

MOTION OF DISCRETE INTERFACES IN LOW-CONTRAST RANDOM ENVIRONMENTS

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Abstract. We study the asymptotic behavior of a discrete-in-time minimizing movement scheme for square lattice interfaces when both the lattice spacing and the time step vanish. The motion is assumed to be driven by minimization of a weighted random perimeter functional with an additional deterministic dissipation term. We consider rectangular initial sets and lower order random perturbations of the perimeter functional. In case of stationary, α -mixing perturbations we prove a stochastic homogenization result for the interface velocity. We also provide an example which indicates that only stationary, ergodic perturbations might not yield a spatially homogenized limit velocity for this minimizing movement scheme.

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1. INTRODUCTION

In 1993 Almgren, Taylor and Wang introduced a notion of minimizing movements suitable to describe geometric motions of interfaces driven by curvature effects (see [5]). In a nutshell it can be summarized as follows: given a fixed time step $\tau > 0$ and an initial set $A_0 \subset \mathbb{R}^d$, one constructs recursively a sequence of sets $\{A_k^\tau\}_k$ minimizing an energy functional of the form

$$A \mapsto E^\tau(A, A_{k-1}^\tau) = \int_{\mathcal{F}A} \varphi(\nu(x)) \, d\mathcal{H}^{d-1} + \frac{1}{2\tau} \int_{A \Delta A_{k-1}^\tau} \text{dist}(x, \partial A_{k-1}^\tau) \, dx, \quad (1.1)$$

where $\nu(x)$ is the normal vector at the point x in the reduced boundary $\mathcal{F}A$ (we refer to [6] for a precise definition) and φ is a suitable surface density. The basic idea behind this approach is the following: while minimizing the surface functional shrinks the set, the bulk term forces the boundary of the minimizer to be close to the boundary of the previous set. Passing to the limit as $\tau \rightarrow 0$ for the piecewise constant interpolations one obtains a time dependent family $A(t)$ of sets that evolves by a weighted curvature (depending on φ), provided the initial set A_0 is regular enough and φ is elliptic and smooth. In the isotropic case one obtains the well-known motion by mean curvature. This minimizing movement procedure was later on exported to random environments by Yip in [21] as follows: at each discrete time step, a minimizer of the energy in (1.1) is computed and then this set is perturbed by a random diffeomorphism.

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In the recent paper [14] Braides, Gelli and Novaga applied the above minimizing movement scheme within a deterministic, discrete environment. In this setting the environment is the scaled two-dimensional lattice $\varepsilon\mathbb{Z}^2$. The surface term in (1.1) is replaced by a discrete interfacial energy which, in its simplest form, is derived from the classical nearest neighbor Ising model for spin systems and can be written formally as

$$P_\varepsilon(u) = \frac{1}{4} \sum_{\substack{\varepsilon i, \varepsilon j \in \varepsilon\mathbb{Z}^2 \\ |i-j|=1}} \varepsilon |u(\varepsilon i) - u(\varepsilon j)|, \quad (1.2)$$

where $u : \varepsilon\mathbb{Z}^2 \rightarrow \{\pm 1\}$ is the spin variable. Note that the energy in (1.2) takes into account only nearest neighbor interactions. Therefore it can be interpreted as a discrete perimeter of the set $\{u = 1\}$ and the relationship to the continuum model is given by identifying the spin variable with this level set. The distance-function in the bulk term in (1.1) is replaced by a discrete version of the l^∞ -distance to the boundary precisely defined in (2.2). From a physical point of view this setup can be seen as a simplified model to describe the motion of boundaries of the level sets of the spin variables, which represent the magnetic domain walls at the discrete level. Since the discrete perimeter inherits the anisotropy of the lattice, this minimizing movement scheme is related to crystalline motions, where φ is not smooth (see [4, 7, 18] in the continuum case). Note that the continuum limit (or Γ -limit; see [10]) of the energies in (1.2) is given by the crystalline perimeter, that is

$$P(u) = \int_{S_u} |\nu(x)|_1 \, d\mathcal{H}^1 = \int_{\mathcal{F}\{u=1\}} |\nu(x)|_1 \, d\mathcal{H}^1, \quad (1.3)$$

where $|\nu|_1$ denotes the l^1 -norm of ν (see [1]). In [14] the authors observed that the asymptotic behavior of the discrete flows depends heavily on the ratio between ε, τ when $\varepsilon, \tau \rightarrow 0$ simultaneously. When $\varepsilon/\tau \rightarrow 0$ fast enough the motion is governed by the Γ -limit, that means one obtains the continuum motion by crystalline curvature. If, on the other hand, $\varepsilon/\tau \rightarrow +\infty$ fast enough, the motions are pinned by the presence of many local minimizers in the discrete environment. This phenomenon is similar to any gradient flow that starts in a local minimum. We remark that in general such *a priori* results are abstract and the necessary speed of convergence/divergence might be unknown. We refer the reader to Chapter 8 in [11] for a parade of further examples on this issue. However the exact behavior was found in [14]. The critical scaling for the discrete perimeter energies (1.2) is $\varepsilon \sim \tau$, where pinning effects due to discreteness as well as a quantized crystalline motion can occur (see also Thm. 2.9 below).

In this paper we start studying the effect of a random discrete environment on the continuum limit flow. We take a different approach compared to [21] and associate the random effects directly to the lattice points. In [3] Alicandro, Cicalese and the author performed a discrete-to-continuum analysis for a large class of ferromagnetic Ising-type energies including (1.2), but where the interacting particles are located at the points of a so-called stochastic lattice $\varepsilon\mathcal{L}(\omega)$ instead of the periodic $\varepsilon\mathbb{Z}^2$ (or more generally $\varepsilon\mathbb{Z}^d$). In particular, assuming the stochastic lattice to be stationary with respect to translations, one can prove the existence of a homogeneous limit surface energy that turns out to be deterministic under some ergodicity assumptions. In that case the continuum limit takes the form

$$P_{\text{hom}}(u) = \int_{S_u} \varphi_{\text{hom}}(\nu(x)) \, d\mathcal{H}^1.$$

Hence the continuum limit resembles the surface term in (1.1) even though in general φ_{hom} might be non-smooth. Our aim is to include dynamical effects in order to describe the curvature-driven motion of magnetic domain walls. The natural approach in the spirit of [3, 12] would be to replace the periodic lattice in the definition of P_ε by a stationary random lattice $\varepsilon\mathcal{L}(\omega)$ with suitable short-range interactions. This seems to be a very challenging problem. Thus we start with a much simpler model by adding very small random perturbations directly to the periodic lattice model, that means we will study the minimizing movement of a random discrete perimeter of

the form

$$P_\varepsilon^\omega(u) = \frac{1}{4} \sum_{\substack{\varepsilon i, \varepsilon j \in \varepsilon \mathbb{Z}^2 \\ |i-j|=1}} \varepsilon(1 + \varepsilon c_{ij}(\omega)) |u(\varepsilon i) - u(\varepsilon j)|.$$

For the precise assumptions on the random field c_{ij} we refer to Section 2. Note that in this scaling the random perturbations are a lower order term as they are scaled by ε . Nevertheless it turns out that they may influence at least the velocity of the limit motion. The reason why we don't let the bulk term be affected by the randomness comes from the physical interpretation that we give to this model as motion of aligned spins, which differs from lattice particle models: while the interaction between particles may be affected by some random noise deriving from microscopic fluctuations, the energy to flip a spin should be constant, depending only on how many boundary layers are flipped in one time step. The interpretation of the bulk term in the energy in this setting is the following: flipping the first layer of spins costs the least energy while the following layers are energetically more expensive. Of course this interpretation makes sense only if one can prove that in presence of randomness sets shrink by flipping spins close to the boundary by a certain number of layers. This is the case in the deterministic setting considered in [14].

For the sake of simplicity, we investigate the evolution when the initial set is a coordinate rectangle, that means a rectangle with all sides parallel to one of the coordinate axes. In Theorem 3.1 we prove that under stationarity and quantified mixing assumptions as well as a suitable uniform bound on the random field c_{ij} , the limit motion law is deterministic and coincides with the quantized crystalline flow obtained in [14]. This however depends strongly on the fact that the random field is stationary with respect to the translation group on \mathbb{Z}^2 . In Section 4 we show that the velocity changes if we restrict stationarity to a subgroup of the form $m\mathbb{Z}^2$ with $m \geq 2$. Anyhow, we stress that our results should be seen as a stability result of the deterministic problem rather than an exhaustive description of the possible effects of randomness on the limit flow. Indeed, randomness can influence the motion drastically. For example, in [3] it is proven that when we replace the square lattice \mathbb{Z}^2 by a suitable isotropic stochastic lattice $\mathcal{L}(\omega)$, then, up to a multiplicative constant, the discrete perimeters Γ -converge to the Euclidean instead of the crystalline perimeter. Thus, with an appropriate choice of discrete distance, one should not expect a crystalline motion anymore in the limit but rather some type of motion by mean curvature, at least if $\varepsilon \ll \tau$ and the initial sets $A_0^{\varepsilon, \tau}$ converge to a smooth set. To highlight possible difficulties even in the present very weak random setting, we provide an example of stationary, ergodic perturbations, for which the functional describing the pointwise motion may not converge (Exp. 3.5). This example also indicates strong non-uniqueness effects when we drop the mixing hypothesis (see Rem. 3.6). Moreover, in Remark 3.4 we briefly discuss what might happen when we consider random fields c_{ij} satisfying only a generic L^∞ -bound.

2. NOTATION AND PRELIMINARIES

In this section we introduce our model and recall some definitions from probability theory as well as existing results in the deterministic setting.

2.1. The random model

First we set some notation. We define $Q_\delta(x) = x + [-\frac{\delta}{2}, \frac{\delta}{2})^2$ as the half-open coordinate square centered at x with side length δ . For a real number $y \in \mathbb{R}$, we let $[y]$ be its integer part and $\lceil y \rceil := [y] + 1$. By $|\cdot|$ we denote the Euclidean norm on \mathbb{R}^2 . If $B \subset \mathbb{R}^2$ is a Borel set we denote by $|B|$ its Lebesgue measure and by $\mathcal{H}^1(B)$ its one-dimensional Hausdorff measure. Moreover, we set $d_{\mathcal{H}}(A, B)$ as the Hausdorff metric between two sets A, B . The symmetric difference of two sets A, B is denoted by $A \Delta B$. We set $\mathbf{1}_B$ as the characteristic function of B , and we denote by $\mathbb{E}[X]$ the first moment of a random variable X .

We now specify the framework for our model. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. As pointed out in the introduction, we consider the easiest type of normalized ferromagnetic energies accounting only for

nearest neighbor interactions. Given $\omega \in \Omega$ and a function $u : \varepsilon\mathbb{Z}^2 \rightarrow \{\pm 1\}$ we set

$$P_\varepsilon^\omega(u) = \frac{1}{4} \sum_{\substack{i,j \in \mathbb{Z}^2 \\ |i-j|=1}} \varepsilon (1 + \varepsilon c_{ij}(\omega)) |u(\varepsilon i) - u(\varepsilon j)|,$$

where the $c_{ij} : \Omega \rightarrow \mathbb{R}$ are uniformly bounded random variables satisfying a suitable α -mixing assumption specified in (2.5) below. Note that without loss of generality we may assume that $c_{ij} = c_{ji}$ for all $|i - j| = 1$. We define

$$\mathcal{A}_\varepsilon := \left\{ A \subset \mathbb{R}^2 : A = \bigcup_{i \in \mathcal{I}} Q_\varepsilon(i) \quad \text{for some} \quad \mathcal{I} \subset \varepsilon\mathbb{Z}^2 \right\}.$$

The family of sets \mathcal{A}_ε is closed under unions and intersections. Identifying the level set $\{u = 1\}$ of a function $u : \varepsilon\mathbb{Z}^2 \rightarrow \{\pm 1\}$ with the set $A \in \mathcal{A}_\varepsilon$ given by

$$A := \bigcup_{i \in \mathbb{Z}^2 : u(\varepsilon i) = +1} Q_\varepsilon(i),$$

we can interpret P_ε^ω as a random perimeter defined on \mathcal{A}_ε via $P_\varepsilon^\omega(A) := P_\varepsilon^\omega(u)$.

If $\mathbb{Z}^2 := \{\xi = \frac{i+j}{2} : i, j \in \mathbb{Z}^2, |i - j| = 1\}$ denotes the dual lattice of \mathbb{Z}^2 , we can rewrite the random perimeter as a sum over points on the boundary ∂A via

$$P_\varepsilon^\omega(A) = \sum_{\substack{\xi \in \mathbb{Z}^2 \\ \varepsilon\xi \in \partial A}} \varepsilon (1 + \varepsilon c_\xi(\omega)), \tag{2.1}$$

where with a slight abuse of notation we set $c_\xi(\omega) := c_{ij}(\omega)$. From now on we assume the random variables to be indexed by the dual lattice. Given $A \in \mathcal{A}_\varepsilon$ it will be useful to define the properly scaled random perimeter also on portions of the boundary $\Gamma \subset \partial A$ setting

$$p_\varepsilon^\omega(\Gamma) = \sum_{\xi \in \mathbb{Z}^2 : \varepsilon\xi \in \Gamma} \varepsilon c_\xi(\omega).$$

With this notion, it holds that $P_\varepsilon^\omega(A) = \mathcal{H}^1(\partial A) + \varepsilon p_\varepsilon^\omega(\partial A)$.

In order to adapt the idea of Almgren, Taylor and Wang for studying curvature-driven motions, we have to define a suitable discrete distance between sets. As in [14] we take a discrete version of the l_∞ -distance. To this end, first note that for every $x \in \mathbb{R}^2$ there exists a unique point $i \in \varepsilon\mathbb{Z}^2$ such that $x \in Q_\varepsilon(i)$. Given a set $A \subset \mathcal{A}_\varepsilon$ we define the value of the measurable function $d_\infty^\varepsilon(\cdot, \partial A) : \mathbb{R}^2 \rightarrow [0, +\infty)$ at $x \in Q_\varepsilon(i)$ by

$$d_\infty^\varepsilon(x, \partial A) := \begin{cases} \inf\{\|i - j\|_\infty : j \in \varepsilon\mathbb{Z}^2 \setminus A\} & \text{if } i \in A, \\ \inf\{\|i - j\|_\infty : j \in \varepsilon\mathbb{Z}^2 \cap A\} & \text{if } i \notin A. \end{cases} \tag{2.2}$$

Observe that by definition $d_\infty^\varepsilon(x, \partial A) \in \varepsilon\mathbb{N}$ for all $x \in \mathbb{R}^2$.

