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EXISTENCE RESULT FOR DEGENERATE CROSS-DIFFUSION SYSTEM WITH APPLICATION TO SEAWATER INTRUSION

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Abstract. In this paper we study a degenerate parabolic system, which is strongly coupled. We prove general existence result, but the uniqueness question remains open. Our proof of existence is based on a crucial entropy estimate which controls the gradient of the solution together with its non-negativity. Our system is of porous medium type which is applicable to models in seawater intrusion.

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1. Introduction

For the sake of simplicity, we will work on the torus $\Omega := \mathbb{T}^N = (\mathbb{R}/\mathbb{Z})^N$ with $N \geq 1$. Let $\Omega_T := (0,T) \times \Omega$ with $T > 0$ and $m \geq 1$ be an integer. Our purpose is to study a class of degenerate strongly coupled parabolic system of the form

$$\partial_t u^i = \operatorname{div}\left(u^i \sum_{j=1}^{m} A_{ij} \nabla u^j\right) \qquad \text{in} \quad \Omega_T, \quad \text{for} \quad i = 1, \ldots, m. \tag{1.1}$$

with the initial condition

$$u^i(0,x) = u_0^i(x) \geq 0 \qquad \text{a.e. in} \quad \Omega, \quad \text{for} \quad i = 1, \ldots, m. \tag{1.2}$$

In the core of the paper we will assume that $A = (A_{ij})_{1 \leq i,j \leq m}$ is a real $m \times m$ matrix (not necessarily symmetric) that satisfies the following positivity condition: there exists $\delta_0 > 0$, such that

$$\xi^T A \, \xi \geq \delta_0 |\xi|^2 \quad \text{for all} \quad \xi \in \mathbb{R}^m. \tag{1.3}$$

This condition can be weaken, see Section 12. Problem (1.1) appears naturally in the modeling of seawater intrusion (see Sect. 3).

Keywords and phrases. Degenerate parabolic system, entropy estimate; porous medium like systems.

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2. Main results

To introduce our main result, we define a nonnegative entropy function Ψ:

$$\Psi(a) - \frac{1}{e} = \begin{cases} a \ln a & \text{for} \quad a > 0, \\ 0 & \text{for} \quad a = 0, \\ +\infty & \text{for} \quad a < 0, \end{cases} \tag{2.1}$$

which is minimal when $a = \dfrac{1}{e}$.

Theorem 2.1 (Existence for system (1.1))**.** *Assume that A satisfies (1.3). For $i = 1, \ldots, m$, let $u_0^i \in L^2(\Omega)$ satisfying $u_0^i \geq 0$ in Ω and*

$$\sum_{i=1}^{m} \int_{\Omega} \Psi(u_0^i) < +\infty, \tag{2.2}$$

where Ψ is given in (2.1). Then there exists a solution $u = (u^i)_{1\leq i\leq m} \in (L^2(0,T;H^1(\Omega)) \cap C([0,T);(W^{1,\infty}(\Omega))'))^m$ of (1.1), (1.2) in distribution sense, with $u^i \geq 0$ a.e. in Ω_T for $i = 1, \ldots, m$. Denote $u^i(t_j) = u^i(t_j, .)$ for $j = 1, 2$. This solution satisfies the following entropy estimate for a.e. $t_1, t_2 \in (0,T)$

$$\sum_{i=1}^{m} \int_{\Omega} \Psi(u^i(t_2)) + \delta_0 \sum_{i=1}^{m} \int_{t_1}^{t_2} \int_{\Omega} |\nabla u^i|^2 \leq \sum_{i=1}^{m} \int_{\Omega} \Psi(u_0^i), \tag{2.3}$$

where Ψ is given in (2.1).

The entropy estimate (2.3) guarantees that $\nabla u^i \in L^2(0,T;L^2(\Omega))$, and therefore allows us to define the product $u^i \sum_{i=1}^{m} A_{ij} \nabla u^j$ in (1.1). After our proofs were obtained, we realized that a similar entropy estimate has been obtained in [4, 8, 10].

Remark 2.2 (Decreasing energy)**.** If A is a symmetric matrix then a solution u of system (1.1) satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\sum_{i=1}^{m} \sum_{j=1}^{m} \int_{\Omega} \frac{1}{2} A_{ij} u^i u^j \right) = -\sum_{i=1}^{m} \int_{\Omega} u^i \left| \sum_{j=1}^{m} A_{ij} \nabla u^j \right|^2.$$

In this paper we let $|\cdot|$ be the usual Euclidean norm in $\mathbb{R}^m$ and let

$$\|A\| = \sup_{|\xi|=1} |A\xi|. \tag{2.4}$$

3. Application to seawater intrusion

In this section, we describe briefly a model of seawater intrusion, which is a particular case of our system (1.1). An aquifer is an underground layer of a porous and permeable rock through which water can move. Coastal aquifers contain freshwater and saltwater from the sea with a sharp interface between them. We refer to [2] for a general overview on seawater intrusion models.

Let $\nu = 1 - \epsilon_0 \in (0,1)$,

$$\epsilon_0 = \frac{\gamma_s - \gamma_f}{\gamma_s},$$

and the constants γ_s and γ_f are the specific weight of the saltwater and freshwater, respectively.

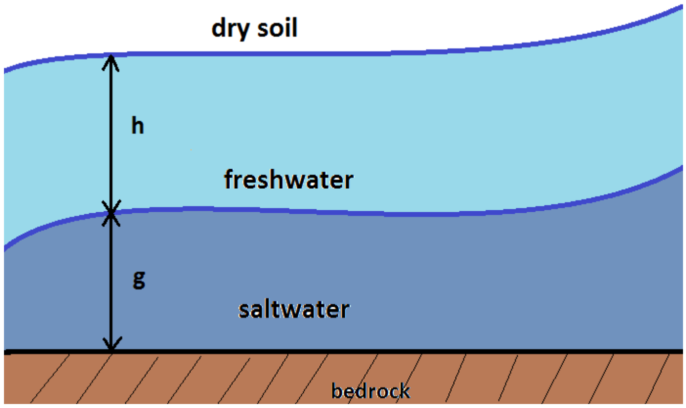

FIGURE 1. Seawater intrusion in coastal aquifer.

In this porous medium, we assume the interface between the saltwater and the bedrock is $z = 0$; the interface between the saltwater and the freshwater, which are assumed to be immiscible, can be written as $z = g(t, x)$; and the interface between the freshwater and the dry soil is $z = h(t, x)+g(t, x)$ (see Fig. 1). Then the evolutions of h and g are given by a coupled nonlinear parabolic system [16]

$$\begin{cases} \partial_t h = \operatorname{div}\{h\nabla(\nu(h+g))\} & \text{in} \quad \Omega_T, \\ \partial_t g = \operatorname{div}\{g\nabla(\nu h+g)\} & \text{in} \quad \Omega_T. \end{cases} \tag{3.1}$$

This is a particular case of (1.1), with a 2×2 matrix

$$A = \begin{pmatrix} \nu & \nu \\ \nu & 1 \end{pmatrix} \tag{3.2}$$

satisfying (1.3).

4. Brief review of the literature

The cross-diffusion systems, in particular the strongly coupled ones (for which the equations are coupled in the highest derivatives terms), are widely employed in diverse domains such as biology, chemistry, ecology, fluid mechanics and others. They are difficult to treat. Many of the standard techniques, such as the maximum principle, cannot be applied for such problems. Hereafter, we cite several models where our method is applicable (see Sect. 13 for more generalizations).

In [27], Shigesada, Kawasaki and Teramoto proposed a two-species SKT model in one-dimensional space which arises in population dynamics. It can be written in a generalized form with m-species as

$$\partial_t u^i - \Delta\left[\left(\beta_i + \sum_{j=1}^m \alpha_{ij}u^j\right)u^i\right] = \left(a_i - \sum_{j=1}^m b_{ij}u^j\right)u^i, \quad \text{in} \qquad \Omega\times(0,T), \tag{4.1}$$

where u^i, for $i = 1, \dots, m$, denotes the population density of the i-th species and β_i, α_{ij}, a_i, b_{ij} are nonnegative constants. In the case where β_i is positive (4.1) is not of degenerate type. The existence of a global solution for such problem in arbitrary space dimension is studied in [30], where the quadratic form of the diffusion matrix is supposed positive definite. On the other hand, the two-species case was frequently studied, see for instance [14,17,23,28,31] for dimensions 1, 2, and [4,5,25,26] for arbitrary dimension and appropriate conditions.

In [20], Lepoutre, Pierre and Rolland studied a relaxed model, without a term source of the form

$$\begin{cases} \partial_t u_i = \Delta\left[a_i(\tilde{u})u_i\right], \quad \tilde{u} = (\tilde{u}_i)_{1\leq i\leq I}, & \text{for} \quad i = 1,\dots,I,\\ \tilde{u}_i - \delta_i \Delta \tilde{u}_i = u_i \text{ with } \delta_i > 0, & \text{for} \quad i = 1,\dots,I, \end{cases}$$

in any dimension and for general nonlinearities a_i, which are only assumed to be continuous and bounded from below. They show the existence of a weak solution. Moreover, if the functions a_i are locally Lipschitz continuous then it is shown that this solution has more regularity and then is unique.

Another example of such problems is the electochemistry model studied by Choi, Huan and Lui in [7] where they consider the general form

$$\partial_t u^i = \sum_{\ell=1}^{n}\sum_{j=1}^{m} \frac{\partial}{\partial x_\ell}\left(a_\ell^{ij}(u)\frac{\partial u^j}{\partial x_\ell}\right), \quad u = (u^i)_{1\leq i\leq m} \qquad \text{for} \quad i = 1,\dots,m, \tag{4.2}$$

and prove the existence of a weak solution of (4.2) under assumptions on the matrices $A_l(u) = (a_l^{ij}(u))_{1\leq i,j\leq m}$: it is continuous in u, its components are uniformly bounded with respect to u and its symmetric part is definite positive. Their strategy of proof seeks to use Galerkin method to prove the existence of solutions to the linearized system and then to apply Schauder fixed-point theorem. Then they apply the results obtained to an electrochemistry model.

A fourth example of cross-diffusion models is the chemotaxis model introduced in [21]. The global existence for classical solutions of this model is studied by Hillen and Painter in [15] where they considered

$$\begin{cases} \partial_t u = \nabla\cdot(\nabla u - \chi(u,v)\nabla v), & t>0, x\in\Omega\\ \partial_t v = \mu\Delta v + g(u,v), & t>0, x\in\Omega, \end{cases}$$

on a C^3- differentiable compact Riemannian manifold without boundary, where the function u describes the particle density, v is the density of the external signal, the chemotactic cross-diffusion χ is assumed to be bounded, and the function g describes production and degradation of the external stimulus.

Another problem is the Muskat Problem for Thin Fluid Layers of the form

$$\begin{cases} \partial_t f = (1+R)\partial_x(f\partial_x f) + R\partial_x(f\partial_x g),\\ \partial_t g = R_\mu\partial_x(g\partial_x f) + R_\mu\partial_x(g\partial_x g). \end{cases}$$

It models, [11], the motion of two fluids with different densities and viscosities in a porous meduim in one dimension, where f and g are the thickness of the two fluids and R, $R_\mu > 0$ depend on the densities and the viscosities of the fluids. The classical solutions of such problem are studied in [11]. The existence of a weak solution and its exponential stability are established in [10] on a bounded interval $(0,L)$. Where the existence of weak solutions on $\mathbb{R}$ are established in [18] by using a gradient flow approach. A key argument in [10, 18] is the availability of two Liapunov functionals.

