T D^2 w_\delta(y)) + o_R(1) + G(x, y) + E(x, y) + F(x, y) + D(x, y), \tag{5.46}$$

where we used (5.43) and (5.44) to estimate the ψ_R-terms and where we denote

$$D(x, y) = \delta w_\delta(y) - \delta w_\delta(x);$$

$$L(x, y) = (s, (x-y)) - (b-y, r_y) - (b-x, r_x);$$

$$G(x, y) = |\tau^T(s + r_x)|^2 - |\tau^T(s - r_y)|^2;$$

$$E(x, y) = (2\tau\sigma(\bar{x}, x)^T\bar{p}, s + r_x) - (2\tau\sigma(\bar{x}, y)^T\bar{p}, s - r_y);$$

$$F(x, y) = |\sigma^T(\bar{x}, x)\bar{p}|^2 - |\sigma^T(\bar{x}, y)\bar{p}|^2.$$

We estimate each term in (5.46). The most important terms is L since it gives rise to the left order term $C|x-y|$ in (5.40). Indeed by (5.45), we have

$$L(x, y) \ge C|x-y| - (\mu - y)\cdot r_y - (\mu - x)\cdot r_x$$

and notice that by (5.41) and (5.42) we have

$$x\cdot r_x = R^{-1}|x|^2\psi'\left(\frac{\sqrt{|x|^2+1}}{R}\right)(\sqrt{|x|^2+1})^{-1}$$

and

$$y\cdot r_y = R^{-1}|y|^2\psi'\left(\frac{\sqrt{|y|^2+1}}{R}\right)y(\sqrt{|y|^2+1})^{-1}$$

and since $\psi' \ge 0$ by definition of ψ_R, we have

$$x\cdot r_x \ge 0, \quad y\cdot r_y \ge 0. \tag{5.47}$$

By (5.47) and (5.43), we get

$$-(b-y)\cdot r_y - (b-x)\cdot r_x \ge o_R(1),$$

and then

$$L(x, y) \ge C|x-y| + o_R(1).$$

Then by the previous estimates we get

$$C|x-y| \le \text{tr}(\tau\tau^T D^2 w_\delta(x)) - \text{tr}(\tau\tau^T D^2 w_\delta(y)) + o_R(1) + G(x, y) + E(x, y) + F(x, y) + D(x, y). \tag{5.48}$$

Now we estimate the remaining terms in the right-hand side of (5.48). First note that

$$D(x, y) = \delta w_\delta(y) - \delta w_\delta(x) \leq 0.$$

By (5.43) and (5.45), we have

$$G(x, y) \leq o_R(1). \tag{5.49}$$

Next, by (S1) and (S2) and the boundedness of σ, we have

$$E(x, y) \leq CK_1 g(x, y)|x - y| + o_R(1),$$

where $K_1 > 0$ depends on $\bar{p}, ||\tau||_\infty, ||\sigma||_\infty$.

By the Lipschitz continuity and boundedness of σ, we have

$$F(x, y) \leq K_2 |x - y|,$$

where K_2 depends on $||\sigma||_\infty$ and the Lipschitz constant of σ and on $\bar{p}$.

Finally we estimate the second order terms in (5.48) as follows

$$\text{tr}(\tau\tau^T D^2 w_\delta(x)) - \text{tr}(\tau\tau^T D^2 w_\delta(y)) \leq o_R(1). \tag{5.50}$$

where by $o_R(1)$ we mean that $\lim_{R\to+\infty} o_R(1) = 0$. The proof of (5.50) is analogous to the proof of (5.17), Lemma 5.5, Proposition 5.3 and even simpler. Indeed, we use again the following property: if e_i, $i = 1, \ldots m$ is an orthonormal basis of $\mathbb{R}^m$ and A is a matrix $m \times m$, we have

$$\text{tr}(A) = \sum_{i=1}^{m} (Ae_i, e_i),$$

then for any orthonormal basis e_i, $i = 1, \ldots m$ of $\mathbb{R}^m$, we can write

$$\text{tr}(\tau\tau^T D^2 w_\delta(x)) = \sum_{i=1}^{m}(\tau\tau^T D^2 w_\delta(x)e_i, e_i) = \sum_{i=1}^{m}(D^2 w_\delta(x)\tau e_i, \tau e_i). \tag{5.51}$$