Now we can define the total energy to be considered in the minimizing movement scheme. Given a mesh size $\varepsilon > 0$, a time step $\tau > 0$, sets $A, F \in \mathcal{A}_\varepsilon$ and $\omega \in \Omega$ we set

$$E_{\varepsilon,\tau}^\omega(A, F) := P_\varepsilon^\omega(A) + \frac{1}{\tau} \int_{A \Delta F} d_\infty^\varepsilon(x, \partial F) dx.$$

For a fixed (possibly random) initial set $A_\varepsilon^0(\omega)$, we introduce the following discrete-in-time minimization scheme:

- (i) $A_{\varepsilon,\tau}^0(\omega) := A_\varepsilon^0(\omega)$,
- (ii) $A_{\varepsilon,\tau}^{k+1}(\omega)$ minimizes $A \mapsto E_{\varepsilon,\tau}^\omega(A, A_{\varepsilon,\tau}^k(\omega))$.

Note that this procedure might not be unique. The discrete flat flow is defined as the piecewise constant interpolation

$$A_{\varepsilon,\tau}(t)(\omega) := A_{\varepsilon,\tau}^{\lfloor t/\tau \rfloor}(\omega).$$

As a by-product of the analysis performed in [14], the most interesting regime is $\tau \sim \varepsilon$. Hence throughout this paper we assume for simplicity that

$$\tau = \gamma \varepsilon \quad \text{for some } \gamma > 0$$

and omit the dependence on τ in the notation introduced above. For a complete analysis we have to require that the coefficient field satisfies the bound

$$\sup_{\xi} |c_{\xi}(\omega)| =: c_{\infty}(\omega) < \frac{1}{4\gamma} \quad \mathbb{P}\text{-almost surely.} \tag{2.3}$$

Remark 2.1. Some of the results in this paper are valid for a generic L^{∞} -bound $c_{\infty}(\omega) \leq C$ but unfortunately these are not enough to characterize the motion (see also Rem. 3.4). The bound (2.3) allows to control oscillations in the random part of the perimeter by the bulk term on small scales where we cannot apply the law of large numbers. Roughly speaking this bound compensates the fact that the shape of minimal paths in \mathbb{Z}^2 can change drastically if one adds a very small perturbation to the crystalline perimeter, so that we need to control this deviation by the bulk term. Therefore it is clear that when $\tau \gg \varepsilon$, our random perturbations are required to be very small. The precise value comes from a reverse isoperimetric inequality on $\mathcal{A}_{\varepsilon}$: for all $A \in \mathcal{A}_{\varepsilon}$ it holds that

$$\varepsilon \mathcal{H}^1(\partial A) \leq 4|A|. \tag{2.4}$$

Remark 2.2. Using the boundedness of the random coefficients it is easy to see that $P_{\varepsilon}^{\omega}(A)$ has the same Γ -limit in the L^1 -topology as P_{ε} defined in (1.2), so that it converges to the crystalline perimeter (1.3).

Now we introduce several stochastic properties of the random field $\{c_{\xi}\}_{\xi \in \mathbb{Z}^2}$. In what follows, given a subset $I \subset \mathbb{Z}^2$ we set $\mathcal{F}_I = \sigma(c_{\xi} : \xi \in I)$ as the σ -algebra generated by the random variables $\{c_{\xi}\}_{\xi \in I}$. We recall the following definitions from ergodic/probability theory:

Definition 2.3. We say that a family of measurable functions $\{\tau_z\}_{z \in \mathbb{Z}^2}, \tau_z : \Omega \rightarrow \Omega$, is an additive group action on Ω if

$$\tau_{z_1+z_2} = \tau_{z_2} \circ \tau_{z_1} \quad \forall z_1, z_2 \in \mathbb{Z}^2.$$

Such an additive group action is called measure preserving if

$$\mathbb{P}(\tau_z B) = \mathbb{P}(B) \quad \forall B \in \mathcal{F}, z \in \mathbb{Z}^2.$$

Moreover $\{\tau_z\}_{z \in \mathbb{Z}^2}$ is called ergodic if, in addition, for all $B \in \mathcal{F}$ we have

$$(\tau_z(B) = B \quad \forall z \in \mathbb{Z}^2) \quad \Rightarrow \quad \mathbb{P}(B) \in \{0, 1\}.$$

Definition 2.4. Let $\{\tau_z\}_{z \in \mathbb{Z}^2} : \Omega \rightarrow \Omega$ be a measure preserving group action. We say that the random field $\{c_{\xi}\}_{\xi \in \mathbb{Z}^2}$ is

- (i) *stationary*, if $c_{\xi} \circ \tau_z = c_{\xi+z} \quad \forall z \in \mathbb{Z}^2$ and $\forall \xi \in \mathbb{Z}^2$;
- (ii) *ergodic*, if it is stationary and $\{\tau_z\}_z$ is ergodic;
- (iii) *strongly mixing (in the ergodic sense)*, if it is stationary and

$$\lim_{|z| \rightarrow +\infty} \mathbb{P}(A \cap (\tau_z B)) = \mathbb{P}(A)\mathbb{P}(B) \quad \forall A, B \in \mathcal{F};$$

(iv) α -mixing, if there exists a sequence $\alpha(n) \rightarrow 0$ such that for all sets $I_1, I_2 \in \mathcal{Z}^2$ with $\text{dist}(I_1, I_2) \geq n$ we have

$$\sup\{|\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)| : A \in \mathcal{F}_{I_1}, B \in \mathcal{F}_{I_2}\} \leq \alpha(n).$$

Remark 2.5. In contrast to ergodicity, the property of α -mixing is conserved on subsets of \mathcal{Z}^2 . In general, it might seem natural to start with the easiest case of an independent and identically distributed random field $\{c_\xi\}$. However this does not simplify the arguments since our approach heavily relies on Theorem 2.7 below. Indeed, in addition to this theorem we only use stationarity. We recall that in general, up to switching to a product space, we have the following implications (always assuming stationarity):

$$\text{independence} \Rightarrow \alpha\text{-mixing} \Rightarrow \text{strongly mixing in the ergodic sense} \Rightarrow \text{ergodicity}.$$

In our analysis we have to study averages of sequences of the form $\{c_{\xi_j}\}_j$, where the sequence $\{\xi_j\}_j$ describes a single side in the boundary of a rectangle. For sequences the definition of α -mixing reads as follows:

Definition 2.6. A sequence $\{X_j\}_{j \in \mathbb{N}}$ of random variables is said to be α -mixing if there exists a sequence $\alpha(n) \rightarrow 0$ such that for all sets $I_1, I_2 \subset \mathbb{N}$ with $\text{dist}(I_1, I_2) \geq n$ it holds that

$$\sup\{|\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)| : A \in \mathcal{F}_{I_1}, B \in \mathcal{F}_{I_2}\} \leq \alpha(n).$$

Here again \mathcal{F}_I denotes the σ -algebra generated by the random variables $\{X_j\}_{j \in I}$.

Similar to independent random variables (which are α -mixing with $\alpha(n) = 0$), α -mixing allows for quantitative estimates for the error probabilities in the law of large numbers. We will need the following polynomial decay theorem for bounded α -mixing sequences, proved by Berbee in [8].

Theorem 2.7. *Let $p > 1$ and $\{X_j\}_{j \in \mathbb{N}}$ be an α -mixing sequence of random variables bounded by 1 such that $\mathbb{E}[X_j] = 0$ for all j . If*

$$\sum_{n \geq 1} n^{p-2} \alpha(n) < +\infty,$$

then, setting $S_k = \sum_{j=1}^k X_j$, for all $\delta > 0$ it holds that

$$\sum_{n \geq 1} n^{p-2} \mathbb{P}\left(\sup_{k \geq n} |S_k/k| > \delta\right) < +\infty.$$

While for static problems stationarity and ergodicity are often enough to prove stochastic homogenization results for variational models (see for example [2, 3, 12, 15]), in this minimizing movement setting we make use of mixing properties. More precisely, motivated by Theorem 2.7 we require that the random field is α -mixing with coefficients satisfying

$$\sum_{n \geq 1} \alpha(n) < +\infty. \tag{2.5}$$

Remark 2.8. There are stronger notions of mixing in the literature, however we prefer to choose α -mixing with a suitable decay rate of $\alpha(n)$ rather than a general ϕ -mixing condition since the generalization of ϕ -mixing conditions to two-dimensional random fields is not trivial and many of them already imply a finite range dependence assumption (see [9]). Moreover, in general α -mixing is much weaker than ϕ -mixing.

2.2. Results for deterministic models

Let us collect some results obtained in the deterministic setting. Within a discrete, deterministic environment, the problem we are interested in has first been studied by Braides, Gelli and Novaga in [14] in the case $c_\xi(\omega) = 0$. For coordinate rectangles as initial sets they prove the following:

Theorem 2.9 (Braides, Gelli, Novaga). *Let $A_\varepsilon^0 \in \mathcal{A}_\varepsilon$ be a coordinate rectangle with sides $S_{1,\varepsilon}, \dots, S_{4,\varepsilon}$. Assume that A_ε^0 converges in the Hausdorff metric to a coordinate rectangle A . Then, up to subsequences, $A_\varepsilon(t)$ converges locally in time to $A(t)$, where $A(t)$ is a coordinate rectangle with sides $S_i(t)$ such that $A(0) = A$ and any side S_i moves inward with velocity $v_i(t)$ given by*

$$v_i(t) \begin{cases} = \frac{1}{\gamma} \left\lfloor \frac{2\gamma}{L_i(t)} \right\rfloor & \text{if } \frac{2\gamma}{L_i(t)} \notin \mathbb{N}, \\ \in \frac{1}{\gamma} \left[\left(\frac{2\gamma}{L_i(t)} - 1 \right), \frac{2\gamma}{L_i(t)} \right] & \text{if } \frac{2\gamma}{L_i(t)} \in \mathbb{N}, \end{cases}$$

where $L_i(t) := \mathcal{H}^1(S_i(t))$ denotes the length of the side $S_i(t)$, until the extinction time when $L_i(t) = 0$.

Assume in addition that the lengths L_1^0, L_2^0 of A satisfy one of the three following conditions (assuming that $L_1^0 \leq L_2^0$):

- (i) $L_1^0, L_2^0 > 2\gamma$ (total pinning),
- (ii) $L_1^0 < 2\gamma$ and $L_2^0 \leq 2\gamma$ (vanishing in finite time with shrinking velocity larger than $1/\gamma$),
- (iii) $L_1^0 < 2\gamma$ such that $2\gamma/L_1^0 \notin \mathbb{N}$ and $L_2^0 > 2\gamma$ (partial pinning).

Then $A_\varepsilon(t)$ converges locally in time to $A(t)$ as $\varepsilon \rightarrow 0$, where $A(t)$ is the unique rectangle with side lengths $L_1(t)$ and $L_2(t)$ solving the following system of ordinary differential equations

$$\begin{cases} \frac{d}{dt} L_1(t) = -\frac{2}{\gamma} \left\lfloor \frac{2\gamma}{L_2(t)} \right\rfloor, \\ \frac{d}{dt} L_2(t) = -\frac{2}{\gamma} \left\lfloor \frac{2\gamma}{L_1(t)} \right\rfloor \end{cases}$$

for almost every t with initial conditions $L_1(0) = L_1^0$ and $L_2(0) = L_2^0$.

It is the aim of this paper to extend these results to small random perturbations of the perimeter. While in [14] more general classes of sets are studied, we restrict ourselves to rectangles as the analysis of these sets already contains the main features deriving from randomness. We mention that some effects of periodic perturbations have already been studied in [16, 19]. In [16] the authors treat the following type of high-contrast periodicity: let $N_a, N_b \in \mathbb{N}$ and $N_{ab} = N_a + N_b$. The coefficients c_ξ are N_{ab} periodic and on the periodicity cell $0 \leq \xi_1, \xi_2 < N_{ab}$ they satisfy

$$c_\xi = \begin{cases} b & \text{if } 0 \leq \xi_1, \xi_2 \leq N_b, \\ a & \text{otherwise,} \end{cases}$$

with weights $a < b$. It is shown that minimizers avoid the b -interactions and thus the limit velocity does not depend on b but only on the geometric proportions N_a, N_b of the periodicity cell. It would be interesting to see how random interactions acting on this scale influence the minimizing sets, since without periodicity it might be impossible to take only a -interactions. However, in this paper we take the same scaling as the periodic perturbations considered in [19]. These are so called low-contrast perturbations since they vanish when $\varepsilon \rightarrow 0$. It is shown in [19] that the right scaling to obtain also b -interactions is $b - a \sim \varepsilon$. More precisely, one has to require that $|b - a| < \frac{\varepsilon}{2\gamma}$. Note that this bound agrees with (2.3). Hence with $a = 1$ this model corresponds to

a deterministic version of (2.1). In this sense, up to the bound (2.3), our Theorem 4.2 generalizes the results of [19] to the most general periodic interactions as well as to the random case. While in the above deterministic setting coefficients with $|b - a| \geq \frac{\varepsilon}{2\gamma}$ lead to rectangular interfaces using only a -interactions, in the random case it is not clear what happens. We leave this issue as well as the high-contrast case open for future studies. For the interested reader we mention the recent papers [13, 17], where the minimizing movements have been studied for other discrete surface-type models.

3. HOMOGENIZED LIMIT MOTION OF A RECTANGLE

In the sequel we study the case, when the initial data A_ε^0 is a coordinate rectangle. We further assume for the rest of this paper that

$$\sup_\varepsilon \mathcal{H}^1(\partial A_\varepsilon^0(\omega)) = C < +\infty. \tag{3.1}$$

This bound implies that, for ε small enough, any sequence chosen by the minimizing movement has equibounded perimeter. Indeed, by minimality we have

$$P_\varepsilon^\omega(A_\varepsilon^{k+1}(\omega)) \leq E_\varepsilon^\omega(A_\varepsilon^{k+1}(\omega), A_\varepsilon^k(\omega)) \leq E_\varepsilon^\omega(A_\varepsilon^k(\omega), A_\varepsilon^k(\omega)) = P_\varepsilon^\omega(A_\varepsilon^k(\omega)),$$

so that by induction and L^∞ -boundedness of the random perturbations we infer for ε small enough that

$$\mathcal{H}^1(\partial A_\varepsilon^k(\omega)) \leq 2P_\varepsilon^\omega(A_\varepsilon^k(\omega)) \leq 2P_\varepsilon^\omega(A_\varepsilon^0(\omega)) \leq 4\mathcal{H}^1(\partial A_\varepsilon^0(\omega)). \tag{3.2}$$

The following theorem is the main result of Section 3.