5. Strategy of the proof

In (1.1), the elliptic part of the equation does not have a Lax-Milgram structure. Otherwise, our existence result can make use of the entropy estimate (2.3). It is difficult to get this entropy estimate directly (we do not have enough regularity to do it), so we proceed by approximations.

Approximation 1.

We discretize in time system (1.1), with a time step $\Delta t = T/K$, where $K \in \mathbb{N}$. Then for a given $u^n = (u^{i,n})_{1\le i\le m} \in (H^1(\Omega))^m$, we consider the implicit scheme which is an elliptic system:

$$\frac{u^{i,n+1}-u^{i,n}}{\Delta t} = \operatorname{div}\left\{u^{i,n+1}\sum_{j=1}^{m} A_{ij}\nabla u^{j,n+1}\right\}. \tag{5.1}$$

Approximation 2.

We regularize the right-hand term of (5.1). To do that, we take $\eta > 0$, $0 < \epsilon < 1 < \ell$, and choose the following regularization

$$\frac{u^{i,n+1}-u^{i,n}}{\Delta t} = \operatorname{div}\left\{T^{\epsilon,\ell}(u^{i,n+1})\sum_{j=1}^{m} A_{ij}\nabla\rho_\eta\star\rho_\eta\star u^{j,n+1}\right\}, \tag{5.2}$$

where $T^{\epsilon,\ell}$ is truncation operator defined as

$$T^{\epsilon,\ell}(a) := \begin{cases} \epsilon & \text{if} \quad a \le \epsilon, \\ a & \text{if} \quad \epsilon \le a \le \ell, \\ \ell & \text{if} \quad a \ge \ell, \end{cases} \tag{5.3}$$

and the mollifier $\rho_\eta(x) = \eta^{-N}\rho(x/\eta)$ with $\rho \in C_c^\infty(\mathbb{R}^N)$, $\rho \ge 0$, $\int_{\mathbb{R}^N}\rho = 1$ and $\rho(-x) = \rho(x)$.

Now, with the convolution by ρ_η in (5.2), the term $\nabla\rho_\eta\star\rho_\eta\star u^{j,n+1}$ behaves like $u^{j,n+1}$.

Note that, after a $\mathbb{Z}^N$- periodic extension of $u^{j,n+1}$ to $\mathbb{R}^N$, the convolution $\rho_\eta\star u^{j,n+1}$ is possible over $\mathbb{R}^N$.

Approximation 3.

We modify equation (5.2) to make it uniformly elliptic. Let $\delta > 0$, we add $\operatorname{div}\left(\delta T^{\epsilon,\ell}(u^i)\nabla u^i\right)$ to its right hand side to preserve the entropy estimate. Then we freeze coefficients $u^{i,n+1}$ on the right-hand side (these coefficients are now called $\delta T^{\epsilon,\ell}(v^{i,n+1})$) and obtain the following linear modified system:

$$\frac{u^{i,n+1}-u^{i,n}}{\Delta t} = \operatorname{div}\left\{T^{\epsilon,\ell}(v^{i,n+1})\left(\sum_{j=1}^{m}A_{ij}\nabla\rho_\eta\star\rho_\eta\star u^{j,n+1} + \delta\nabla u^{i,n+1}\right)\right\}. \tag{5.4}$$

We will look for fixed points of this modified system ie. when $v^{i,n+1} = u^{i,n+1}$. Finally, we derive the expected result by taking appropriate limits to get rid of all the approximations.

6. Organization of the paper

In the next section we recall some useful tools. Then we prove Theorem 2.1; by discretizing our problem on time in Section 7, we obtain an elliptic problem. We use the Lax-Milgram theorem to show the existence of a unique solution to the linear problem (5.4). We demonstrate, in Section 8, the existence of a solution of the nonlinear problem, using the Schaefer's fixed point theorem. Then we pass to the limit in the following order: $(\Delta t, \epsilon) \to (0,0)$ in Section 9, $(\ell, \eta) \to (\infty, 0)$ in Section 10 and $\delta \to 0$ in Section 11.

Generalizations (including more general matrices A or tensors) will be presented in Sections 11 and 12. We end with an Appendix showing some technical results.

Preliminary tools

Theorem 6.1 (Schaefer's fixed point Theorem ([13], Thm. 4 p. 504)).

Let X be a real Banach space. Suppose that

$$\Phi : X \to X$$

is a continuous and compact mapping. Assume further that the set

$$\{u \in X, \quad u = \lambda\Phi(u) \quad \textit{for some} \quad \lambda \in [0, 1]\}$$

is bounded. Then Φ has a fixed point.

Proposition 6.2 (Aubin's lemma [29])**.** *For any $T > 0$, and $\Omega = \mathbb{T}^N$, let E denote the space*

$$E := \left\{g \in L^2((0,T); H^1(\Omega)) \textit{ and } \partial_t g \in L^2((0,T); H^{-1}(\Omega))\right\},$$

endowed with the Hilbert norm

$$\|\omega\|_E = \left(\|\omega\|^2_{L^2(0,T;H^1(\Omega))} + \|\partial_t\omega\|^2_{L^2(0,T;H^{-1}(\Omega))}\right)^{\frac{1}{2}}.$$

The embedding

$$E \hookrightarrow L^2((0,T); L^2(\Omega)) \quad \textit{is compact.}$$

On the other hand, it follows from ([22], Prop. 2.1 and Thm. 3.1, Chap. 1) that the embedding

$$E \hookrightarrow C([0,T]; L^2(\Omega)) \quad \text{is continuous.}$$

Lemma 6.3 (Simon's Lemma [29])**.**
Let X, B and Y three Banach spaces, where $X \hookrightarrow B$ with compact embedding and $B \hookrightarrow Y$ with continuous embedding. If $(g^n)_n$ is a sequence such that

$$\|g^n\|_{L^q(0,T;B)} + \|g^n\|_{L^1(0,T;X)} + \|\partial_t g^n\|_{L^1(0,T;Y)} \leq C,$$

where $1 < q \leq \infty$, and C is a constant independent of n, then $(g^n)_n$ is relatively compact in $L^p(0,T;B)$ for all $1 \leq p < q$.

Now we will present the variant of the original result of Simon's lemma ([29], Cor. 6, p. 87). First of all, let us define the norm $\|.\|_{\mathrm{Var}([a,b);Y)}$ where Y is a Banach space with the norm $\|.\|_Y$.

For a function $g : [a,b) \to Y$, we set

$$\|g\|_{\mathrm{Var}([a,b);Y)} = \sup \sum_j \|g(a_{j+1}) - g(a_j)\|_Y \tag{6.1}$$

over all possible finite partitions:

$$a \leq a_0 < \cdots < a_k < b$$

Theorem 6.4 (Variant of Simon's Lemma)**.**
Let X, B and Y three Banach spaces, where $X \hookrightarrow B$ with compact embedding and $B \hookrightarrow Y$ with continuous embedding. Let $(g^n)_n$ be a sequence such that

$$\|g^n\|_{L^1(0,T;X)} + \|g^n\|_{L^q(0,T;B)} + \|g^n\|_{\mathrm{Var}([0,T);Y)} \leq C, \tag{6.2}$$

where $1 < q < \infty$, and C is a constant independent of n. Then $(g^n)_n$ is relatively compact in $L^p(0,T;B)$ for all $1 \leq p < q$.

Proof.

Step 1. Regularization of the sequence

Let $\bar\rho \in C_c^\infty(\mathbb{R})$ with $\bar\rho \geq 0$, $\int_{\mathbb{R}} \bar\rho = 1$ and supp $\bar\rho \subset (-1,1)$. For $\varepsilon > 0$, we set

$$\bar\rho_\varepsilon(x) = \varepsilon^{-1}\bar\rho(\varepsilon^{-1}x).$$

We extend $g^n = g^n(t)$ by zero outside the time interval $[0,T)$. Because $q < +\infty$, we see that for each n, we choose some $0 < \varepsilon_n \to 0$ as $n \to +\infty$ such that

$$\|\bar g^n - g^n\|_{L^q(0,T;B)} \to 0 \quad \text{as} \quad n \to +\infty, \quad \text{with} \quad \bar g^n = \bar\rho_{\varepsilon_n} \star g^n \tag{6.3}$$

For any $\delta > 0$ small enough, we also have for n large enough (such that $\varepsilon_n < \delta$):

$$\|\bar g^n\|_{L^1(\delta,T-\delta;X)} \leq \|g^n\|_{L^1(0,T;X)} \leq C$$

and

$$\|\partial_t \bar g^n\|_{L^1(\delta,T-\delta;Y)} \leq \|g^n\|_{\mathrm{Var}([0,T);Y)} \leq C \tag{6.4}$$

Step 2. Checking (6.4)

By (6.2) there exists a sequence of step functions f_η which approximates uniformly g^n on $[0,T)$ as $\eta \to 0$, with moreover satisfies

$$\|f_\eta\|_{\mathrm{Var}([0,T);Y)} \to \|g^n\|_{\mathrm{Var}([0,T);Y)}.$$

Therefore we get easily (for $\varepsilon_n < \delta$)

$$\|\partial_t(\bar\rho_{\varepsilon_n} \star f_\eta)\|_{L^1(\delta,T-\delta;Y)} \leq \|f_\eta\|_{\mathrm{Var}([0,T);Y)}$$

which implies (6.4), when we pass to the limit as η goes to zero.

Step 3. Conclusion

We can then apply Corollary 6 in [29] to deduce that $\bar g^n$ is relatively compact in $L^p(0,T;B)$ for all $1 \leq p < q$. Because of (6.3), we deduce that this is also the case for the sequence $(g^n)_n$, which ends the proof of the Theorem. □

Besides the previous statement, several compactness results have been developed recently for piecewise constant functions of time resulting from a time discretization, see [1, 6, 9], Prop. 3.3.1).

Proof of the main theorem

Our goal is to prove Theorem 2.1 in order to get the existence of a solution for system (1.1).

7. Existence for the approximate linear elliptic problem

In this section we prove the existence, *via* Lax-Milgram theorem, of the unique solution for the linear elliptic system (5.4).

Let us recall our linear elliptic system. Assume that A is any $m \times m$ real matrix. Let $v^{n+1} = (v^{i,n+1})_{1\leq i \leq m} \in (L^2(\Omega))^m$ and $u^n = (u^{i,n})_{1\leq i\leq m} \in (H^1(\Omega))^m$. Then for all Δt, ϵ, ℓ, η, $\delta > 0$, with $\epsilon < 1 < \ell$ and $\Delta t < \tau$ where τ is given in (7.2), we look for the solution $u^{n+1} = (u^{i,n+1})_{1\leq i\leq m}$ of the following system:

$$\begin{cases} \dfrac{u^{i,n+1} - u^{i,n}}{\Delta t} = \operatorname{div}\left\{ J^i_{\epsilon,\ell,\eta,\delta}(v^{n+1}, u^{n+1}) \right\} \quad \text{in} \quad \mathcal{D}'(\Omega), \\ J^i_{\epsilon,\ell,\eta,\delta}(v^{n+1}, u^{n+1}) = T^{\epsilon,\ell}(v^{i,n+1}) \left\{ \sum_{j=1}^m A_{ij} \nabla\rho_\eta \star \rho_\eta \star u^{j,n+1} + \delta \nabla u^{i,n+1} \right\}, \end{cases} \tag{7.1}$$

where $T^{\epsilon,\ell}$ is given in (5.3).

