MATHEMATICAL ANALYSIS OF THE STABILIZATION OF LAMELLAR PHASES BY A SHEAR STRESS

V. Torri[1]

Abstract. We consider a 2D mathematical model describing the motion of a solution of surfactants submitted to a high shear stress in a Couette−Taylor system. We are interested in a stabilization process obtained thanks to the shear. We prove that, if the shear stress is large enough, there exists global in time solution for small initial data and that the solution of the linearized system (controlled by a nonconstant parameter) tends to 0 as t goes to infinity. This explains rigorously some experiments.

1. Introduction

1.1. Description of the experiment

Lamellar phases are structures that are created by adding a large quantity of surfactants in water. Molecules of surfactants (like soap) have a particular form: an hydrophile part and an hydrophobe one. When the concentration of surfactants in water is large enough, the molecules organise themselves in order to minimize the interaction with water. Some bilayers of molecules appear in order to form the lamellar phase as showed in figure 1.

When such structures are submitted to a shear in a Couette−Taylor experiment, different cases are observed: if the shear stress is large enough, then these structures are stable. Otherwise some instabilities appear and lead to the formation of more complicated structures: the layers break themselves and are reorganised in concentric spheres which size is of order of the micrometer and that are called spherulits or "onions" (they have the structure of an onion!) (*cf.* [5,6]). Recently, a mathematical model has been written in order to try to explain this instability (see [3]). It relies on the fact that the Couette−Taylor system (*cf.* Fig. 2: two coaxial cylinders: the solution being between the cylinders) is never perfect and that the cylinders are never really coaxial. Even if the strain is not large (1%), this can induce huge forces acting on the layers since the thickness of the layers is that of two molecules. The numerical results of [3] show that this is in fact the right explanation for this instability and correspond to the experiments. It is also observed in the experiments that at high shear, the instability does not develop and that the small scales are always stable.

The aim of this paper is to prove these facts from the model. We show that the small scales are always linearly stable (implied by Prop. 3.6) and that when the shear stress is large enough, then the lamellar phase is linearly and nonlinearly stable (*cf.* Th. 3.1 and Th. 4.1).

[1] Mathématiques Appliquées de Bordeaux, Université de Bordeaux 1 et UMR 5466 du CNRS, 351 cours de la Libération, 33405 Talence Cedex, France; e-mail: `torri@math.u-bordeaux.fr`
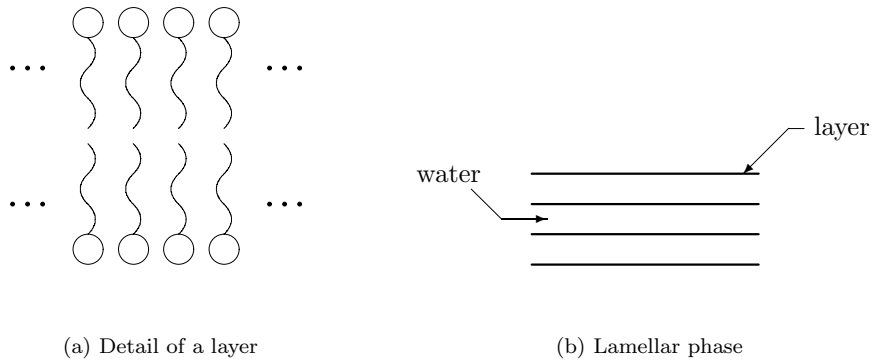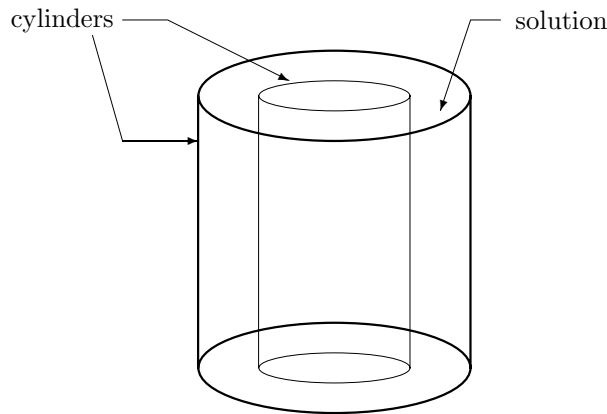
FIGURE 1. Structure of lamellar phases.

(a) Detail of a layer (b) Lamellar phase

FIGURE 2. Couette−Taylor system.

1.2. Description of the model

We briefly give the physical model that decribes the facts presented in the previous section.

As we neglect the boundary effects, we consider an horizontal section of the Couette−Taylor system. In Figure 3a is represented the coordinate system. The deplacement field h and the velocity field (u, v) are given by Figure 3b (the grey rectangles are the inner and the outer cylinders of the Couette−Taylor system, the black lines are the lamellar phase at equilibrium and one layer is represented around its equilibrium position).

The Navier−Stokes equations link u, v and h and the system is closed by the motion of a layer:

$$\rho(\partial_t U + (U.\nabla)U) = -\nabla p + F_{\text{visq}} + G_z(h)$$
$$\nabla.U = 0$$
$$\partial_t + (U.\nabla)h = v + \lambda_p G_z(h),$$

(a)   Coordinate   system

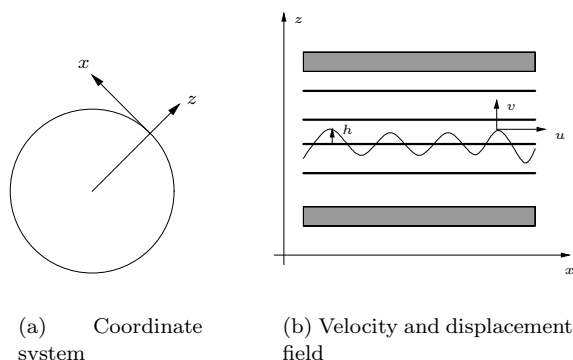(b) Velocity and displacement   field

FIGURE 3

where F_{visq} is a viscous force, λ_p a small constant and G_z the force describing the interaction between the layers and the water:

$$G_z(h) = -\frac{\partial F}{\partial h}$$

where $F(h) = \frac{B}{2}\left(\partial_z h - \frac{1}{2}(\partial_x h)^2\right)^2 + \frac{1}{2}K(\partial_x^2 h)^2$. B is the compression modulus (that is how the layers react when they are only compressed) and K the curvature modulus (that is how the layers react when they are only bent).

We consider the system around a steady solution of these equations. After some hypothesis, one of the two components of the velocity field can be deduced from the other. Then the system is reduced to a system of two equations with the unkowns $v(t, x, z)$ (the radial velocity) and $h(t, x, z)$ (the relative displacement of a layer with respect to the state of the rest). Finally, the system is written in a nondimensionnal form (*cf.* [3] for a detailed obtention of these equations):

$$
\begin{aligned}
\partial_t v + z\partial_x v &= \lambda\alpha\partial_x^2 v + \lambda\beta\partial_z^2 v + \lambda^2\gamma_1\partial_z^2 h - \lambda^2\delta_1\partial_x^2 h - \lambda^2\epsilon_1\partial_x^4 h \\
&\quad + \mu\lambda^2\theta_1\partial_x\left(\partial_x h\partial_z h\right) + \mu\lambda^2\theta_2\partial_z\left(\partial_x h\right)^2 + \nu^2\lambda^3\theta_3\partial_x\left(\partial_x h\right)^3 \\
\partial_t h + z\partial_x h &= v + \lambda\gamma_2\partial_z^2 h - \lambda\delta_2\partial_x^2 h - \lambda\epsilon_2\partial_x^4 h.
\end{aligned}
$$

We note by λ^{-1} the velocity of the outer cylinder. It is suppose to be great, so λ is small. The physical parameters α, β, γ_i, δ_i, ϵ_i ($i = 1, 2$), θ_i ($i = 1, 2, 3$), μ and ν are positive and depend on physical constants and datas (such as density, compression modulus...) (*cf.* [3]). The last two parameters depend on the velocity of the shear.

We conclude this description by the boundary conditions:

- v and h are periodic in the x direction (as the solution is between two cylinders);
- They vanish at $z = 0, 1$.

The main question concerning this system is the stability of the zero solution, that is, do we have global existence of a bounded solution for small data? The first question is therefore: is zero linearly stable? By changing h in $\tilde{h} = \frac{1}{\lambda}h$ and t by λt, the linearized version (around zero) of this system is:

$$
\begin{aligned}
\partial_t v + \frac{1}{\lambda}z\partial_x v &= \alpha\partial_x^2 v + \beta\partial_z^2 v + \gamma_1\partial_z^2 h - \delta_1\partial_x^2 h - \epsilon_1\partial_x^4 h \\
\partial_t h + \frac{1}{\lambda}z\partial_x h &= v + \gamma_2\partial_z^2 h - \delta_2\partial_x^2 h - \epsilon_2\partial_x^4 h.
\end{aligned}
$$

Recall that $\frac{1}{\lambda}$ corresponds to the angular velocity of the outer cylinder of the Couette system. If one takes $\frac{1}{\lambda} = 0$, expanding h and v in Fourier series with respect to x and z gives (k and l are the Fourier variables corresponding respectively to x and z):

$$\begin{aligned} \partial_t v &= -\left(\alpha k^2 + \beta l^2\right) v + \left(-\gamma_1 l^2 + \delta_1 k^2 - \epsilon_1 k^4\right) h \\ \partial_t h &= v + \left(-\gamma_2 l^2 + \delta_2 k^2 - \epsilon_2 k^4\right) h. \end{aligned}$$

The determinant of the matrix of the right-hand-side is equal to

$$\left(\alpha k^2 + \beta l^2\right)\left(\gamma_2 l^2 - \delta_2 k^2 + \epsilon_2 k^4\right) + \left(\gamma_1 l^2 - \delta_1 k^2 + \epsilon_1 k_x^4\right),$$

which can be negative (for small k and small l). Hence, the solution can be unbounded as t goes to infinity.

The aim of this paper is to prove that if λ is small enough, then these small frequencies are stable. Recent results (*cf.* [2,7,8]) show that skew-symmetric operators with large coefficients have some smoothing effects in nonlinear problems. Here, since the operator $\frac{1}{\lambda} z \partial_x$ does not have constant coefficient, this smoothing appears in the linear case, see Section 2.3 for a simple example.

2. STATEMENT OF THE MAIN RESULTS

2.1. **Functionnal spaces and notations**

The system of coordinates is $(x, z) \in \Omega = (-1, 1) \times (0, 1)$. We will consider periodic functions in the x direction which vanish at $z = 0, 1$.

Let us define $B = \mathbb{R} \times (0, 1)$. Let

$$L(\Omega) = \left\{f \in L_{\text{loc}}^2\left(\overline{B}\right),\ 2\text{-periodic in the } x \text{ direction a.e. in } z\right\},$$

$$H(\Omega) = \left\{f \in L(\Omega), f_{|\Omega} \in H^1\left(\Omega\right),\ 2\text{-periodic in } x, f\left(x, 0\right) = f\left(x, 1\right) = 0\right\}$$

and

$$K(\Omega) = \left\{f \in H(\Omega),\ \partial_x^2 f \in L^2\left(\Omega\right)\right\}.$$

We will denote $|u|_p = \left(\int_\Omega |u|^p\right)^{\frac{1}{p}}$ and H^s the usual Sobolev spaces. We note by C_p the Poincaré's constant, which verifies

$$|h|_2 \leqslant C_p\,|\partial_x h|_2\,,$$

for $h \in H(\Omega)$.

2.2. **Statement of the main results**

Let us recall that the system reads as follows:

$$\partial_t v + z\partial_x v = \lambda\alpha\partial_x^2 v + \lambda\beta\partial_z^2 v + \lambda^2\gamma_1\partial_z^2 h - \lambda^2\delta_1\partial_x^2 h - \lambda^2\epsilon_1\partial_x^4 h$$
$$+ \mu\lambda^2\theta_1\partial_x\left(\partial_x h\partial_z h\right) + \mu\lambda^2\theta_2\partial_z\left(\partial_x h\right)^2 + \nu^2\lambda^3\theta_3\partial_x\left(\partial_x h\right)^3 \tag{1}$$
$$\partial_t h + z\partial_x h = v + \lambda\gamma_2\partial_z^2 h - \lambda\delta_2\partial_x^2 h - \lambda\epsilon_2\partial_x^4 h \tag{2}$$

(*cf.* Sect. 1). By changing h in $\tilde{h} = \lambda h$ and scaling in time (replacing t by λt), we obtain:

$$\partial_t v + \frac{1}{\lambda} z \partial_x v = \alpha \partial_x^2 v + \beta \partial_z^2 v + \gamma_1 \partial_z^2 h - \delta_1 \partial_x^2 h - \epsilon_1 \partial_x^4 h$$

$$+ \frac{\mu}{\lambda} \theta_1 \partial_x (\partial_x h \partial_z h) + \frac{\mu}{\lambda} \theta_2 \partial_z (\partial_x h)^2 + \frac{\nu^2}{\lambda} \theta_3 \partial_x (\partial_x h)^3$$

$$\partial_t h + \frac{1}{\lambda} z \partial_x h = v + \gamma_2 \partial_z^2 h - \delta_2 \partial_x^2 h - \epsilon_2 \partial_x^4 h,$$

and $(x, z) \in \Omega$.