Denote $f(z) = |z|$. We recall that the function in (5.32) is smooth at $(x, y) = (x_R, y_R)$ for R large enough by Lemma 5.6. Then, since x, y is a maximum point of the function in (5.32) and by (5.44), we get

$$(D^2 w_\delta(x)p, p) - (D^2 w_\delta(y)q, q) \leq C(D^2 f(x - y)(p - q), (p - q)) + o_R(1) \tag{5.52}$$

for any $p, q \in \mathbb{R}^m$. Then, in order to prove the claim, it is enough to choose in (5.52) for all $i \in \{1, \ldots m\}$

$$p = \tau e_i, \quad q = \tau e_i.$$

Then we get

$$(D^2 w_\delta(x)\tau e_i, \tau e_i) - (D^2 w_\delta(y)\tau e_i, \tau e_i) \leq o_R(1) \quad \text{for all } i \in \{1, \ldots m\},$$

and by summing the previous equation on i, we get

$$\sum_{i=1}^{m}(D^2 w_\delta(x)\tau e_i, \tau e_i) - \sum_{i=1}^{m}(D^2 w_\delta(y)\tau e_i, \tau e_i) \leq o_R(1)$$

from which we conclude (5.50). By coupling all the previous estimates, we get (5.40) and we conclude the proof of Lemma 5.7. $\qquad\qquad\square$

Now we conclude the the argument as follows. We use assumptions (S1) and (S2) and by taking $\bar{R} > R_1$ large enough, we consider $|x|, |y|$ large enough, such that

$$K_1 g(x, y) \leq \frac{1}{2}. \tag{5.53}$$

Now we send $R \to +\infty$ in (5.40) and divide by $|x - y|$ thanks to Lemma 5.6, and we get

$$C \leq \frac{C}{2} + K_2, \tag{5.54}$$

Then, to get a contradiction with (5.54), it is enough to take C large enough such that

$$C > 2K_2. \tag{5.55}$$

Note that $C, \bar{R}$ depend respectively only on K_2, R_1 and in particular, they are independent on δ.

Then the proof follows by taking C in (5.31), such that $C > \max\{C_{\bar{R}}, C_{\bar{R}+1}, 2K_2\}$, where $\bar{R} > R_1$ is such that (5.53) holds. $\qquad\square$

6. The comparison principle

In this section we provide the comparison principle for the limit PDE

$$v_t - \bar{H}(x, Dv) = 0 \quad \text{in } (0, T) \times \mathbb{R}^n, \tag{6.1}$$

where $\bar{H}$ is defined in Proposition 3.3 for $\alpha = 2$ and in Proposition 4.4 for $\alpha > 2$.

Note that the comparison principle for the limit problem is a crucial ingredient in the proof of the convergence, which we address in the following section.

Theorem 6.1. *Let Assumption* (U) *hold. Let* $u \in BUSC([0, T] \times \mathbb{R}^n)$ *and* $v \in BLSC([0, T] \times \mathbb{R}^n)$ *be, respectively, a subsolution and a supersolution to* (6.1) *such that* $u(0, x) \leq v(0, x)$ *for all* $x \in \mathbb{R}^n$. *Then* $u(x, t) \leq v(x, t)$ *for all* $x \in \mathbb{R}^n$ *and* $0 \leq t \leq T$.

Proof. The proof is exactly the same to [7] Theorem 3.5, in particular is based on the properties (a), (b), (c), (d) of Proposition 3.6 satisfied by the effective Hamiltonian $\bar{H}$. $\qquad\square$

7. The convergence result

In this section we prove the convergence of the v^ε to the unique solution of the limit problem (7.3). Throughout this section, let Assumptions (U), (S1) and (S2) hold. Let $\alpha \geq 2$. We recall that v^ε denotes the unique bounded viscosity solution of

$$\begin{cases} \partial_t v^\varepsilon - H^\varepsilon \left(x, y, D_x v^\varepsilon, \dfrac{D_y v^\varepsilon}{\varepsilon^{\alpha-1}}, D_{xx}^2 v^\varepsilon, \dfrac{D_{yy}^2 v^\varepsilon}{\varepsilon^{\alpha-1}}, \dfrac{D_{xy}^2 v^\varepsilon}{\varepsilon^{\frac{\alpha-1}{2}}} \right) = 0 & \text{in} \quad [0, T] \times \mathbb{R}^n \times \mathbb{R}^m, \\ v^\varepsilon(0, x, y) = h(x) & \text{in} \quad \mathbb{R}^n \times \mathbb{R}^m. \end{cases} \tag{7.1}$$

where

$$\begin{aligned} H^\varepsilon(x, y, p, q, X, Y, Z) :=\ & |\sigma^T p|^2 + b \cdot q + \text{tr}(\tau\tau^T Y) + \varepsilon \left(\text{tr}(\sigma\sigma^T X) + \phi \cdot p \right) \\ & + 2\varepsilon^{\frac{\alpha}{2}-1}(\tau\sigma^T p) \cdot q + 2\varepsilon^{\frac{1}{2}}\text{tr}(\sigma\tau^T Z) + \varepsilon^{\alpha-2}|\tau^T q|^2. \end{aligned}$$