Proposition 7.1 (Existence for system (7.1)).
Assume that A is any $m \times m$ real matrix. Let Δt, ϵ, ℓ, η, $\delta > 0$, with $\epsilon < 1 < \ell$, such that

$$\Delta t < \frac{\delta \epsilon \eta^2}{{C_0}^2 \ell^2 \|A\|^2} := \tau, \tag{7.2}$$

where

$$C_0 = \|\nabla \rho\|_{L^1(\mathbb{R}^N)}. \tag{7.3}$$

Then for $n \in \mathbb{N}$, for a given $v^{n+1} = (v^{i,n+1})_{1 \le i \le m} \in (L^2(\Omega))^m$ and $u^n = (u^{i,n})_{1 \le i \le m} \in (H^1(\Omega))^m$, there exists a unique function $u^{n+1} = (u^{i,n+1})_{1 \le i \le m} \in (H^1(\Omega))^m$ solution of system (7.1). Moreover, this solution u^{n+1} satisfies the following estimate

$$\left(1 - \frac{\Delta t}{\tau}\right) \left\|u^{n+1}\right\|^2_{(L^2(\Omega))^m} + \Delta t \epsilon \delta \left\|\nabla u^{n+1}\right\|^2_{(L^2(\Omega))^m} \le \|u^n\|^2_{(L^2(\Omega))^m}, \tag{7.4}$$

where τ is given in (7.2).

Proof. The proof is done in four steps using Lax-Milgram theorem.

First of all, let us define for all $u^{n+1} = (u^{i,n+1})_{1 \le i \le m}$ and $\varphi = (\varphi^i)_{1 \le i \le m} \in (H^1(\Omega))^m$, the following bilinear form:

$$\begin{aligned} a(u^{n+1}, \varphi) = \sum_{i=1}^m \int_\Omega u^{i,n+1} \varphi^i + \Delta t \sum_{i,j=1}^m \int_\Omega T^{\epsilon,\ell}(v^{i,n+1}) A_{ij} \left(\nabla \rho_\eta \star \rho_\eta \star u^{j,n+1}\right) \cdot \nabla \varphi^i \\ + \Delta t \delta \sum_{i=1}^m \int_\Omega T^{\epsilon,\ell}(v^{i,n+1}) \nabla u^{i,n+1} \cdot \nabla \varphi^i, \end{aligned}$$

which can be also rewritten as

$$\begin{aligned} a(u^{n+1}, \varphi) = \left\langle u^{n+1}, \varphi \right\rangle_{(L^2(\Omega))^m} + \Delta t \left\langle T^{\epsilon,\ell}(v^{n+1}) \nabla \varphi, A \nabla \rho_\eta \star \rho_\eta \star u^{n+1} \right\rangle_{(L^2(\Omega))^m} \\ + \Delta t \delta \left\langle T^{\epsilon,\ell}(v^{n+1}) \nabla \varphi, \nabla u^{n+1} \right\rangle_{(L^2(\Omega))^m}, \end{aligned} \tag{7.5}$$

where $\langle \cdot, \cdot \rangle_{(L^2(\Omega))^m}$ denotes the scalar product on $(L^2(\Omega))^m$ and $T^{\epsilon,\ell}(v^{n+1}) \nabla \varphi = \left(T^{\epsilon,\ell}(v^{i,n+1}) \nabla \varphi^i\right)_{1 \le i \le m}$. Also we define the following linear form:

$$L(\varphi) = \sum_{i=1}^m \int_\Omega u^{i,n} \varphi^i = \langle u^n, \varphi \rangle_{(L^2(\Omega))^m}. \tag{7.6}$$

Step 1. Continuity of a

For every $n \in \mathbb{N}$, u^{n+1} and $\varphi \in (H^1(\Omega))^m$, we have

$$\begin{aligned} |a(u^{n+1}, \varphi)| &\le \|u^{n+1}\|_{(L^2(\Omega)^m} \|\varphi\|_{(L^2(\Omega))^m} + \Delta t \ell \|A\| \|\nabla \rho_\eta \star \rho_\eta \star u^{n+1}\|_{(L^2(\Omega))^m} \|\nabla \varphi\|_{(L^2(\Omega))^m} \\ &\quad + \Delta t \delta \ell \|\nabla u^{n+1}\|_{(L^2(\Omega))^m} \|\nabla \varphi\|_{(L^2(\Omega))^m} \\ &\le 3 \max(1, \Delta t \ell \|A\|, \Delta t \delta \ell) \|u^{n+1}\|_{(H^1(\Omega))^m} \|\varphi\|_{(H^1(\Omega))^m}. \end{aligned}$$

where $\|A\|$ is given in (2.4) and we have used the fact that

$$\left\|\nabla \rho_\eta \star \rho_\eta \star u^{n+1}\right\|_{(L^2(\Omega))^m} \le \left\|\nabla u^{n+1}\right\|_{(L^2(\Omega))^m}, \tag{7.7}$$

and

$$\epsilon \le T^{\epsilon,\ell}(s) \le \ell, \qquad \text{for all } s \in \mathbb{R}. \tag{7.8}$$

Step 2. Coercivity of a
For all $\varphi \in (H^1(\Omega))^m$, we have that $a(\varphi,\varphi) = a_0(\varphi,\varphi) + a_1(\varphi,\varphi)$, where

$$a_0(\varphi,\varphi) = \|\varphi\|^2_{(L^2(\Omega))^m} + \Delta t\delta \left\langle T^{\epsilon,\ell}(\varphi)\nabla\varphi, \nabla\varphi\right\rangle_{(L^2(\Omega))^m}$$

and

$$a_1(\varphi,\varphi) = \Delta t \left\langle T^{\epsilon,\ell}(\varphi)\nabla\varphi, A\nabla\rho_\eta \star \rho_\eta \star \varphi\right\rangle_{(L^2(\Omega))^m}.$$

On the one hand, we already have the coercivity of a_0:

$$a_0(\varphi,\varphi) \ge \|\varphi\|^2_{(L^2(\Omega))^m} + \Delta t\delta\epsilon\|\nabla\varphi\|^2_{(L^2(\Omega))^m}.$$

On the other hand, we have

$$\begin{aligned}
|a_1(\varphi,\varphi)| &\le \Delta t\ell \|A\| \|\nabla\rho_\eta \star \rho_\eta \star \varphi\|_{(L^2(\Omega))^m} \|\nabla\varphi\|_{(L^2(\Omega))^m} \\
&\le \Delta t\ell \|A\| \left(\frac{1}{2\alpha}\|\nabla\rho_\eta \star \rho_\eta \star \varphi\|^2_{(L^2(\Omega))^m} + \frac{\alpha}{2}\|\nabla\varphi\|^2_{(L^2(\Omega))^m}\right) \\
&\le \frac{\Delta t\ell^2 \|A\|^2 C_0^2}{2\delta\epsilon\eta^2}\|\varphi\|^2_{(L^2(\Omega))^m} + \frac{\Delta t\epsilon\delta}{2}\|\nabla\varphi\|^2_{(L^2(\Omega))^m},
\end{aligned}$$

where in the second line we have used Young's inequality, and chosen $\alpha = \dfrac{\delta\epsilon}{\|A\|\,\ell}$ in the third line, with C_0 is given in (7.3) and $\|A\|$ is given in (2.4). So we get that

$$a(\varphi,\varphi) \ge \left(1 - \frac{\Delta t}{2\tau}\right)\|\varphi\|^2_{(L^2(\Omega))^m} + \frac{\Delta t\epsilon\delta}{2}\|\nabla\varphi\|^2_{(L^2(\Omega))^m} \tag{7.9}$$

is coercive, since $\Delta t < \tau$ where τ is given in (7.2).

Step 3. Existence by Lax-Milgram
It is clear that L is linear and continuous on $(H^1(\Omega))^m$. Then by Step 1, Step 2 and Lax-Milgram theorem there exists a unique solution, u^{n+1}, of system (7.1).

Step 4. Proof of estimate (7.4)

Using (7.9) and the fact that $a(u^{n+1}, u^{n+1}) = L(u^{n+1})$ we get

$$\begin{aligned}
\left(1 - \frac{\Delta t}{2\tau}\right)\left\|u^{n+1}\right\|^2_{(L^2(\Omega))^m} + \frac{\Delta t\epsilon\delta}{2}\left\|\nabla u^{n+1}\right\|^2_{(L^2(\Omega))^m} &\le \left\langle u^{i,n}, u^{i,n+1}\right\rangle_{(L^2(\Omega))^m} \\
&\le \frac{1}{2}\|u^n\|^2_{(L^2(\Omega))^m} + \frac{1}{2}\left\|u^{n+1}\right\|^2_{(L^2(\Omega))^m},
\end{aligned}$$

which gives us the estimate (7.4). □

8. Existence for the nonlinear time-discrete problem

In this section we prove the existence, using Schaefer's fixed point theorem, of a solution for the nonlinear time discrete-system (8.3) given below. Moreover, we also show that this solution satisfies a suitable entropy estimate.

First, to present our result we need to choose a function $\Psi_{\epsilon,\ell}$ which is continuous, convex and satisfies that $\Psi''_{\epsilon,\ell}(x) = \dfrac{1}{T^{\epsilon,\ell}(x)}$, where $T^{\epsilon,\ell}$ is given in (5.3). So let

$$\Psi_{\epsilon,\ell}(a) - \frac{1}{e} = \begin{cases} \dfrac{a^2}{2\epsilon} + a \quad \ln \quad \epsilon - \dfrac{\epsilon}{2} & \text{if} \quad a \leq \epsilon, \\ a \quad \ln \quad a & \text{if} \quad \epsilon < a \leq \ell, \\ \dfrac{a^2}{2\ell} + a \quad \ln \quad \ell - \dfrac{\ell}{2} & \text{if} \quad a > \ell. \end{cases} \tag{8.1}$$

Let us introduce our nonlinear time discrete system: Assume that A satisfies (1.3). Let $u_0 := (u^{i,0})_{1\leq i\leq m} := (u_0^i)_{1\leq i\leq m}$ that satisfies

$$C_1 := \sum_{i=1}^{m} \int_{\Omega} \Psi_{\epsilon,\ell}(u_0^i) < +\infty, \tag{8.2}$$

such that $u_0^i \geq 0$ in Ω for $i = 1, \ldots, m$. Then for all Δt, ϵ, ℓ, η, $\delta > 0$, with $\epsilon < 1 < \ell$ and $\Delta t < \tau$ where τ is given in (7.2), for $n \in \mathbb{N}$, we look for a solution $u^{n+1} = (u^{i,n+1})_{1\leq i\leq m}$ of the following system:

$$\begin{cases} \dfrac{u^{i,n+1} - u^{i,n}}{\Delta t} = \operatorname{div}\left\{J^i_{\epsilon,\ell,\eta,\delta}(u^{n+1}, u^{n+1})\right\} & \text{in} \quad \mathcal{D}'(\Omega), \text{ for } n \geq 0 \\ u^{i,0}(x) \qquad = u_0^i(x) & \text{in} \quad \Omega, \end{cases} \tag{8.3}$$

where $J^i_{\epsilon,\ell,\eta,\delta}$ is given in system (7.1), and $T^{\epsilon,\ell}$ is given in (5.3).