For sake of simplicity, we omit the coefficients of the nonlinear terms (θ_1, θ_2 and θ_3) and those of the viscous terms (α, β, γ_1, γ_2, ϵ_1 and ϵ_2). They do not have an important role in the proofs and give complicated formulae. We just keep δ_1 and δ_2. Here is the system we will study:

$$\partial_t v + \frac{1}{\lambda} z \partial_x v = \Delta v + \partial_z^2 h - \delta_1 \partial_x^2 h - \partial_x^4 h$$

$$+ \frac{\mu}{\lambda} \partial_x (\partial_x h \partial_z h) + \frac{\mu}{\lambda} \partial_z (\partial_x h)^2 + \frac{\nu^2}{\lambda} \partial_x (\partial_x h)^3 \tag{3}$$

$$\partial_t h + \frac{1}{\lambda} z \partial_x h = v + \partial_z^2 h - \delta_2 \partial_x^2 h - \partial_x^4 h. \tag{4}$$

The main result of this paper is the stabilization of the lamellar phase at high shear, *i.e.* the global existence of a solution of (3, 4) (therefore of (1, 2)) for small λ and small initial data and convergence to 0 of the solution of the linearised system as t goes to infinity. More precisely, we have:

Theorem 2.1. *There exists $\lambda_0 > 0$ such that for all λ in $(0, \lambda_0)$, for all (v_0, h_0) in $\mathrm{L}(\Omega) \times \mathrm{H}(\Omega)$, the linearised system of (3, 4) has a unique solution (v, h) defined on $\mathbb{R}_+$ and which satisfies $v(0) = v_0$, $h(0) = h_0$ and*

$$(v, h)(t) \xrightarrow[t \to +\infty]{} 0 \text{ in } \mathrm{L}(\Omega) \times \mathrm{K}(\Omega).$$

This stabilization result is obtained thanks to energy estimates.

Theorem 2.2. *There exists $0 < \lambda_0 < 1$ and $M > 0$ such that, for all $0 < \lambda < \lambda_0$, v_0 in $\mathrm{L}(\Omega)$, h_0 in $\mathrm{K}(\Omega)$ such that $\left(|v_0|_2^2 + |h_0|_2^2 + \left| \partial_x^2 h_0 \right|_2^2 + \lambda \left| \partial_z h_0 \right|_2^2 \right) < M$, for all $\mu < \lambda^{\frac{5}{2}}$, $\nu < \lambda^{\frac{5}{4}}$, there exists $v, h : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ such that*

- $v \in \mathrm{L}^\infty(\mathbb{R}_+; \mathrm{L}(\Omega)) \cap \mathrm{L}^2(\mathbb{R}_+; \mathrm{H}(\Omega)) \cap \mathrm{C}^0(\mathbb{R}_+; \mathrm{L}^2(\Omega))$;
- $h \in \mathrm{L}^\infty(\mathbb{R}_+; \mathrm{K}(\Omega)) \cap \mathrm{L}^2(\mathbb{R}_+; \mathrm{H}(\Omega) \cap \mathrm{H}^2(\Omega)) \cap \mathrm{C}^0(\mathbb{R}_+; \mathrm{L}^2(\Omega))$;
- $\partial_x^2 \partial_z h \in \mathrm{L}^2(\mathbb{R}_+ \times \Omega)$;
- $\partial_x^4 h \in \mathrm{L}^2(\mathbb{R}_+ \times \Omega)$;

solutions of (3) and (4) such that $v(0) = v_0$ and $h(0) = h_0$.

Remark 2.3. It is not easy to estimate λ_0 in terms of the physical parameters. However, in [3], the numerical value of λ_0 corresponds to that of the experiment.

We also give a local existence theorem whatever the velocity of the shear and the initial data are:

Theorem 2.4. *There exists $T_e > 0$ such that, for all v_0 in $\mathrm{L}(\Omega)$ and h_0 in $\mathrm{K}(\Omega)$, there exists $v, h : [0, T_e[\times \Omega \to \mathbb{R}$ such that for all T in $]0, T_e[$,*

- $v \in \mathrm{L}^\infty(0, T; \mathrm{L}(\Omega)) \cap \mathrm{L}^2(0, T; \mathrm{H}(\Omega)) \cap \mathrm{C}^0([0, T]; \mathrm{L}^2(\Omega))$;
- $h \in \mathrm{L}^\infty(0, T; \mathrm{K}(\Omega)) \cap \mathrm{L}^2(0, T; \mathrm{H}(\Omega) \cap \mathrm{H}^2(\Omega)) \cap \mathrm{C}^0([0, T]; \mathrm{L}^2(\Omega))$;

- $\partial_x^2 \partial_z h \in \mathrm{L}^2\left(\mathbb{R}_+ \times \Omega\right)$;
- $\partial_x^4 h \in \mathrm{L}^2\left(\mathbb{R}_+ \times \Omega\right)$;

solutions of (3) and (4) such that $v(0) = v_0$ and $h(0) = h_0$.

This paper is organised as follows:

- the Section 2.3 gives a short example of an equation which shows the main ideas used to prove Theorem 2.2;
- Section 3 is devoted to energy estimates for the linear case and to the proof of Theorem 2.1. As the system is periodic in the x direction, we compute the fourier coefficients of v and h. We deal with both local and global in time cases;
- Section 4 is devoted to *a priori* estimates of the nonlinear terms that appear in equation (3). Here again we give estimates for the global and the local in time cases;
- Sections 5 and 6 are devoted respectively to the proof of Theorem 2.2 and Theorem 2.4.

2.3. **Example**

One of the main difficulties of this paper will be reviewed on the following example: find a bounded solution of the equation

$$\partial_t u + \frac{1}{\lambda} z \partial_x u = \Delta u + \alpha^2 u \tag{5}$$

provided λ small, where α and λ are two positive constants, $u : \Omega \to \mathbb{R}$ 2-periodic in x and vanishing at $z = 0, 1$, with a mean value with respect to x equal to zero. The method used here is the Galerkin method which requires *a priori* estimates.

In this section, if u is a solution of (5), then we consider $N(t) = |u|_2^2 + |\partial_x u|_2^2 + \lambda |\partial_z u|_2^2$ and $\tilde{N}(t) = N(t) - C\sqrt{\lambda} \int_\Omega \partial_x u \partial_z u$, where C is a constant wich will be precised later. Then we have the following result:

Theorem 2.5. *There exists $\lambda_0 > 0$ such that, for all λ in $(0, \lambda_0)$, for all u_0 in $\mathrm{H}^1(\Omega)$ with a mean value with respect to x equal to zero, there exists u in $\mathrm{L}^\infty\left(\mathbb{R}_+; \mathrm{H}^1(\Omega)\right) \cap \mathrm{C}^0\left(\mathbb{R}_+; \mathrm{H}^1(\Omega)\right)$ such that u verifies (5) and $u(0) = u_0$. Moreover,*

$$|u(t)|_{H^1} \xrightarrow[t \to +\infty]{} 0.$$

Proof. This result is a straightforward corollary of the following lemma:

Lemma 2.6. *Let $C < \frac{1}{\sqrt{2}}$. There exists $A > 0$ and $\lambda_0 > 0$ such that, for all solution u of (5), for all λ in $(0, \lambda_0)$ and for all t in $[0, T)$,*

$$\tilde{N}'(t) + \frac{1}{2}|\partial_z u|_2^2 + |\partial_x \partial_z u|_2^2 + \lambda \left|\partial_z^2 u\right|_2^2 + \frac{C}{2\sqrt{\lambda}}|\partial_x u|_2^2 \leqslant -A N(t).$$

Proof. We first perform a Fourier transform with respect to x, $\hat{u}(k, z) = \frac{1}{2}\int_{-1}^{1} \exp\left(-ik\pi s\right) u(s, z)\,\mathrm{d}s$, still denoted by u. Then we get the following equation:

$$\partial_t u + \frac{ik}{\lambda} z u = -k^2 u + \partial_z^2 u + \alpha^2 u. \tag{6}$$

Let $\Omega_z = (0, 1)$ and $N_k(t) = |u|_2^2 + k^2 |u|_2^2 + \lambda |\partial_z u|_2^2$ and $\tilde{N}_k(t) = N_k(t) - Ck\sqrt{\lambda}\mathrm{Im}\int_{\Omega_z} u \partial_z \overline{u}$. We compute $\tilde{N}'_k(t)$:

$$\frac{1}{2}\partial_t |u|_2^2 = \left(\alpha^2 - k^2\right)|u|_2^2 - |\partial_z u|_2^2,$$

as the resulting imaginary term is equal to zero. In the same way,

$$\frac{1}{2}\partial_t |\partial_z u|_2^2 = \frac{k}{\lambda}\mathrm{Im}\int_{\Omega_z} u \partial_z \overline{u} + \left(\alpha^2 - k^2\right)|\partial_z u|_2^2 - \left|\partial_z^2 u\right|_2^2.$$

Finally,

$$
\begin{aligned}
\frac{1}{2}\partial_t \mathrm{Im}\int_{\Omega_z}\partial_z\overline{u}u &= -\frac{k}{\lambda}\mathrm{Re}\int_{\Omega_z}zu\partial_z\overline{u}+\left(\alpha^2-k^2\right)\mathrm{Im}\int_{\Omega_z}u\partial_z\overline{u}+\mathrm{Im}\int_{\Omega_z}\partial_z^2u\partial_z\overline{u} \\
&= \frac{k}{2\lambda}\left|u\right|_2^2+\left(\alpha^2-k^2\right)\mathrm{Im}\int_{\Omega_z}u\partial_z\overline{u}+\mathrm{Im}\int_{\Omega_z}\partial_z^2u\partial_z\overline{u},
\end{aligned}
$$

by integrating by parts the first term of the right member. This term is the one which will be significant in the proof (thanks to the $\frac{1}{\lambda}\left|u\right|_2$ term). We therefore obtain

$$
\begin{aligned}
\frac{1}{2}\tilde{N}_k'(t) &= \left(\alpha^2-k^2\right)\left|u\right|_2^2-\left|\partial_z u\right|_2^2+k^2\left(\alpha^2-k^2\right)\left|u\right|_2^2-k^2\left|\partial_z u\right|_2^2 \\
&\quad +k\mathrm{Im}\int_{\Omega_z}u\partial_z\overline{u}+\lambda\left(\alpha^2-k^2\right)\left|\partial_z u\right|_2^2-\lambda\left|\partial_z^2 u\right|_2^2 \\
&\quad -\frac{Ck^2}{2\sqrt{\lambda}}\left|u\right|_2^2-Ck\sqrt{\lambda}\left(\alpha^2-k^2\right)\mathrm{Im}\int_{\Omega_z}u\partial_z\overline{u}-Ck\sqrt{\lambda}\mathrm{Im}\int_{\Omega_z}\partial_z^2u\partial_z\overline{u} \\
&\leqslant \left(\left(\alpha^2-\frac{k^2}{2}\right)+\left(\frac{C^2}{2}-1\right)k^4+k^2\alpha^2\left(1-C^2\right)+\frac{1}{2}C^2\alpha^4\right)\left|u\right|_2^2 \\
&\quad -\frac{1}{2}\left|\partial_z u\right|_2^2+\lambda\left(\alpha^2-\frac{k^2}{2}\right)\left|\partial_z u\right|_2^2-\frac{\lambda}{2}\left|\partial_z^2 u\right|_2^2-\frac{Ck^2}{2\sqrt{\lambda}}\left|u\right|_2. \tag{7}
\end{aligned}
$$

We now deal separately with high frequencies (k large) and small ones (k small).
- Study for high frequencies: since $C<\frac{1}{\sqrt{2}}<1$, there exists $K>0$ (depending only on α and C) so that, for all $k>K$,

$$
\tilde{N}_k'(t)\leqslant -\frac{k^2}{2}N_k(t)-\left|\partial_z u\right|_2^2-k^2\left|\partial_z u\right|_2-\lambda\left|\partial_z^2 u\right|_2^2-\frac{Ck^2}{2\sqrt{\lambda}}\left|u\right|_2^2. \tag{8}
$$

- Study for small frequencies: since the mean value of u is equal to zero (it is easy to show, by computing $\left|u\right|_2$ and by using Gronwall's lemma, that the mean value with respect to x remains equal to 0), we only need to study $\tilde{N}_k'(t)$ for k between 1 and K. By (7),

$$
\begin{aligned}
\tilde{N}_k(t) &\leqslant -k^2 N_k(t)+\left(2\lambda\alpha^2-1\right)\left|\partial_z u\right|_2^2-k^2\left|\partial_z u\right|_2^2-\lambda\left|\partial_z^2 u\right|_2^2 \\
&\quad +\left(2\alpha^2-2k^2\left(2C^2\alpha^2-1\right)+C^2\alpha^4-\frac{Ck^2}{\sqrt{\lambda}}\right)\left|u\right|_2.
\end{aligned}
$$

So for small λ, for all k between 1 and K, we have

$$
\tilde{N}_k'(t)\leqslant -N_k(t)-\frac{1}{2}\left|\partial_z u\right|_2^2-k^2\left|\partial_z u\right|_2^2-\lambda\left|\partial_z^2 u\right|_2^2-\frac{Ck^2}{2\sqrt{\lambda}}\left|u\right|_2. \tag{9}
$$

Hence, using (8) and (9), we have, for all $k>0$

$$
\tilde{N}_k(t)\leqslant -AN_k(t)-\frac{1}{2}\left|\partial_z u\right|_2^2-k^2\left|\partial_z u\right|_2^2-\lambda\left|\partial_z^2 u\right|_2^2-\frac{Ck^2}{2\sqrt{\lambda}}\left|u\right|_2,
$$

where $A=\min\left(1,\frac{K^2}{2}\right)$. We have the same result for $k<0$ by taking the conjugate expression of (6). Then the Parseval–Bessel's identity finishes the proof of the lemma. $\square$

To conclude the proof of Theorem 2.5, it suffices to show that $\tilde{N}(t)$ and $N(t)$ are equivalent, which is obvious if C is sufficiently small (possible as the lemma is valid since $C < \frac{1}{\sqrt{2}}$), and to use a Galerkin method, which is quite easy to apply as the equation is linear. $\qquad\square$

Remark 2.7. This stabilization process (obtained thanks to the linear term $\frac{z}{\lambda}\partial_x u$) is the one that will be used in the physical case.