Theorem 3.1. *Assume that the random field $\{c_\varepsilon\}_\varepsilon$ is stationary and α -mixing with coefficients satisfying (2.5). Then with probability 1 the following holds: let $\varepsilon_j \downarrow 0$ and let $A_{\varepsilon_j}^0(\omega) \in \mathcal{A}_{\varepsilon_j}$ be a coordinate rectangle with sides $S_{1,j}(\omega), \dots, S_{4,j}(\omega)$. Assume that $A_{\varepsilon_j}^0(\omega)$ converges in the Hausdorff metric to a coordinate rectangle $A(\omega)$. Then we can choose a subsequence (not relabeled) such that $A_{\varepsilon_j}(t)(\omega)$ converges locally in time to $A(t)(\omega)$, where $A(t)(\omega)$ is a coordinate rectangle with sides $S_i(t)(\omega)$ such that $A(0)(\omega) = A(\omega)$ and any side $S_i(t)(\omega)$ moves inward with velocity $v_i(t)(\omega)$ solving the following differential inclusions:*

$$v_i(t)(\omega) \begin{cases} = \frac{1}{\gamma} \left\lfloor \frac{2\gamma}{L_i(t)(\omega)} \right\rfloor & \text{if } \frac{2\gamma}{L_i(t)(\omega)} \notin \mathbb{N}, \\ \in \frac{1}{\gamma} \left[\left(\frac{2\gamma}{L_i(t)(\omega)} - 1 \right), \frac{2\gamma}{L_i(t)(\omega)} \right] & \text{if } \frac{2\gamma}{L_i(t)(\omega)} \in \mathbb{N}, \end{cases}$$

where $L_i(t)(\omega) := \mathcal{H}^1(S_i(t)(\omega))$ denotes the length of the side $S_i(t)(\omega)$. The differential inclusions are valid until the extinction time when $L_i(t)(\omega) = 0$.

Following word by word the proof of ([14], Thm. 2) we obtain unique limit motions in many cases:

Corollary 3.2. *Let $A_{\varepsilon_j}^0(\omega)$ and $\{c_\varepsilon\}_\varepsilon$ be as in Theorem 3.1. Assume in addition that the lengths $L_1^0(\omega), L_2^0(\omega)$ of $A(\omega)$ satisfy one of the three following conditions (assuming that $L_1^0(\omega) \leq L_2^0(\omega)$):*

- (i) $L_1^0(\omega), L_2^0(\omega) > 2\gamma$ (total pinning),
- (ii) $L_1^0(\omega) < 2\gamma$ and $L_2^0(\omega) \leq 2\gamma$ (vanishing in finite time with shrinking velocity larger than $1/\gamma$),
- (iii) $L_1^0(\omega) < 2\gamma$ and $2\gamma/L_1^0(\omega) \notin \mathbb{N}$, and $L_2^0(\omega) > 2\gamma$ (partial pinning).

The sequence $A_{\varepsilon_j}(t)(\omega)$ converges locally in time to $A(t)(\omega)$, where $A(t)(\omega)$ is the unique coordinate rectangle with sides $S_1(t)(\omega)$ and $S_2(t)(\omega)$ such that $A(0)(\omega) = A(\omega)$ and the side lengths $L_1(t)(\omega)$ and $L_2(t)(\omega)$ solve

the following differential equations for all but countably many times until the extinction time:

$$\begin{cases} \frac{d}{dt}L_1(t)(\omega) = -\frac{2}{\gamma} \left\lfloor \frac{2\gamma}{L_2(t)(\omega)} \right\rfloor, \\ \frac{d}{dt}L_2(t)(\omega) = -\frac{2}{\gamma} \left\lfloor \frac{2\gamma}{L_1(t)(\omega)} \right\rfloor \end{cases}$$

with initial condition $L_1(0)(\omega) = L_1^0(\omega)$ and $L_2(0)(\omega) = L_2^0(\omega)$.

We postpone the proof of Theorem 3.1 and first derive some auxiliary results.

3.1. Qualitative behavior

In this part we prove that during the minimization scheme coordinate rectangles remain sets of the same type as long as its sides don't degenerate to a point. As we will see later, this property is enough to derive the equation of motion at a fixed time t . The argument splits into two steps. First we prove that any minimizer must be connected and second, using the precise bound in (2.3), we conclude that this component has to be a coordinate rectangle. The idea to prove connectedness is as follows: first we compare the energy with a fast flow of a deterministic functional to conclude that the minimizer must contain a very large rectangle. Then the remaining components are ruled out using the isoperimetric inequality.

Proposition 3.3. *Assume that $\{c_\xi\}_\xi$ fulfills (2.3). Let $\eta > 0$ and suppose $A_\varepsilon^k(\omega)$ is a coordinate rectangle with all side lengths greater than η . Then, for ε small enough, $A_\varepsilon^{k+1}(\omega)$ is again a coordinate rectangle contained in $A_\varepsilon^k(\omega)$.*

Proof. As explained above we divide the proof into two steps. As the arguments are purely deterministic we drop the ω -dependence of the sets.

Step 1. Connectedness of minimizers

We consider the minimizing movement for an auxiliary deterministic functional that turns out to yield a faster flow. Given $0 < \delta \ll 1$, we define

$$G_\varepsilon^\delta(A, F) := \mathcal{H}^1(\partial A) + \frac{\delta}{\gamma\varepsilon} \int_{A\Delta F} d_\infty^\varepsilon(x, \partial F) \, dx.$$

By the uniform bound in (2.3) the random weights defining P_ε^ω are nonnegative for ε small enough. Thus for any sets $A, B, F \in \mathcal{A}_\varepsilon$ we have the (in)equalities

$$P_\varepsilon^\omega(A \cup B) + P_\varepsilon^\omega(A \cap B) \leq P_\varepsilon^\omega(A) + P_\varepsilon^\omega(B),$$

$$\int_{F\Delta(A\cap B)} d_\infty^\varepsilon(x, \partial F) \, dx + \int_{F\Delta(A\cup B)} d_\infty^\varepsilon(x, \partial F) \, dx = \int_{F\Delta A} d_\infty^\varepsilon(x, \partial F) \, dx + \int_{F\Delta B} d_\infty^\varepsilon(x, \partial F) \, dx.$$

The inequality also holds for the standard perimeter, which implies the two general estimates

$$\begin{aligned} E_\varepsilon^\omega(A \cap B, F) + E_\varepsilon^\omega(A \cup B, F) &\leq E_\varepsilon^\omega(A, F) + E_\varepsilon^\omega(B, F), \\ G_\varepsilon^\delta(A \cap B, F) + G_\varepsilon^\delta(A \cup B, F) &\leq G_\varepsilon^\delta(A, F) + G_\varepsilon^\delta(B, F). \end{aligned} \tag{3.3}$$

Now let $R_\varepsilon^\delta \in \mathcal{A}_\varepsilon$ be the smallest minimizer of $G_\varepsilon^\delta(\cdot, A_\varepsilon^k)$ with respect to set inclusion. This is well-defined due to (3.3). From the analysis in [14] we know that $R_\varepsilon^\delta \subset A_\varepsilon^k$ is a coordinate rectangle and, denoting by $N_{i,\varepsilon}$ the distance between corresponding sides of R_ε^δ and A_ε^k , for ε small enough it holds that

$$\left(\frac{2\gamma}{\delta L_{i,\varepsilon}} - 1 \right) \varepsilon \leq N_{i,\varepsilon} \leq \left(\frac{2\gamma}{\delta L_{i,\varepsilon}} + 1 \right) \varepsilon,$$

where $L_{i,\varepsilon}$ denotes the length of the side $S_{i,\varepsilon}$ of A_ε^k . In particular, using (3.1), (3.2) and the assumptions on the sides of A_ε^k , we infer the two-sided bound

$$\left(\frac{\gamma}{C\delta} - 1\right)\varepsilon \leq N_{i,\varepsilon}\varepsilon \leq \left(\frac{2\gamma}{\delta\eta} + 1\right)\varepsilon. \tag{3.4}$$

In particular $N_{i,\varepsilon}$ is positive for small δ . We argue that, upon further reducing δ , it holds that $R_\varepsilon^\delta \subset A_\varepsilon^{k+1}$. Assume by contradiction that $R_\varepsilon^\delta \setminus A_\varepsilon^{k+1} \neq \emptyset$. Since (3.4) implies that

$$d_\infty^\varepsilon(x, \partial A_\varepsilon^k) \geq \left(\frac{\gamma}{C\delta} - 1\right)\varepsilon \quad \forall x \in R_\varepsilon^\delta, \tag{3.5}$$

using (3.3) and that both A_ε^{k+1} and R_ε^δ are minimizers of the corresponding functionals, we obtain

$$\begin{aligned} 0 &\geq E_\varepsilon^\omega(A_\varepsilon^{k+1}, A_\varepsilon^k) - E_\varepsilon^\omega(R_\varepsilon^\delta \cup A_\varepsilon^{k+1}, A_\varepsilon^k) \geq E_\varepsilon^\omega(R_\varepsilon^\delta \cap A_\varepsilon^{k+1}, A_\varepsilon^k) - E_\varepsilon^\omega(R_\varepsilon^\delta, A_\varepsilon^k) \\ &= \varepsilon (p_\varepsilon^\omega(\partial(R_\varepsilon^\delta \cap A_\varepsilon^{k+1})) - p_\varepsilon^\omega(\partial R_\varepsilon^\delta)) + \frac{1-\delta}{\gamma\varepsilon} \int_{R_\varepsilon^\delta \setminus A_\varepsilon^{k+1}} d_\infty^\varepsilon(x, \partial A_\varepsilon^k) dx \\ &\quad + \mathcal{H}^1(\partial(R_\varepsilon^\delta \cap A_\varepsilon^{k+1})) - \mathcal{H}^1(\partial R_\varepsilon^\delta) + \frac{\delta}{\gamma\varepsilon} \int_{R_\varepsilon^\delta \setminus A_\varepsilon^{k+1}} d_\infty^\varepsilon(x, \partial A_\varepsilon^k) dx \\ &= \varepsilon (p_\varepsilon^\omega(\partial(R_\varepsilon^\delta \cap A_\varepsilon^{k+1})) - p_\varepsilon^\omega(\partial R_\varepsilon^\delta)) + \frac{1-\delta}{\gamma\varepsilon} \int_{R_\varepsilon^\delta \setminus A_\varepsilon^{k+1}} d_\infty^\varepsilon(x, \partial A_\varepsilon^k) dx \\ &\quad + G_\varepsilon^\delta(R_\varepsilon^\delta \cap A_\varepsilon^{k+1}, A_\varepsilon^k) - G_\varepsilon^\delta(R_\varepsilon^\delta, A_\varepsilon^k) \\ &\geq \varepsilon (p_\varepsilon^\omega(\partial(R_\varepsilon^\delta \cap A_\varepsilon^{k+1})) - p_\varepsilon^\omega(\partial R_\varepsilon^\delta)) + \frac{1-\delta}{\gamma\varepsilon} \int_{R_\varepsilon^\delta \setminus A_\varepsilon^{k+1}} d_\infty^\varepsilon(x, \partial A_\varepsilon^k) dx, \end{aligned}$$

where we used several times that $R_\varepsilon^\delta \subset A_\varepsilon^k$ to simplify the symmetric differences. In combination with (3.5), for $\delta \leq \frac{1}{2}$ the last estimate yields

$$\left(\frac{1}{2C\delta} - \frac{1}{2\gamma}\right) |R_\varepsilon^\delta \setminus A_\varepsilon^{k+1}| \leq \varepsilon (p_\varepsilon^\omega(\partial R_\varepsilon^\delta) - p_\varepsilon^\omega(\partial(R_\varepsilon^\delta \cap A_\varepsilon^{k+1}))). \tag{3.6}$$

In order to use this inequality, we need to analyze which boundary contributions cancel in the last difference. Given $\xi = \frac{i+j}{2} \in \mathcal{Z}^2$ we distinguish two exhaustive cases:

- (i) $i \in R_\varepsilon^\delta, j \notin R_\varepsilon^\delta$: if $i \in A_\varepsilon^{k+1}$ we have $i \in R_\varepsilon^\delta \cap A_\varepsilon^{k+1}$ and $j \notin R_\varepsilon^\delta \cap A_\varepsilon^{k+1}$ which implies $\xi \in \partial(R_\varepsilon^\delta \cap A_\varepsilon^{k+1})$ and thus this contribution cancels. Otherwise $i \notin A_\varepsilon^{k+1}$ and consequently $\xi \in \partial(R_\varepsilon^\delta \setminus A_\varepsilon^{k+1})$;
- (ii) $i \in R_\varepsilon^\delta \cap A_\varepsilon^{k+1}, j \notin R_\varepsilon^\delta \cap A_\varepsilon^{k+1}$: if $j \notin R_\varepsilon^\delta$, then $\xi \in \partial R_\varepsilon^\delta$ and the contribution cancels, while $j \in R_\varepsilon^\delta$ yields $j \notin A_\varepsilon^{k+1}$ and therefore $\xi \in \partial(R_\varepsilon^\delta \setminus A_\varepsilon^{k+1})$.

From those two cases and (2.3) we infer that

$$\varepsilon p_\varepsilon^\omega(\partial R_\varepsilon^\delta) - \varepsilon p_\varepsilon^\omega(\partial(R_\varepsilon^\delta \cap A_\varepsilon^{k+1})) \leq c_\infty(\omega)\varepsilon \mathcal{H}^1(\partial(R_\varepsilon^\delta \setminus A_\varepsilon^{k+1})).$$

Using the reverse isoperimetric inequality (2.4), we can put together the last estimate and (3.6) to deduce

$$\left(\frac{1}{2C\delta} - \frac{1}{2\gamma}\right) |R_\varepsilon^\delta \setminus A_\varepsilon^{k+1}| \leq 4c_\infty(\omega) |R_\varepsilon^\delta \setminus A_\varepsilon^{k+1}|.$$

Choosing δ small enough this yields a contradiction. Hence we proved that $R_\varepsilon^\delta \subset A_\varepsilon^{k+1}$ for δ small enough.

Next we rule out any other connected component except the one containing R_ε^δ . Note that estimate (3.4) implies

$$d_\infty^\varepsilon(x, \partial A_\varepsilon^k) \leq \left(\frac{2\gamma}{\delta\eta} + 1\right)\varepsilon \quad \forall x \in A_\varepsilon^k \setminus R_\varepsilon^\delta. \tag{3.7}$$

Consider a connected component A of A_ε^{k+1} not containing R_ε^δ . We set $A' = A_\varepsilon^{k+1} \setminus A$. Due to (3.1) and (3.7) it holds that $|A_\varepsilon^k \cap A| \leq |A_\varepsilon^k \setminus R_\varepsilon^\delta| \leq C_{\delta,\eta}\varepsilon$. Hence, for ε small enough, we obtain from (2.3) that

$$\begin{aligned} E_\varepsilon^\omega(A_\varepsilon^{k+1}, A_\varepsilon^k) - E_\varepsilon^\omega(A', A_\varepsilon^k) &\geq (1 - c_\infty(\omega)\varepsilon)\mathcal{H}^1(\partial A) - \frac{1}{\gamma\varepsilon} \int_{A_\varepsilon^k \cap A} d_\infty^\varepsilon(x, \partial A_\varepsilon^k) \, dx \\ &\geq \frac{\mathcal{H}^1(\partial A)}{2} - \left(\frac{2}{\delta\eta} + \frac{1}{\gamma}\right) |A_\varepsilon^k \cap A| \geq \frac{\mathcal{H}^1(\partial A)}{2} - |A_\varepsilon^k \cap A|^{\frac{1}{2}} \\ &\geq \frac{\mathcal{H}^1(\partial A)}{2} - |A|^{\frac{1}{2}} \geq \frac{1}{2} \left(1 - \frac{1}{\sqrt{\pi}}\right) \mathcal{H}^1(\partial A) > 0, \end{aligned}$$

where we first used the general set inclusion $(F \setminus A)\Delta B \subset F\Delta B \cup (B \cap A)$ to estimate the integrals and then the two-dimensional isoperimetric inequality. This estimate contradicts the minimality of A_ε^{k+1} and we conclude that A_ε^{k+1} has exactly one connected component for ε small enough.