Proposition 8.1 (Existence for system (8.3))**.** *Assume that A satisfies (1.3). Let $u_0 = (u_0^i)_{1\leq i\leq m}$ that satisfies (8.2), such that $u_0^i \geq 0$ a.e. in Ω for $i = 1, \ldots, m$. Then for all Δt, ϵ, ℓ, η, $\delta > 0$, with $\epsilon < 1 < \ell$ and $\Delta t < \tau$ where τ is given in (7.2), there exists a sequence of functions $u^{n+1} = (u^{i,n+1})_{1\leq i\leq m} \in \left(H^1(\Omega)\right)^m$ for $n \in \mathbb{N}$, solution of system (8.3), that satisfies the following entropy estimate:*

$$\sum_{i=1}^{m} \int_{\Omega} \Psi_{\epsilon,\ell}(u^{i,n+1}) + \delta \Delta t \sum_{i=1}^{m} \sum_{k=0}^{n} \int_{\Omega} |\nabla u^{i,k+1}|^2 + \delta_0 \Delta t \sum_{i=1}^{m} \sum_{k=0}^{n} \int_{\Omega} |\nabla \rho_\eta \star u^{i,k+1}|^2 \leq \sum_{i=1}^{m} \int_{\Omega} \Psi_{\epsilon,\ell}(u_0^i), \tag{8.4}$$

where $\Psi_{\epsilon,\ell}$ is given in (8.1).

Proof. Our proof is based on the Schaefer's fixed point theorem. So we need to define, for a given $w := u^n = (u^{i,n})_{1\leq i\leq m} \in (L^2(\Omega))^m$ and $v := v^{n+1} = (v^{i,n+1})_{1\leq i\leq m} \in (L^2(\Omega))^m$, the map Φ as:

$$\begin{array}{rccc} \Phi : & (L^2(\Omega))^m & \to & (L^2(\Omega))^m \\ & v & \mapsto & u \end{array}$$

where $u := u^{n+1} = (u^{i,n+1})_{1\leq i\leq m} = \Phi(v^{n+1}) \in (H^1(\Omega))^m$ is the unique solution of system (7.1), given by Proposition 7.1.

Step 1. Continuity of Φ
Let us consider the sequence v_k such that

$$\begin{cases} v_k \in (L^2(\Omega))^m, \\ v_k \longrightarrow v \quad \text{in} \quad (L^2(\Omega))^m. \end{cases} \tag{8.5}$$

We want to prove that the sequence $u_k = \Phi(v_k) \longrightarrow u = \Phi(v)$ to get the continuity of Φ. From the estimate (7.4), we deduce that u_k is bounded in $(H^1(\Omega))^m$. Therefore, up to a subsequence, we have

$$\begin{cases} u_k \rightharpoonup u & \text{weakly in} \quad (H^1(\Omega))^m, \\ \text{and} & \\ u_k \to u & \text{strongly in} \quad (L^2(\Omega))^m, \end{cases}$$

where the strong convergence arises because Ω is compact. Thus, by the definition of the truncation operator $T^{\epsilon,\ell}$, we can see that $T^{\epsilon,\ell}$ is Lipschitz continuous together with (8.5) yield to

$$T^{\epsilon,\ell}(v_k^i) \longrightarrow T^{\epsilon,\ell}(v^i) \qquad \text{in} \qquad L^2(\Omega), \quad \text{for} \quad i = 1, \dots, m.$$

Now we have

$$\frac{u_k^i - w^i}{\Delta t} = \operatorname{div}\left\{J^i_{\epsilon,\ell,\eta,\delta}(v_k, u_k)\right\} \qquad \text{in} \qquad \mathcal{D}'(\Omega). \tag{8.6}$$

This system also holds in $H^{-1}(\Omega)$, because $J^i_{\epsilon,\ell,\eta,\delta}(v_k, u_k) \in L^2(\Omega)$. Hence by multiplying this system by a test function in $(H^1(\Omega))^m$ and integrating over Ω for the bracket $\langle \cdot, \cdot \rangle_{H^{-1}(\Omega)\times H^1(\Omega)}$, we can pass directly to the limit in (8.6) as k tends to ∞, and we get

$$\frac{u^i - w^i}{\Delta t} = \operatorname{div}\{J^i_{\epsilon,\ell,\eta,\delta}(v, u)\} \quad \text{in} \quad \mathcal{D}'(\Omega). \tag{8.7}$$

where we used in particular the weak L^2 - strong L^2 convergence in the product $T^{\epsilon,\ell}(v_k)\nabla u_k$. Then $u = (u^i)_{1\le i\le m} = \Phi(v)$ is a solution of system (7.1). Finally, by uniqueness of the solutions of (7.1), we deduce that the limit u does not depend on the choice of the subsequence, and then that the full sequence converges:

$$u_k \to u \qquad \text{strongly in} \quad (L^2(\Omega))^m, \quad \text{with} \quad u = \Phi(v).$$

Step 2. Compactness of Φ

By the definition of Φ we can see that for a bounded sequence $(v_k)_k$ in $(L^2(\Omega))^m$, $\Phi(v_k) = u_k$ converges strongly in $(L^2(\Omega))^m$ up to a subsequence, which implies the compactness of Φ.

Step 3. *A priori* bounds on the solutions of $v = \lambda\Phi(v)$

Assume that $v \in (L^2(\Omega))^m$ such that

$$v = \lambda\Phi(v) \quad \text{for some} \quad \lambda \in (0, 1].$$

Then $\frac{v}{\lambda} = \Phi(v)$; or, in other words, $v \in (L^2(\Omega))^m$ and

$$a(v, v) = \lambda L(v)$$

where a and L are the bilinear and linear operators defined in (7.5) and (7.6) respectively. Using the same strategy as in the proof of (7.4) with the fact that $\lambda \in [0, 1]$ we obtain that $\|v\|_{(L^2(\Omega)))^m} \le C_2$, where C_2 does not depend on λ.

Step 4. Existence of a solution

Now, we can apply Schaefer's fixed point Theorem (Thm. 6.1), to deduce that Φ has a fixed point u^{n+1} on $(L^2(\Omega))^m$. This implies the existence of a solution u^{n+1} of system (8.3).

Step 5. Proof of estimate (8.4)

We have,

$$
\begin{aligned}
&\sum_{i=1}^{m}\int_{\Omega}\frac{\Psi_{\epsilon,\ell}(u^{i,n+1})-\Psi_{\epsilon,\ell}(u^{i,n})}{\Delta t}\\
&\le \sum_{i=1}^{m}\int_{\Omega}\left(\frac{u^{i,n+1}-u^{i,n}}{\Delta t}\right)\Psi'_{\epsilon,\ell}(u^{i,n+1})\\
&= \sum_{i=1}^{m}\left\langle\frac{u^{i,n+1}-u^{i,n}}{\Delta t},\Psi'_{\epsilon,\ell}(u^{i,n+1})\right\rangle_{H^{-1}(\Omega)\times H^{1}(\Omega)}\\
&= -\sum_{i=1}^{m}\left\langle \delta T^{\epsilon,\ell}(u^{i,n+1})\nabla u^{i,n+1}+T^{\epsilon,\ell}(u^{i,n+1})\sum_{j=1}^{m}A_{ij}\nabla\rho_\eta\star\rho_\eta\star u^{j,n+1},\Psi''_{\epsilon,\ell}(u^{i,n+1})\nabla u^{i,n+1}\right\rangle_{L^2(\Omega)}\\
&= -\sum_{i=1}^{m}\left\{\delta\int_{\Omega}|\nabla u^{i,n+1}|^2+\int_{\Omega}\sum_{j=1}^{m}\nabla\rho_\eta\star u^{i,n+1}A_{ij}\nabla\rho_\eta\star u^{j,n+1}\right\}\\
&\le -\sum_{i=1}^{m}\delta\int_{\Omega}|\nabla u^{i,n+1}|^2-\delta_0\sum_{i=1}^{m}\int_{\Omega}|\nabla\rho_\eta\star u^{i,n+1}|^2,
\end{aligned}
$$

where we have used, in the second line, the convexity inequality on $\Psi_{\epsilon,\ell}$. In the third line, we used the fact that $\dfrac{u^{i,n+1}-u^{i,n}}{\Delta t}\in H^{-1}(\Omega)$ and $\nabla\Psi'_{\epsilon,\ell}(u^{i,n+1})=\Psi''_{\epsilon,\ell}(u^{i,n+1})\nabla u^{i,n+1}\in L^2(\Omega)$ coming from the fact that $\Psi'_{\epsilon,\ell}\in C^1(\mathbb{R})$, ([3], Prop. IX.5, p. 155), $\Psi''_{\epsilon,\ell}(u^{i,n+1})\in L^\infty(\Omega)$ and $\nabla u^{i,n+1}\in L^2(\Omega)$ for all $i=1,\dots,m$. Thus, in the fourth line we use that $u^{i,n+1}$ is a solution for system (8.3) where we have applied an integration by parts. In the fifth line, we used the transposition of the convolution (see for instance [3], Prop. IV.16, p. 67), and the fact that $\check{\rho}_\eta(x)=\rho_\eta(-x)=\rho_\eta(x)$. Finally, in the last line we use that A satisfies (1.3).

Then by a straightforward recurrence we get estimate (8.4). This ends the proof of Proposition 8.1. □

9. Passage to the limit as $(\Delta t,\epsilon)\to(0,0)$

In this section we pass to the limit as $(\Delta t,\epsilon)\to(0,0)$ in system (8.3) to get the existence of a solution for the continuous approximate system (9.3) given below.

First, let us define the function $\Psi_{0,\ell}$ as

$$
\Psi_{0,\ell}(a)-\frac{1}{e}:=\begin{cases}+\infty & \text{if}\quad a<0,\\ 0 & \text{if}\quad a=0,\\ a\ln a & \text{if}\quad 0<a\le\ell,\\ \dfrac{a^2}{2\ell}+a\ln\ell-\dfrac{\ell}{2} & \text{if}\quad a>\ell.\end{cases}\tag{9.1}
$$

Now let us introduce our continuous approximate system. Assume that A satisfies (1.3). Let $u_0=(u_0^i)_{1\le i\le m}$ satisfying

$$
C_3:=\sum_{i=1}^{m}\int_{\Omega}\Psi_{0,\ell}(u_0^i)<+\infty,\tag{9.2}
$$

which implies that $u_0^i \geq 0$ a.e. in Ω for $i = 1, \dots, m$. Then for all ℓ, η, $\delta > 0$, with $1 < \ell < +\infty$, we look for a solution $u = (u^i)_{1 \leq i \leq m}$ of the following system:

$$\begin{cases} \partial_t u^i = \mathrm{div}\left\{ J^i_{0,\ell,\eta,\delta}(u) \right\} & \text{in} \quad \mathcal{D}'(\Omega_T), \\ J^i_{0,\ell,\eta,\delta}(u) = T^{0,\ell}(u^i) \left\{ \displaystyle\sum_{j=1}^m A_{ij} \nabla \rho_\eta \star \rho_\eta \star u^j + \delta \nabla u^i \right\}, & \\ u^i(0,x) = u_0^i(x) & \text{in} \quad \Omega. \end{cases} \tag{9.3}$$

where $T^{0,\ell}$ is given in (5.3) for $\epsilon = 0$, and we recall here $\Omega_T := (0,T) \times \Omega$.