3. LINEAR CASE

In this section, we study the linearized system of (3, 4), *i.e.*

$$\partial_t v + \frac{1}{\lambda}z\partial_x v \;=\; \Delta v + \partial_z^2 h - \delta_1\partial_x^2 h - \partial_x^4 h \tag{10}$$

$$\partial_t h + \frac{1}{\lambda}z\partial_x h \;=\; v + \partial_z^2 h - \delta_2\partial_x^2 h - \partial_x^4 h \tag{11}$$

$$v(0) \;=\; v_0 \tag{12}$$

$$h(0) \;=\; h_0. \tag{13}$$

We will show local existence of solutions of such systems and linear stability when λ is sufficently small. As the arguments are based on the Galerkin method, we will derive energy estimates envolving quantities like $|v|_2^2$, $|h|_2^2$, $\left|\partial_x^2 h\right|_2^2$ and $\lambda\left|\partial_z h\right|_2^2$. Hence the two following results hold:

Theorem 3.1. *Let (v_0, h_0) in $\mathrm{L}(\Omega) \times \mathrm{H}(\Omega)$. There exists a unique solution (v, h) to (10–13) such that (v, h) in $\mathrm{C}^0\left(\mathbb{R}_+; \mathrm{L}(\Omega) \times \mathrm{K}(\Omega)\right)$ and $v \in \mathrm{L}_{\mathrm{loc}}^2\left(\mathbb{R}_+; \mathrm{H}^1\right)$, $\partial_x^2 h, \partial_z h \in \mathrm{L}_{\mathrm{loc}}^2\left(\mathbb{R}_+; \mathrm{L}^2(\Omega)\right)$.*

Theorem 3.2. *There exists $\lambda_0 > 0$ such that for all λ in $(0, \lambda_0)$, for all (v_0, h_0) in $\mathrm{L}(\Omega) \times \mathrm{H}(\Omega)$, the solution (v, h) given by Theorem 3.1 satisfies*

$$(v, h)(t) \xrightarrow[t \to +\infty]{} 0 \ \text{ in } \mathrm{L}(\Omega) \times \mathrm{K}(\Omega).$$

Remark 3.3. This result is the linear version of what is observed in the experiments: the lamellar phase is stable at high shear (see [5, 6]). It is also what is observed numerically (see [3]).

As the most important point is the linear stability (Th. 3.2), we begin this section by proving this fact. We postone to the next paragraph the proof of Theorem 3.1.

3.1. **Linear stability**

In this section, we apply the stabilization process described in Section 2.3 in order to obtain bounded in time solutions, provided λ is small enough.

This result need energy estimates on the following quantity: if (v, h) is a solution of (10–13) defined on $[0, T]$, sufficently smooth, let

$$N(t) = |v|_2^2 + E|h|_2^2 + \left|\partial_x^2 h\right|_2^2 + \lambda\left|\partial_z h\right|_2^2 + F\left(|V|_2^2 + |\partial_z H|_2^2\right)$$

and

$$\tilde{N}(t) = N(t) - C\sqrt{\lambda}\int_\Omega \partial_x h\, \partial_z h,$$

where C, E and F are constants that will be precised later on, $V(t, z) = \frac{1}{2}\int_{-1}^{1} v(t, x, z)\,\mathrm{d}x$, $\partial_z H(t, z) = \frac{1}{2}\int_{-1}^{1}\partial_z h(t, x, z)\,\mathrm{d}x$ are the mean values with respect to x of v and $\partial_z h$ respectively.

Remark 3.4. We have to introduce in this Lyapounov function the quantities V and H (whereas it has not been the case in Sect. 2.3) because we can not prove *a priori* that the mean value of v and h are constant, contrary to the exemple in Section 2.3.

The proof of Theorem 3.2 will follow easily from this proposition:

Proposition 3.5. Let $E > 4$, $F > \max\left(\frac{E}{\pi^4}, \frac{E}{\pi^2}\right)$ and $C < \min\left(\frac{1}{2}, \frac{2}{C_p}\right)$. Then, there exist $A > 0$ and $\lambda_0 > 0$ such that, for all $\lambda < \lambda_0$,

$$\tilde{N}'(t) \leqslant -AN(t). \tag{14}$$

The scheme of the proof will be:
- expansion in Fourier series;
- analogous result of (14) is found for large frequencies;
- then for small frequencies;
- finally for the mean value.

3.1.1. *Expansion in Fourier series*

As in Section 2.3, we expand the equations (10, 11) in Fourier series with respect to x, and we get

$$\partial_t v + \frac{ikz}{\lambda} v = -k^2 v + \partial_z^2 v + \partial_z^2 h - k^2 \left(k^2 - \delta_1\right) h \tag{15}$$

$$\partial_t h + \frac{ikz}{\lambda} h = v + \partial_z^2 h - k^2 \left(k^2 - \delta_2\right) h. \tag{16}$$

We note $\Omega_z = (0, 1)$. For sake of simplicity, v still denotes $\hat{v}(k) = \frac{1}{2} \int_{-1}^{1} v(s) \exp(-i\pi ks) \, ds$ and h denotes $\hat{h}(k)$. Let denote, for $k \neq 0$,

$$N_k(t) = |v|_2^2 + E|h|_2^2 + k^4|h|_2^2 + \lambda|\partial_z h|_2^2,$$

$$\tilde{N}_k(t) = N_k(t) - Ck\sqrt{\lambda}\,\mathrm{Im}\int_{\Omega_z} h\partial_z\overline{h}$$

and

$$\tilde{N}_0(t) = N_0(t) = |v|_2^2 + E|h|_2^2 + \lambda|\partial_z h|_2^2 + F\left(|v|_2^2 + |\partial_z h|_2^2\right).$$

We compute $\tilde{N}_k'(t)$ and $\tilde{N}_0'(t)$:

$$\frac{1}{2}\partial_t |v|_2^2 = -k^2|v|_2^2 - |\partial_z v|_2^2 - \mathrm{Re}\int_{\Omega_z} \partial_z h\partial_z\overline{v} - k^2\left(k^2 - \delta_1\right)\mathrm{Re}\int_{\Omega_z} h\overline{v}, \tag{17}$$

as $\mathrm{Re}\int_{\Omega_z} izv\overline{v} = \mathrm{Re}\int_{\Omega_z} iz|v|^2 = 0$. The same computation gives:

$$\frac{1}{2}\partial_t |h|_2^2 = \mathrm{Re}\int_{\Omega_z} h\overline{v} - |\partial_z h|_2^2 - k^2\left(k^2 - \delta_2\right)|h|_2^2. \tag{18}$$

$$\begin{aligned}
\frac{1}{2}\partial_t |\partial_z h|_2^2 &= -\mathrm{Re}\int_{\Omega_z} \frac{-izk}{\lambda} h\partial_z^2\overline{h} - \mathrm{Re}\int_{\Omega_z} v\partial_z^2\overline{h} - \left|\partial_z^2 h\right|_2^2 - k^2\left(k^2 - \delta_2\right)|\partial_z h|_2^2 \\
&= \frac{k}{\lambda}\mathrm{Im}\int_{\Omega_z} h\partial_z\overline{h} + \mathrm{Re}\int_{\Omega_z} \partial_z v\partial_z\overline{h} - \left|\partial_z^2 h\right|_2^2 - k^2\left(k^2 - \delta_2\right)|\partial_z h|_2^2, \tag{19}
\end{aligned}$$

by integrating by parts the imaginary term.

$$\frac{1}{2}\partial_t \mathrm{Im} \int_{\Omega_z} h\partial_z \overline{h} \;=\; \mathrm{Im} \int_{\Omega_z} \frac{-ikz}{\lambda} h\partial_z \overline{h} + \mathrm{Im} \int_{\Omega_z} v\partial_z \overline{h} + \int_{\Omega_z} \partial_z^2 h\partial_z \overline{h}$$
$$-k^2\left(k^2-\delta_2\right)\mathrm{Im}\int_{\Omega_z} h\partial_z \overline{h}.$$

Integrating by parts the first term of the right member gives

$$\frac{1}{2}\partial_t \mathrm{Im}\int_{\Omega_z} h\partial_z \overline{h} = \frac{k}{2\lambda}\left|h\right|_2^2 + \mathrm{Im}\int_{\Omega_z} v\partial_z \overline{h} + \mathrm{Im}\int_{\Omega_z}\partial_z^2 h\partial_z \overline{h} - k^2\left(k^2-\delta_2\right)\mathrm{Im}\int_{\Omega_z} h\partial_z \overline{h}. \tag{20}$$

As in Section 2.3, we study separately high frequencies and small ones. Due to its special behaviour, $\tilde{N}_0$ is treated separately. By using (17–20) for $k \neq 0$, we have

$$\frac{1}{2}\tilde{N}_k'(t) \;=\; -k^2\left|v\right|_2^2 - \left|\partial_z v\right|_2^2 + \delta_1 k^2 \mathrm{Re}\int_{\Omega_z} h\overline{v} + E\mathrm{Re}\int_{\Omega_z} h\overline{v} - E\left|\partial_z h\right|_2^2$$
$$-Ek^2\left(k^2-\delta_2\right)\left|h\right|_2^2 - k^4\left|\partial_z h\right|_2^2 - k^6\left(k^2-\delta_2\right)\left|h\right|_2^2$$
$$+k\mathrm{Im}\int_{\Omega_z} h\overline{\partial_z h} + \lambda\mathrm{Re}\int_{\Omega_z}\partial_z v\overline{\partial_z h} - \lambda\left|\partial_z^2 h\right|_2^2$$
$$-\lambda k^2\left(k^2-\delta_2\right)\left|\partial_z h\right|_2^2 - \frac{Ck^2}{2\sqrt{\lambda}}\left|h\right|_2^2 - Ck\sqrt{\lambda}\mathrm{Im}\int_{\Omega_z} v\overline{\partial_z h}$$
$$-Ck\sqrt{\lambda}\mathrm{Im}\int_{\Omega_z}\partial_z^2 h\overline{\partial_z h} - C\sqrt{\lambda}k^3\left(k^2-\delta_2\right)\mathrm{Im}\int_{\Omega_z} h\overline{\partial_z h}, \tag{21}$$

and, using (17–19), for $k = 0$,

$$\frac{1}{2}\tilde{N}_0'(t) \;=\; -\left|\partial_z v\right|_2^2 + E\mathrm{Re}\int_{\Omega_z} h\overline{v} - E\left|\partial_z h\right|_2^2 + \lambda\mathrm{Re}\int_{\Omega_z}\partial_z v\overline{\partial_z h}$$
$$-\lambda\left|\partial_z^2 h\right|_2^2 - F\left|\partial_z v\right|_2^2 - F\left|\partial_z^2 h\right|_2^2. \tag{22}$$

Let note that for this last equation, V is equal to v (let recall that v means $\hat{v}(0)$) and is a real function. The same remarks hold for h.

3.1.2. *High frequencies*

As in Section 2.3, the main idea if that, for k large enough, the viscous terms are dominant. By using the fact that the sign of a quadratic form does not change if its discriminant is negative, we take each term of (21) which is not negative and find an associated quadratic form which controls them for large k. Hence, we get the following result for linear stability for large frequencies:

Proposition 3.6. *Let* $\Lambda > 0$, *then there exists* K *in* $\mathbb{N}$, *depending only on* Λ, C *and* E, *and* $A_1 > 0$ *such that, for all* λ *in* $(0,\Lambda)$, *for all* $k > K$ *and for all* t *in* $[0,T)$, $\tilde{N}_k'(t) \leqslant -A_1 N_k(t)$.

Proof. First, we choose k sufficiently large so that $k^2 - \delta_2$ is positive.

1. For $\mathrm{Re}\int_{\Omega_z} h\overline{v}$:

$$Q_1 = -\frac{k^2}{4}\left|v\right|_2^2 + \left(\delta_1 k^2 + E\right)\mathrm{Re}\int_{\Omega_z} h\overline{v} - \frac{k^6}{8}\left(k^2-\delta_2\right)\left|h\right|_2^2.$$

Using Cauchy$-$Schwarz's inequality leads to

$$Q_1 \leqslant -\frac{k^2}{4}\,|v|_2^2 + \left(\delta_1 k^2 + E\right)|v|_2\,|h|_2 - \frac{k^6}{8}\left(k^2 - \delta_2\right)|h|_2^2.$$

The second member is a quadratic form whose discriminant is $\Delta_1 = \left(\delta_1 k^2 + E\right)^2 - \frac{k^8}{8}\left(k^2 - \delta_2\right)$. So for k sufficiently large, $\Delta_1 < 0$ and $Q_1 \leqslant 0$. The five following quantities will be treated in the same way.

2. For $\mathrm{Re}\int_{\Omega_z}\partial_z h \partial_z \overline{v}$:

$$
\begin{aligned}
Q_2 &= -\frac{1}{2}\,|\partial_z v|_2^2 + (\lambda - 1)\,\mathrm{Re}\int_{\Omega_z}\partial_z h \partial_z \overline{v} - \frac{1}{8}k^4\,|\partial_z h|_2^2, \\
&\leqslant -\frac{1}{2}\,|\partial_z v|_2^2 + (\lambda - 1)\,|\partial_z h|_2\,|\partial_z v|_2 - \frac{1}{8}k^4\,|\partial_z h|_2^2.
\end{aligned}
$$

The discriminant of the second member is $\Delta_2 = (\lambda - 1)^2 - \frac{1}{4}k^4$. Let $\Lambda > 0$. Then, for k sufficiently large and for all $\lambda < \Lambda$, $\Delta_2 < 0$ and $Q_2 \leqslant 0$.