Step 2. Reduction to coordinate rectangles

First note that if we replace an arbitrary set $A \in \mathcal{A}_\varepsilon$ by the set $A \cap A_\varepsilon^k$ we strictly reduce the energy if the sets are not equal. To see this, we observe that

$$\begin{aligned} E_\varepsilon^\omega(A, A_\varepsilon^k) - E_\varepsilon^\omega(A \cap A_\varepsilon^k, A_\varepsilon^k) &\geq \frac{1}{\gamma\varepsilon} \int_{A \setminus A_\varepsilon^k} d_\infty^\varepsilon(x, \partial A_\varepsilon^k) \, dx + P_\varepsilon^\omega(A) - P_\varepsilon^\omega(A \cap A_\varepsilon^k) \\ &\geq \frac{|A \setminus A_\varepsilon^k|}{\gamma} + P_\varepsilon^\omega(A) - P_\varepsilon^\omega(A \cap A_\varepsilon^k). \end{aligned} \tag{3.8}$$

Again we need to analyze which interactions cancel due to the random perimeter difference in (3.8). As A_ε^k is a coordinate rectangle, by elementary geometric considerations one can prove that $\mathcal{H}^1(\partial A) \geq \mathcal{H}^1(\partial(A \cap A_\varepsilon^k))$. On the other hand, reasoning similar to the lines succeeding (3.6) one can show that all random interactions cancel except those coming from $\partial(A \setminus A_\varepsilon^k)$. In case this set is non-empty, by the precise bound in (2.3) we conclude that (3.8) can be further estimated *via* the strict inequality

$$E_\varepsilon^\omega(A, A_\varepsilon^k) - E_\varepsilon^\omega(A \cap A_\varepsilon^k, A_\varepsilon^k) > \frac{|A \setminus A_\varepsilon^k|}{\gamma} - \frac{1}{4\gamma} \varepsilon \mathcal{H}^{d-1}(\partial(A \setminus A_\varepsilon^k)) \geq 0,$$

where we used again the reverse isoperimetric inequality in \mathcal{A}_ε . Whenever A is a minimizer we obtain a contradiction which shows that $\partial(A_\varepsilon^{k+1} \setminus A_\varepsilon^k) = \emptyset$, or equivalently $A_\varepsilon^{k+1} \subset A_\varepsilon^k$.

To conclude we assume by contradiction that A_ε^{k+1} is not a coordinate rectangle. Consider then the minimal coordinate rectangle R containing A_ε^{k+1} (see Fig. 1). Then again by elementary geometric arguments it holds that $\mathcal{H}^1(\partial A_\varepsilon^{k+1} \setminus \partial R) \geq \mathcal{H}^1(\partial R \setminus \partial A_\varepsilon^{k+1})$. Moreover, as $A_\varepsilon^{k+1} \subset R$ one can check that

$$\left(\partial A_\varepsilon^{k+1} \setminus \partial R\right) \cup \left(\partial R \setminus \partial A_\varepsilon^{k+1}\right) = \partial(R \setminus A_\varepsilon^{k+1}). \tag{3.9}$$

Since $R \subset A_\varepsilon^k$ by the previous argument, using (2.3) the difference of the energies can be estimated by

$$\begin{aligned} 0 &\geq E_\varepsilon^\omega(A_\varepsilon^{k+1}, A_\varepsilon^k) - E_\varepsilon^\omega(R, A_\varepsilon^k) \geq \frac{1}{\gamma} |R \setminus A_\varepsilon^{k+1}| + P_\varepsilon^\omega(A_\varepsilon^{k+1}) - P_\varepsilon^\omega(R) \\ &\geq \frac{1}{\gamma} |R \setminus A_\varepsilon^{k+1}| + (1 - c_\infty(\omega)\varepsilon)\mathcal{H}^1(\partial A_\varepsilon^{k+1} \setminus \partial R) - (1 + c_\infty(\omega)\varepsilon)\mathcal{H}^1(\partial R \setminus \partial A_\varepsilon^{k+1}) \\ &\geq \frac{1}{\gamma} |R \setminus A_\varepsilon^{k+1}| - c_\infty(\omega)\varepsilon \left(\mathcal{H}^1(\partial A_\varepsilon^{k+1} \setminus \partial R) + \mathcal{H}^1(\partial R \setminus \partial A_\varepsilon^{k+1})\right) \geq \left(\frac{1}{\gamma} - 4c_\infty(\omega)\right) |R \setminus A_\varepsilon^{k+1}|, \end{aligned}$$

where in the last inequality we used (3.9) and (2.4). We conclude the proof by the strict bound in (2.3). □

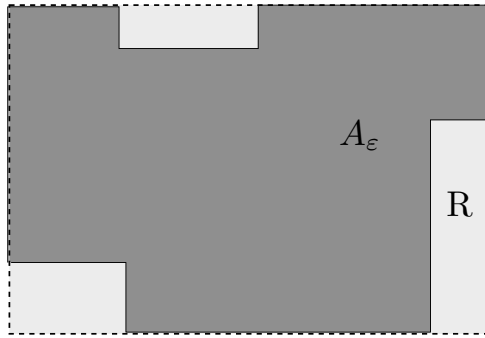


FIGURE 1. The minimal coordinate rectangle R containing a connected set $A_\varepsilon \in \mathcal{A}_\varepsilon$.

Remark 3.4. We want to comment on the L^∞ -bound (2.3): note that we used the precise bound by $\frac{1}{4\gamma}$ only in Step 2. If we assume a generic bound, then Step 2 would probably need probabilistic arguments. We believe that, starting from a coordinate rectangle and using the law of large numbers, one can show that any minimizer must have the same deterministic perimeter as the minimal coordinate rectangle containing it. In particular the new set must be contained in the previous one and can differ from a rectangle only at the corners. However when (2.3) is violated, then with positive probability this minimal rectangle can have higher energy since cutting the edges leaves the deterministic perimeter unchanged and leads to a competition between the random part and the bulk term. Even though the bulk term yields a control on the behavior at the corners, cuts can exist and that causes difficulties. Indeed we are not able to rule out that cuts grow with the number of time steps, but for an inductive argument (which is necessary also for connectedness) one needs to control the deviation from a rectangle.

3.2. Computation of the velocity

As a next step we derive a precise formula for the velocity of the discrete motion still assuming that the side lengths of $A_\varepsilon^k(\omega)$ are bounded away from zero. We follow [14] and express the functional to be minimized by the distance from each side of the optimal rectangle to the corresponding side of the previous set $A_\varepsilon^k(\omega)$. Let $A_\varepsilon^{k+1}(\omega)$ be a minimizer. To reduce notation, we let $s_{i,\varepsilon}$ and $s'_{i,\varepsilon}$ ($i = 1, \dots, 4$) be the sides of $A_\varepsilon^k(\omega)$ and $A_\varepsilon^{k+1}(\omega)$ respectively and set $l_{i,\varepsilon} = \mathcal{H}^1(s_{i,\varepsilon})$. We define $N_{i,\varepsilon}^{k+1}\varepsilon$ as the distance from the side $s_{i,\varepsilon}$ to the side $s'_{i,\varepsilon}$. Observe that for ε small enough $A_\varepsilon^{k+1}(\omega)$ must contain the center of the previous rectangle $A_\varepsilon^k(\omega)$. Otherwise the bulk term would blow up, while choosing $A = A_\varepsilon^k(\omega)$ as a candidate yields equibounded energy. Rewriting the functional $E_\varepsilon^\omega(A, A_\varepsilon^k)$ in terms of the four integer numbers $N_{i,\varepsilon}^{k+1}$, we obtain that these are minimizers of the function $\tilde{f}_\varepsilon^\omega : \mathbb{N}_0^4 \rightarrow \mathbb{R}$ defined by

$$\begin{aligned} \tilde{f}_\varepsilon^\omega(N) &:= \sum_{i=1}^4 (l_{i,\varepsilon} - 2N_i\varepsilon) + \sum_{i=1}^4 p_\varepsilon^\omega(s_{i,\varepsilon} + N_i\varepsilon\hat{n}_i)\varepsilon - \varepsilon^2 e_\varepsilon^{per} + \frac{\varepsilon}{\gamma} \sum_{i=1}^4 \sum_{n=1}^{N_i} l_{i,\varepsilon} n - \varepsilon^2 e_\varepsilon^{bulk} \\ &= \varepsilon \sum_{i=1}^4 \left(\left(\frac{l_{i,\varepsilon}}{\varepsilon} - 2N_i \right) + p_\varepsilon^\omega(s_{i,\varepsilon} + N_i\varepsilon\hat{n}_i) + \frac{(N_i + 1)N_i l_{i,\varepsilon}}{2\gamma} \right) - \varepsilon^2 (e_\varepsilon^{per} + e_\varepsilon^{bulk}), \end{aligned}$$

where $\hat{n}_i \in \{\pm e_1, \pm e_2\}$ denotes the vector representing the inward motion of each side and the error terms e_ε^{per} , e_ε^{bulk} account for the fact that we neglect the shrinking effect on the random part of the energy and that we count twice the bulk part in the corners (one time with the wrong distance). For these errors we have the following bounds:

$$|e_\varepsilon^{per}| \leq \frac{2}{\gamma} \max_i N_i, \quad |e_\varepsilon^{bulk}| \leq \frac{4}{\gamma} (\max_i N_i)^3. \tag{3.10}$$

We argue that the error terms are negligible as $\varepsilon \rightarrow 0$. To this end we show that $\max_i N_{i,\varepsilon}$ is equibounded with respect to ε as long as $l_{i,\varepsilon} \geq \eta > 0$ for some $\eta > 0$. Indeed, suppose without loss of generality that $\bar{N} := \max_i N_i$ corresponds to the right vertical side $s_{i,\varepsilon}$. Let us denote by P the center of $A_\varepsilon^k(\omega)$. Then, for ε small enough, one can easily prove that

$$\left\{ x \in A_\varepsilon^k(\omega) : \frac{\bar{N}}{2}\varepsilon \leq \text{dist}(x, s_{i,\varepsilon}) \leq \bar{N}\varepsilon, |\langle x - P, e_2 \rangle| \leq \frac{\eta}{4} \right\} \subset A_\varepsilon^k(\omega) \setminus A_\varepsilon^{k+1}(\omega).$$

Thus for the bulk term we obtain the lower bound

$$\frac{1}{\gamma\varepsilon} \int_{A_\varepsilon^k(\omega) \setminus A_\varepsilon^{k+1}(\omega)} d_\infty^\varepsilon(x, \partial A_\varepsilon^k(\omega)) \, dx \geq \frac{\min\{\frac{\eta}{4}, \frac{\bar{N}}{2}\varepsilon\} \bar{N}\eta}{\gamma} \frac{\bar{N}\eta}{4}.$$

Using (2.3) and (3.1), for ε small enough, we deduce a lower bound for the random perimeter via

$$\begin{aligned} P_\varepsilon^\omega(A_\varepsilon^{k+1}(\omega)) &\geq P_\varepsilon^\omega(A_\varepsilon^k(\omega)) - 8\bar{N}\varepsilon - \frac{1}{2\gamma}\varepsilon\mathcal{H}^1(\partial A_\varepsilon^k(\omega)) - \frac{2}{\gamma}\varepsilon^2\bar{N} \\ &\geq E_\varepsilon^\omega(A_\varepsilon^k(\omega), A_\varepsilon^k(\omega)) - \varepsilon \left(9\bar{N} + \frac{C}{2\gamma} \right). \end{aligned}$$

Assuming that $\bar{N} \geq \frac{C}{2\gamma}$, we infer that such \bar{N} can't yield a minimizer as soon as

$$-10\bar{N}\varepsilon + \frac{\min\{\frac{\eta}{4}, \frac{\bar{N}}{2}\varepsilon\} \bar{N}\eta}{\gamma} \frac{\bar{N}\eta}{4} > 0. \tag{3.11}$$

From (3.11) we deduce that \bar{N} has to be bounded when $\varepsilon \rightarrow 0$.

It follows from (3.10) that, asymptotically, we can instead minimize the functional

$$f_\varepsilon^\omega(N) = \sum_{i=1}^4 \left(-2N_i + p_\varepsilon^\omega(s_{i,\varepsilon} + N_i\varepsilon\hat{n}_i) + \frac{1}{2\gamma}(N_i + 1)N_i l_{i,\varepsilon} \right), \tag{3.12}$$

provided that the minimizer of the limit is unique. In particular, as in [14] each side moves independently from the remaining ones. More precisely, we have to study the minimizers of the one-dimensional random function

$$v_{i,\varepsilon}^\omega(N) := -2N + p_\varepsilon^\omega(s_{i,\varepsilon} + N\varepsilon\hat{n}_i) + \frac{1}{2\gamma}(N + 1)N l_{i,\varepsilon}. \tag{3.13}$$

We have a good *a priori* estimate for the minimizers which is valid without any stationarity assumption. Indeed, by (2.3) it holds that

$$\sup_{N,N'} |p_\varepsilon^\omega(s_{i,\varepsilon} + N\varepsilon\hat{n}_i) - p_\varepsilon^\omega(s_{i,\varepsilon} + N'\varepsilon\hat{n}_i)| \leq \frac{1}{2\gamma}l_{i,\varepsilon},$$

while (one of) the integer minimizers for the polynomial $P(x) = -2x + \frac{l_{i,\varepsilon}}{2\gamma}(x + 1)x$ is given by $x^* = \lfloor \frac{2\gamma}{l_{i,\varepsilon}} \rfloor$. We deduce the estimate

$$|P(x^* \pm 2) - P(x^*)| = \begin{cases} \frac{3l_{i,\varepsilon}}{\gamma} - 4 + \frac{2l_{i,\varepsilon}}{\gamma}x^* \geq \frac{l_{i,\varepsilon}}{\gamma}, \\ \frac{l_{i,\varepsilon}}{\gamma} + 4 - \frac{2l_{i,\varepsilon}}{\gamma}x^* \geq \frac{l_{i,\varepsilon}}{\gamma}. \end{cases}$$

We infer that for minimizing $v_{i,\varepsilon}^\omega$ we need only to consider three values, namely

$$\min_N v_{i,\varepsilon}^\omega(N) = \min \{ v_{i,\varepsilon}^\omega(x^*), v_{i,\varepsilon}^\omega(x^* + 1), v_{i,\varepsilon}^\omega(x^* - 1) \}. \tag{3.14}$$

Thus the randomness can only cause one additional jump forwards or backwards. The precise asymptotic behavior of the stochastic term in (3.13) is more involved since the segment $s_{i,\varepsilon}$ can vary along infinitely many different lattice positions as $\varepsilon \rightarrow 0$. Thus a direct application of Birkhoff’s ergodic theorem to prove the existence of a limit is not possible. Indeed, in Example 3.5 below we construct a stationary, ergodic field of perturbations where for at least one side $s_{i,\varepsilon}$ the term $p_\varepsilon^\omega(s_{i,\varepsilon} + N\varepsilon\hat{n}_i)$ does not converge with probability 1.