Proposition 9.1 (Existence for system (9.3)**.**

Assume that A satisfies (1.3). Let $u_0 = (u_0^i)_{1 \leq i \leq m}$ satisfying (9.2). Then for all ℓ, η, $\delta > 0$ with $1 < \ell < +\infty$ there exists a function $u = (u^i)_{1 \leq i \leq m} \in (L^2(0,T; H^1(\Omega)) \cap C([0,T); L^2(\Omega)))^m$, with $u^i \geq 0$ a.e. in Ω_T, solution of system (9.3) that satisfies the following entropy estimate for a.e. $t_1, t_2 \in (0,T)$ with $u^i(t_1) = u^i(t_1, \cdot)$

$$\int_\Omega \sum_{i=1}^m \Psi_{0,\ell}(u^i(t_2)) + \delta \int_{t_1}^{t_2} \int_\Omega \sum_{i=1}^m \left| \nabla u^i \right|^2 + \delta_0 \int_{t_1}^{t_2} \int_\Omega \sum_{i=1}^m \left| \nabla \rho_\eta \star u^i \right|^2 \leq \int_\Omega \sum_{i=1}^m \Psi_{0,\ell}(u_0^i). \tag{9.4}$$

Proof. Our proof is based on the variant of Simon's Lemma (Thm. 6.4). Recall that $\Delta t = \dfrac{T}{K}$ where $K \in \mathbb{N}^*$ and $T > 0$ is given. We denote by C a generic constant independent of Δt and ϵ. For all $n \in \{0, \dots, K-1\}$ and $i = 1, \dots, m$, set $t_n = n\Delta t$ and let the piecewise constant in time function:

$$U^{i,\Delta t}(t,x) := u^{i,n+1}(x), \quad \text{for } t \in (t_n, t_{n+1}], \tag{9.5}$$

with $U^{i,\Delta t}(0,x) := u_0^i(x)$ satisfying (8.2).

Step 1. Upper bound on $\left\| \mathbf{U}^{\Delta t} \right\|_{(L^2(0,T;H^1(\Omega)))^m}$

We will prove that $U^{\Delta t} = (U^{i,\Delta t})_{1 \leq i \leq m}$ satisfies

$$\int_0^T \| \nabla U^{\Delta t}(t) \|^2_{(L^2(\Omega))^m} \leq C.$$

For all $n \in \{0, \dots, K-1\}$ and $i = 1, \dots, m$ we have

$$\nabla U^{i,\Delta t}(t,x) = \nabla u^{i,n+1}(x), \quad \text{for } t \in (t_n, t_{n+1}].$$

Then

$$\int_{t_n}^{t_{n+1}} \| \nabla U^{i,\Delta t}(t) \|^2_{L^2(\Omega)} = \Delta t \| \nabla u^{i,n+1} \|^2_{L^2(\Omega)}.$$

Hence

$$\begin{aligned} \int_0^T \| \nabla U^{\Delta t}(t) \|^2_{(L^2(\Omega))^m} &= \Delta t \sum_{k=0}^{K-1} \| \nabla u^{k+1} \|^2_{(L^2(\Omega)^m} \\ &\leq \frac{C_1}{\delta}, \end{aligned}$$

where we have used the entropy estimate (8.4) with C_1 is given in (8.2). Hence, using Poincaré−Wirtinger's inequality we can get similarly an upper bound on $\int_0^T \left\|U^{i,\Delta t}\right\|^2_{(L^2(\Omega))^m}$ independently of Δt (using the fact that $\int_\Omega u^{i,n+1} = \int_\Omega u^{i,n} = \int_\Omega u^{i,0}$ by Eq. (8.3)) .

Step 2. Upper bound on $\left\|\mathbf{U}^{\Delta t}\right\|_{(\mathrm{Var}([0,T);H^{-1}(\Omega)))^m}$

We will prove that

$$\left\|U^{\Delta t}\right\|_{(\mathrm{Var}([0,T);H^{-1}(\Omega)))^m} \leq C.$$

We have for $i = 1, \ldots, m$

$$\begin{aligned}
\left\|U^{i,\Delta t}\right\|_{\mathrm{Var}([0,T);H^{-1}(\Omega))} &= \sum_{n=0}^{K-1} \left\|U^{i,\Delta t}(t_{n+1}) - U^{i,\Delta t}(t_n)\right\|_{H^{-1}(\Omega)} \\
&= \sum_{n=0}^{K-1} \left\|u^{i,n+1} - u^{i,n}\right\|_{H^{-1}(\Omega)} \\
&= \Delta t \sum_{n=0}^{K-1} \left\|\frac{u^{i,n+1} - u^{i,n}}{\Delta t}\right\|_{H^{-1}(\Omega)} \\
&\leq \Delta t \sum_{n=0}^{K-1} \left\|T^{\epsilon,\ell}(u^{i,n+1}) \left(\sum_{j=1}^{m} A_{ij} \nabla\rho_\eta \star \rho_\eta \star u^{j,n+1} + \delta \nabla u^{i,n+1}\right)\right\|_{L^2(\Omega)} \\
&\leq \ell \Delta t \sum_{n=0}^{K-1} \left\{ \|A\|_\infty \sum_{j=1}^{m} \left\|\nabla\rho_\eta \star u^{j,n+1}\right\|_{L^2(\Omega)} + \delta \left\|\nabla u^{i,n+1}\right\|_{L^2(\Omega)} \right\} \\
&\leq C,
\end{aligned}$$

where

$$\|A\|_\infty = \max_{1\leq i\leq m} \sum_{j=1}^{m} |A_{ij}|\,, \tag{9.6}$$

and we have used in the last inequality the entropy estimate (8.4), and the fact that

$$\Delta t \sum_{n=0}^{K-1} \left\|\nabla u^{i,n+1}\right\|_{L^2(\Omega)} \leq \sqrt{T} \left(\Delta t \sum_{n=0}^{K-1} \left\|\nabla u^{i,n+1}\right\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}}.$$

Step 3. $\mathbf{U}^{\mathbf{i},\Delta t} \in \mathbf{L}^{\mathbf{p}}(\mathbf{0},\mathbf{T},\mathbf{L}^{\mathbf{2}}(\Omega))$ with $\mathbf{p} > 2$

The estimate (8.4) gives us that $U^{i,\Delta t} \in L^\infty(0,T;L^1(\Omega)) \cap L^2(0,T;H^1(\Omega))$ for $i = 1, \ldots, m$. Using Sobolev injections we get $H^1(\Omega) \hookrightarrow L^{2+\alpha(N)}(\Omega)$, with $\alpha(N) > 0$, and then $U^{i,\Delta t} \in L^2(0,T;L^{2+\alpha(N)}(\Omega))$. Hence by interpolation, we find that $U^{i,\Delta t} \in L^p(0,T;L^2(\Omega))$ with $(\frac{1}{p},\frac{1}{2}) = (1-\theta)(\frac{1}{\infty},\frac{1}{2}) + \theta(\frac{1}{2},\frac{1}{2+\alpha(N)})$ and $\theta \in (0,1)$, *i.e.* for

$$p = \frac{4+4\alpha(N)}{2+\alpha(N)} > 2. \tag{9.7}$$

Step 4. Passage to the limit as $(\Delta \mathbf{t}, \epsilon) \to (0,0)$

By Steps 1,2 and 3 we have

$$\left\|U^{i,\Delta t}\right\|_{L^p(0,T;L^2(\Omega))} + \left\|U^{i,\Delta t}\right\|_{L^2(0,T;H^1(\Omega))} + \left\|U^{i,\Delta t}\right\|_{\mathrm{Var}([0,T);H^{-1}(\Omega))} \leq C.$$

Then by noticing that $H^1(\Omega) \overset{\text{compact}}{\hookrightarrow} L^2(\Omega) \overset{\text{continous}}{\hookrightarrow} H^{-1}(\Omega)$, and applying the variant of Simon's Lemma (Thm. 6.4), we deduce that $(U^{i,\Delta t})_{\Delta t}$ is relatively compact in $L^2(0,T;L^2(\Omega))$, and there exists a function $U = (U^i)_{1\le i\le m} \in (L^2(0,T;H^1(\Omega)))^m$ such that, as $(\Delta t,\epsilon) \to (0,0)$, we have (up to a subsequence)

$$U^{i,\Delta t} \to U^i \text{ strongly in } L^2(0,T;L^2(\Omega)).$$

By Step 1, we have $\nabla U^{i,\Delta t} \rightharpoonup \nabla U^i$ weakly in $L^2(0,T;L^2(\Omega))$. Now system (8.3) can be written as

$$\frac{U^{i,\Delta t}(t+\Delta t) - U^{i,\Delta t}(t)}{\Delta t} = \operatorname{div}\left\{J^i_{\epsilon,\ell,\eta,\delta}(U^{i,\Delta t}(t+\Delta t), U^{i,\Delta t}(t+\Delta t))\right\} \qquad \text{in} \quad \mathcal{D}'(\Omega_T). \tag{9.8}$$

Multiplying this system by a test function in $\mathcal{D}(\Omega_T)$ and integrating over Ω_T, we can pass directly to the limit as $(\Delta t,\epsilon) \to (0,0)$ in (9.8) to get

$$\partial_t U^i = \operatorname{div}\left(T^{0,\ell}(U^i)\left(\sum_{j=1}^m A_{ij}\nabla\rho_\eta \star \rho_\eta \star U^j + \delta\nabla U^i\right)\right) \qquad \text{in} \quad \mathcal{D}'(\Omega_T),$$

where we used the weak L^2 - strong L^2 convergence in the products such $T^{\epsilon,\ell}(U^{i,\Delta t})\nabla U^{i,\Delta t}$ to get the existence of a solution of system (9.3).

Step 5. Recovering the initial condition
Let $\bar\rho \in C_c^\infty(\mathbb{R})$ with $\bar\rho \ge 0$, $\int_{\mathbb{R}} \bar\rho = 1$ and supp $\bar\rho \subset (-\frac{1}{2},\frac{1}{2})$. We set

$$\bar\rho_{\Delta t}(t) = \Delta t^{-1}\bar\rho(\Delta t^{-1}t), \text{ with } \bar\rho(t) = \bar\rho(-t).$$