We state and prove the convergence result. We will make use of the relaxed semi-limits which we define as follows. We denote

$$v_1(t,x) := \inf \left\{ \liminf_{\varepsilon \to 0} \{ v^\varepsilon(t_\varepsilon, x_\varepsilon, y_\varepsilon) \,|\, x_\varepsilon \to x,\, t_\varepsilon \to t,\, y_\varepsilon \text{ bounded} \} \right\},$$

$$v_2(t,x) := \sup \left\{ \limsup_{\varepsilon \to 0} \{ v^\varepsilon(t_\varepsilon, x_\varepsilon, y_\varepsilon) \,|\, x_\varepsilon \to x,\, t_\varepsilon \to t,\, y_\varepsilon \text{ bounded} \} \right\},$$

and we define the lower semi-limit as $\underline{v} = (v_1)_*$ the lower semicontinuous regularization of v_1 and the upper semi-limit as $\bar{v} = v_2^*$ the upper semicontinuous regularization of v_2.

Since h is bounded, the family v^ε is equibounded and we have $\bar{v} \in BUSC([0,T] \times \mathbb{R}^n)$ and $\underline{v} \in BLSC([0,T] \times \mathbb{R}^n)$. Notice that by definition, we have

$$\bar{v}(x,t) \geq \underline{v}(x,t). \tag{7.2}$$

Theorem 7.1. *Let Assumptions* (U), (S1) *and* (S2) *hold. Recall the effective problem*

$$v_t - \bar{H}(x, Dv) = 0 \text{ in } (0,T) \times \mathbb{R}^n \quad v(0,x) = h(x) \text{ on } \mathbb{R}^n \tag{7.3}$$

where $\bar{H}$ is defined by Proposition 3.3 *for $\alpha = 2$ and Proposition* 4.4 *for $\alpha > 2$. Then*

(a) *the upper limit $\bar{v}$ of v^ε is a subsolution of* (7.3);
(b) *the lower limit $\underline{v}$ is a supersolution of* (7.3);
(c) *v^ε converges uniformly on the compact subsets of $[0,T] \times \mathbb{R}^n \times \mathbb{R}^m$ to the unique viscosity solution of* (7.3).

Proof. Note that, once (a) and (b) proved, by the definition of semilimits and by the comparison principle (Thm. 6.1) for the effective equation (7.3), we get

$$\bar{v} = \underline{v} = v \quad \text{in } [0,T] \times \mathbb{R}^n$$

and then, thanks to the properties of semilimits, we get that v^ε converges locally uniformly to the unique bounded solution of (7.3). Therefore, the main claims which we have to prove are (a) and (b). We prove only (a) since the proof of (b) is analogous. Moreover, since the proofs for the critical and supercritical case are similar with some minor (and standard) adaptations, we treat only the case $\alpha = 2$.

We take a smooth function ψ, and without loss of generality we assume that ψ is coercive in the variable x and for all compact $K \subset [0,T] \times \mathbb{R}^n$ there exists a constant $C_K > 0$ such that

$$|\partial_t \psi(t,x)| \leq C_K \quad \forall (t,x) \in K. \tag{7.4}$$

Let $(\bar{t}, \bar{x})$ be a point of strict maximum of $\bar{v}(t,x) - \psi(t,x)$. Let $\eta > 0$ and consider the function

$$\Phi(t,x,y) = v^\varepsilon(t,x,y) - \psi(t,x) - \varepsilon(w(y) + \eta\chi(y)), \tag{7.5}$$

where w is the corrector, solution to the ergodic problem (3.1) for $\lambda = \bar{H}(\bar{x}, D_x\psi(\bar{t}, \bar{x}))$ and χ is the Liapounov function, that is

$$\chi = a|y|^2, \quad a < \frac{1}{2T}, \tag{7.6}$$

for some $T > 0$ depending on $||\tau||_\infty$ which we defined in (2.11).