3. For $\mathrm{Im}\int_{\Omega_z} h \partial_z \overline{h}$:

$$
\begin{aligned}
Q_3 &= -\frac{k^6}{8}\left(k^2 - \delta 2\right)|h|_2^2 + k\,\mathrm{Im}\int_{\Omega_z} h \partial_z \overline{h} - \frac{k^4}{8}\,|\partial_z h|_2^2 \\
&\leqslant -\frac{k^6}{8}\left(k^2 - \delta 2\right)|h|_2^2 + k\,|h|_2^2\,|\partial_z h|_2^2 - \frac{k^4}{8}\,|\partial_z h|_2^2.
\end{aligned}
$$

The discriminant of the second member is $\Delta_3 = k^2 - \frac{1}{16}k^{10}\left(k^2 - \delta 2\right)$, so that for k large enough, $\Delta_3 < 0$ and $Q_3 \leqslant 0$.

4. For $k\sqrt{\lambda}\,\mathrm{Im}\int_{\Omega_z} v \partial_z \overline{h}$:

$$
\begin{aligned}
Q_4 &= -\frac{1}{4}k^2\,|v|_2^2 - Ck\sqrt{\lambda}\,\mathrm{Im}\int_{\Omega_z} v \partial_z \overline{h} - \frac{\lambda}{4}k^2\left(k^2 - \delta_2\right)|\partial_z h|_2^2 \\
&\leqslant -\frac{1}{4}k^2\,|v|_2^2 + Ck\sqrt{\lambda}\,|v|_2^2\,|\partial_z h|_2^2 - \frac{\lambda}{4}k^2\left(k^2 - \delta_2\right)|\partial_z h|_2^2.
\end{aligned}
$$

The discriminant of the right member is $\Delta_4 = \lambda k^2\left(C^2 - \frac{1}{4}k^2\left(k^2 - \delta_2\right)\right)$. So for k sufficiently large, $\Delta_4 < 0$ and $Q_4 \leqslant 0$.

5. For $k\sqrt{\lambda}\,\mathrm{Im}\int_{\Omega_z}\partial_z^2 h \partial_z \overline{h}$:

$$
\begin{aligned}
Q_5 &= -\frac{\lambda}{2}\,|\partial_z^2 h|_2^2 - Ck\sqrt{\lambda}\,\mathrm{Im}\int_{\Omega_z}\partial_z^2 h \partial_z \overline{h} - \frac{1}{4}k^4\,|\partial_z h|_2^2 \\
&\leqslant -\frac{\lambda}{2}\,|\partial_z^2 h|_2^2 + Ck\sqrt{\lambda}\,|\partial_z h|_2^2\,|\partial_z^2 h|_2^2 - \frac{1}{4}k^4\,|\partial_z h|_2^2.
\end{aligned}
$$

The discriminant of the second member is $\Delta_5 = \lambda k^2\left(C^2 - \frac{1}{2}k^2\right)$. So for k large enough, $\Delta_5 < 0$ and $Q_5 \leqslant 0$.

6. For $k^3\sqrt{\lambda}\left(k^2 - \delta_2\right)\mathrm{Im}\int_{\Omega_z} h \partial_z \overline{h}$:

$$
\begin{aligned}
Q_6 &= \left(k^2 - \delta_2\right)\left(-\frac{1}{4}k^6\,|h|_2^2 + Ck^3\sqrt{\lambda}\,\mathrm{Im}\int_{\Omega_z} h \partial_z \overline{h} - \frac{\lambda}{4}k^2\,|\partial_z h|_2^2\right) \\
&\leqslant k^2\left(k^2 - \delta_2\right)\left(-\frac{1}{4}k^4\,|h|_2^2 + Ck\sqrt{\lambda}\,|h|_2^2\,|\partial_z h|_2^2 - \frac{\lambda}{4}\,|\partial_z h|_2^2\right).
\end{aligned}
$$

The discriminant of the second member is $\Delta_6 = \lambda k^2 \left(C^2 - \frac{1}{4}k^2\right)$. So for k sufficiently large, $\Delta_6 < 0$ and $Q_6 \leqslant 0$.

Then, using the fact that for k large enough, Q_1, Q_2, Q_3, Q_4, Q_5 and Q_6 are non positive, we have

$$\tilde{N}'(t) + k^2 \left|v\right|_2^2 + \left|\partial_z v\right|_2^2 + \left(2Ek^2 + k^6\right)\left(k^2 - \delta_2\right)\left|h\right|_2^2 + \frac{Ck^2}{\sqrt{\lambda}}\left|h\right|_2^2$$
$$+ \left(2E + k^4 + \lambda k^2\left(k^2 - \delta_2\right)\right)\left|\partial_z h\right|_2^2 + \lambda\left|\partial_z^2 h\right|_2^2 \leqslant 0.$$

As $\frac{1}{2}k^2 - \delta_2 > 0$, for k large enough,

$$\begin{aligned}
\tilde{N}'(t) &\leqslant -\frac{k^2}{2}\left(\left|v\right|_2^2 + \left(Ek^2 + \frac{1}{2}k^6\right)\left|h\right|_2^2 + \lambda k^2\left|\partial_z h\right|_2^2\right) \\
&\quad -\frac{1}{2}k^2\left|v\right|_2^2 - \left|\partial_z v\right|_2^2 - \frac{1}{4}k^4\left(2E - k^4\right)\left|h\right|_2^2 \\
&\quad -\frac{Ck^2}{\sqrt{\lambda}}\left|h\right|_2^2 - \left(2E - k^4\right)\left|\partial_z h\right|_2^2 - \lambda\left|\partial_z^2 h\right|_2^2.
\end{aligned}$$

Let k large enough such that $k^2 > 1$ and $k^2\left(2E + k^4\right) > 2\left(E + k^4\right)$. Then there exists $K > 0$ and μ (depending on E and C) such that, for all $\lambda < \Lambda$ and for all $k > K$ we have

$$\tilde{N}'_k(t) + \mu\left(k^2\left|v\right|_2^2 + \left|\partial_z v\right|_2^2 + \left(k^4 + k^8\right)\left|h\right|_2^2 + \left|\partial_z h\right|_2^2 + k^4\left|\partial_z h\right|_2^2\right) + \frac{Ck^2}{\sqrt{\lambda}}\left|h\right|_2^2 + \lambda\left|\partial_z^2 h\right|_2^2 \leqslant -\frac{K}{2}N_k(t). \quad (23)$$

Hence the proof is finished by choosing $A_1 = \frac{K}{2}$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

Remark 3.7. Let note that this result has a physical interpretation: high frequencies are always stable, so that the structures that are created by the instability and that are observed in the experiments have a size that corresponds to the small waves numbers.

3.1.3. *Low frequencies*

In this section, we study the frequencies k between 1 and K, the case $k = 0$ being studied in the next paragraph. Here, the viscous terms can not control the non negative terms of (21). We however have an *a priori* estimates thanks to the term of order $\lambda^{-\frac{1}{2}}$. The method is strictly the same: we use the same property of quadratic forms as in the previous lemma.

Lemma 3.8. *There exists $\lambda_0 > 0$ and $A_2 > 0$ depending on E and C such that, for all λ in $(0, \lambda_0)$, for all k between 1 and K, $\tilde{N}'_k(t) \leqslant -A_2 N_k(t)$.*

Proof. As in the previous lemma, we control each non negative term by negative terms:

1. For $\mathrm{Re}\int_{\Omega_z} h\overline{v}$:

$$\begin{aligned}
Q_1 &= -\frac{k^2}{4}\left|v\right|_2^2 + \left(\delta_1 k^2 + E\right)\mathrm{Re}\int_{\Omega_z} h\overline{v} - \frac{Ck^2}{16\sqrt{\lambda}}\left|h\right|_2^2 \\
&\leqslant -\frac{k^2}{4}\left|v\right|_2^2 + \left(\delta_1 k^2 + E\right)\left|v\right|_2\left|h\right|_2 - \frac{Ck^2}{16\sqrt{\lambda}}\left|h\right|_2^2.
\end{aligned}$$

The second member is a quadratic form and its discriminant is $\Delta_1 = \left(\delta_1 k^2 + E\right)^2 - \frac{Ck^4}{16\sqrt{\lambda}}$. So for λ small enough ($\lambda < \Lambda$, Λ fixed in Prop. 3.6) and for all $k \leqslant K$, $\Delta_1 < 0$ and $Q_1 \leqslant 0$.

2. For $\mathrm{Re}\int_{\Omega_z}\partial_z h\partial_z\overline{v}$:

$$
\begin{aligned}
Q_2 &= -\frac{1}{2}\left|\partial_z v\right|_2^2 + (\lambda - 1)\,\mathrm{Re}\int_{\Omega_z}\partial_z h\partial_z\overline{v} - \frac{E}{8}\left|\partial_z h\right|_2^2 \\
&\leqslant -\frac{1}{2}\left|\partial_z v\right|_2^2 + (\lambda - 1)\left|\partial_z h\right|_2\left|\partial_z v\right|_2 - \frac{E}{8}\left|\partial_z h\right|_2^2 .
\end{aligned}
$$

The second member is a quadratic form and its discriminant is $\Delta_2 = (\lambda - 1)^2 - \frac{E}{4}\xrightarrow[\lambda\to 0]{}1 - \frac{E}{4} < 0$, by choice of E. So for λ small enough and for all $k\leqslant K$, $\Delta_2 < 0$ and $Q_2\leqslant 0$.

3. For $\mathrm{Im}\int_{\Omega_z} h\partial_z\overline{h}$:

$$
\begin{aligned}
Q_3 &= -\frac{Ck^2}{16\sqrt{\lambda}}\left|h\right|_2^2 + k\,\mathrm{Im}\int_{\Omega_z} h\partial_z\overline{h} - \frac{E}{8}\left|\partial_z h\right|_2^2 \\
&\leqslant -\frac{Ck^2}{16\sqrt{\lambda}}\left|h\right|_2^2 + k\left|h\right|_2^2\left|\partial_z h\right|_2^2 - \frac{E}{8}\left|\partial_z h\right|_2^2 .
\end{aligned}
$$

The second member is a quadratic form and its discriminant is $\Delta_3 = k^2\left(1 - \frac{CE}{32\sqrt{\lambda}}\right)$, so that for all $k\leqslant K$ and for λ small enough, $\Delta_3 < 0$ and $Q_3\leqslant 0$.

4. For $k\sqrt{\lambda}\,\mathrm{Im}\int_{\Omega_z} v\partial_z\overline{h}$:

$$
\begin{aligned}
Q_4 &= -\frac{1}{4}k^2\left|v\right|_2^2 - Ck\sqrt{\lambda}\,\mathrm{Im}\int_{\Omega_z} v\partial_z\overline{h} - \frac{E}{4}\left|\partial_z h\right|_2^2 \\
&\leqslant -\frac{1}{4}k^2\left|v\right|_2^2 + Ck\sqrt{\lambda}\left|v\right|_2^2\left|\partial_z h\right|_2^2 - \frac{E}{4}\left|\partial_z h\right|_2^2 .
\end{aligned}
$$

The second member is a quadratic form and its discriminant is $\Delta_4 = k^2\left(C^2\lambda - \frac{E}{4}\right)$, so that for all $k\leqslant K$ and for λ small enough, $\Delta_4 < 0$ and $Q_4\leqslant 0$.

5. For $k\sqrt{\lambda}\,\mathrm{Im}\int_{\Omega_z}\partial_z^2 h\partial_z\overline{h}$:

$$
\begin{aligned}
Q_5 &= -\frac{\lambda}{2}\left|\partial_z^2 h\right|_2^2 - Ck\sqrt{\lambda}\,\mathrm{Im}\int_{\Omega_z}\partial_z^2 h\partial_z\overline{h} - \frac{1}{8}k^4\left|\partial_z h\right|_2^2 \\
&\leqslant -\frac{\lambda}{2}\left|\partial_z^2 h\right|_2^2 + Ck\sqrt{\lambda}\left|\partial_z h\right|_2^2\left|\partial_z^2 h\right|_2^2 - \frac{1}{8}k^4\left|\partial_z h\right|_2^2 .
\end{aligned}
$$

The second member is a quadratic form and its discriminant is $\Delta_5 = \lambda k^2\left(C^2 - \frac{1}{4}k^2\right)$ (as we choose $C < \frac{1}{2}$), so that for all $k\leqslant K$ and for λ small enough, $\Delta_5 < 0$ and $Q_5\leqslant 0$.

6. For $k^3\sqrt{\lambda}\left(k^2 - \delta_2\right)\mathrm{Im}\int_{\Omega_z} h\partial_z\overline{h}$:

$$
\begin{aligned}
Q_6 &= -\frac{1}{8}k^4\left|\partial_z h\right|_2^2 + Ck^3\sqrt{\lambda}\left(k^2 - \delta_2\right)\mathrm{Im}\int_{\Omega_z} h\partial_z\overline{h} - \frac{Ck^2}{16\sqrt{\lambda}}\left|h\right|_2^2 \\
&\leqslant -\frac{1}{8}k^4\left|\partial_z h\right|_2^2 + Ck^3\sqrt{\lambda}\left(k^2 - \delta_2\right)\left|h\right|_2^2\left|\partial_z h\right|_2^2 - \frac{Ck^2}{16\sqrt{\lambda}}\left|h\right|_2^2 .
\end{aligned}
$$

The second member is a quadratic form and its discriminant is

$$\Delta_6 = C^2 \lambda k^6 \left(k^2 + \delta_2\right)^2 - \frac{Ck^6}{32\sqrt{\lambda}} \leqslant C^2 \lambda K^6 \left(K^2 + \delta_2\right)^2 - \frac{C}{32\sqrt{\lambda}},$$

so that for all $k \leqslant K$ and for λ small enough, $\Delta_6 < 0$ and $Q_6 \leqslant 0$.