Then we have

$$\begin{aligned}
&\left\|\partial_t U^{\Delta t} \star \bar\rho_{\Delta t}\right\|^2_{(L^2(0,T;H^{-1}(\Omega)))^m} = \sum_{i=1}^m \int_0^T \left\|\sum_{n=0}^{K-1}(u^{i,n+1} - u^{i,n})\delta_{t_{n+1}} \star \bar\rho_{\Delta t}\right\|^2_{H^{-1}(\Omega)} \\
&= \sum_{i=1}^m\sum_{n=0}^{K-1}\int_0^T (\Delta t\bar\rho_{\Delta t}(t-t_{n+1}))^2 \left\|\frac{u^{i,n+1}-u^{i,n}}{\Delta t}\right\|^2_{H^{-1}(\Omega)} \\
&\le C_4\Delta t\sum_{i=1}^m\sum_{n=0}^{K-1}\left\|\frac{u^{i,n+1}-u^{i,n}}{\Delta t}\right\|^2_{H^{-1}(\Omega)} \\
&\le C_4\Delta t\sum_{n=0}^{K-1}\sum_{i=1}^m\int_\Omega\left|T^{\epsilon,\ell}(u^{i,n+1})\left(\sum_{j=1}^m A_{ij}\nabla\rho_\eta\star\rho_\eta\star u^{j,n+1} + \delta\nabla u^{i,n+1}\right)\right|^2 \\
&\le 2\ C_4\ell^2\Delta t\sum_{n=0}^{K-1}\sum_{i=1}^m\int_\Omega\left\{\left(\sum_{j=1}^m A_{ij}\nabla\rho_\eta\star\rho_\eta\star u^{j,n+1}\right)^2 + \delta^2\left|\nabla u^{i,n+1}\right|^2\right\} \\
&\le 2\ C_4\ell^2\Delta t\sum_{n=0}^{K-1}\left\{\|A\|^2\left\|\nabla\rho_\eta\star\rho_\eta\star u^{n+1}\right\|^2_{(L^2(\Omega))^m} + \delta^2\left\|\nabla u^{n+1}\right\|^2_{(L^2(\Omega))^m}\right\} \\
&\le 2\ C_4\ell^2\Delta t\sum_{n=0}^{K-1}\left\{\|A\|^2\|\nabla\rho_\eta\star u^{n+1}\|^2_{(L^2(\Omega))^m} + \delta^2\left\|\nabla u^{n+1}\right\|^2_{(L^2(\Omega))^m}\right\} \\
&\le 2\ C_4\ell^2 C_1\left(\frac{\|A\|^2}{\delta_0} + \delta\right) \le 2\ C_4\ell^2 C_3\left(\frac{\|A\|^2}{\delta_0} + \delta\right),
\end{aligned}$$

where $\delta_{t_{n+1}}$ is Dirac mass in $t = t_{n+1}$, C_1 as in (8.2), C_3 as in (9.2), $C_4 := C(\int_0^T \bar{\rho}(t)\,\mathrm{d}t)$, and we have used in the last line the entropy estimate (8.4). Clearly, $\bar{\rho}_{\Delta t} \star U_t^{i,\Delta t} \rightharpoonup U_t^i$ weakly in $L^2(0,T;H^{-1}(\Omega))$ as $(\Delta t, \epsilon) \to 0$. Similarly we have that $\bar{\rho}_{\Delta t} \star U^{i,\Delta t} \to U^i$ strongly in $L^2(0,T;L^2(\Omega))$ since $U^{i,\Delta t} \to U^i$ in $L^2(0,T;L^2(\Omega))$. Then we deduce that $U^i \in \{g \in L^2(0,T;H^1(\Omega)); g_t \in L^2(0,T;H^{-1}(\Omega))\}$. And now $U^i(0,x)$ has sense, by Proposition 6.2, and we have that $U^i(0,x) = u_0^i(x)$ by Proposition A.1.

Step 6. Proof of estimate (9.4)

By Step 4, there exists a function $U^i \in L^2(0,T;H^1(\Omega))$ such that the following holds true as $(\Delta t, \epsilon) \to (0,0)$

$$\left\{\begin{array}{ll} U^{i,\Delta t} & \to U^i \\ \nabla U^{i,\Delta t} & \rightharpoonup \nabla U^i \\ \nabla\rho_\eta \star U^{i,\Delta t} & \to \nabla\rho_\eta \star U^i \end{array}\right| \quad \text{in} \quad L^2(0,T;L^2(\Omega)).$$

Now using the fact that the norm L^2 is weakly lower semicontinuous, with a sequence of integers n_2 (depending on Δt) such that $t_{n_2+1} \to t_2 \in (0,T)$ and

$$U^{i,\Delta t}(t_2) = U^{i,\Delta t}(t_{n_2+1}) = u^{n_2+1},$$

we get for $t_1 < t_2$

$$\int_{t_1}^{t_2}\int_\Omega \left|\nabla U^i\right|^2 \le \int_0^{t_2}\int_\Omega \left|\nabla U^i\right|^2 \le \liminf_{(\Delta t,\epsilon)\to(0,0)} \int_0^{t_{n_2+1}}\int_\Omega \left|\nabla U^{i,\Delta t}\right|^2 = \liminf_{(\Delta t,\epsilon)\to(0,0)} \Delta t \sum_{k=0}^{n_2}\int_\Omega |\nabla u^{i,k+1}|^2, \quad (9.9)$$

and

$$\int_{t_1}^{t_2}\int_\Omega \left|\nabla\rho_\eta \star U^i\right|^2 \le \int_0^{t_2}\int_\Omega \left|\nabla\rho_\eta \star U^i\right|^2 \le \liminf_{(\Delta t,\epsilon)\to(0,0)} \Delta t \sum_{k=0}^{n_2}\int_\Omega |\nabla\rho_\eta \star u^{i,k+1}|^2. \quad (9.10)$$

Moreover, since we have $U^{i,\Delta t} \to U^i$ in $L^2(0,T;L^2(\Omega))$, we get that for a.e. $t \in (0,T)$ (up to a subsequence) $U^{i,\Delta t}(t,\cdot) \to U^i(t,\cdot)$ in $L^2(\Omega)$. For such t we have (up to a subsequence) $U^{i,\Delta t}(t,\cdot) \to U^i(t,\cdot)$ for a.e. in Ω. Moreover, by applying Lemma A.2 we get that for a.e. $t \in (0,T)$

$$\Psi_{0,\ell}(U^i(t)) \le \liminf_{(\Delta t,\epsilon)\to(0,0)} \Psi_{\epsilon,\ell}(U^{i,\Delta t}(t)). \quad (9.11)$$

Integrating over Ω then applying Fatou's Lemma we get for a.e. $t_1 < t_2$

$$\sum_{i=1}^m \int_\Omega \Psi_{0,\ell}(U^i(t_2)) \le \int_\Omega \liminf_{(\Delta t,\epsilon)\to(0,0)} \sum_{i=1}^m \Psi_{\epsilon,\ell}(U^{i,\Delta t}(t_2)) \le \liminf_{(\Delta t,\epsilon)\to(0,0)} \sum_{i=1}^m \int_\Omega \Psi_{\epsilon,\ell}(u^{i,n_2+1}). \quad (9.12)$$

(9.9), (9.10) and (9.12) with the entropy estimate (8.4) give us that for a.e. $t_1 < t_2 \in (0,T)$

$$\begin{aligned} &\sum_{i=1}^m \int_\Omega \Psi_{0,\ell}(U^i(t_2)) + \delta \sum_{i=1}^m \int_{t_1}^{t_2}\int_\Omega \left|\nabla U^i\right|^2 + \delta_0 \sum_{i=1}^m \int_{t_1}^{t_2}\int_\Omega \left|\nabla\rho_\eta \star U^i\right|^2 \\ &\le \liminf_{(\Delta t,\epsilon)\to(0,0)} \sum_{i=1}^m \int_\Omega \Psi_{\epsilon,\ell}(u^{i,n_2+1}) + \liminf_{(\Delta t,\epsilon)\to(0,0)} \delta\Delta t \sum_{i=1}^m\sum_{k=0}^{n_2}\int_\Omega |\nabla u^{i,k+1}|^2 \\ &\quad + \liminf_{(\Delta t,\epsilon)\to(0,0)} \delta_0\Delta t \sum_{i=1}^m\sum_{k=0}^{n_2}\int_\Omega |\nabla\rho_\eta \star u^{i,k+1}|^2 \\ &\le \sum_{i=1}^m \int_\Omega \Psi_{\epsilon,\ell}(u_0^i) \le \sum_{i=1}^m \int_\Omega \Psi_{0,\ell}(u_0^i), \end{aligned}$$

which is estimate (9.4).

Step 7. Non-negativity of $\mathbf{U^i}$
Let $\Omega_i^\epsilon(t) := \left\{x \in \Omega : U^{i,\Delta t}(t,x) \le \epsilon\right\}$ for all $t \in [0,T]$ and $i = 1,\dots,m$. By estimate (8.4), there exists a positive constant C independent of ϵ and Δt such that for all $i = 1,\dots,m$ we have

$$\begin{aligned}
C &\ge \int_\Omega \Psi_{\epsilon,\ell}(U^{i,\Delta t}) \\
&\ge \int_{\Omega_i^\epsilon(t)} \Psi_{\epsilon,\ell}(U^{i,\Delta t}) \\
&= \int_{\Omega_i^\epsilon(t)} \frac{1}{e} + \frac{(U^{i,\Delta t})^2}{2\epsilon} + U^{i,\Delta t}\ln\epsilon - \frac{\epsilon}{2} \\
&\ge \int_{\Omega_i^\epsilon(t)} \frac{1}{e} + \frac{(U^{i,\Delta t})^2}{2\epsilon} + \epsilon\ln\epsilon - \frac{1}{2} \\
&\ge \int_{\Omega_i^\epsilon(t)} \frac{(U^{i,\Delta t})^2}{2\epsilon} - \frac{1}{2},
\end{aligned}$$

i.e.

$$\int_{\Omega_i^\epsilon(t)} \frac{(U^{i,\Delta t})^2}{2\epsilon} \le C + \frac{1}{2}. \tag{9.13}$$

Now by passing to the limit as $(\Delta t, \epsilon) \to (0,0)$ in (9.13) we deduce that $\int_{\Omega_i^-(t)} \left|U^i\right|^2 = 0$, where $\Omega_i^-(t) := \left\{x \in \Omega : U^i(t,x) \le 0\right\}$ for all $t \in [0,T]$ and $i = 1,\dots,m$, which gives us that $(U^i)^- = 0$ in $L^2(\Omega)$, where $(U^i)^- = \min(0, U^i)$. □

Remark 9.2 (Another method following [22])**.**
Note that it would be also possible to use a theorem in Lions-Magenes ([22], Chap. 3, Thm. 4.1, p. 257). This would prove in particular the existence of a unique solution for the following system:

$$\left\{\begin{aligned}
\partial_t u^i &= \operatorname{div}\left\{J^i_{\epsilon,\ell,\eta,\delta}(v,u)\right\} && \text{in} \quad \mathcal{D}'(\Omega_T), \\
J^i_{\epsilon,\ell,\eta,\delta}(v,u) &= T^{\epsilon,\ell}(v^i)\left\{\sum_{j=1}^m A_{ij}\nabla\rho_\eta \star \rho_\eta \star u^j + \delta\nabla u^i\right\}, && \\
u^i(0,x) &= u_0^i(x) && \text{in} \quad \Omega,
\end{aligned}\right. \tag{9.14}$$

where $T^{\epsilon,\ell}$ is given in (5.3).

It would then be possible to find a fixed point solution $v = u$ of (9.14) to recover a solution of (9.3). We would have to justify again the entropy inequality (9.4).

10. Passage to the limit as $(\ell,\eta) \to (\infty, 0)$

In this section we pass to the limit as $(\ell,\eta) \to (\infty,0)$ in system (9.3) to get the existence of a solution for system (10.1) given below (system independent of ℓ and η).