By (3.8) and the definition (2.10) of χ, we have for η fixed

$$w(y) + \eta\chi(y) \to +\infty \text{ as } |y| \to +\infty.$$

Then, there exists $(t_{\varepsilon,\eta}, x_{\varepsilon,\eta}, y_{\varepsilon,\eta}) \in [0,T] \times \mathbb{R}^n \times \mathbb{R}^m$ point of maximum of Φ defined in (7.5). We denote

$$(t_{\varepsilon,\eta}, x_{\varepsilon,\eta}, y_{\varepsilon,\eta}) =: (t,x,y).$$

Since v_ε is a solution of equation (7.1), we test it as a subsolution with the function $\psi + \varepsilon(w + \eta\chi)$ and by writing

$$|\tau(y)^T(Dw(y) + \eta D\chi(y))|^2 = |\tau(y)^T Dw(y)|^2 + \eta^2 |\tau(y)^T D\chi(y)|^2 + 2\eta(\tau(y)^T Dw(y), D\chi(y)),$$

we get

$$\psi_t(t,x) - \varepsilon \text{tr}(\sigma\sigma(x,y)^T D_{xx}^2\psi(t,x)) - \varepsilon\phi(x,y) \cdot D_x\psi(t,x) - |\sigma(x,y)^T D_x\psi(t,x)|^2$$
$$- b(y) \cdot Dw(y) - \text{tr}(\tau(y)\tau(y)^T D^2 w(y)) - 2\tau(y)^T\sigma(x,y)^T D_x\psi(t,x) \cdot Dw(y) - |\tau(y)^T Dw(y)|^2$$
$$+ \eta G_{\varepsilon,\eta}(x,y) \le 0, \quad (7.7)$$

where, for convenience of notations, we denote

$$G_{\varepsilon,\eta}(x,y) = -b(y) \cdot D\chi(y) - \text{tr}(\tau(y)\tau(y)^T D^2\chi(y)) - \eta|\tau(y)^T D\chi(y)|^2$$
$$- 2\tau(y)^T Dw(y) \cdot D\chi(y) - 2\tau(y)\sigma(x,y)^T D_x\psi(t,x) \cdot D\chi(y). \quad (7.8)$$

We recall that the corrector w is solution of the ergodic problem (3.1) for $\lambda = \bar{H}(\bar{x}, D_x\psi(\bar{t},\bar{x}))$ (see Prop. 3.3), that is, w satisfies

$$\bar{H}(\bar{x}, D\psi(\bar{t},\bar{x})) = b(y) \cdot Dw(y) + \text{tr}(\tau(y)\tau(y)^T D^2 w(y)) + |\tau(y)^T Dw(y)|^2$$
$$+ 2(\tau(y)\sigma(\bar{x},y)^T D_x\psi(\bar{t},\bar{x})) \cdot Dw(y) + |\sigma(\bar{x},y)^T D_x\psi(\bar{t},\bar{x})|^2. \quad (7.9)$$

We use (7.9) in (7.7) and we get

$$\psi_t(t,x) - \varepsilon\text{tr}(\sigma\sigma(x,y)^T D_{xx}^2\psi(t,x)) - \varepsilon\phi(x,y) \cdot D_x\psi(t,x) + \eta G_{\varepsilon,\eta}(x,y) + F_\varepsilon(x,y) - \bar{H}(\bar{x}, D\psi(\bar{t},\bar{x})) \le 0, \quad (7.10)$$

where we denote

$$F_\varepsilon(x,y) = (-2\tau(y)\sigma(x,y)^T D_x\psi(t,x) + 2\tau(y)\sigma(\bar{x},y)^T D_x\psi(\bar{t},\bar{x})) \cdot Dw(y) - |\sigma(x,y)^T D_x\psi(t,x)|^2 + |\sigma(\bar{x},y)^T D_x\psi(\bar{t},\bar{x})|^2. \quad (7.11)$$

In the following lemma we prove that (x,t,y) are uniformly bounded in ε and that $x, t \to \bar{x}, \bar{t}$ as $\varepsilon \to 0$. Note that we split the proof of the equiboundedness of (t,x,y) into (i) and (ii) in the following lemma only for convenience of exposition.

Lemma 7.2. *Let $\eta > 0$ be fixed. Under the above notations and under the assumptions of Theorem 7.1, we have*

(i) *(x,t) are uniformly bounded in ε;*
(ii) *y is uniformly bounded in ε;*
(iii) *$(x,t) \to (\bar{x},\bar{t})$ as $\varepsilon \to 0$.*

We split the proof into three steps; in Step 1 we prove (i), in Step 2 we prove (ii) and in Step 3 we prove (iii).