By using the fact that, for λ small enough and for all $k \leqslant K$, $Q_1, \ldots, Q_6$ are non positive, we have

$$\tilde{N}_k'(t) \leqslant -k^2 |v|_2^2 - |\partial_z v|_2^2 - 2k^2 \left(k^2 - \delta_2\right) \left(E + k^4\right) |h|_2^2 - \frac{3Ck^2}{4\sqrt{\lambda}} |h|_2^2$$
$$- \left(E + \frac{3}{2}k^4\right) |\partial_z h|_2^2 - 2\lambda k^2 \left(k^2 - \delta_2\right) |\partial_z h|_2^2 - \lambda \left|\partial_z^2 h\right|_2^2.$$

It remains some terms (in $|h|_2^2$ and in $|\partial_z h|_2^2$) which are not necessarily negative. As usual, we control them by using the term of order $\lambda^{-\frac{1}{2}}$:

For λ small enough, we have $\frac{C}{8\sqrt{\lambda}} > 2E\delta_2 + 2\delta_2 K^4$, then, for all $k \leqslant K$,

$$2E\delta_2 k^2 + 2\delta_2 k^6 - \frac{Ck^2}{8\sqrt{\lambda}} \leqslant 0.$$

For λ small enough, we have $2\lambda\delta_2 < \frac{1}{2}$, then, for all $k \leqslant K$, $2\lambda\delta_2 k^2 - \frac{1}{2}\gamma_2 k^4 < 0$.

Hence, there exists $\mu > 0$ such that for λ small enough, for all $k \leqslant K$,

$$\tilde{N}_k'(t) + \mu \left(k^2 |v|_2^2 + |\partial_z v|_2^2 + \left(k^4 + k^8 + \frac{Ck^2}{2\sqrt{\lambda}}\right) |h|_2^2 + \left(1 + k^4\right) |\partial_z h|_2^2\right) + \lambda \left|\partial_z^2 h\right|_2^2 \leqslant$$
$$- \frac{1}{2}k^2 |v|_2^2 - Ek^4 |h|_2^2 - k^8 |h|_2^2 - \frac{Ck^2}{8\sqrt{\lambda}} |h|_2^2 - 2\lambda k^4 |\partial_z h|_2^2.$$

It is easy to show that, for λ small enough and for all k in $\{1, \ldots, K\}$, the right member of this inequality is less or equal than $-A_2 N_k(t)$, for some $A_2 > 0$ depending on E, C and λ_0.

Then, we have the following result:

$$\tilde{N}_k'(t) + \mu \left(k^2 |v|_2^2 + |\partial_z v|_2^2 + \left(k^4 + k^8 + \frac{Ck^2}{2\sqrt{\lambda}}\right) |h|_2^2 + \left(1 + k^4\right) |\partial_z h|_2^2\right) + \lambda \left|\partial_z^2 h\right|_2^2 \leqslant -A_2 N_k(t), \qquad (24)$$

which obviously concludes the proof. $\square$

3.1.4. *The mean value*

As N_0 (and of course $\tilde{N}_0$) has a different form than N_k, for $k \neq 0$, we study this case here and we have the following lemma:

Lemma 3.9. *There exists $\lambda_0 > 0$ and $A_3 > 0$ such that, for all λ in $(0, \lambda_0)$, $\tilde{N}_0'(t) \leqslant -A_3 N_0(t)$.*

Proof. This result is based on the symmetry of the system. Indeed, the evolution of $|v|_2$ and $|\partial_z h|_2$ is given by

$$\frac{1}{2}\partial_t |v|_2^2 = -|\partial_z v|_2^2 - \mathrm{Re} \int_{\Omega_z} \partial_z h \partial_z \overline{v},$$

and

$$\frac{1}{2}\partial_t \left|\partial_z h\right|_2^2 = \mathrm{Re} \int_{\Omega_z} \partial_z h \partial_z \overline{v} - \left|\partial_z^2 h\right|_2^2.$$

It is obvious that by taking a linear combination, we eliminate the $\mathrm{Re} \int_{\Omega_z} \partial_z h \partial_z \overline{v}$ term, allowing us to control $|v|_2^2$ and $\left|\partial_z h\right|_2^2$ and the other terms of N_0. As v and h verify $|v|_2 \leqslant C_p \left|\partial_z v\right|_2$, $|h|_2 \leqslant C_p^2 \left|\partial_z^2 h\right|_2$ and $\left|\partial_z h\right|_2 \leqslant C_p \left|\partial_z^2 h\right|_2$, the choice of F and the fact that λ_0 can be as small as we want (here $\lambda_0 < 1$) imply that (22) becomes:

$$\tilde{N}_0'(t) + \frac{1}{2}\left|\partial_z v\right|_2^2 + E\left|\partial_z h\right|_2^2 + (\lambda+1)\left|\partial_z^2 h\right|_2^2 \leqslant -A_3 N_0(t), \tag{25}$$

where A_3 is a positif real. The proof is finished. $\qquad\square$

3.1.5. *Conclusion*

To establish the stability result, we need the following embedding lemma:

Lemma 3.10. *Let* $\mathrm{H}_1 \hookrightarrow \mathrm{H} \hookrightarrow \mathrm{H}_2$ *three hilbert spaces so that the first embedding is compact. Then for any* $p > 1$, *the embedding*

$$\left\{ f \in \mathrm{L}^\infty(0,T;\mathrm{H}_1), \ \frac{\mathrm{d}f}{\mathrm{d}t} \in \mathrm{L}^p(0,T;\mathrm{H}_2) \right\} \hookrightarrow \mathrm{C}^0([0,T];H)$$

is compact.

(For a proof, see [12], p. 85.) This lemma will also be used in the proof of Theorems 2.2 and 2.4.

Proof of Theorem 3.2. By Proposition 3.6, Lemmas 3.8 and 3.9, choosing $A = \min(A_1, A_2, A_3)$ implies that for all λ in $(0, \lambda_0)$ and for all $k \geqslant 0$, then $\tilde{N}_k'(t) \leqslant -AN_k(t)$. By taking the conjugate expression of (15) and (16), we have the same result for $k < 0$ (the proof is exactly the same). The choice of C implies that $\tilde{N}_k(t)$ and $N_k(t)$ are equivalent. Then, by adding the last inequality for k in $\mathbb{Z}$ and by using the Parseval−Bessel's identity, one get $\tilde{N}'(t) \leqslant -A\tilde{N}(t)$.

More precisely, using inequalities (23–25), for some $B > 0$, we have

$$\tilde{N}'(t) + B\left(\left|\partial_x v\right|_2^2 + \left|\partial_z v\right|_2^2 + \left|\partial_x^2 h\right|_2^2 + \left|\partial_x^4 h\right|_2^2 + \frac{1}{\sqrt{\lambda}}\left|\partial_x h\right|_2^2 + \left|\partial_z h\right|_2^2 + \left|\partial_x^2 \partial_z h\right|_2^2 + \lambda\left|\partial_z^2 h\right|_2^2\right) \leqslant -A\tilde{N}(t). \tag{26}$$

This inequality will be also used to perform the limit in the Galerkin method for the global in time existence theorem.

As the system is linear, the Galerkin method gives a sequence of approximated solution of (10−13), denoted (v_n, h_n), which verify

$$v_n \text{ is in a bounded set of } \mathrm{L}^\infty(\mathbb{R}_+; \mathrm{L}(\Omega)) \cap \mathrm{L}^2(\mathbb{R}_+; \mathrm{H}(\Omega)),$$
$$h_n \text{ is in a bounded set of } \mathrm{L}^\infty(\mathbb{R}_+; \mathrm{K}(\Omega)) \cap \mathrm{L}^2(\mathbb{R}_+; \mathrm{H}^2(\Omega) \cap \mathrm{H}(\Omega)).$$

Then there exists

$$v \in \mathrm{L}^\infty(\mathbb{R}_+; \mathrm{L}(\Omega)) \cap \mathrm{L}^2(\mathbb{R}_+; \mathrm{H}(\Omega))$$

and

$$h \in \mathrm{L}^\infty(\mathbb{R}_+; \mathrm{K}(\Omega)) \cap \mathrm{L}^2(\mathbb{R}_+; \mathrm{H}^2(\Omega) \cap \mathrm{H}(\Omega))$$

such that

$$v_n \ \rightharpoonup \ v \text{ weak * in } \mathrm{L}^\infty(\mathbb{R}_+; \mathrm{L}(\Omega)) \text{ and weakly in } \mathrm{L}^2(\mathbb{R}_+; \mathrm{H}(\Omega)),$$
$$h_n \ \rightharpoonup \ h \text{ weak * in } \mathrm{L}^\infty(\mathbb{R}_+; \mathrm{K}(\Omega)) \text{ and}$$
$$\text{weakly in } \mathrm{L}^2(\mathbb{R}_+; \mathrm{H}^2(\Omega) \cap \mathrm{H}(\Omega)).$$

254 V. TORRI

The system being linear, it is straightforward to conclude that (v, h) is a solution. We now deal with the fact that $(v, h)(t)$ tends to 0. The proof is the same for v and h, so we only show the result for v. It is easy to see that $\partial_t v_n$ and $\partial_t h_n$ are bounded in $\mathrm{L}^2\left(0, T ; \mathrm{H}^{-1}\right)$. So Lemma 3.10 implies that

$$(v_n, h_n) \rightarrow (v, h) \text{ in } \mathrm{C}^0\left([0, T] ; \mathrm{L}^2(\Omega) \times \mathrm{L}^2(\Omega)\right). \tag{27}$$

Let $\epsilon > 0$. There exists $T > 0$ such that, for all $t > T$,

$$\|v_n(t)\|_{\mathrm{L}(\Omega)} \leqslant N(0) \exp(-At) < \frac{\epsilon}{2}.$$

The fact that this *a priori* estimate is independant of n is the most important argument. Let $t > T$. Then (27) implies that

$$\sup_{[0, t+1]} \|v_n - v\|_{\mathrm{L}(\Omega)} \underset{n \rightarrow +\infty}{\longrightarrow} 0,$$

i.e. there exists n such that $\|v_n - v\|_{\mathrm{L}(\Omega)} < \frac{\epsilon}{2}$. So

$$\|v(t)\|_{\mathrm{L}(\Omega)} \leqslant \|v_n - v\|_{\mathrm{L}(\Omega)} + \|v_n(t)\|_{\mathrm{L}(\Omega)} \leqslant \frac{\epsilon}{2} + N(0) \exp(-At) \leqslant \epsilon.$$

$\square$

Remark 3.11. This result can be obtained without expandind v and h in Fourier series, but this method gives us a better understanding of the physical processes that are involved in the experiment:
  • The symmetry of the system controls the mean value:

$$\frac{1}{2} \partial_t \left(|V|_2^2 + |\partial_z H|_2^2\right) \leqslant -|\partial_z V|_2^2 - \left|\partial_z^2 H\right|_2^2,$$

   (*cf.* Lem. 3.9).
  • The instabilities can not have a size corresponding to large waves numbers (*cf.* Prop. 3.6).
  • Small frequencies are controlled by the skew-symmetric operator $\frac{z}{\lambda} \partial_x$ when λ is small (*cf.* Lem. 3.8), that is to say by the shear.

3.2. Construction of the semi-group

The aim of this paragraph is to prove Theorem 3.1. At this stage, we are not concerned with bounded in time solutions, so we do not try to apply the method described in Section 2.3.

Proof of Theorem 3.1. As usual, we work on some energy estimates. By computing $N(t) = |v|_2^2 + \epsilon_1 \left|\partial_x^2 h\right|_2^2 + |\partial_z h|_2^2$ with (10) and (11) we have

$$
\begin{aligned}
\frac{1}{2} N'(t) = & -|\partial_x v|_2^2 - |\partial_z v|_2^2 - \int_\Omega \partial_z h \partial_z v + \delta_1 \int_\Omega \partial_x h \partial_x v \\
& - \int_\Omega v \partial_x^4 h + \int_\Omega v \partial_x^4 h - \left|\partial_x^2 \partial_z h\right|_2^2 \\
& - \delta_2 \int_\Omega \partial_x^2 h \partial_x^4 h - \left|\partial_x^4 h\right|_2^2 + \frac{1}{\lambda} \int_\Omega \partial_x h \partial_z h \\
& - \int_\Omega v \partial_z^2 h - \left|\partial_z^2 h\right|_2^2 + \delta_2 \int_\Omega \partial_x^2 h \partial_z^2 h - \left|\partial_x^2 \partial_z h\right|_2^2.
\end{aligned}
$$

Here, C_2 will be a constant. Using Young's inequality, one obtains the following inequalities:

- $-\int_\Omega \partial_z h \partial_z v \leqslant \frac{1}{2} |\partial_z v|_2^2 + C_2 |\partial_z h|_2^2$;
- $\delta_1 \int_\Omega \partial_x h \partial_x v \leqslant \frac{1}{2} |\partial_x v|_2^2 + C_2 |\partial_x h|_2^2$;
- $-\delta_2 \int_\Omega \partial_x^2 h \partial_x^4 h \leqslant \frac{1}{2} |\partial_x^4 h|_2^2 + C_2 |\partial_x^2 h|_2^2$;
- $\frac{1}{\lambda} \int_\Omega \partial_x h \partial_z h \leqslant C_2 |\partial_x h|_2^2 + C_2 |\partial_z h|_2^2 \leqslant C_2 |\partial_x^2 h|_2^2 + C_2 |\partial_z h|_2^2$, by Poincaré's inequality;
- $-\int_\Omega v \partial_z^2 h \leqslant \frac{1}{4} |\partial_z^2 h|_2^2 + C_2 |v|_2^2$;
- $\delta_2 \int_\Omega \partial_x^2 h \partial_z^2 h \leqslant \frac{1}{4} |\partial_z^2 h|_2^2 + C_2 |\partial_x^2 h|_2^2$.