Let us introduce the system independant of ℓ and η. Asume that A satisfies (1.3). Let $u_0 = (u_0^i)_{1\le i\le m}$ satisfying (2.2). Then for all $\delta > 0$ we look for a solution $u = (u^i)_{1\le i\le m}$ of the following system:

$$\left\{\begin{aligned}
\partial_t u^i &= \operatorname{div}\left\{u^i\sum_{j=1}^m A_{ij}\nabla u^j + \delta u^i\nabla u^i\right\} && \text{in} \quad \mathcal{D}'(\Omega_T), \\
u^i(0,x) &= u_0^i(x) && \text{a.e.} \quad \text{in} \quad \Omega.
\end{aligned}\right. \tag{10.1}$$

Proposition 10.1 (Existence for system (10.1))**.** *Assume that A satisfies (1.3). Let $u_0 = (u_0^i)_{1\le i\le m}$ satisfying (2.2). Then for all $\delta > 0$ there exists a function $u = (u^i)_{1\le i\le m} \in (L^2(0,T;H^1(\Omega))\cap C([0,T);(W^{1,\infty}(\Omega))'))^m$, with $u^i \ge 0$ a.e. on Ω_T, solution of system (10.1), that satisfies the following entropy estimate for a.e. $t_1, t_2 \in (0,T)$ with $u^i(t_2) = u^i(t_2,.)$:*

$$\int_\Omega \sum_{i=1}^m \Psi(u^i(t_2)) + \delta \int_{t_1}^{t_2}\int_\Omega \sum_{i=1}^m \left|\nabla u^i\right|^2 + \delta_0 \int_{t_1}^{t_2}\int_\Omega \sum_{i=1}^m \left|\nabla u^i\right|^2 \le \int_\Omega \sum_{i=1}^m \Psi(u_0^i), \tag{10.2}$$

with Ψ is given in (2.1).

Proof. Let C be a generic constant independent of ℓ and η, and $u^\ell := (u^{i,\ell})_{1\le i\le m}$ a solution of system (9.3), where we drop the indices η and δ to keep light notations. The proof is accomplished by passing to the limit as $(\ell,\eta)\to(\infty,0)$ in (9.3) and using Simon's lemma (Lem. 6.3), in order to get the existence result.

Step 1. Upper bound on $\partial_t \mathbf{u}^{i,\ell}$

As in Step 3 of the proof of Proposition 9.1, estimate (9.4) gives us that $u^{i,\ell} \in L^p(0,T,L^2(\Omega))$ with $p > 2$ is given in (9.7). Let $q = \dfrac{2p}{p+2} > 1$. It remains to prove that for $i = 1,\dots,m$

$$\left\|\partial_t u^{i,\ell}\right\|_{L^q(0,T;(W^{1,\infty})'(\Omega))} < C.$$

We have

$$\begin{aligned}
&\|\partial_t u^{i,\ell}\|_{L^q(0,T;(W^{1,\infty}(\Omega))')} = \left(\int_0^T \left\|\partial_t u^{i,\ell}\right\|^q_{(W^{1,\infty}(\Omega))'}\right)^{\frac{1}{q}}\\
&= \left(\int_0^T \left\|\operatorname{div}\left\{T^{0,\ell}(u^{i,\ell})\left(\sum_{j=1}^m A_{ij}\nabla\rho_\eta \star \rho_\eta \star u^{j,\ell} + \delta\nabla u^{i,\ell}\right)\right\}\right\|^q_{(W^{1,\infty}(\Omega))'}\right)^{\frac{1}{q}}\\
&\le \left(\int_0^T \left(\int_\Omega \left|T^{0,\ell}(u^{i,\ell})\left(\sum_{j=1}^m A_{ij}\nabla\rho_\eta \star \rho_\eta \star u^{j,\ell} + \delta\nabla u^{i,\ell}\right)\right|\right)^q\right)^{\frac{1}{q}}\\
&\le \left(\int_0^T \left(\int_\Omega \left|u^{i,\ell}\right|\left(\left|\sum_{j=1}^m A_{ij}\nabla\rho_\eta \star \rho_\eta \star u^{j,\ell}\right| + \delta\left|\nabla u^{i,\ell}\right|\right)\right)^q\right)^{\frac{1}{q}}\\
&\le \left\|u^{i,\ell}\right\|_{L^p(0,T;L^2(\Omega))}\left\|\sum_{j=1}^m A_{ij}\nabla\rho_\eta \star \rho_\eta \star u^{j,\ell}\right\|_{L^2(0,T;L^2(\Omega))} + \delta\left\|u^{i,\ell}\right\|_{L^p(0,T;L^2(\Omega))}\left\|\nabla u^{i,\ell}\right\|_{L^2(0,T;L^2(\Omega))}\\
&\le \left\|u^{i,\ell}\right\|_{L^p(0,T;L^2(\Omega))}\left(\|A\|_\infty \sum_{j=1}^m \left\|\nabla\rho_\eta \star u^{j,\ell}\right\|_{L^2(0,T;L^2(\Omega))} + \delta\left\|\nabla u^{i,\ell}\right\|_{L^2(0,T;L^2(\Omega))}\right) \le C,
\end{aligned}$$

where we have used in the fifth line Holder's inequality (since we have $\dfrac{1}{q} = \dfrac{1}{p} + \dfrac{1}{2}$) and in the last line the entropy estimate (9.4).

Step 2. Passage to the limit as $(\ell,\eta)\to(\infty,0)$
In view of Step 1 of this proof and (9.4) we have that

$$\left\|u^{i,\ell}\right\|_{L^p(0,T;L^2(\Omega))}+\left\|u^{i,\ell}\right\|_{L^2(0,T;H^1(\Omega))}+\left\|\partial_t u^{i,\ell}\right\|_{L^q(0,T;(W^{1,\infty}(\Omega))')}\le C,$$

where $p>2$ is given in (9.7) and $q=\dfrac{2p}{p+2}>1$. Then by noticing that $H^1(\Omega)\overset{\text{compact}}{\hookrightarrow}L^2(\Omega)\overset{\text{continous}}{\hookrightarrow}(W^{1,\infty}(\Omega))'$, and applying Simon's Lemma (Lem. 2.3), we deduce that $(u^{i,\ell})_\ell$ is relatively compact in $L^2(0,T;L^2(\Omega))$, and there exists a function $u^i\in L^2(0,T;H^1(\Omega))$ such that, as $(\ell,\eta)\to(\infty,0)$, we have (up to a subsequence)

$$u^{i,\ell}\to u^i \text{ strongly in } L^2(0,T;L^2(\Omega)).$$

In addition, since $u^{i,\ell}\to u^i$ a.e., u^i is nonnegative a.e. hence $T^{0,\ell}(u^{i,\ell})\to u^i$ strongly in $L^2(0,T;L^2(\Omega))$. Multiplying system (9.3) by a test function in $\mathcal{D}(\Omega_T)$ and integrating over Ω_T we can pass directly to the limit as $(\ell,\eta)\to(\infty,0)$, and we get

$$\partial_t u^i=\operatorname{div}\left\{u^i\sum_{j=1}^m A_{ij}\nabla u^j+\delta u^i\nabla u^i\right\}\qquad\text{in } \mathcal{D}'(\Omega_T).$$

where we used in particular the weak L^2 - strong L^2 convergence in the products such $T^{0,\ell}(u^{i,\ell})\nabla u^{i,\ell}$. Therefore, $u=(u^i)_{1\le i\le m}$ is a solution of system (10.1).

Step 3. Recovering the initial condition
Using Step 1 of this proof with the fact that $W^{1,1}(0,T;(W^{1,\infty}(\Omega))')\hookrightarrow C([0,T);(W^{1,\infty}(\Omega))')$ then $u^i(0,x)$ makes sense and $u^i(0,x)=u_0^i(x)$ for all $i=1,\dots,m$, by Proposition A.1.

Step 5. Proof of the estimate (10.2)
The proof is similar to Step 6 of the proof of Proposition 9.1. □

11. Passage to the limit as $\delta\to 0$

Proof. Let C be a generic constant independent of δ and $u^\delta:=(u^{i,\delta})_{1\le i\le m}$ a solution of system (10.1). We follow the lines of proof of Proposition 10.1.

An upper bound on $u_t^{i,\delta}$ and estimate (10.2) allow us to apply Simon's Lemma (Lem. 2.3), then $(u^{i,\delta})_\delta$ is relatively compact in $L^2(0,T;L^2(\Omega))$, and there exists a function $u^i\in L^2(0,T;H^1(\Omega))$ such that, as $\delta\to 0$, we have (up to a subsequence)

$$u^{i,\delta}\to u^i \text{ strongly in } L^2(0,T;L^2(\Omega)),$$

and

$$\partial_t u^i=\operatorname{div}\left\{u^i\sum_{j=1}^m A_{ij}\nabla u^j\right\}\qquad\text{in } \mathcal{D}'(\Omega_T).$$

Similarly to Step 4 of the proof of Proposition 10.1 the initial condition is recoverd. Also estimate (2.3) can be easily obtained. □

Remark 11.1 (Passage to the limit as $(\ell,\eta,\delta)\to(\infty,0,0)$)**.**
It is possible to pass to the limit in system (9.3) as $(\ell,\eta,\delta)\to(\infty,0,0)$ at the same time: By using the entropy estimate (9.4) and applying Simon's Lemma on the sequence $\rho_\eta\star u^{i,\ell}$ instead of $u^{i,\ell}$. Moreover, to get the entropy estimate (2.3) it is sufficient to use the fact that $\int_\Omega\Psi_{0,\ell}(\rho_\eta\star u^{i,\ell})\le\int_\Omega\rho_\eta\star\Psi_{0,\ell}(u^{i,\ell})$.

Generalizations

12. Generalization on the matrix A

Assumption (1.3) can be weaken. Indead, we can assume that $A = (A_{ij})_{1\leq i,j\leq m}$ is a real $m\times m$ matrix that satisfies a positivity condition, in the sense that there exist two positive definite diagonal $m\times m$ matrices L and R and $\delta_0 > 0$, such that we have

$$\zeta^T LAR\,\zeta \geq \delta_0|\zeta|^2, \quad \text{for all} \quad \zeta\in\mathbb{R}^m. \tag{12.1}$$

Remark 12.1 (Comments on the positivity condition (12.1))**.** The assumption of positivity condition (12.1), generalize our problem for A not necessarily having a symmetric part positive definite. Here is an example of such a matrix, whose symmetric part is not definite positive, but the symmetric part of $L\,A\,R$ is definite positive for some suitable positive diagonal matrices L and R.

We consider

$$A = \begin{pmatrix} 1 & -a \\ 2a & 1 \end{pmatrix} \text{ with } |a| > 2.$$

Indeed,

$$A^{\text{sym}} = \frac{A^T + A}{2} = \begin{pmatrix} 1 & \frac{a}{2} \\ \frac{a}{2} & 1 \end{pmatrix},$$

satisfying $\det(A^{\text{sym}}) = 1 - \dfrac{a^2}{4} < 0$. And let

$$L = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad R = I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

On the other hand,

$$B = L.A.R = \begin{pmatrix} 2 & -2a \\ 2a & 1 \end{pmatrix},$$

satisfies that

$$B^{\text{sym}} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix},$$

is definite positive.