Example 3.5. Let $\gamma = 1$ and let $\{X_i\}_{i \in \mathbb{Z}}$ be a sequence of non-constant independent and identically distributed random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a measure-preserving, ergodic map $\tau : \Omega \rightarrow \Omega$ such that $X_k(\omega) = X_0(\tau^k\omega)$, where τ^k denotes the k -times iterated composition of the map τ (this whole setting can be realized on a suitable product space with the shift operator). Moreover assume that $\|X_i\|_\infty < \frac{1}{4}$ and set $c_\xi(\omega) = X_{\lfloor \xi_1 \rfloor}(\omega)$. Then c_ξ is a stationary, ergodic random field. If the initial coordinate rectangles A_ε^0 converge in the Hausdorff metric to a coordinate rectangle A^0 , then for at least one of the vertical sides we have that, for all $N \in \mathbb{N}_0$,

$$\mathbb{P}\left(\left\{\omega : \lim_{\varepsilon \rightarrow 0} v_{i,\varepsilon}^\omega(N) \text{ exists}\right\}\right) = 0.$$

The above can be seen as follows: when $\varepsilon_n \rightarrow 0$, note that for at least one vertical side the x -component of $\frac{s_{i,\varepsilon_n}}{\varepsilon_n}$ diverges to $\pm\infty$. Then so does the x -component of $\frac{s_{i,\varepsilon_n}}{\varepsilon_n} + N\varepsilon_n\hat{n}_i$. Without loss of generality we assume that these x -components form a sequence of positive numbers $\{k_n + \frac{1}{2}\}_n \rightarrow +\infty$ with $k_n \in \mathbb{N}$. Passing to a subsequence (not relabeled) we can assume that this sequence is monotone increasing. Since l_{i,ε_n} converges to the vertical side length l_i of A^0 , we only have to take into account the random term. Since $A_{\varepsilon_n}^0 \in \mathcal{A}_{\varepsilon_n}$, we have

$$p_{\varepsilon_n}^\omega(s_{i,\varepsilon_n} + N\varepsilon_n\hat{n}_i) = X_{k_n}(\omega)l_{i,\varepsilon_n}.$$

For l_{i,ε_n} converging to $l_i \neq 0$, the asymptotic behavior of $p_{\varepsilon_n}^\omega(s_{i,\varepsilon_n} + N\varepsilon_n\hat{n}_i)$ is characterized by $X_{k_n}(\omega)$. Since these variables are non-constant and independent, it follows from Kolmogorov’s 0-1 law that

$$\mathbb{P}(\{\omega : \lim_n X_{k_n}(\omega) \text{ exists}\}) = 0.$$

Moreover we can define the measure preserving group action $\tau_z : \Omega \rightarrow \Omega$ as

$$\tau_z\omega := \tau^{z_1}\omega.$$

From the construction of the random field, it follows immediately that $\{c_\xi\}_\xi$ is stationary. By assumption the group action is ergodic, too.

Remark 3.6. In addition to the lack of convergence for the random field in Example 3.5, it is not clear whether for this example one can neglect the error terms in (3.10). This is due to possible non-uniqueness of minimizers: denote by x_j the x -component of a left vertical side $s_{i,\varepsilon_j}(\omega)/\varepsilon_j$ and set $N_j^* = \lfloor 2/l_{i,\varepsilon_j} \rfloor$ and $k_j := \lfloor x_j + N_j^* \rfloor$. Recall that we assume $\gamma = 1$. Comparing the three possibilities in (3.14), one can show that minimizers of the function $v_{i,\varepsilon_j}^\omega$ are not unique if and only if

$$\frac{2}{l_{i,\varepsilon_j}} - N_j^* \in \{(X_{k_j}(\omega) - X_{k_j-1}(\omega)), 1 + (X_{k_j+1}(\omega) - X_{k_j}(\omega))\}.$$

Now take X_k to be uniformly distributed on the interval $(0, \rho)$ with $\rho \in (0, \frac{1}{4})$. Then for every k the random variable $Y_k = X_k - X_{k-1}$ have a triangular distribution on $(-\rho, \rho)$. Hence the sequence $\{Y_{2k}\}_k$ is an independent and identically distributed sequence of random variables, which implies

$$\mathbb{P}(\omega : \{Y_{2k}(\omega)\}_k \text{ is not dense in } (-\rho, \rho)) = 0.$$

Indeed, given $q \in \mathbb{Q} \cap (-\rho, \rho)$ and $n \in \mathbb{N}$, from independence we infer

$$\mathbb{P}\left(\omega : Y_{2k}(\omega) \notin q + \left(-\frac{1}{n}, \frac{1}{n}\right) \quad \forall k\right) = 0.$$

In order to obtain the convergence of the sequence $v_{i,\varepsilon}^\omega$ when $\varepsilon \rightarrow 0$, we need a stronger condition than ergodicity, which in particular has to be preserved on one-dimensional sections of \mathcal{Z}^2 . It turns out that the α -mixing condition introduced in (2.5) is enough. Indeed, we have the following crucial result:

Proposition 3.7. *Assume that the random field $\{c_\xi\}_\xi$ is stationary and α -mixing with coefficients satisfying (2.5) holds and set $\mu := \mathbb{E}[c_\xi]$. Let $\varepsilon_j \downarrow 0$. There exists a set $\Omega' \subset \Omega$ of full probability (independent of the particular sequence ε_j) such that for every $\omega \in \Omega'$ and every sequence of sides $\{S_j\}_{j \in \mathbb{N}}$ such that S_j converges in the Hausdorff metric to a segment S , we have*

$$\lim_j p_{\varepsilon_j}^\omega(S_j) = \mathcal{H}^1(S)\mu.$$

Proof. We assume that the side is a vertical side, the case of horizontal sides works the same way with another set of full measure. Moreover it is not restrictive to consider the case $\mu = 0$. To reduce notation we let $[x]^* := [x] + \frac{1}{2}$. Given $q \in \mathbb{Q} \cap (0, +\infty)$ we define the following sequences of random variables:

$$X_n^{q,\pm}(\omega) := \sup_{k \geq qn} \left| \frac{1}{2k+1} \sum_{l=-k}^k c_{([\pm n]^*, l)}(\omega) \right|.$$

Given $\delta > 0$, by stationarity and an elementary fact about average sums we have

$$\begin{aligned} \mathbb{P}(|X_n^{q,\pm}| > \delta) &= \mathbb{P}\left(\sup_{k \geq qn} \left| \frac{1}{2k+1} \sum_{l=-k}^k c_{([0]^*, l)} \right| > \delta\right) \\ &\leq \mathbb{P}\left(\sup_{k \geq qn} \left| \frac{1}{k+1} \sum_{l=0}^k c_{([0]^*, l)} \right| > \delta\right) + \mathbb{P}\left(\sup_{k \geq qn} \left| \frac{1}{k} \sum_{l=1}^k c_{([0]^*, -l)} \right| > \delta\right). \end{aligned} \tag{3.15}$$

Upon rescaling c_ξ we can apply Theorem 2.7 with $p = 2$ to the two bounded and α -mixing sequences $\{c_{([0]^*, l)}\}_{l \in \mathbb{N}_0}$ and $\{c_{([0]^*, -l)}\}_{l \in \mathbb{N}}$ and deduce from (3.15) that

$$\begin{aligned} \sum_{n \geq 1} \mathbb{P}(|X_n^{q,\pm}| > \delta) &\leq \sum_{n \geq 1} \mathbb{P}\left(\sup_{k \geq qn} \left| \frac{1}{k+1} \sum_{l=0}^k c_{([0]^*, l)} \right| > \delta\right) + \mathbb{P}\left(\sup_{k \geq qn} \left| \frac{1}{k} \sum_{l=1}^k c_{([0]^*, -l)} \right| > \delta\right) \\ &\leq [q^{-1}] \sum_{i \geq 1} \mathbb{P}\left(\sup_{k \geq i} \left| \frac{1}{k+1} \sum_{l=0}^k c_{([0]^*, l)} \right| > \delta\right) + \mathbb{P}\left(\sup_{k \geq i} \left| \frac{1}{k} \sum_{l=1}^k c_{([0]^*, -l)} \right| > \delta\right) < +\infty. \end{aligned}$$

Hence by the Borel-Cantelli Lemma there exists a set of full probability Ω^q such that both $X_n^{q,+}$ and $X_n^{q,-}$ converge to 0 pointwise on Ω^q . We set $\Omega'' := \bigcap_q \Omega^q$.

Next we check that we can relate the random length of the side S_j to one of the random variables $X_n^{q,\pm}$. Let S_j converge to a segment S in the Hausdorff metric and denote by $x \in \mathbb{R}$ the x -coordinate of S . We start with the case $x > 0$. Fix $\beta > 0$ and let $x_j \in \mathbb{Z} + \frac{1}{2}$ be the x -component of S_j/ε_j . Then there exists $j_0 = j_0(\beta)$ such that for all $j \geq j_0$ we have $x + \beta \geq \varepsilon_j x_j$ and $\varepsilon_j \#\{\xi \in S_j/\varepsilon_j \cap \mathcal{Z}^2\} \geq \mathcal{H}^1(S) - \beta$. For such j we infer that

$$\#\left\{\xi \in \frac{S_j}{\varepsilon_j} \cap \mathcal{Z}^2\right\} \geq \frac{\mathcal{H}^1(S) - \beta}{x + \beta} x_j.$$

For β small enough, there exists $q \in \mathbb{Q}$ such that $\frac{\mathcal{H}^1(S) - \beta}{x + \beta} > 3q > 0$. Now for every j we let $n_j \in \mathbb{N}$ satisfying $[n_j]^* = x_j$ (we may assume that $x_j > 0$ for all j). Then

$$\#\left\{\xi \in \frac{S_j}{\varepsilon_j} \cap \mathcal{Z}^2\right\} > 3qn_j. \tag{3.16}$$

Let us first assume that $S = \{x\} \times \frac{1}{2}[-\mathcal{H}^1(S), \mathcal{H}^1(S)]$ is a centered side. Then for j large enough it holds that

$$\# \left(\left\{ \xi \in \frac{S_j}{\varepsilon_j} \cap \mathcal{Z}^2 \right\} \Delta \left\{ \xi = (x_j, l) \in \mathcal{Z}^2 : |l| \leq \frac{\mathcal{H}^1(S_j)}{2\varepsilon_j} \right\} \right) \leq \frac{\beta}{\varepsilon_j}, \tag{3.17}$$

so that by (2.3) we have

$$\begin{aligned} \left| \sum_{\varepsilon_j \xi \in S_j} \varepsilon_j c_\xi(\omega) \right| &= \mathcal{H}^1(S_j) \left| \frac{1}{\#\{\varepsilon_j \xi \in S_j\}} \sum_{\varepsilon_j \xi \in S_j} c_\xi(\omega) \right| \\ &\leq C\beta + C \left| \frac{1}{\#\{\varepsilon_j \xi \in S_j\}} \sum_{2\varepsilon_j |l| \leq \mathcal{H}^1(S_j)} c_{(x_j, l)}(\omega) \right| \leq C\beta + CX_{n_j}^{q,+}(\omega), \end{aligned} \tag{3.18}$$

where we used that $[\mathcal{H}^1(S_j)/(2\varepsilon_j)] \geq qn_j$ for all but finitely many j by (3.16). Since $\beta > 0$ is arbitrary and $X_{n_j}^{q,+}(\omega) \rightarrow 0$ for all $\omega \in \Omega''$ we conclude in this special case.

Now assume that $S = \{x\} \times \frac{1}{2}[y - \mathcal{H}^1(S), y + \mathcal{H}^1(S)]$ with $y > 0$ (the other case is similar). We aim to transfer the variables pointwise with the help of the group action. For $\beta > 0$ and q fixed as above, we define the events

$$\mathcal{Q}_N := \left\{ \omega \in \Omega : \forall n \geq \frac{N}{2} \text{ it holds that } |X_n^{q,+}(\omega)| \leq \beta \right\}.$$

By the arguments hitherto we know that the function $\mathbb{1}_{\mathcal{Q}_N}$ converges to $\mathbb{1}_\Omega$ almost surely. Let us denote by \mathcal{J}_{e_2} the (maybe non-trivial) σ -algebra of invariant sets for the measure preserving map τ_{e_2} . Fatou’s lemma for the conditional expectation yields

$$\mathbb{1}_\Omega = \mathbb{E}[\mathbb{1}_\Omega | \mathcal{J}_{e_2}] \leq \liminf_{N \rightarrow +\infty} \mathbb{E}[\mathbb{1}_{\mathcal{Q}_N} | \mathcal{J}_{e_2}].$$

Hence we know that, given $\delta > 0$, almost surely we find $N_0 = N_0(\omega, \delta)$ such that

$$1 \geq \mathbb{E}[\mathbb{1}_{\mathcal{Q}_{N_0}} | \mathcal{J}_{e_2}](\omega) \geq 1 - \delta.$$

Due to Birkhoff’s ergodic theorem, almost surely, there exists $n_0 = n_0(\omega, \delta)$ such that, for any $m \geq \frac{1}{2}n_0$,

$$\left| \frac{1}{m} \sum_{i=1}^m \mathbb{1}_{\mathcal{Q}_{N_0}}(\tau_{ie_2}\omega) - \mathbb{E}[\mathbb{1}_{\mathcal{Q}_{N_0}} | \mathcal{J}_{e_2}](\omega) \right| \leq \delta.$$

Note that the set we exclude will be a countable union of null sets (depending only on the sequences $X_n^{q,\pm}$ and rational β). With a slight abuse of notation we still denote the smaller set by Ω'' .