Proof of Lemma 7.2.

Step. 1 (Proof of (i)).
For all $x' \in \mathbb{R}^n, y' \in \mathbb{R}^m$ and $t' \in (0,T)$ we have

$$v^\varepsilon(t,x,y) - \psi(t,x) - \varepsilon(w(y) + \eta\chi(y)) \ge v^\varepsilon(t',x',y') - \psi(t',x') - \varepsilon(w(y') + \eta\chi(y')),$$

that is

$$\psi(t,x) + \varepsilon(w(y) + \eta\chi(y)) \le 2\sup_\varepsilon \|v^\varepsilon\|_\infty + \sup_\varepsilon [\psi(t',x') + \varepsilon(w(y') + \eta\chi(y'))]$$

then

$$\sup_{\varepsilon} \left[\psi(t,x) + \varepsilon(w(y) + \eta\chi(y))) \right] < \infty. \tag{7.12}$$

Note that (7.12) implies

$$\sup_{\varepsilon} \psi(t,x) < \infty. \tag{7.13}$$

Indeed, (7.13) follows immediately from (7.12) if $|y|$ is bounded in ε; when $|y| \to +\infty$ it follows since $\varepsilon(w(y) + \eta\chi(y)))$ is positive thanks to the definition of χ and the logarithmic growth of w proved in (3.8). Then the uniform boundedness of x and t follows from (7.13) and the coercivity of ψ.

Step. 2 (Proof of (ii)).

We proceed by contradiction, supposing $|y| \to +\infty$ as $\varepsilon \to 0$ and we get a contradiction with the equation (7.10) by applying Lemma 7.3, whose proof is postponed at the end of the proof of (a). We just observe that it essentially relies on (i) of Lemma 7.2 proved in step 1, on the quadratic growth of the Liapounov function χ and on the uniform estimate of the gradient of the corrector w (Prop. 5.2).

Lemma 7.3. *Let assumptions of Theorem* 7.1 *hold. Let* $G_{\varepsilon,\eta}(x,y)$ *and* $F_\varepsilon(x,y)$ *be defined respectively in* (7.8) *and* (7.11) *and let* $\eta > 0$ *be fixed. Then, if*

$$|y| \to +\infty \ \text{as} \ \varepsilon \to 0, \tag{7.14}$$

then we have

(1) $\lim_{\varepsilon \to 0} G_{\varepsilon,\eta}(x,y) = +\infty.$
(2) $|\lim_{\varepsilon \to 0} F_\varepsilon(x,y)| \leq C'$, *for some constant* $C' > 0$.

Then the uniform boundedness of y follows by coupling (1) and (2) of Lemma 7.3 with equation (7.10) and observing that ϕ and σ are bounded, t, x are uniformly bounded in ε and the time derivative of ψ is bounded by (7.4).

Step. 3 (Proof of (iii)).

Note that, by Steps 1 and 2, we can suppose that there exists $(\tilde{t}, \tilde{x}, \tilde{y})$ such that, up to subsequences

$$(t,x,y) \to (\tilde{t}, \tilde{x}, \tilde{y}) \quad \text{as} \ \varepsilon \to 0. \tag{7.15}$$

Since, for all t', x', y',

$$v^\varepsilon(t,x,y) - \psi(t,x) - \varepsilon(w(y) + \eta\chi(y)) \geq v^\varepsilon(t',x',y') - \psi(t',x') - \varepsilon(w(y') - \eta\chi(y')),$$

using the uniform boundedness of y and the definition of upper semi-limit we get

$$\bar{v}(\tilde{t}, \tilde{x}) - \psi(\tilde{t}, \tilde{x}) \geq \bar{v}(t',x') - \psi(t',x') \quad \forall t', x'.$$

Then

$$\tilde{x} = \bar{x}, \quad \tilde{t} = \bar{t}$$

and

$$t \to \bar{t}, \quad x \to \bar{x} \quad \text{as} \ \varepsilon \to 0, \tag{7.16}$$

concluding the proof of the lemma. $\square$

Now we conclude the proof of Theorem 7.1 (a).

Note that from now on when we do the limit as $\varepsilon \to 0$, we mean the limit along the subsequences such that (7.15) (and then also (7.16)) hold.