By taking and $A = \max\left(C_2, \frac{1}{2}\right)$, we get

$$N'(t) + |\partial_x v|_2^2 + |\partial_z v|_2^2 + |\partial_z^2 h|_2^2 + |\partial_x^2 \partial_z h|_2^2 + |\partial_x^4 h|_2^2 \leqslant A N(t) \tag{28}$$

which finishes the proof. $\qquad\square$

4. Nonlinear results

We deal now with *a priori* estimates of the nonlinear terms, for both local and global in time cases. This last one require bounds on

$$N(t) = |v|_2^2 + E |h|_2^2 + |\partial_x^2 h|_2^2 + \lambda |\partial_z h|_2^2 + F\left(|V|_2^2 + |\partial_z H|_2^2\right).$$

Here is the result:

Theorem 4.1. *Let $E > 4$, $F > \max\left(\frac{E}{\pi^4}, \frac{E}{\pi^2}\right)$ and $C < \min\left(\frac{1}{2}, \frac{2}{C_p}\right)$. Then there exists λ_0, $A > 0$, $B > 0$ (all depending on E, F, C and the parameters of the system) and Q a 2-valuated polynomial with positive coefficients, so that for all (v, h) being a solution of the problem (3, 4) defined on $[0, T)$, for all t in $[0, T)$ and for all λ in $(0, \lambda_0)$,*

$$\tilde{N}'(t) + B\left(|\partial_x v|_2^2 + |\partial_z v|_2^2 + |\partial_x^2 h|_2^2 + |\partial_x^4 h|_2^2 + |\partial_z h|_2^2 + |\partial_x^2 \partial_z h|_2^2 + \lambda |\partial_z^2 h|_2^2\right) \leqslant -A\tilde{N}(t) + Q\left(\tilde{N}(t)\right).$$

The local in time case can be interpreted as a particular form of the global one, as λ does not play an important role and can be considered just as a physical parameter. Let

$$N(t) = |v|_2^2 + |\partial_x^2 h|_2^2 + |\partial_z h|_2^2.$$

Then we have the

Theorem 4.2. *Let $T > 0$. Suppose that (v, h) is a solution of the problem (3, 4). Then for all $\lambda > 0$, there exists $A > 0$ and Q a 1-valuated polynomial with positive coefficients, so that for all $t \in [0, T)$,*

$$N'(t) + A\left(|\partial_x v|_2^2 + |\partial_z v|_2^2 + |\partial_x^4 h|_2^2 + |\partial_x^2 \partial_z h|_2^2 + |\partial_z^2 h|_2^2\right) \leqslant Q(N(t)).$$

Hence, we devote this section mainly to the proof of Theorem 4.1 which needs some inequalities we give in the next paragraph.

4.1. **Some inequalities**

The inequalities used to prove those theorems are Propositions 4.5 and 4.6 and its Corollary 4.7. Their proof are based on the Young's inequality (if $p, q > 1$, $p^{-1} + q^{-1} = 1$, $a, b \geqslant 0$ and $\epsilon > 0$ then $ab \leqslant \epsilon a^p + b^q/(4\epsilon)$) and the two following lemmas. Proposition 4.9 will be used later, to give compactness to the Galerkin sequence.

Lemma 4.3. *For all $p > 1$, there exists $C > 0$ such that, for all $u \in \mathrm{C}^\infty(\overline{B})$, 2-periodic in x, vanishing at $z = 0, 1$,*

$$|u|_{2p}^{2p} \leqslant C \, |u|_{2p-2}^{2p-2} \, |\nabla u|_2^2 \, .$$

The proof is based on the one given by Ladyzhenskaya in [9], which can be easily adapted for periodic functions.

Due to the anisotropic nature of the energy involved in the studied system (1, 2), we need the following Sobolev-type inequality:

Lemma 4.4. *There exists $C > 0$ such that for all $u \in \mathrm{C}^\infty(\overline{B})$, 2-periodic in x, vanishing at $z = 0, 1$ and for all $p \in [2, 6]$,*

$$|\partial_x u|_p \leqslant C \left(|\partial_x u|_2 + |\partial_z u|_2 + \left|\partial_x^2 u\right|_2 \right) .$$

For a proof, *cf.* [1] (p. 323). This kind of inequality is also used for the K.P. equation in [4].

Proposition 4.5. *For all $C_1 > 0$, there exists $C_2 > 0$ such that for all $u \in \mathrm{C}^\infty\left(\overline{\Omega}\right)$, 2-periodic in x and vanishing at $z = 0, 1$,*

$$|\partial_z u|_4^3 \leqslant C_2 \, |\partial_z u|_2^6 + C_1 \left(\left|\partial_x^2 u\right|_2^2 + \left|\partial_z^2 u\right|_2^2 \right) .$$

Proof. First, the result of the Lemma 4.3 remains for $\partial_z u$ as its mean value with respect to z is equal to zero. Then it suffices to reuse the arguments used for the x variable. The Lemma 4.3 and Young's inequality leads to

$$|\partial_z u|_4^3 \leqslant C_2 \, |\partial_z u|_2^6 + C_1 \left(|\partial_x \partial_z u|_2^2 + \left|\partial_z^2 u\right|_2^2 \right) .$$

$$|\partial_x \partial_z u|_2^2 = \int_\Omega \partial_x^2 u \partial_z^2 u \leqslant \frac{1}{2} \left(\left|\partial_x^2 u\right|_2^2 + \left|\partial_z^2 u\right|_2^2 \right) ,$$

by Cauchy−Schwartz's inequality. This concludes the proof. $\qquad\square$

Proposition 4.6. *There exists $C > 0$ such that for all $u \in \mathrm{C}^\infty\left(\overline{\Omega}\right)$, 2-periodic in x and vanishing at $z = 0, 1$ and for all $p \in [2, 6]$,*

$$|\partial_x u|_p \leqslant C \left(|\partial_z u|_2 + \left|\partial_x^2 u\right|_2 \right) .$$

Proof. We use the Lemma 4.4 and the Poincaré's inequality. $\qquad\square$

Corollary 4.7. *There exists $C > 0$ such that for all $u \in \mathrm{C}^\infty\left(\overline{\Omega}\right)$, 2-periodic in x and vanishing at $z = 0, 1$ and for all $p \in [2, 6]$ and $q \geqslant 0$,*

$$|\partial_x u|_p^{2q} \leqslant C \left(|\partial_z u|_2^2 + \left|\partial_x^2 u\right|_2^2 \right)^q .$$

Proof. We use the Proposition 4.6 and the equivalence of norms in finite dimension. $\qquad\square$

Remark 4.8. By changing the constant C of this corollary, we can adjust the coefficients of $|\partial_z u|_2^2$ and $\left|\partial_x^2 u\right|_2^2$.

Proposition 4.9. *There exists $C > 0$ such that for all $u \in \mathrm{C}^\infty\left(\overline{\Omega}\right)$, 2-periodic in x and vanishing at $z = 0, 1$,*

$$\left|\partial_x^2 u\right|_4 \leqslant C \, |\partial_x u|_4^{\frac{1}{2}} \, \left|\partial_x^2 u\right|_2^{\frac{1}{16}} \, \left|\partial_x^4 u\right|_2^{\frac{1}{16}} \, \left|\partial_x^3 u\right|_6^{\frac{3}{8}} \, .$$

Proof. Integrating by parts and using Hölder's inequality give

$$\left|\partial_x^2 u\right|_4^4 = -3 \int_\Omega \partial_z u \partial_x^3 u \left|\partial_x^2 u\right|^2 \leqslant 3 \, |\partial_x u|_4 \, \left|\partial_x^3 u\right|_4 \, \left|\partial_x^2 u\right|_4^2 \, .$$

Classical results on interpolation inequalities give $\left|\partial_x^3 u\right|_4 \leqslant \left|\partial_x^3 u\right|_2^{\frac{1}{4}} \, \left|\partial_x^3 u\right|_6^{\frac{3}{4}}$. By integration by parts and using Cauchy−Schwartz's inequality we have $\left|\partial_x^3 u\right|_2 \leqslant \left|\partial_x^2 u\right|_2^{\frac{1}{2}} \, \left|\partial_x^4 u\right|_2^{\frac{1}{2}}$. This concludes the proof, with $C = \sqrt{3}$. $\qquad\square$

4.2. **Nonlinear** *a priori* **estimates for global existence**

We complete in this paragraph the linear *a priori* estimate found in Section 3 (*cf.* Th. 3.2), by adding the one of nonlinear terms.

Basically, we will bound all nonlinear terms by polynomial in N, which will lead to the estimate of Theorem 4.1:

$$\tilde{N}'(t) + B\left(|\partial_x v|_2^2 + |\partial_z v|_2^2 + \left|\partial_x^2 h\right|_2^2 + \left|\partial_x^4 h\right|_2^2 + |\partial_z h|_2^2 + \left|\partial_x^2 \partial_z h\right|_2^2 + \lambda \left|\partial_z^2 h\right|_2^2\right) \leqslant -A\tilde{N}(t) + Q\left(\tilde{N}(t)\right)$$

where Q is the polynomial.

Remark 4.10. • It is equivalent to work on $N(t)$ or $\tilde{N}(t)$ as they are equivalent.
  • The fact that Q is a 2-valuated polynomial is quite important: the linear part of the inequality of Theorem 4.1 as a negative coefficient. This is this fact that will give existence of the solution on $\mathbb{R}_+$.
  • The others terms on the left-hand-side of this inequality are necessary to prove the convergence of the Galerkin sequence (see 5.3).

Theorem 4.1 is easily proved if the three nonlinear terms of (3) are bounded in a suitable way. The following proposition gives these bounds.

Proposition 4.11. *Let $C_1 > 0$. Then there exists $C_2 > 0$ such that, for all (v, h) being a solution of the problem (3, 4) and for $\mu < \lambda^2$ and $\nu^2 < \lambda^{\frac{5}{2}}$,*

$$-\frac{\mu}{\lambda}\int_\Omega \partial_x v \partial_x h \partial_z h - \frac{\mu}{\lambda}\int_\Omega \partial_z v \left(\partial_x h\right)^2 + \frac{\nu^2}{\lambda}\int_\Omega \partial_x v \left(\partial_x h\right)^3 \leqslant C_1\left(|\partial_x v|_2^2 + |\partial_z v|_2^2 + \left|\partial_x^2 h\right|_2^2 + \lambda\left|\partial_z^2 h\right|_2^2\right)$$

$$+ C_2\left(\left(\left|\partial_x^2 h\right|_2^2 + \lambda\left|\partial_z h\right|_2^2\right)^2 + \left(\left|\partial_x^2 h\right|_2^2 + \lambda\left|\partial_z h\right|_2^2\right)^3\right).$$

Proof. We study separately the three terms of the left-hand-side.
  • $\underline{\text{The } \int_\Omega \partial_x v \partial_x h \partial_z h \text{ term:}}$
    Let $\mu = \lambda^2$. $I = \frac{\mu}{\lambda}\int_\Omega \partial_x v \partial_x h \partial_z h \leqslant \lambda |\partial_x v|_2 |\partial_z h|_4 |\partial_x h|_4$. By Proposition 4.6 and Young's inequality, we have

$$I \leqslant C_1 |\partial_x v|_2^2 + \lambda^{\frac{3}{2}} |\partial_z h|_4^3 + C_2\lambda^3 |\partial_x h|_4^6.$$

With Proposition 4.5 and Corollary 4.7 we obtain:

$$I \leqslant C_1\left(|\partial_x v|_2^2 + \left|\partial_x^2 h\right|_2^2 + \lambda\left|\partial_z^2 h\right|_2^2\right) + C_2\left(\lambda^3 |\partial_z h|_2^6 + \left(\left|\partial_x^2 h\right|_2^2 + \lambda\left|\partial_z h\right|_2^2\right)^3\right).$$

So, with the equivalence of norms in finite dimension:

$$I \leqslant C_1\left(|\partial_x v|_2^2 + \left|\partial_x^2 h\right|_2^2 + \lambda\left|\partial_z^2 h\right|_2^2\right) + C_2\left(\left|\partial_x^2 h\right|_2^2 + \lambda|\partial_z h|_2^2\right)^3.$$

  • $\underline{\text{The } \int_\Omega \partial_z v \left(\partial_x h\right)^2 \text{ term:}}$
    Let $\mu = \lambda^2$.

$$I = \frac{\mu}{\lambda}\int_\Omega \partial_z v \left(\partial_x h\right)^2 \leqslant \lambda |\partial_z v|_2 |\partial_x h|_4^2 \leqslant C_1 |\partial_z v|_2^2 + C_2\lambda^2 |\partial_x h|_4^4,$$

by Young's inequality. By Corollary 4.7, we obtain:

$$I \leqslant C_1 |\partial_z v|_2^2 + C_2\left(\lambda |\partial_z h|_2^2 + \left|\partial_x^2 h\right|_2^2\right)^2.$$

- The $\int_\Omega \partial_x v \, (\partial_x h)^3$ term:

  Let $\nu^2 = \lambda^{\frac{5}{2}}$.

$$I = \frac{\nu^2}{\lambda} \int_\Omega \partial_x v \, (\partial_x h)^3 \leqslant \lambda^{\frac{3}{2}} \, |\partial_x v|_2 \, |\partial_x h|_6^3 \leqslant C_1 \, |\partial_x v|_2^2 + C_2 \lambda^3 \, |\partial_x h|_6^6 \,,$$

by Young's inequality. With Corollary 4.7 we have:

$$I \leqslant C_1 \, |\partial_x v|_2^2 + C_2 \left(\left|\partial_x^2 h\right|_2^2 + \lambda \, |\partial_z h|_2^2 \right)^3 .$$

$\square$

Proof of Theorem 4.1. Thanks to Theorem 3.2, we get (26) and A_1, B, and λ_0. Let $C_1 < B$. Proposition 4.11 gives $C_2 > 0$ such that nonlinear terms are bounded by $C_2 \left(N^2 + N^3 \right)$. So Theorem 4.1 is proved with $Q = C_2 \left(X^2 + X^3 \right)$. $\square$

4.3. **Nonlinear** *a priori* **estimates for local existence**

The result for the local case needs few modifications. As it was said, λ does not play any important role and is considered as a physical parameter. The proof is quite simple:

Proof of Theorem 4.2. The conclusion of Proposition 4.11 can be reused to obtain the result in combination to the inequality (28) of Theorem 3.1. $\square$

5. Proof of the global existence

In this section, we apply the Galerkin method (*cf.* [10]) to prove Theorem 2.2. This proof is completed in 4 points:

- we found a sequence of solutions of approximated PDE's of (3, 4). These solutions are denoted by (v_n, h_n);
- we use *a priori* estimates found in Section 4 with (v_n, h_n) to show that v_n and h_n are in a bounded set of some spaces;
- we show then the existence of a limit of a subsequence of (v_n, h_n), denoted (v, h), as well as their spatial derivatives. The difficult point is to show the strong convergence of nonlinecar terms;
- we conclude by the fact that (v, h) is a solution of (3, 4).