We now fix $\omega \in \Omega''$. For $m \geq \max\{n_0(\omega, \delta), N_0(\omega, \delta)\}$ we denote by R the maximal integer such that for all $i = m + 1, \dots, m + R$ we have $\tau_{ie_2}(\omega) \notin \mathcal{Q}_{N_0}$. In order to bound R let \tilde{m} be the number of non-zero elements in the sequence $\{\mathbb{1}_{\mathcal{Q}_{N_0}}(\tau_{ie_2}(\omega))\}_{i=1}^m$. By definition of R we have

$$\delta \geq \left| \frac{\tilde{m}}{m + R} - \mathbb{E}[\mathbb{1}_{\mathcal{Q}_{N_0}} | \mathcal{J}_{e_2}](\omega) \right| = \left| 1 - \mathbb{E}[\mathbb{1}_{\mathcal{Q}_{N_0}} | \mathcal{J}_{e_2}](\omega) + \frac{\tilde{m} - m - R}{m + R} \right| \geq \frac{R + m - \tilde{m}}{m + R} - \delta.$$

Since $m - \tilde{m} \geq 0$ and without loss of generality $\delta \leq \frac{1}{4}$, this provides an upper bound by $R \leq 4m\delta$. Thus for an arbitrary $m \geq \max\{n_0(\omega, \delta), N_0(\omega, \delta)\}$ and $\tilde{R} = 6m\delta$ we find $l_m \in [m + 1, m + \tilde{R}]$ such that $\tau_{l_m e_2}(\omega) \in \mathcal{Q}_{N_0}$. Then for all $n \geq \frac{N_0}{2}$ it holds that

$$|X_n^{q,+}(\tau_{l_m e_2}\omega)| \leq \beta. \tag{3.19}$$

For j large enough we have $\lfloor y/\varepsilon_j \rfloor \geq \max\{n_0(\omega, \delta), N_0(\omega, \delta)\}$, so that there exists $l_j \in \mathbb{N}$ satisfying (3.19) and moreover

$$|\lfloor y/\varepsilon_j \rfloor - l_j| \leq 6\delta \lfloor y/\varepsilon_j \rfloor. \tag{3.20}$$

In addition we can assume that $|\mathcal{H}^1(S) - \mathcal{H}^1(S_j)| \leq \beta$. Note that (3.20) is the analogue of (3.17). Thus from (3.19), stationarity and the definition of $X_n^{q,+}$ we deduce that

$$|p_{\varepsilon_j}^\omega(S_j)| \leq C_y(\beta + \delta) \tag{3.21}$$

for all j large enough. By the arbitrariness of β and δ we proved the claim.

The case $x < 0$ can be proved the same way using the random variables $X_n^{q,-}$ instead. It remains the case when $x = 0$. For fixed $z \in \mathbb{Z}$ we consider the following sequences of random variables:

$$Y_n^z(\omega) := \sup_{k \geq n} \left| \frac{1}{2k+1} \sum_{l=-k}^k c_{(\lfloor z \rfloor^*, l)}(\omega) \right|.$$

With essentially the same arguments as above one can show that there exists a set Ω^z of full probability such that for every sequence of sides S_j contained in $\lfloor z \rfloor^* \times \mathbb{R}$ and all $\omega \in \Omega^z$ we have

$$p_{\varepsilon_j}^\omega(S_j) \rightarrow 0,$$

where Ω^z does not depend on the sequence ε_j . We finally set $\Omega' := \Omega'' \cap \bigcap_z \Omega^z$. Let us fix $\omega \in \Omega'$. Note that if $x = 0$, then for every subsequence of ε_j there exists a further subsequence ε_{j_k} , such that either

- (i) $x_{j_k} \rightarrow \pm\infty$,
- (ii) $x_{j_k} = \lfloor z \rfloor^*$ for all k and for some $z \in \mathbb{Z}$

In the first case we can use the construction for $x \neq 0$ with arbitrary $q \in \mathbb{Q} \cap (0, +\infty)$ since $\omega \in \Omega''$ and in the second case we use that $\omega \in \Omega^z$ to conclude. □

Remark 3.8. It is straightforward to check that the limit relation of Proposition 3.7 holds for convergence in probability even under the weaker assumption that both the σ -algebras invariant with respect to the two group actions τ_{e_1}, τ_{e_2} are trivial.

With Proposition 3.7 at hand we are now in a position to prove the main result of this section.

Proof of Theorem 3.1. Let Ω' be the set of full probability given by Proposition 3.7. We fix $\omega \in \Omega'$. Since $A_j^0(\omega)$ converges to a coordinate rectangle we can assume that the sides of $A_j^0(\omega)$ are larger than $\eta > 0$ for some η independent of j . Therefore we can apply Proposition 3.3 for all j large enough. For fixed j and $i = 1, \dots, 4$, the minimizing movement procedure yields two random sequences $L_{i,\varepsilon_j}^k(\omega), N_{i,\varepsilon_j}^k(\omega)$ of side-lengths and displacements. Let us denote by $L_i^j(t)(\omega) = L_{i,\varepsilon_j}^{\lfloor \tau_j/t \rfloor}(\omega)$ and $N_i^j(t)(\omega) = N_{i,\varepsilon_j}^{\lfloor \tau_j/t \rfloor}(\omega)$ the piecewise constant interpolations. Note that the function $L_i^j(t)(\omega)$ is decreasing in t . Set

$$t^* := \min_i \left\{ \inf\{t > 0 : \liminf_j L_i^j(t)(\omega) = 0\} \right\} \in [0, +\infty].$$

Recall that we already deduced from (3.11) that the discrete velocity, that is the distance between two corresponding sides between two time steps is equibounded by $C_\eta \varepsilon$ for some constant C_η depending on the minimal side-length. Thus it follows that

$$\min_i \liminf_j L_i^j(t^*)(\omega) = 0$$

and consequently $t^* > 0$. Without changing notation we consider the subsequence realizing the \liminf . Then, by monotonicity, one can verify that for all $t < t^*$ we have

$$\min_i \liminf_j L_i^j(t)(\omega) > 0.$$

Now fix $t_1 < t^*$. Taking i into account modulo 4, by construction it holds that

$$\frac{L_{i,\varepsilon_j}^{k+1}(\omega) - L_{i,\varepsilon_j}^k(\omega)}{\tau} = -\frac{1}{\gamma}(N_{i-1,\varepsilon_j}^k(\omega) + N_{i+1,\varepsilon_j}^k(\omega)). \tag{3.22}$$

Hence on $[0, t_1]$ the piecewise affine interpolations $t \mapsto L_i^{j,a}(t)(\omega)$ are uniformly Lipschitz-continuous and decreasing whereas $N_i^j(t)(\omega)$ is locally bounded in L^∞ . Thus, by a diagonal argument, we can find a further subsequence such that $L_i^j(t) \rightarrow L_i(t)$ pointwise and locally uniformly on $[0, t^*)$ for some locally Lipschitz-continuous, decreasing function $L_i(t)(\omega)$ and additionally $N_i^j(t)(\omega)$ weakly*-converges in $L^\infty_{loc}((0, t^*))$ to some function $N_i(t)(\omega)$. It follows that, up to a subsequence, $A_{\varepsilon_j}(t)(\omega)$ converges in the Hausdorff metric to a coordinate rectangle $A(t)(\omega)$ for all $0 \leq t < t^*$.

We conclude the proof by computing the velocity of each side $L_i(t)(\omega)$. Again we fix $0 < t < t^*$. Then $\liminf_j L_i^j(t)(\omega) > 0$ for all i . Therefore we have that the minimizers $N_{i,\varepsilon_j}^k(\omega)$ of the functional $f_{\varepsilon_j}^\omega(N)$ introduced in (3.12) are uniformly bounded if $|k\tau_j - t|$ is small enough. Hence they converge (up to subsequences) to minimizers of the Γ -limit of $f_{\varepsilon_j}^\omega$, which in case of a discrete domain coincides with the pointwise limit. By Proposition 3.7 and the precedent discussion we know that for any $N \in \mathbb{N}_0^4$

$$f_{\varepsilon_j}^\omega(N) \rightarrow \sum_{i=1}^4 -2N_i + L_i(t)(\omega)\mu + \frac{1}{2\gamma}(N_i + 1)N_iL_i(t)(\omega)$$

with $\mu = \mathbb{E}[c_\xi]$. A straightforward calculation shows that the minimizers are given by

$$N_i \begin{cases} = \left\lfloor \frac{2\gamma}{L_i(t)(\omega)} \right\rfloor & \text{if } \frac{2\gamma}{L_i(t)(\omega)} \notin \mathbb{N}, \\ \in \left\{ \frac{2\gamma}{L_i(t)(\omega)} - 1, \frac{2\gamma}{L_i(t)(\omega)} \right\} & \text{otherwise.} \end{cases}$$

Summing the equality (3.22) we further infer that

$$\begin{aligned} L_i^j(t)(\omega) &= L_i^j(0)(\omega) - \frac{1}{\gamma} \sum_{k=0}^{\lfloor t/\tau_j \rfloor} \tau_j (N_{i-1}^j(k\tau_j)(\omega) + N_{i+1}^j(k\tau_j)(\omega)) \\ &= L_i^j(0)(\omega) - \frac{1}{\gamma} \int_0^t (N_{i-1}^j(s)(\omega) + N_{i+1}^j(s)(\omega)) \, ds + \mathcal{O}(\tau_j). \end{aligned} \tag{3.23}$$

Passing to the limit as $j \rightarrow +\infty$ in (3.23), we deduce from local weak convergence that

$$L_i(t)(\omega) = L_i(0)(\omega) - \frac{1}{\gamma} \int_0^t (N_{i-1}(s)(\omega) + N_{i+1}(s)(\omega)) \, ds. \tag{3.24}$$

To conclude, we note that if t is such that $2\gamma/L_i(t)(\omega) \notin \mathbb{N}$, then by continuity we have that $2\gamma/L_i(t')(\omega) \notin \mathbb{N}$ for $|t - t'| \leq \delta$ and some $\delta > 0$. It follows from comparing pointwise convergence with weak*-convergence that

$$N_i(t')(\omega) = \gamma v_i(t')(\omega) \quad \text{for almost all } |t - t'| \leq \delta.$$

In particular $N_i(\cdot)(\omega)$ has a constant representative on $(t - \delta, t + \delta)$ so that the velocity of the side $S_i(t)(\omega)$ given by

$$\lim_{h \rightarrow 0} \frac{1}{2} \frac{L_{i-1}(t+h)(\omega) - L_{i-1}(t)(\omega)}{h} = -v_i(t)(\omega)$$

exists by (3.24) whenever $2\gamma/L_i(t)(\omega) \notin \mathbb{N}$. Note that the formula for the velocity is true because if $2\gamma/L_i(t)(\omega) \notin \mathbb{N}$, then at least in every short time interval opposite sides move with the same velocities. The claim for $2\gamma/L_i(t)(\omega) \in \mathbb{N}$ follows from well known properties of weak*-convergence (note that for these values of t the velocity may not be a classical derivative). \square

Remark 3.9. Without any assumptions on the distribution of the random field except the bound (2.3), up to subsequences, we can still obtain a rectangular limit motion. Due to (3.14) we can also give an estimate of the velocity via

$$v_i(t)(\omega) \in \frac{1}{\gamma} \left[\left\lfloor \frac{2\gamma}{L_i(t)(\omega)} \right\rfloor - 1, \left\lfloor \frac{2\gamma}{L_i(t)(\omega)} \right\rfloor + 1 \right].$$

Note that the subsequence may depend on ω . This estimate shows that for any possible limit velocity we have

$$\lim_{\gamma \rightarrow +\infty} v_i(t)(\omega) = \frac{2}{L_i(t)},$$

which corresponds to the velocity of the continuum flow. However observe that (2.3) requires vanishing randomness when $\gamma \rightarrow +\infty$.

Remark 3.10. For the continuum flow it is known that rectangles always shrink to a point; see for example the more general result contained in ([20], Prop. 3.1). The same occurs for any possible limit motion in our discrete model provided the sets vanish in finite time. Indeed, assume by contradiction that $L_i(t^*)(\omega) = 0$ and $L_{i+1}(t^*)(\omega) = a > 0$. Then, for any $t < t^*$, by monotonicity of the side-lengths and the velocity estimate in Remark 3.9 there exists a constant $c > 0$ such that

$$L_i(t^*)(\omega) - L_i(t)(\omega) \geq -c(t^* - t).$$

By definition of t^* we obtain the bound $L_i(t) \leq c(t^* - t)$. Inserting this bound in the estimate of Remark 3.9 we conclude that, again for any $0 < t < t^*$ and a slightly larger constant $c > 0$,

$$L_{i+1}(t)(\omega) - L_{i+1}(0)(\omega) \leq - \int_0^t \frac{c}{t^* - s} ds = c \log(1 - t/t^*).$$

Letting $t \uparrow t^*$ we obtain a contradiction.

4. DEPENDENCE ON THE RANGE OF STATIONARITY

In the previous section we proved that the velocity of the limit motion is the same as in the unperturbed deterministic case. For the homogenization of the velocity we used in particular Theorem 2.7 which required normalized random variables with zero expectation. For stationary random fields this is not a restriction since they all have the same expectation. However the situation changes if we require stationarity only for shifts belonging to a subgroup of the form $m\mathbb{Z}^2$ since the distributions then are only m -periodic. We will prove that in this case we can indeed obtain a velocity that differs from the homogeneous case. In particular this highlights that the results obtained hitherto are not only due to the additional scaling of the random terms but indeed due to homogenization. Note that in a deterministic setting the results of this section corresponds to replacing constant weights by $m\mathbb{Z}^2$ -periodic ones. A possible example with m -stationary coefficients (see the Definition below) can be constructed by taking $m\mathbb{Z}^2$ -periodic perturbations and adding an independent and identically distributed random field.