Note that, by (iii) of Lemma 7.2 and by definition of the corrector w, we have

$$\lim_{\varepsilon \to 0} F_\varepsilon(x, y) = 0, \tag{7.17}$$

where F_ε is defined in (7.11). Then, we let $\varepsilon \to 0$ in (7.10) and use again (7.16), (7.15) and (7.17) to get

$$\psi_t(\bar{t}, \bar{x}) + \eta G_\eta(\bar{x}, \tilde{y}) - \bar{H}(\bar{x}, D\psi(\bar{t}, \bar{x})) \le 0. \tag{7.18}$$

where

$$G_\eta(\bar{x}, \tilde{y}) := \lim_{\varepsilon \to 0} G_{\varepsilon, \eta}(x, y),$$

where and $G_{\varepsilon, \eta}$ is defined in (7.8).

Note that

$$G_\eta(\bar{x}, \tilde{y}) = -b(\tilde{y}) \cdot D\chi(\tilde{y}) - \operatorname{tr}(\tau(\tilde{y})\tau(\tilde{y})^T D^2\chi(\tilde{y})) - \eta|\tau(\tilde{y})^T D\chi(\tilde{y})|^2$$
$$- 2\tau(\tilde{y})^T Dw(\tilde{y}) \cdot D\chi(\tilde{y}) - 2\tau(\tilde{y})\sigma(\bar{x}, \tilde{y})^T D_x\psi(\bar{t}, \bar{x}) \cdot D\chi(\tilde{y}).$$

We observe that if $\tilde{y}$ is uniformly bounded in η, we send $\eta \to 0$ and we conclude

$$\psi_t - \bar{H}(\bar{x}, D\psi(\bar{t}, \bar{x})) \le 0. \tag{7.19}$$

Otherwise, if

$$|\tilde{y}| \to +\infty \text{ as } \eta \to 0,$$

we prove analogously as in Lemma 7.3 (1) that for any η small enough

$$\lim_{\eta \to 0} G_\eta(\bar{x}, \tilde{y}) = +\infty.$$

Then we can suppose for η small

$$\eta G_\eta(\bar{x}, \tilde{y}) \ge 0 \tag{7.20}$$

and by coupling (7.20) with (7.18), we conclude again (7.19). □

Finally we prove Lemma 7.3.

Proof of Lemma 7.3. First we prove Lemma 7.3 (1). Take $\eta, \varepsilon < 1$ and consider $|y| \ge R_1$, where R_1 is defined in (U). We analyse $G_{\varepsilon, \eta}$ term by term:

$$-b(y) \cdot D_y\chi(y) - \eta|\tau(y)^T D_y\chi(y)|^2 \ge 2a|y|^2 - 2a|b||y| - 4a^2T|y|^2,$$

by (7.6) and assumption (U);

$$-\operatorname{tr}(\tau(y)\tau(y)^T D^2\chi(y)) - 2\tau(y)\sigma(x, y)^T D_x\psi(t, x) \cdot D_y\chi(y) \ge -2aK|D_x\psi(t, x)||y| - 2aK,$$

where from now on we denote by $K > 0$ a constant depending only on $||\tau||_\infty, ||\sigma||_\infty$ which may change from line to line. Note that $|D_x\psi(t, x)|$ is bounded uniformly in ε by Lemma 7.2 (i) and the smoothness of ψ. We control the growth of the gradient of w by the global estimate (5.2) proved in Proposition 5.2 and we get

$$-2\tau(y)^T D_y\chi(y) \cdot D_y w(y) \ge -4aCK|y|,$$

where C is defined in (5.2). Then, by coupling all the previous estimates, we get

$$G_{\varepsilon, \eta}(x, y) \ge (2a - 4a^2T)|y|^2 - 2a|b||y| - 4aCK|y| - 2aK|D_x\psi(t, x)||y| - 2aK.$$

and by the second of (7.6), we complete the proof of Lemma 7.3 (1).

In order to prove Lemma 7.3 (2), we use again (5.2) of Proposition 5.2 to get

$$\tau(y)\sigma(\bar{x}, y)^T D_x \psi(t, x)) \cdot D_y w(y) \geq -KC|D_x \psi(t, x))|,$$

where $C > 0$ is defined in (5.2). Then, since (t, x) are bounded in ε by Lemma 7.2 (i), τ, σ are bounded, and by the definition of F_ε in (7.11), we conclude

$$F_\varepsilon(x, y) \geq -C',$$

for some constant $C' > 0$. The upper bound in Lemma 7.3 (2) is deduced by similar arguments. $\qquad\square$

8. Applications to option pricing and implied volatility

In this section we derive a large deviation principle for the process X_t^ε defined in (2.3). As applications, we give estimates of option prices near maturity and an asymptotic formula for the implied volatility. Similar financial applications were also given in [7], in particular we refer to Theorem 7.1, Corollaries 8.1 and 8.2. However, at least to our point of view, these financial applications take on particular relevance in the present paper due to the unbounded setting, which provides a more realistic model for volatility as appearing in financial markets. We refer to the introduction of the present paper for further discussions.