5.1. **Approximation of the solution**

Let $e_{n,p} = \sin(n\pi z) \exp(ip\pi x)$ the usual complete orthonormal basis of $L^2(\Omega)$, eigenvectors of Δ with boundary conditions used in this paper: periodic in x and vanishing at $z = 0, 1$. For sake of simplicity, these eigenvectors will be indexed by $\mathbb{N}$. Let $n \in \mathbb{N}$, $E_n = \mathrm{Span}(e_0, \ldots, e_n)$ and Π_n the orthogonal projector on E_n. Let $v_n = \sum_{i=0}^n a_i(t) e_i$, $h_n = \sum_{i=0}^n b_i(t) e_i$, where a_i and b_i are smooth real valued functions, $v_0 \in L(\Omega)$ and $h_0 \in K(\Omega)$.

So, we can note $\Pi_n(v_0) = \sum_{i=0}^n a_i^0 e_i$ and $\Pi_n(h_0) = \sum_{i=0}^n b_i^0 e_i$, where a_i^0 and b_i^0 are real constants.

We consider the Galerkin approximation of (3) and (4):

$$\partial_t v_n + \frac{1}{\lambda}\Pi_n\left(z\partial_x v_n\right) = \Delta v_n + \partial_z^2 h_n - \delta_1\partial_x^2 h_n - \partial_x^4 h_n$$
$$+ \Pi_n\left(\frac{\mu}{\lambda}\partial_x\left(\partial_x h_n \partial_z h_n\right) + \frac{\mu}{\lambda}\partial_z\left(\partial_x h_n\right)^2 + \frac{\nu^2}{\lambda}\partial_x\left(\partial_x h_n\right)^3\right) \tag{29}$$

$$\partial_t h_n + \frac{1}{\lambda}\Pi_n\left(z\partial_x h_n\right) = v_n + \partial_z^2 h_n - \delta_2\partial_x^2 h_n + \partial_x^4 h_n, \tag{30}$$

with the following initial conditions:

$$v_n\left(0\right) = \Pi_n\left(v_0\right) \tag{31}$$

and

$$h_n\left(0\right) = \Pi_n\left(h_0\right). \tag{32}$$

Let $j \in \{0,\ldots,n\}$; we multiply (29) and (30) by e_j, then, since $(e_n)_{n\in\mathbb{N}}$ is an orthonormal system and Π_n is self-adjoint,

$$\partial_t a_j + \frac{1}{\lambda}\sum_{k=0}^{n} a_k\int_\Omega z e_j \partial_x e_k = \sum_{k=0}^{n} a_k\int_\Omega e_j\partial_x^2 e_k + \sum_{k=0}^{n} a_k\int_\Omega e_j\partial_z^2 e_k$$
$$+ \sum_{k=0}^{n} b_k\int_\Omega e_j\partial_z^2 e_k - \delta_1\sum_{k=0}^{n} b_k\int_\Omega e_j\partial_x^2 e_k$$
$$+ \sum_{k=0}^{n} b_k\int_\Omega e_j\partial_x^4 e_k - F\left(\frac{\mu}{\lambda},\frac{\mu}{\lambda},\frac{\nu}{\lambda}\right)(X) \tag{33}$$

$$\partial_t b_j + \frac{1}{\lambda}\sum_{k=0}^{n} b_k\int_\Omega z e_j \partial_x e_k = a_j + \sum_{k=0}^{n} b_k\int_\Omega e_j\partial_z^2 e_k - \delta_2\sum_{k=0}^{n} b_k\int_\Omega e_j\partial_x^2 e_k$$
$$+ \sum_{k=0}^{n} b_k\int_\Omega e_j\partial_x^4 e_k, \tag{34}$$

where X is the column vector $(b_0\ldots b_n)$ and

$$F\left(r,s,t\right)(X) = \sum_{k,l=0}^{n} b_k b_l\left(r\int_\Omega \partial_x e_k \partial_z e_l \partial_x e_j + s\int_\Omega \partial_x e_k \partial_x e_l \partial_z e_j\right) + \sum_{k,l,m=0}^{n} b_k b_l b_m t\int_\Omega \partial_x e_k \partial_x e_l \partial_x e_m \partial_x e_j.$$

In the same way, we obtain the following initial conditions:

$$a_j\left(0\right) = a_j^0 \tag{35}$$

and

$$b_j\left(0\right) = b_j^0. \tag{36}$$

Hence, by Cauchy−Lipschitz's theorem, we can conclude that there is a unique maximal solution of (33–36), denoted by v_n and h_n according to the previous definitions, defined on $[0, t_n[$, and which verify $v_n(0) = \Pi_n(v_0)$ and $h_n(0) = \Pi_n(h_0)$.

5.2. *A priori* **estimates**

In this section, we show that v_n and h_n (built in the previous section) are defined on $\mathbb{R}$ provided the initial conditions are small enough, and are uniformly bounded in n. As in Section 3.1, we note V_n and H_n the mean value with respect to x of the respective functions v_n and h_n.

Proposition 5.1. *Let $E > 4$, $F > \max\left(\frac{E}{\pi^4}, \frac{E}{\pi^2}\right)$, v_0 in $\mathrm{L}(\Omega)$, h_0 in $\mathrm{K}(\Omega)$, and $N_n(t)$ the following quantity:*

$$N_n(t) = |v_n|_2^2 + E\,|h_n|_2^2 + \left|\partial_x^2 h_n\right|_2^2 + \lambda\,|\partial_z h_n|_2^2 + F\left(|V_n|_2^2 + |H_n|_2^2\right).$$

Then there exists $X_0 > 0$, $\lambda_0 > 0$ such that, if the quantity

$$N_0 = |v_0|_2^2 + \left|\partial_x^2 h_0\right|_2^2 + E\,|h_0|_2^2 + \lambda_0\,|\partial_z h_0|_2^2 + F\left(|V_0|_2^2 + |H_0|_2^2\right)$$

is less that X_0, then for all n in $\mathbb{N}$, N_n is defined on $\mathbb{R}$ and there exists $M > 0$ such that, for all λ in $(0, \lambda_0)$, for all $t > 0$ and for all n in $\mathbb{N}$, $N_n(t) \leqslant M N_0$.

This proposition needs two lemmas:

Lemma 5.2. *Let A a positive real, Q a 2-valuated polynomial with positive coefficients and X_0 the first positive root of $f(z) = -Az + Q(z)$. For all z_0 in $]0, X_0[$, let $z(t)$ be the maximal solution of the Cauchy problem*

$$\begin{cases} z'(t) = f(z(t)) \\ z(0) = z_0. \end{cases}$$

Then z is defined on $\mathbb{R}_+$, is decreasing and tends to 0 as t tends to $+\infty$.

Lemma 5.3. *For all n in $\mathbb{N}$, $N_n(0) \leqslant N_0$.*

Proof. The norm of Π_n is equal to 1. So, $|\Pi_n(v_0)|_2 \leqslant |v_0|_2$ and $|\Pi_n(h_0)|_2 \leqslant |h_0|_2$. As $\left[\Pi_n, \partial_x^2\right] = 0$, $\left|\partial_x^2 \Pi_n h_0\right|_2 = \left|\Pi_n \partial_x^2 h_0\right|_2 \leqslant \left|\partial_x^2 h_0\right|_2$.

$|\partial_z \Pi_n h_0|_2^2 = -\int_\Omega \Pi_n h_0 \partial_z^2 \Pi_n h_0$. As $\left[\Pi_n, \partial_z^2\right] = 0$ and Π_n is a self-adjoint projector,

$$|\partial_z \Pi_n h_0|_2^2 = -\int_\Omega \Pi_n h_0 \partial_z^2 h_0 = \int_\Omega \partial_z \Pi_n h_0 \partial_z h_0 \leqslant |\partial_z \Pi_n h_0|_2\, |\partial_z h_0|_2.$$

This concludes the proof.   □

Proof of Proposition 5.1. Thanks to Section 5.1, $N_n(t)$ is defined on $[0, t_n[$. We apply Theorem 4.1 with $C < \min\left(\frac{1}{2}, \frac{2}{C_p}\right)$ and we get A, λ_0 and Q, independant of n such that, for all n, for all t in $[0, t_n[$ and for all λ in $(0, \lambda_0)$,

$$\tilde{N}_n'(t) \leqslant -A\tilde{N}_n(t) + Q\left(\tilde{N}_n(t)\right).$$

Let $y = \tilde{N}_n$. As $\tilde{N}_n$ and N_n are equivalent, there exists $M > 0$ such that $y(0) \leqslant M N_n(0)$. By Lemma 5.3, $y(0) \leqslant M N_0$. So, y satisfies:

$$\begin{cases} y'(t) \leqslant f(y(t)) \\ y(0) \leqslant M N_0. \end{cases}$$

Let $z_0 = MN_0$ and z the solution of the differential equation of Lemma 5.2. If we take $X_0 = \frac{x_0}{M}$, where x_0 is the first positive root of $-Az + Q(z)$, the conclusion of Lemma 5.2 remains. Then classical results on differential equations give y defined on $\mathbb{R}$ and for all $t \in \mathbb{R}$, $0 \leqslant y(t) \leqslant z(t)$. As z is decreasing, for all $t \in \mathbb{R}$, $y(t) \leqslant MN_0$. The fact that N_n and $\tilde{N}_n$ are equivalent finishes the proof.   □

5.3. Existence of the limit

We define here two functions v and h which are the limit of v_n and h_n, thanks to compactness of the weak* topology.

From Proposition 5.1 and from the viscous terms given by (26), we get the following bounds:

$$v_n \text{ is in a bounded set of } \mathrm{L}^\infty\left(\mathbb{R}_+; \mathrm{L}(\Omega)\right) \cap \mathrm{L}^2\left(\mathbb{R}_+; \mathrm{H}(\Omega)\right), \tag{37}$$

$$h_n \text{ is in a bounded set of } \mathrm{L}^\infty\left(\mathbb{R}_+; \mathrm{K}(\Omega)\right) \cap \mathrm{L}^2\left(\mathbb{R}_+; \mathrm{H}^2(\Omega) \cap \mathrm{H}(\Omega)\right), \tag{38}$$

$$\partial_x^2 \partial_z h_n \text{ is in a bounded set of } \mathrm{L}^2\left(\mathbb{R}_+ \times \Omega\right), \tag{39}$$

$$\partial_x^4 h_n \text{ is in a bounded set of } \mathrm{L}^2\left(\mathbb{R}_+ \times \Omega\right). \tag{40}$$

Then, up to an extraction of a subsequence, inclusions (37–40) imply that there exists

$$v \in \mathrm{L}^\infty\left(\mathbb{R}_+; \mathrm{L}(\Omega)\right) \cap \mathrm{L}^2\left(\mathbb{R}_+; \mathrm{H}(\Omega)\right)$$

and

$$h \in \mathrm{L}^\infty\left(\mathbb{R}_+; \mathrm{K}(\Omega)\right) \cap \mathrm{L}^2\left(\mathbb{R}_+; \mathrm{H}^2(\Omega) \cap \mathrm{H}(\Omega)\right)$$

such that $\partial_x^2 \partial_z h \in \mathrm{L}^2\left(\mathbb{R}_+ \times \Omega\right)$ and $\partial_x^4 h \in \mathrm{L}^2\left(\mathbb{R}_+ \times \Omega\right)$ and

$$v_n \;\rightharpoonup\; v \text{ weak * in } \mathrm{L}^\infty\left(\mathbb{R}_+; \mathrm{L}(\Omega)\right) \text{ and weakly in } \mathrm{L}^2\left(\mathbb{R}_+; \mathrm{H}(\Omega)\right), \tag{41}$$

$$h_n \;\rightharpoonup\; h \text{ weak * in } \mathrm{L}^\infty\left(\mathbb{R}_+; \mathrm{K}(\Omega)\right) \text{ and}$$
$$\text{weakly in } \mathrm{L}^2\left(\mathbb{R}_+; \mathrm{H}^2(\Omega) \cap \mathrm{H}(\Omega)\right), \tag{42}$$

$$\partial_x^2 \partial_z h_n \;\rightharpoonup\; \partial_x^2 \partial_z h \text{ weakly in } \mathrm{L}^2\left(\mathbb{R}_+ \times \Omega\right), \tag{43}$$

$$\partial_x^4 h_n \;\rightharpoonup\; \partial_x^4 h \text{ weakly in } \mathrm{L}^2\left(\mathbb{R}_+ \times \Omega\right). \tag{44}$$

To conclude that v_n and h_n verify (3–4), we will use compactness in $\mathrm{L}^2(0, \mathrm{T}; X)$, that is to say:

Proposition 5.4. *Up to an extraction of a subsequence, for $T > 0$,*

$$v_n \;\rightarrow\; v \tag{45}$$

$$h_n \;\rightarrow\; h \tag{46}$$

$$\partial_x h_n \;\rightarrow\; \partial_x h \tag{47}$$

$$\partial_z h_n \;\rightarrow\; \partial_z h \tag{48}$$

strong in $\mathrm{L}^2((0, \mathrm{T}) \times \Omega)$ *and a.e.*

This proposition requires the fact that $\partial_t v_n$, $\partial_t h_n$, $\partial_t \partial_x h_n$ and $\partial_t \partial_z h_n$ are bounded in $\mathrm{L}^2(0, \mathrm{T}; \mathrm{H}(\Omega)')$ and needs the following lemma:

Lemma 5.5. *Let $\mathrm{H}_1 \hookrightarrow \mathrm{H} \hookrightarrow \mathrm{H}_2$ three hilbert spaces so that the first embedding is compact. Then for any p_1 and p_2 in $(1, +\infty)$, the embedding*

$$\left\{ f \in \mathrm{L}^{p_1}\left(0, T; \mathrm{H}_1\right), \; \frac{\mathrm{d}f}{\mathrm{d}t} \in \mathrm{L}^{p_2}\left(0, T; \mathrm{H}_2\right) \right\} \hookrightarrow \mathrm{L}^{p_1}\left(0, T; \mathrm{H}\right)$$

is compact.