Definition 4.1. Let $m \in \mathbb{N}$. We say that the random field $\{c_\xi\}_\xi$ is m -stationary if

$$c_\xi \circ \tau_{mz} = c_{\xi+mz} \quad \forall z \in \mathbb{Z}^2 \text{ and } \forall \xi \in \mathcal{Z}^2.$$

As we will see there are $2m$ quantities that can affect the velocity. For $k = 0, \dots, m - 1$ set $[k]^* = k + \frac{1}{2}$ and consider the following averaged random variables:

$$c_{k,+}^{\text{eff}}(\omega) := \frac{1}{m} \sum_{j=0}^{m-1} c_{([k]^*,j)}(\omega), \quad c_{k,-}^{\text{eff}}(\omega) := \frac{1}{m} \sum_{j=0}^{m-1} c_{(j,[k]^*)}(\omega).$$

These averages represent the random weights of a vertical or horizontal side inside a periodicity cell of the distribution. We denote their expectations by

$$\mu_k = \mathbb{E}[c_{k,+}^{\text{eff}}], \quad \lambda_k = \mathbb{E}[c_{k,-}^{\text{eff}}]. \tag{4.1}$$

Before we state the main theorem on the motion for m -stationary perturbations we need to introduce the singular side lengths where the velocity may not be well-defined. We will see later that they are given by

$$\begin{aligned} \mathcal{S}_l^l &:= \{L \in (0, +\infty) : 2\gamma/L \in \mathbb{N}_0 + \gamma(\mu_k - \mu_{k-1}) \text{ for some } k\}, \\ \mathcal{S}_l^r &:= \{L \in (0, +\infty) : 2\gamma/L \in \mathbb{N}_0 - \gamma(\mu_k - \mu_{k-1}) \text{ for some } k\}, \\ \mathcal{S}_l^d &:= \{L \in (0, +\infty) : 2\gamma/L \in \mathbb{N}_0 + \gamma(\lambda_k - \lambda_{k-1}) \text{ for some } k\}, \\ \mathcal{S}_l^u &:= \{L \in (0, +\infty) : 2\gamma/L \in \mathbb{N}_0 - \gamma(\lambda_k - \lambda_{k-1}) \text{ for some } k\}. \end{aligned} \tag{4.2}$$

Here the indices mean left, right, down and up, which describe the type of side. Whenever it is clear from the context, we associate to a side S_i the corresponding set $\mathcal{S}_i \in \{\mathcal{S}_l^l, \mathcal{S}_l^r, \mathcal{S}_l^d, \mathcal{S}_l^u\}$. Given $L_0 \in \mathcal{S}_i$ and a non-increasing function $v_i : (0, +\infty) \setminus \mathcal{S}_i \rightarrow \mathbb{R}$ we set

$$v_i^{(-)}(L_0) = \lim_{L \downarrow L_0} v_i(L), \quad v_i^{(+)}(L_0) = \lim_{L \uparrow L_0} v_i(L).$$

Now we are in a position to state the main theorem for m -stationary fields under the same α -mixing hypothesis as in Theorem 3.1.

Theorem 4.2. Assume that the random field $\{c_\xi\}_\xi$ satisfies (2.3), is m -stationary and α -mixing with coefficients satisfying (2.5). For $i = 1, \dots, 4$ there exist non-increasing functions $v_i^{\text{eff}} : (0, +\infty) \setminus \mathcal{S}_i \rightarrow [0, +\infty)$ such that with probability 1 the following holds: let $\varepsilon_j \downarrow 0$ and let $A_j^0(\omega) \in \mathcal{A}_{\varepsilon_j}$ be a coordinate rectangle with sides $S_{1,j}(\omega), \dots, S_{4,j}(\omega)$. Assume that $A_j^0(\omega)$ converges in the Hausdorff metric to a coordinate rectangle $A(\omega)$. Then we can choose a subsequence (not relabeled), such that $A_{\varepsilon_j}(t)(\omega)$ converges locally in time to $A(t)(\omega)$, where $A(t)(\omega)$ is a coordinate rectangle with sides $S_i(t)(\omega)$ such that $A(0)(\omega) = A(\omega)$. Each side $S_i(t)(\omega)$ moves inward with velocity $v_i(t)(\omega)$ solving the following inclusions:

$$v_i(t)(\omega) \begin{cases} = \frac{1}{\gamma} v_i^{\text{eff}}(L_i(t)(\omega)) & \text{if } L_i(t)(\omega) \notin \mathcal{S}_i, \\ \in \frac{1}{\gamma} \left[(v_i^{\text{eff}})^{(-)}(L_i(t)(\omega)), (v_i^{\text{eff}})^{(+)}(L_i(t)(\omega)) \right] & \text{otherwise,} \end{cases}$$

where $L_i(t)(\omega) := \mathcal{H}^1(S_i(t)(\omega))$ denotes the length of the side $S_i(t)(\omega)$. The inclusions are valid until the extinction time when $L_i(t)(\omega) = 0$.

Remark 4.3.

- (i) We are not able to provide an explicit formula for the effective velocities v_i^{eff} . However for a given set of perturbations the value can be calculated using the algorithm described before Lemma 4.5. Due to (4.7) the minimization problem therein is solved by comparing three values. We will show that this algorithm stops after at most m steps. Moreover by construction the velocity depends only on the expectations in (4.1). Lemma 4.5 contains some further properties of the velocities.
- (ii) In contrast to the deterministic environments considered in [16, 19], in our setting the effective velocity of two opposite sides can be different. However this is not due to random effects but can already be caused by a slightly more complex periodic structure as shown in Example 4.7.

Again we postpone the proof of Theorem 4.2. First we construct the effective velocities. To this end we need the following generalization of Proposition 3.7.

Proposition 4.4. *Assume that the random field $\{c_\xi\}_\xi$ is m -stationary and α -mixing with coefficients satisfying (2.5). Then there exists a set $\Omega' \subset \Omega$ of full probability such that for all $\omega \in \Omega'$ the following holds: suppose that a vertical side S_j converges in the Hausdorff sense to a limit side S and that for all j we have that the x -component x_j of S_j/ε_j fulfills*

$$x_j = k + \frac{1}{2} \pmod m \quad \forall j. \tag{4.3}$$

Let μ_k be given by (4.1). Then it holds that

$$\lim_j p_{\varepsilon_j}^\omega(S_j) = \mathcal{H}^1(S)\mu_k.$$

Moreover the convergence is locally uniform in the following weak sense: there exists $j_0 = j_0(\omega)$ such that for all $j \geq j_0$ and all sequences of vertical sides S'_j satisfying (4.3) and $d_H(S_j, S'_j) \leq \delta$ we have

$$\left| p_{\varepsilon_j}^\omega(S'_j) - \mathcal{H}^1(S'_j)\mu_k \right| \leq C\delta \tag{4.4}$$

for some positive constant $C > 0$ independent of S'_j .

The same statement holds for horizontal sides with the condition on the y -component and μ_k replaced by λ_k .

Proof. The argument to show convergence is very similar to the one used in Proposition 3.7 restricted to a thinned dual lattice. We therefore only provide the main steps. We fix k as in (4.3) and define the two-sided sequence of random variables $\{z_i\}_{i \in \mathbb{Z}}$ via

$$z_i(\omega) := c_{k,|i|}^{\text{eff}}(\tau_{ime_2}\omega).$$

Note that this sequence is stationary and α -mixing such that (2.5) holds. For $q \in \mathbb{Q} \cap (0, +\infty)$ we define the following average sequences:

$$Z_n^{q,\pm}(\omega) := \sup_{i \geq qn} \left| \frac{1}{2i+1} \sum_{l=-i}^i z_l(\tau_{\pm nme_1}\omega) - \mu_k \right|$$

Using m -stationarity and the mixing property we can argue as in the proof of Proposition 3.7 to show that there exists a set Ω'' of full probability such that all the sequences $Z_n^{q,\pm}$ converge to 0 pointwise on Ω'' . Up to minor changes the proof of convergence now is the same as for Proposition 3.7. We omit the details.

In order to prove (4.4) we have to distinguish two cases: first assume that the x -coordinate (also denoted by x) of S is positive (the case of negative x -coordinate works the same way). Then, for δ small enough (otherwise (4.4) is trivial), we have $x'_j > 0$ for j large enough depending only on S_j . The key is to show that we can compare S'_j to one of the sequences of random variables $Z_n^{q,+}$ as in the proof of Proposition 3.7, where q can be chosen

only depending on the sequence S_j . Then the speed of convergence is determined by the one of $Z_n^{q,+}$ for one particular q .

We start with the case of a vertically centered side S , that means $S = \{x\} \times [-\mathcal{H}^1(S)/2, \mathcal{H}^1(S)/2]$. Given $0 < \beta \ll \delta$ there exists j_0 such that for all $j \geq j_0$ we have $x + \beta \geq \varepsilon_j x_j$ and $\varepsilon_j \#\{\xi \in S_j/\varepsilon_j \cap \mathcal{Z}^2\} \geq \mathcal{H}^1(S) - \beta$. Using the assumption $d_H(S_j, S'_j) \leq \delta$, a straightforward computation yields

$$\#\left\{\xi \in \frac{S'_j}{\varepsilon_j} \cap \mathcal{Z}^2\right\} \geq \frac{\mathcal{H}^1(S) - 2\delta - \beta}{x + \delta + \beta} x'_j$$

for all $j \geq j_0$. Therefore we have to chose $\frac{\mathcal{H}^1(S) - 2\delta - \beta}{x + \delta + \beta} > 3q$ which can be done uniformly for small δ . Moreover, from our assumptions we deduce

$$\#\left(\left\{\xi \in \frac{S'_j}{\varepsilon_j} \cap \mathcal{Z}^2\right\} \Delta \left\{\xi = (x_j, l) : |l| \leq \frac{\mathcal{H}^1(S'_j)}{2\varepsilon_j}\right\}\right) \leq \frac{4\delta}{\varepsilon_j}.$$

Assuming (3.1) we deduce that $\sup_j \mathcal{H}^1(S'_j) \leq C$. Hence we can argue as in (3.18) to prove that

$$\left|p_{\varepsilon_j}^\omega(S'_j) - \mathcal{H}^1(S'_j)\mu_k\right| \leq C\delta + Z_{n_j}^{q,+}(\omega), \tag{4.5}$$

where $[n_j]^* = x'_j$. Since $|x_j - x'_j| \leq \delta/\varepsilon_j$ and $\varepsilon_j x_j \rightarrow x$, for every $n \in \mathbb{N}$ we can find j_0 (depending only on S_j) such that for all $j \geq j_0$ we have

$$x'_j \geq \frac{x/2 - \delta}{\varepsilon_j} \geq n.$$

Hence $n_j \rightarrow +\infty$ and since $Z_n^{q,+}$ converges to 0 on Ω'' , the claim (4.4) holds in this particular case.

The case of a general side $S = \{x\} \times [y - \mathcal{H}^1(S)/2, y + \mathcal{H}^1(S)/2]$ with $x, y > 0$ can be treated with the same arguments as in the derivation of (3.21) since this construction is uniformly with respect to small displacements of the limit side. We leave out the details here.

We are left with the case when $x = 0$. Again it is enough to consider a centered side S since the other cases can be deduced from this one. Let us take q small enough such that

$$\frac{\mathcal{H}^1(S) - 4\delta}{2\delta} > 3q.$$

By construction there exists j_0 such that for every sequence S'_j fulfilling the assumptions we have $\mathcal{H}^1(S'_j)/2\varepsilon_j > q|x'_j|$ for all $j \geq j_0$. Thus, if x'_j is not bounded we can control the speed of convergence with the random variables $Z_n^{q,\pm}$ as in (4.5). Perhaps after enlarging j_0 , we obtain that

$$|Z_j^{q,\pm}(\omega)| \leq \delta \quad \forall j \geq j_0.$$

The estimate (4.4) now follows from distinguishing the case where $|x'_j| > j_0$ for which we can use the above bound and (4.5) or $|x'_j| \leq j_0$, where we have to control finitely many sequences of random variables that converge to 0 as $S_j \rightarrow S$. □

Now we can derive a candidate for the velocity. We remark that due to Proposition 4.4 the argument is similar to the deterministic case treated in [16]. In what follows we identify indices modulo m whenever necessary.

We have to minimize the function $v_{i,\varepsilon}^\omega(N)$ given by (3.13) which is the correct one describing the velocity if the limit function as $\varepsilon_j \rightarrow 0$ has a unique minimizer. For the moment we restrict the analysis to the left vertical side. Up to a subsequence, we have that the x -component of $s_{i,\varepsilon_j}/\varepsilon_j$ is constant modulo m , that is there exists $n \in 0, \dots, m - 1$ such that

$$x_j^i = n + \frac{1}{2} \pmod m \quad \forall j.$$

If s_{i,ε_j} converges to a limit side of length L , then by Proposition 4.4 we have that along this particular subsequence, it holds that

$$v_{i,\varepsilon_j}^\omega(N) \rightarrow v_i^{n,L}(N) := -2N + L\mu_k + \frac{L}{2\gamma}(N + 1)N \quad \text{if } N + n = k \pmod m. \tag{4.6}$$

As we will show in the following, we can define an effective velocity which does not depend on the particular subsequence. Setting $N^* = \lfloor 2\gamma/L \rfloor$, as an analogue of (3.14) we have

$$\min_N v_i^{n,L}(N) = \min \left\{ v_i^{n,L}(N^*), v_i^{n,L}(N^* + 1), v_i^{n,L}(N^* - 1) \right\}. \tag{4.7}$$

Since a precise analysis of the minimization process is only possible if the limit functional has a unique minimizer, we check when this is the case. There are three equivalences that turn out to be useful to characterize the lack of uniqueness. Write $N^* = 2\gamma/L - \xi$ with $\xi \in [0, 1)$ and suppose that $N^* + n = k^* \pmod m$. Then it holds that

$$\begin{aligned} v_i^{n,L}(N^*) \leq v_i^{n,L}(N^* + 1) &\iff \xi \leq 1 + \gamma(\mu_{k^*+1} - \mu_{k^*}), \\ v_i^{n,L}(N^*) \leq v_i^{n,L}(N^* - 1) &\iff \xi \geq \gamma(\mu_{k^*} - \mu_{k^*-1}), \\ v_i^{n,L}(N^* + 1) \leq v_i^{n,L}(N^* - 1) &\iff \xi \geq \frac{1}{2} + \frac{\gamma}{2}(\mu_{k^*+1} - \mu_{k^*-1}). \end{aligned} \tag{4.8}$$

Thus minimizers are not unique if and only if

- (i) $\gamma(\mu_{k^*+1} - \mu_{k^*}) + 1 = \xi \geq \frac{\gamma}{2}(\mu_{k^*+1} - \mu_{k^*-1}) + \frac{1}{2}$,
- (ii) $\gamma(\mu_{k^*} - \mu_{k^*-1}) = \xi \leq \frac{\gamma}{2}(\mu_{k^*+1} - \mu_{k^*-1}) + \frac{1}{2}$,
- (iii) $\xi = \frac{1}{2} = \gamma(\mu_{k^*} - \mu_{k^*+1}) = \gamma(\mu_{k^*} - \mu_{k^*+1})$,

where we left out those inequalities with no information. Due to (2.3) the third possibility cannot occur and also the inequalities in (i) and (ii) are always fulfilled since $\mu_{k^*+1} - \mu_{k^*-1} = (\mu_{k^*+1} - \mu_{k^*}) + (\mu_{k^*} - \mu_{k^*-1})$. In particular the set of side lengths where the minimization problem (4.7) has not a unique solution is discrete and is given by \mathcal{S}_1^l . The same analysis for the remaining sides yields that the singular side lengths are indeed the sets introduced in (4.2).