Proposition 12.2 (The case where $L = I_2$)**.** *Let A be a matrix that satisfies the positivity condition (12.1) with $L = I_2$. Then $\bar{u}$ is a solution for system (1.1) with the matrix $\bar{A} = A\,R$ (instead of A) if and only if $u^i = R_{ii}\,\bar{u}^i$ is a solution for system (1.1) with the matrix A.*

Proposition 12.3 (The case where $R = I_2$)**.** *Let $u^{n+1} = (u^{i,n+1})_{1\leq i\leq m}$ be a solution of system (8.3) with a matrix A satisfying the positivity condition (12.1) with $R = I_2$ and L a positive diagonal matrix. Then u^{n+1} satisfies the following entropy estimate*

$$\sum_{i=1}^m \int_\Omega L_{ii}\Psi_{\epsilon,\ell}(u^{i,n+1}) + \delta\Delta t \min_{1\leq i\leq m}\{L_{ii}\}\sum_{i=1}^m\sum_{k=0}^n \int_\Omega |\nabla u^{i,k+1}|^2 \\ + \delta_0\Delta t\sum_{i=1}^m\sum_{k=0}^n\int_\Omega |\nabla\rho_\eta \star u^{i,k+1}|^2 \leq \sum_{i=1}^m\int_\Omega L_{ii}\Psi_{\epsilon,\ell}(u_0^i)$$

Proof. Similarly to Step 5 of the proof of Proposition 8.1 we have

$$\begin{aligned}\sum_{i=1}^{m}\int_\Omega L_{ii}\left(\frac{\Psi_{\epsilon,\ell}(u^{i,n+1})-\Psi_{\epsilon,\ell}(u^{i,n})}{\Delta t}\right) &\le -\int_\Omega\sum_{i=1}^{m}\sum_{j=1}^{m}L_{ii}A_{ij}(\nabla\rho_\eta\star\rho_\eta\star u^{j,n+1})\cdot\nabla u^{i,n+1}\\ &\quad -\delta\sum_{i=1}^{m}\int_\Omega L_{ii}|\nabla u^{i,n+1}|^2\\ &\le \int_\Omega\sum_{i=1}^{m}\sum_{j=1}^{m}(\nabla\rho_\eta\star u^{j,n+1})L_{ii}A_{ij}(\nabla\rho_\eta\star u^{i,n+1})\\ &\quad -\delta\sum_{i=1}^{m}\int_\Omega L_{ii}|\nabla u^{i,n+1}|^2\\ &\le -\delta_0\int_\Omega\sum_{i=1}^{m}|\nabla\rho_\eta\star u^{i,n+1}|^2\\ &\quad -\delta\min_{1\le i\le m}\{L_{ii}\}\sum_{i=1}^{m}\int_\Omega|\nabla u^{i,n+1}|^2,\end{aligned}$$

where we have used, in the last line, the fact that the matrix A satisfies (12.1) with $R = I_2$. Then by a straightforward recurrence we get (12.2). □

Corollary 12.4. *Theorem 2.1 still hold true if we replace condition (1.3) by condition (12.1).*

13. Generalisations on the problem

13.1. The tensor case

Our study can be applied on a generalized systems of the form

$$\partial_t u^i = \sum_{j=1}^{m}\sum_{k=1}^{N}\sum_{l=1}^{N}\frac{\partial}{\partial x_k}\left(f_i(u^i)A_{ijkl}\frac{\partial u^j}{\partial x_l}\right)\qquad \text{for } i=1,\dots,m, \tag{13.1}$$

where f_i satisfies

$$\begin{cases} f_i\in C(\mathbb{R}), & \\ 0\le f_i(a)\le C(1+|a|) & \text{for } a\in\mathbb{R} \text{ and } C>0,\\ 0<f_i(a) & \text{for } a\in(0,a_0]\quad \text{with } a_0>0,\\ \displaystyle\int_{a_0}^{A}\frac{1}{f_i(a)}\,da<+\infty & \text{for all } A\ge a_0.\end{cases}$$

An example for such f_i is

$$f_i(a)=\max\left(0,\min\left(a,\sqrt{|a-1|}\right)\right).$$

Moreover, $A=(A_{ijkl})_{i,j,k,l}$ is a tensor of order 4 that satisfies the following positivity condition: there exists $\delta_0>0$ such that

$$\sum_{i,j,k,l}A_{ijkl}\,\eta^i\,\eta^j\,\zeta_k\,\zeta_l\ \ge\delta_0|\eta|^2|\zeta|^2\qquad\text{for all}\quad \eta\in\mathbb{R}^m,\zeta\in\mathbb{R}^N. \tag{13.2}$$

The entropy function Ψ_i is chosen such that Ψ_i is nonnegative, lower semi-continuous, convex and satisfies that $\Psi_i''(a) = \dfrac{1}{f_i(a)}$ for $i = 1, \ldots, m$. Our solution satisfies the following entropy estimate for a.e. $t > 0$

$$\sum_{i=1}^{m} \int_\Omega \Psi_i(u^i(t)) + \delta_0 \sum_{i=1}^{m} \int_0^t \int_\Omega |\nabla u^i|^2 \leq \sum_{i=1}^{m} \int_\Omega \Psi_i(u_0^i). \tag{13.3}$$

To get this entropy we can apply the same strategy announced in Section 5 where $f_i(u^i)$ will be replaced by $T^{\epsilon,\ell}(f_i(v^i))$ with $T^{\epsilon,\ell}$ given in (5.3) and we use the fact that

$$\begin{aligned}
\int_\Omega \sum_{i,j,k,l} \frac{\partial u^i}{\partial x_k} A_{ijkl} \frac{\partial u^j}{\partial x_l} &= \sum_{n\in\mathbb{Z}^N} \sum_{i,j,k,l} \overline{\widehat{\left(\frac{\partial u^i}{\partial x_k}\right)}}(n) A_{ijkl} \widehat{\left(\frac{\partial u^j}{\partial x_l}\right)}(n) \\
&= \sum_{n\in\mathbb{Z}^N} \sum_{i,j,k,l} n_k\, \widehat{u^i}(n)\, A_{ijkl}\, n_l\, \widehat{u^j}(n) \\
&\geq\ \delta_0 \sum_{n\in\mathbb{Z}^N} |n|^2 \,|\widehat{u}|^2 = \delta_0 \,\|\nabla u\|^2_{(L^2(\Omega))^m}\,.
\end{aligned}$$

13.2. The variables coefficients case

Here the coefficients $A_{ij}(x,u)$ may depend continuously of (x,u). Then we have to take $\rho_\eta \star (A_{ij}(x,u)(\nabla \rho_\eta \star u^j))$ instead of $A_{ij}\nabla(\rho_\eta \star \rho_\eta \star u^j)$ in the approximate problem. We can consider a problem

$$\partial_t u^i = \operatorname{div}\left(u^i \sum_{j=1}^{m} A_{ij}(x,u)\nabla u\right) + g^i(x,u),$$

where the source terms are continuous with respect to the variable u and there exists a positive constant c such that

$$-c\,|u| \leq g^i(x,u) \leq c(1+|u|).$$

13.3. Laplace-type equations

Moreover, our method applies to models of the form

$$\partial_t u^i = \Delta(a_i(u)u^i) \quad \text{with } u = (u^i)_{1\leq i\leq m}, \tag{13.4}$$

under these assumptions:

$$\begin{cases}
a_i(u) \geq 0 \quad \text{if} \quad u^j \geq 0 \quad \text{for} \quad j = 1,\ldots,m, \\
a_i \quad \text{is at most linear}, \\
a_i \in C^1(\mathbb{R}), \\
\operatorname{Sym}\left(\left(\dfrac{\partial a_i}{\partial u_j}\right)_{i,j}\right) \geq \delta_0 I \quad \text{with} \quad \delta_0 > 0, \\
\dfrac{\partial a_i}{\partial u_j} \quad \text{are bounded from below for all } i,j = 1,\ldots,m
\end{cases} \tag{13.5}$$

where *Sym* denotes the symmetric part of a matrix. We can consider a particular case of (13.4) where $a_i(u) = \sum_{j=1}^{m} A_{ij}u^j$. Then problem (13.4) can be written as

$$\partial_t u^i = \operatorname{div}\left\{u^i \sum_{j=1}^{m} A_{ij}\nabla u^j + \left(\sum_{j=1}^{m} A_{ij}u_j\right)\nabla u^i\right\}, \tag{13.6}$$

which can be also solved under these assumptions:

$$\begin{cases} A_{ij} \geq 0 \qquad \text{for} \qquad i, j = 1, \ldots, m, \\ \mathrm{Sym}(A) \geq \delta_0 I. \end{cases}$$

Appendix A: Technical results

In this section we will present some technical results that are used in our proofs.

Proposition A.1 (Recovering the initial condition)**.** *Let Y be a Banach space with the norm $\|.\|_Y$. Consider a sequence $(g_m)_m \in C(0,T;Y)$ such that $\partial_t(g_m)$ is uniformly bounded in $L^q(0,T;Y)$ with $1 < q \leq \infty$, and $(g_m)_{|t=0} \to g_0$ in Y. Then there exists $g \in C(0,T;Y)$ such that $g_m \to g$ in $C(0,T;Y)$ and*

$$g_{|t=0} = g_0 \qquad \text{in } Y.$$

Proof. We have that for all $s < t \in (0,T)$

$$\begin{aligned} \|g_m(t) - g_m(s)\|_Y &= \left\| \int_s^t \partial_\tau(g_m)(\tau) \right\|_Y \\ &\leq \int_s^t \|\partial_\tau(g_m)(\tau)\|_Y \,\mathrm{d}s \\ &\leq (t-s)^{\frac{q-1}{q}} \|\partial_\tau(g_m)(\tau)\|_{L^q(0,T;Y)} \leq (t-s)^{\frac{q-1}{q}} C, \end{aligned} \tag{A.1}$$

where we have used in the second line Holder's inequality, and the fact that $(g_m)_\tau$ is uniformly bounded in $L^q(0,T;Y)$. Since (A.1) implies the equicontinuity of $(g_m)_m$, by Arzelà-Ascoli theorem, there exists $g \in C(0,T;Y)$ such that $g_m \to g$ in $C(0,T;Y)$. Moreover, Taking $s = 0$ in (A.1) we get

$$\|g_m(t) - g_m(0)\|_Y \leq t^{\frac{q-1}{q}} C. \tag{A.2}$$

By passing to the limit in m in (A.2), we deduce that

$$\|g(t) - g_0\|_Y \leq t^{\frac{q-1}{q}} C.$$

Particularly, for $t = 0$, we have

$$\|g(0) - g_0\|_Y = 0.$$

This implies the result. ☐

Lemma A.2 (Convergence result)**.** *Let $(a_\epsilon)_\epsilon$ a real sequence such that $a_\epsilon \to a_0$ as $\epsilon \to 0$. Then we have*

$$\Psi_{0,\ell}(a_0) \leq \liminf_{\epsilon \to 0} \Psi_{\epsilon,\ell}(a_\epsilon),$$

where $\Psi_{\epsilon,\ell}$ and $\Psi_{0,\ell}$ are given in (8.1) and (9.1) respectively.

Proof. Consider the case where $a_0 = 0$.

We consider the subsequence $(a_{\epsilon_{k_1}})_{\epsilon_{k_1}} \in (-\infty; \frac{1}{e}]$. Let $(b_\epsilon)_\epsilon \in (-\infty; \frac{1}{e}]$ a sequence that decreases to 0 as $\epsilon \to 0$ with $b_\epsilon > a_\epsilon$. Since $\Psi_{\epsilon,\ell}$ is decreasing on $(-\infty; \frac{1}{e}]$ we have $\Psi_{\epsilon,\ell}(a_\epsilon) \geq \Psi_{\epsilon,\ell}(b_\epsilon)$. Moreover, using the fact that $\Psi_{\epsilon,\ell}(b_\epsilon) \to 0 = \Psi_{0,\ell}(0)$ we get the result.

Otherwise, consider $(a_{\epsilon_{k_2}})_{\epsilon_{k_2}} \in (\frac{1}{e}; +\infty)$ the proof is the same as above but with taking $b_\epsilon < a_\epsilon$ since $\Psi_{\epsilon,\ell}$ is nondecreasing in $(\frac{1}{e}; +\infty)$.

For the other cases, $a_0 < 0$ and $a_0 > 0$, the result is easily obtained. ☐

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