Throughout this section, we assume that there exists $\nu > 0$ such that

$$|\sigma(x, y)^T \xi|^2 \geq \nu|\xi|^2, \qquad \text{for all } x \in \mathbb{R}^n, y \in \mathbb{R}^m, \xi \in \mathbb{R}^n. \tag{8.1}$$

Remark 8.1. With respect to assumption (S1), where the uniform non degeneracy of σ is supposed only the fast variable and for $\alpha > 2$, (8.1) assumes the uniform non degeneracy in both variable and for all ranges of α. Assumption 8.1 is supposed only in this section in order to ensure the coercivity of the effective Hamiltonian and the well-definition of the rate function, see (8.2).

Before giving our results, let us recall the definition of a large deviation principle. Given a family of probability measures $\{\mu_\varepsilon\}$, a large deviation principle characterizes the limiting behaviour as $\varepsilon \to 0$ of $\{\mu_\varepsilon\}$ in terms of a rate function through asymptotic upper and lower exponential bounds on the values that μ_ε assigns to measurable subsets of $\mathbb{R}^n$. The precise definition is recalled in the following.

Definition 8.2. Let $\mathcal{B}$ denotes the Borel σ-field of $\mathbb{R}^n$.

(i) A rate function I is a lower semicontinuous map $I : \mathbb{R}^n \to [0, \infty]$ and it is a good rate function if for all $\alpha \in [0, \infty)$, the level set $\Psi_I(\alpha) := \{x : I(x) \leq \alpha\}$ is compact.

(ii) A family of probability measures $\{\mu_\varepsilon\}$ satisfies the large deviation principle with a rate function I if, for all $B \in \mathcal{B}$,

$$-\inf_{x \in B^\circ} I(x) \leq \liminf_{\varepsilon \to 0} \varepsilon \log \mu_\varepsilon(B) \leq \limsup_{\varepsilon \to 0} \varepsilon \log \mu_\varepsilon(B) \leq -\inf_{x \in \bar{B}} I(x).$$

For each $x_0 \in \mathbb{R}^n$ and $t > 0$, we define

$$I(x; x_0, t) := \inf \left[\int_0^t \bar{L}\left(\xi(s), \dot{\xi}(s)\right) ds \,|\, \xi \in AC(0, t), \xi(0) = x_0, \xi(t) = x \right], \tag{8.2}$$

where $\bar{L}$ is the *effective Lagrangian*, i.e. for $x \in \mathbb{R}^n$, $\bar{L}(x, q) = \max_{p \in \mathbb{R}^n} \{p \cdot q - \bar{H}(x, p)\}$.

Remark 8.3. Note that under (8.1) and by (3.19), the effective Hamiltonian $\bar{H}$ is coercive.

Remark 8.4. The rate function I satisfies the following properties:

(a) I is continuous in the variable x (see, *e.g.* [7, 22]) and is a nonnegative function such that $I(x_0; x_0, t) = 0$.

(b) I satisfies the following growth condition for some $C > 0$ and all $x, x_0 \in \mathbb{R}^n$

$$\frac{1}{4C} \frac{|x - x_0|^2}{t} \leq I(x; x_0, t) \leq \frac{1}{4\nu} \frac{|x - x_0|^2}{t}, \tag{8.3}$$

where ν is defined in (S1). The proof of (8.3) easily follows by (3.19) and by the definition of I (8.2) (see also [7], Remark 7 for more details).

(c) If σ does not depend on x, i.e. $\bar{H} = \bar{H}(p)$, the rate function in (8.2) is

$$I(x; x_0, t) = t\bar{L}\left(\frac{x - x_0}{t}\right).$$

(d) If σ does not depend on x and $n = 1$, I is a monotone non decreasing function of x when $x > x_0$. Analogously, I is a monotone non increasing function of x when $x < x_0$.

The firs result is the following large deviation principle for the process X_t^ε in (2.3).