Now let us analyze the minimization scheme. Again we illustrate the procedure only for the left vertical side. To this end we fix $L \notin \mathcal{S}_1^l$. Setting $X_0 = x_j^i$, we will see that the motion of the corresponding left vertical side will be given locally by the following algorithm:

For $l = 0, 1, \dots$ set

$$\begin{aligned} n_l &:= X_l - \frac{1}{2} \pmod m, \\ N_{l+1} &= \operatorname{argmin}_N v_i^{n_l,L}(N), \\ X_{l+1} &:= X_l + N_{l+1}, \end{aligned}$$

where $v_i^{n,L}$ is defined in (4.6). This algorithm is well-defined for $L \notin \mathcal{S}_1^l$ and after at most m steps it yields an effective velocity as shown in the lemma below:

Lemma 4.5. *There exist nonnegative integer numbers \tilde{l}, T, M such that $\tilde{l} + T \leq m$ and*

$$X_{l+T} - X_l = Mm \quad \forall l \geq \tilde{l}.$$

Moreover, the quotient M/T does not depend on X_0 .

Proof. Observe that the quotient space $\mathbb{Z}/m\mathbb{Z}$ has only m distinct elements so that the first statement holds. For the second statement we first establish a monotonicity property of the orbits with respect to the initial data X_0 . To this end let $X_0 \leq X'_0$. We argue inductively. Due to (4.7) it is clear that $X_1 \leq X'_1$ in case that $X_0 = X'_0$ or $X'_0 - X_0 \geq 2$. It remains the case where $X'_0 - X_0 = 1$. We assume by contradiction that $X_1 > X'_1$. Writing

$N^* = 2\gamma/L - \xi$, the minimizer to determine X_1 would be given by $N^* + 1$, while for X'_1 minimization yields $N^* - 1$. Using minimality one derives that in this case we have

$$\gamma(\mu_{k^*+1} - \mu_{k^*}) \geq \xi \geq \gamma(\mu_{k^*+1} - \mu_{k^*}) + 1,$$

where $k^* = X_0 - \frac{1}{2} + N^* \pmod m$. This gives a contradiction. By iteration we obtain that $X_k \leq X'_k$ for all k . Now we argue as in ([16], Prop. 3.6) by comparing the long-time behavior of the orbits with starting points X_0, X'_0 and $X_0 + m$. For $L, l_0 \in \mathbb{N}$ we let $k = l_0 + LT(x_0)T(x'_0)$. By the first part of the proof and orbit monotonicity, for l_0 large enough it holds that

$$X_{l_0} + LT(x'_0)M(x_0)m \leq X'_{l_0} + LT(x_0)M(x'_0)m \leq X_{l_0} + LT(x'_0)M(x_0)m + m.$$

Dividing this inequality by L and letting $L \rightarrow +\infty$ yields the claim. □

Definition 4.6. For a given type of side with length $L \notin \mathcal{S}_i$, let M_i, T_i be as in Lemma 4.5, where T_i is chosen to be minimal. The effective velocity for a side \mathcal{S}_i is defined as a function $v_i^{\text{eff}} : (0, +\infty) \setminus \mathcal{S}_i \rightarrow [0, +\infty)$ by

$$v_i^{\text{eff}}(L) = \frac{M_i m}{T_i}.$$

In view of Lemma 4.5, this function is well-defined.

The following example shows that the effective velocities can indeed depend on the type of side.

Example 4.7. Let $m = 6$ and let c_ξ be a (maybe deterministic) field such that

$$\mu_0 = -\frac{1}{8\gamma}, \quad \mu_1 = \mu_2 = \mu_5 = \frac{1}{8\gamma}, \quad \mu_3 = \mu_4 = 0.$$

If $2\gamma/L \in (3 - \frac{1}{8}, 3)$, then the left side of a rectangle moves faster than the right side, namely

$$v_i^{\text{eff}}(L) = 3 > 2 = v_{i+2}^{\text{eff}}(L).$$

This follows from a straightforward computation based on the minimality criteria (4.8). Indeed, if the left side starts at $n_0 = 0$, then we have $N_1 = N_2 = 3$. If the right side starts also at $n_0 = 0$ we deduce that $N_1 = N_2 = N_3 = 2$. We leave the details of the computation to the interested reader.

We now compute the pinning threshold, that is the critical side length above which a side does not move after a finite number of time steps (or equivalently $v_i^{\text{eff}}(L) = 0$). Due to (4.7) a necessary condition is given by $L > \gamma$. We then have to compare the values of $N \in \{0, 1, 2\}$. For an arbitrary starting position of a left vertical side we obtain the conditions

$$L > \frac{2\gamma}{1 + \gamma(\mu_{k+1} - \mu_k)}, \quad L > \frac{4\gamma}{3 + \gamma(\mu_{k+2} - \mu_k)}.$$

As we can chose the index k , the pinning threshold for a left vertical side is given by

$$\bar{L}_i = \min_k \left\{ \max \left\{ \frac{2\gamma}{1 + \gamma(\mu_{k+1} - \mu_k)}, \frac{4\gamma}{3 + \gamma(\mu_{k+2} - \mu_k)} \right\} \right\} > \gamma.$$

The pinning thresholds for the other sides are given by

$$\begin{aligned} \bar{L}_{i+1} &= \min_k \left\{ \max \left\{ \frac{2\gamma}{1 + \gamma(\lambda_{k-1} - \lambda_k)}, \frac{4\gamma}{3 + \gamma(\lambda_{k-2} - \lambda_k)} \right\} \right\}, \\ \bar{L}_{i+2} &= \min_k \left\{ \max \left\{ \frac{2\gamma}{1 + \gamma(\mu_{k-1} - \mu_k)}, \frac{4\gamma}{3 + \gamma(\mu_{k-2} - \mu_k)} \right\} \right\}, \\ \bar{L}_{i+3} &= \min_k \left\{ \max \left\{ \frac{2\gamma}{1 + \gamma(\lambda_{k+1} - \lambda_k)}, \frac{4\gamma}{3 + \gamma(\lambda_{k+2} - \lambda_k)} \right\} \right\}, \end{aligned}$$

where the indices rotate clockwise. The next lemma contains some properties of the effective velocities. We remark that the same results have been obtained in [16] but we find it difficult to reproduce the argument in our slightly more complex setting. Therefore we provide a different proof.

Lemma 4.8. *The velocity functions v_i^{eff} satisfy the following properties:*

- (a) v_i^{eff} is constant on each interval contained in $(0, +\infty) \setminus \mathcal{S}_i$.
- (b) $v_i^{\text{eff}}(L) = 0$ if $L > \bar{L}_i$.
- (c) $v_i^{\text{eff}}(\cdot)$ is non-increasing in L .
- (d) It holds that

$$\lim_{\gamma \rightarrow +\infty} \frac{v_i^{\text{eff}}(L)}{\gamma} = \frac{2}{L}.$$

Proof. To prove the first assertion, fix an interval $I \subset (0, +\infty) \setminus \mathcal{S}_i$ and let $L \in I$. We claim that there exists an open interval I_L around L such that for all $n = 0, \dots, m - 1$ and all $L' \in I_L$ the unique minimizers of $v_i^{n, L'}$ agree with the unique minimizer of $v_i^{n, L}$. As I is connected, it then follows that the minimizers are the same for all $L' \in I$ and we conclude by iteration. To prove the claim, it is enough to observe that whenever $L_j \rightarrow L$, it follows that $v_i^{n, L_j}(N) \rightarrow v_i^{n, L}(N)$ pointwise. Due to (4.7) also the minimizers are bounded. By uniqueness they converge to the minimizer of the limit function. Hence the claim follows for any fixed n and then we take a finite intersection of open intervals to conclude.

The second assertion is an immediate consequence of the definition of the pinning threshold.

To prove the monotonicity, take $L > L'$. The claim follows from the fact that, for every n , in a multi-valued sense it holds that

$$\operatorname{argmin}_N v_i^{n, L}(N) \leq \operatorname{argmin}_N v_i^{n, L'}(N). \tag{4.9}$$

Indeed, observe that $N(L) := \lfloor 2\gamma/L \rfloor \leq \lfloor 2\gamma/L' \rfloor =: N(L')$. Then by (4.7) it suffices to treat the two cases $N(L) = N(L')$ and $N(L) + 1 = N(L')$. In any case, again applying (4.7) there are only finitely many options for violating (4.9) that can be ruled out by a direct calculation based on a characterization as in (4.8). We omit the details.

For the final statement it suffices to note that by (4.7) and the definition of $v_i^{\text{eff}}(L)$ we have

$$\frac{2\gamma}{L} - 2 \leq v_i^{\text{eff}}(L) \leq \frac{2\gamma}{L} + 1. \tag{4.10}$$

Now we are in a position to prove the main result of this section.

Proof of Theorem 4.2. Due to Remark 3.9 and the monotonicity proven in Lemma 4.8 (c) we only have to verify the formula for the velocities. We fix $\omega \in \Omega'$ given by Proposition 4.4. Using the same notation as in the proof of Theorem 3.1, we have to identify the weak*-limit N_i of N_i^j on the interval $(0, t^*)$. Therefore we fix $t_1 \in (0, t^*)$ such that $L_i(t_1)(\omega) \notin \mathcal{S}_i$. Given $\delta > 0$ there exists an open interval $I_\delta \ni t_1$ and j_0 such that for all $j \geq j_0$

- (i) $L_i^j(t)(\omega) \notin \mathcal{S}_i \quad \forall t \in I_\delta$,
- (ii) $d_H(S_{i,j}(t)(\omega), S_{i,j}(t_1)(\omega)) \leq \delta \quad \forall t \in I_\delta$.

Hence, by Proposition 4.4 we may assume that for $j \geq j_0$ and $t \in I_\delta$ there exists $n = n(j, t)$ such that for $L = L_i^j(t)(\omega)$ it holds that

$$N_i^j(t)(\omega) = \operatorname{argmin}_N v_i^{n, L}(N),$$

where $v_i^{n,L}$ is defined in (4.6). Since without loss of generality $L_i^j(t)(\omega)$ is in the same interval contained in $(0, +\infty) \setminus \mathcal{S}_i$ as $L_i^j(t_1)(\omega)$, we infer from Lemmata 4.5 and 4.8 (a) that

$$\begin{aligned} \int_{I_\delta} N_i(s)(\omega) \, ds &= \lim_j \int_{I_\delta} N_i^j(s)(\omega) \, ds = \lim_j \sum_{k\tau_j \in I_\delta} \tau_j N_i^j(k\tau_j)(\omega) + \mathcal{O}(\tau_j) \\ &= \lim_j |I_\delta| v_i^{\text{eff}}(L_i(t_1)(\omega)) + \mathcal{O}(\tau_j) = |I_\delta| v_i^{\text{eff}}(L_i(t_1)). \end{aligned}$$

Dividing by $|I_\delta|$ and letting $\delta \rightarrow 0$ we obtain the claim using Lebesgue’s differentiation theorem. Note that similar to the proof of Theorem 3.1 the formula for the velocity holds for every such t_1 since N_i has a constant representative locally near t_1 so that the side positions are differentiable in the classical sense. However here we have to take the side positions and cannot deduce the velocity from the side lengths since the center might move (see Exp. 4.7).

It remains the case where $L_i(t_1)(\omega) \in \mathcal{S}_i$. Note that by (4.9) we still have the monotonicity of orbits. That means if $L^-, L^+ \in (0, +\infty) \setminus \mathcal{S}_i$ are in the two intervals enclosing $L_i(t_1)(\omega)$ such that $L^- < L_i(t_1)(\omega) < L^+$ and we start the algorithm for computing the effective velocity with the same initial datum choosing the minimizer arbitrarily in the case of non-uniqueness, we have $X_k^+ \leq X_k \leq X_k^-$. This yields

$$|I_\delta| v_i^{\text{eff}}(L^+) \leq \int_{I_\delta} N_i(s)(\omega) \, ds \leq |I_\delta| v_i^{\text{eff}}(L^-).$$

The claim follows after dividing by $|I_\delta|$, sending $\delta \rightarrow 0$ and then taking both the limits as $L^- \uparrow L_i(t_1)(\omega)$ and $L^+ \downarrow L_i(t_1)(\omega)$ for which we use monotonicity. \square

As a last result we have several cases where a unique limit motion exists. However the equations differ from Corollary 3.2 since the velocity of two opposite sides may be not equal. We don’t list all possible cases where there is a unique motion.

Corollary 4.9. *Let $A_{\varepsilon_j}^0(\omega)$ and $\{c_\xi\}_\xi$ be as in Theorem 4.2. Assume in addition that the lengths $L_1^0(\omega), L_2^0(\omega)$ of $A(\omega)$ satisfy one of the three following conditions (we assume that $L_1^0(\omega) \leq L_2^0(\omega)$ and $\bar{L}_1 \leq \bar{L}_3$ as well as $\bar{L}_2 \leq \bar{L}_4$):*

- (i) $L_i^0(\omega) > \bar{L}_i$ (total pinning),
- (ii) $L_1^0(\omega) < \bar{L}_1$ and $L_2^0(\omega) \leq \bar{L}_2$ (vanishing in finite time),
- (iii) $\bar{L}_1 < L_1^0(\omega) < \bar{L}_3$ and $L_1^0(\omega) \notin \mathcal{S}_3$, and $L_2^0(\omega) > \bar{L}_4$ (partial pinning).

The sequence $A_{\varepsilon_j}(t)(\omega)$ converges locally in time to $A(t)(\omega)$, where $A(t)(\omega)$ is the unique coordinate rectangle with sides $S_i(t)(\omega)$ such that $A(0)(\omega) = A(\omega)$ and the side lengths $L_i(t)(\omega)$ solve the following differential equations for all but countably many times until the extinction time:

$$\frac{d}{dt} L_i(t)(\omega) = -\frac{1}{\gamma} \left(v_{i-1}^{\text{eff}}(L_{i-1}(t)(\omega)) + v_{i+1}^{\text{eff}}(L_{i+1}(t)(\omega)) \right)$$

with initial condition $L_1(0)(\omega) = L_1^0(\omega)$ and $L_2(0) = L_2^0$.

Proof. (i) and (ii) can be proven as in ([14], Thm. 3.2). In Case (iii) note that the side S_3 moves inward with a strictly positive velocity bounded away from 0. Hence $L_2(t)(\omega)$ is strictly decreasing until it vanishes. Consequently $L_2(t)(\omega) \in \mathcal{S}_2 \cup \mathcal{S}_4$ only for countably many times. Moreover, as soon as $L_2(t)(\omega) < \bar{L}_4$ also the side length $L_1(t)(\omega)$ shrinks strictly since from that time on the side S_4 moves inward with positive velocity. Hence the times when $L_1(t)(\omega) \in \mathcal{S}_1 \cup \mathcal{S}_3$ are discrete, too. Note that by continuity, the values at the critical times are uniquely defined. In between these critical times, one can use general results from ODE-theory to obtain that the rectangular motion is unique. The particular form of the ODE describing the motion is a straightforward consequence of Theorem 4.2. \square

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