Theorem 8.5. *Let Assumption* (U), (S1) *and* (S2) *hold. Let* $(X^\varepsilon, Y^\varepsilon)$ *be the process defined in* (2.3) *with initial position* $X_0^\varepsilon = x_0$ *and* $Y_0^\varepsilon = y_0$. *Then for every* $t > 0$, *a large deviation principle holds for* $\{X_t^\varepsilon : \varepsilon > 0\}$ *with speed* $\frac{1}{\varepsilon}$ *and good rate function* $I(x; x_0, t)$. *In particular, for any open set* $B \subseteq \mathbb{R}^n$

$$\lim_{\epsilon \to 0} \epsilon \log P(X_t^\varepsilon \in B) = -\inf_{x \in B} I(x; x_0, t). \tag{8.4}$$

Proof. The proof follows by the same arguments to Theorem 7.1 [7], namely by proving the exponential tightness of $\{X_t^\varepsilon\}$ and then applying Bryck inverse Varadhan lemma with $\{\mu_\varepsilon\}$ given by the laws of $\{X_\varepsilon\}$ and Λ_h given by v_ε. A crucial step is the convergence of v_ε, which is proved in Theorem 7.1. We omit the details since the proof of the exponential tightness of $\{X_t^\varepsilon\}$ is the same as in [7]. $\square$

In the following we give estimates of option prices near maturity (Cor. 8.6) and an asymptotic formula for the implied volatility (Cor. 8.7). The proofs are omitted since they follow by direct applications of Theorem 8.5 (and moreover, similar arguments were used in [7] Cors. 8.1 and 8.2). We refer also to [28] for a detailed review of the results of the present paper and of [7]. First let us set some notation in order to introduce the results. Let S_t^ε be the asset price, evolving according to the following stochastic differential system

$$\begin{cases} dS_t^\varepsilon = \varepsilon\xi(S_t^\varepsilon, Y_t^\varepsilon)S_t^\varepsilon dt + \sqrt{2\varepsilon}\zeta(S_t^\varepsilon, Y_t^\varepsilon)S_t^\varepsilon dW_t & S_0^\varepsilon = S_0 \in \mathbb{R}_+ \\ dY_t^\varepsilon = \varepsilon^{1-\alpha}b(Y_t^\varepsilon)dt + \sqrt{2\varepsilon^{1-\alpha}}\tau(Y_t^\varepsilon)dW_t & Y_0^\varepsilon = y_0 \in \mathbb{R}^m, \end{cases} \tag{8.5}$$

where $\alpha > 2$, τ, b are as in (2.3) and $\xi : \mathbb{R}_+ \times \mathbb{R}^m \to \mathbb{R}$, $\zeta : \mathbb{R}_+ \times \mathbb{R}^m \to \mathbf{M}^{1,r}$ are Lipschitz continuous bounded functions, periodic in y. Observe that $S_t^\varepsilon > 0$ almost surely if $S_0 > 0$. We define $X_t^\varepsilon = \log S_t^\varepsilon$. Then $(X_t^\varepsilon, Y_t^\varepsilon)$ satisfies (2.3) with

$$\phi(x, y) = \xi(e^x, y) - \zeta(e^x, y)\zeta^T(e^x, y) \qquad \sigma(x, y) = \zeta(e^x, y).$$

We consider out-of-the-money call option by taking

$$S_0 < K \quad \text{or} \quad x_0 < \log K. \tag{8.6}$$

Similarly, by considering out-of-the-money put options, one can obtain the same formula for $S_0 > K$.

Corollary 8.6. *Let Assumptions* (U), (S1) *and* (S2) *hold. Suppose that* $S_0 < K$. *Then, for fixed* $t > 0$

$$\lim_{\varepsilon \to 0^+} \varepsilon \log E\left[(S_t^\varepsilon - K)^+\right] = -\inf_{y > \log K} I(y; x_0, t).$$

We recall that given an observed European call option price for a contract with strike price K and expiration date T, the *implied volatility* $\sigma_\varepsilon(t, \log K, x_0)$ is defined to be the value of the volatility parameter that must go into the Black-Scholes formula to match the observed price. We have the following corollary.

Corollary 8.7. *Let Assumptions* (U), (S1) *and* (S2) *hold. Then*

$$\lim_{\varepsilon \to 0^+} \sigma_\varepsilon^2(t, \log K, x_0) = \frac{(\log K - x_0)^2}{2 \inf_{y > \log K} I(y; x_0, t) t}. \tag{8.7}$$

Remark 8.8. Note that the infimum in the right-hand side of (8.7), is always positive by Assumption (8.6) and by (8.3).

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