For a proof, see [10] (p. 58).

Hence, nonlinear terms must lay in a bounded set of $L^2((0,T) \times \Omega)$:

Lemma 5.6. *The nonlinear terms in equations (29) and (30) verified by v_n and h_n are in a bounded set of* $L^2((0,T) \times \Omega)$.

Proof. For sake of simplicity, we note h instead of h_n. As h is periodic in the direction x, integrating by parts with respect to x does not add any boundary terms. We first show three inequalities we will use throughout this proof.

We recall that h verifies (38–40). This fact and Lemma 4.4 show that there exists $M > 0$ such that, for all $p \in [2,6]$ and for all $t \in [0,T]$,

$$|\partial_x h|_p \leqslant M. \tag{49}$$

In the same way, Lemma 4.3 implies that, for $p \geqslant 1$, there exists $M > 0$ such that, for all $t \in [0,T]$,

$$|h|_p^p \leqslant M |h|_{p-2}^{p-2} \left(|\partial_x h|_2^2 + |\partial_z h|_2^2 \right) \leqslant M |h|_{p-2}^{p-2}, \tag{50}$$

and that there exists $M > 0$ such that

$$|\partial_z h|_4^{\frac{15}{4}} \leqslant M \left(|\partial_z h|_2^{30} + |\partial_x^2 h|_2^2 + |\partial_z^2 h|_2^2 \right). \tag{51}$$

- First, let show the result on $\partial_z \left((\partial_x h)^2 \right)$:

    As $\partial_z \left((\partial_x h)^2 \right) = 2\partial_x \partial_z h \partial_x h$, let

$$I = \int_\Omega (\partial_x \partial_z h)^2 (\partial_x h)^2 = -2 \int_\Omega \partial_z h \partial_x \partial_z h \partial_x^2 h \partial_x h - 2 \int_\Omega \partial_z h \partial_x^2 \partial_z h \, (\partial_x h)^2,$$

by integrating by parts with respect to x. Let $-2I_1$ be equal to the first integral and $-2I_2$ to the second. It is sufficient to show that I_1 and I_2 are in a bounded set of $L^1(0,T)$.

By integrating I_1 by parts with respect to x, we get $I_1 = -I_1 - I_3 - I_4$ where $I_3 = \int_\Omega (\partial_z h)^2 \partial_x^3 h \partial_x h$ and $I_4 = \int_\Omega (\partial_z h)^2 \left(\partial_x^2 h \right)^2$.

$I_3 \leqslant |\partial_z h|_4^2 |\partial_x h|_4 |\partial_x^3 h|_4$, and the proof of Proposition 4.9 shows that $\left| \partial_x^3 h \right|_4 \leqslant M \left| \partial_x^2 h \right|_2^{\frac{1}{8}} \left| \partial_x^4 h \right|_2^{\frac{1}{8}} \left| \partial_x^3 h \right|_6^{\frac{3}{4}}$. Hence

$$
\begin{aligned}
I_3 &\leqslant M |\partial_z h|_4^2 |\partial_x h|_4 \left| \partial_x^2 h \right|_2^{\frac{1}{8}} \left| \partial_x^4 h \right|_2^{\frac{1}{8}} \left| \partial_x^3 h \right|_6^{\frac{3}{4}} \\
&\leqslant M \left(|\partial_z h|_4^{\frac{15}{4}} + \left| \partial_x^4 h \right|_2^2 + |\partial_x h|_4^{60} + \left| \partial_x^2 h \right|_2^{80} + \left| \partial_x^3 h \right|_6^2 \right) \\
&\leqslant M \left(|\partial_z h|_4^{\frac{15}{4}} + \left| \partial_x^4 h \right|_2^2 + |\partial_x h|_4^{60} + \left| \partial_x^2 h \right|_2^{80} + \left| \partial_x^2 \partial_z h \right|_2^2 + \left| \partial_x^4 h \right|_2^2 \right),
\end{aligned}
$$

by using Lemma 4.4 for the last term of the second inequality. According to (38–40, 49, 51), I_3 is in a bounded set of $L^1(0,T)$ (we recall that h replaces h_n).

Now, let us show that I_4 is in a bounded set of $L^1(0,T)$ to conclude that I_1 verifies the same property:

$$I_4 \leqslant |\partial_z h|_4^2 \left| \partial_x^2 h \right|_4^2 \leqslant M \left(|\partial_z h|_4^{\frac{15}{4}} + \left| \partial_x^2 h \right|_4^{\frac{30}{7}} \right).$$

Using Proposition 4.9 and the fact that $\left|\partial_x^2 h\right|_2$ is in a bounded set of $\mathrm{L}^\infty(0,T)$ (*cf.* (38)) leads to

$$
\begin{aligned}
I_4 &\leqslant M\left(|\partial_z h|_4^{\frac{15}{4}} + |\partial_x h|_4^{\frac{240}{7}} + \left|\partial_x^4 h\right|_2^2 + \left|\partial_x^3 h\right|_6^2\right) \\
&\leqslant M\left(|\partial_z h|_4^{\frac{15}{4}} + |\partial_x h|_4^{\frac{240}{7}} \left|\partial_x^2 \partial_z h\right|_2^2 + \left|\partial_x^4 h\right|_2^2\right),
\end{aligned}
$$

by using Lemma 4.4 for the last term of the first inequality. Hence it is easy to conclude that I_4 is in a bounded set of $\mathrm{L}^1(0,T)$ with (39, 40, 49, 51), so does I_1.

Finally,

$$
I_2 \leqslant \left|\partial_x^2 \partial_z h\right|_2 |\partial_z h|_4 |\partial_x h|_8^2.
$$

Using Lemma 4.3 leads to

$$
I_2 \leqslant M\left|\partial_x^2 \partial_z h\right|_2 \left(\left|\partial_x^2 h\right|_2^2 + \left|\partial_z^2 h\right|_2^2\right)^{\frac{1}{2}} |\partial_z h|_2^{\frac{1}{2}} |\partial_x h|_6^{\frac{3}{2}}.
$$

Using (49) and (38) leads to

$$
I_2 \leqslant M\left(\left|\partial_x^2 \partial_z h\right|_2^2 + \left|\partial_x^2 h\right|_2^2 + \left|\partial_z^2 h\right|_2^2\right),
$$

which is in a bounded set of $\mathrm{L}^1(0,T)$ thanks to (38) and (39).

- Now we deal with $\partial_x\left(\partial_x h \partial_z h\right)$ which is equal to $\partial_x^2 h \partial_z h + \partial_x \partial_z h \partial_x h$. The last term is the one treated previously and showing the result on the first one is dealing with I_4. Hence $\partial_x\left(\partial_x h \partial_z h\right)$ is in a bounded set of $\mathrm{L}^2((0,T)\times\Omega)$.
- Finally, $\partial_x\left((\partial_x h)^3\right) = 3\partial_x^2 h\left(\partial_x h\right)^2$. Let $I = \int_\Omega \left(\partial_x^2 h\right)^2 \left(\partial_x h\right)^4$. By integrating by parts with respect to x, one has $I = -3I_5 - 2I_6$, where $I_5 = \int_\Omega h\left(\partial_x^2 h\right)^3\left(\partial_x h\right)^2$ and $I_6 = \int_\Omega h\partial_x^3 h\partial_x^2 h\left(\partial_x h\right)^3$. Using Proposition 4.9 leads to

$$
\begin{aligned}
I_5 &\leqslant |h|_{12}|\partial_x h|_6^2\left|\partial_x^2 h\right|_4^3 \\
&\leqslant |h|_{12}|\partial_x h|_6^2|\partial_x h|_4^{\frac{3}{2}}\left|\partial_x^2 h\right|_2^{\frac{3}{16}}\left|\partial_x^4 h\right|_2^{\frac{3}{16}}\left|\partial_x^3 h\right|_6^{\frac{9}{8}}.
\end{aligned}
$$

Using (49) and (50) leads to

$$
\begin{aligned}
I_5 &\leqslant M|h|_2^{\frac{2}{3}}|\partial_x h|_4^{\frac{3}{2}}\left|\partial_x^2 h\right|_2^{\frac{3}{16}}\left|\partial_x^4 h\right|_2^{\frac{3}{16}}\left|\partial_x^3 h\right|_6^{\frac{9}{8}} \\
&\leqslant M\left(\left|\partial_x^2 h\right|_2^2 + \left|\partial_x^4 h\right|_2^2 + \left|\partial_x^3 h\right|_6^2\right) \\
&\leqslant M\left(\left|\partial_x^2 h\right|_2^2 + \left|\partial_x^2 \partial_z h\right|_2^2 + \left|\partial_x^4 h\right|_2^2\right),
\end{aligned}
$$

according to Lemma 4.4. Then, it is easy to conclude that I_5 is in a bounded set of $\mathrm{L}^1(0,T)$ thanks to (38–40).

Now, we deal with I_6 which is equal to $-I_6 - I_7 - I_8 - 2I_9$ by integrating by parts with respect to x, where $I_7 = \int_\Omega h^2\partial_x^4 h\partial_x^2 h\left(\partial_x h\right)^2$, $I_8 = \int_\Omega h^2\left(\partial_x^3 h\right)^2\left(\partial_x h\right)^2$ and $I_9 = \int_\Omega h^2\partial_x h\left(\partial_x^2 h\right)^2\partial_x^3 h$. Then

$$
\begin{aligned}
I_7 &\leqslant |h|_{15}^2\left|\partial_x^4 h\right|_2|\partial_x h|_6^2\left|\partial_x^2 h\right|_5 \\
&\leqslant |h|_{15}^2\left|\partial_x^4 h\right|_2|\partial_x h|_6^2\left|\partial_x^2 h\right|_2^{\frac{1}{10}}\left|\partial_x^2 h\right|_6^{\frac{9}{10}},
\end{aligned}
$$

by using classical results on interpolation. As $|h|_{15}$ is in a bounded set of $L^\infty(0,T)$ (thanks to (50) and (38)) as well as $|\partial_x h|_6$ (thanks to (49) and (38)), $I_7 \leqslant M \left|\partial_x^4 h\right|_2 \left|\partial_x^2 h\right|_2^{\frac{1}{10}} \left|\partial_x^3 h\right|_6^{\frac{9}{10}} \leqslant M \left(\left|\partial_x^4 h\right|_2^2 + \left|\partial_x^2 h\right|_2^2 + \left|\partial_x^3 h\right|_6^2\right)$. To conclude that I_7 is in a bounded set of $L^1(0,T)$, it suffices to apply Lemma 4.4 to the last term of this inequality and to conclude with (38) and (39).

Now we estimate I_8 which is less or equal than $|h|_{12}^2 \left|\partial_x h\right|_6^2 \left|\partial_x^3 h\right|_4^2$. We have already seen that $\left|\partial_x^3 h\right|_4 \leqslant M \left|\partial_x^2 h\right|_2^{\frac{1}{8}} \left|\partial_x^4 h\right|_2^{\frac{1}{8}} \left|\partial_x^3 h\right|_6^{\frac{3}{4}}$. Then

$$I_8 \leqslant M |h|_{12}^2 \left|\partial_x h\right|_6^2 \left|\partial_x^2 h\right|_2^{\frac{1}{4}} \left|\partial_x^4 h\right|_2^{\frac{1}{4}} \left|\partial_x^3 h\right|_6^{\frac{3}{2}}.$$

As usual, $|h|_{12}^2 \left|\partial_x h\right|_6^2$ is in a bounded set of $L^\infty(0,T)$. Then

$$I_8 \leqslant M \left|\partial_x^2 h\right|_2^{\frac{1}{4}} \left|\partial_x^4 h\right|_2^{\frac{1}{4}} \left|\partial_x^3 h\right|_6^{\frac{3}{2}} \leqslant M \left(\left|\partial_x^2 h\right|_2^2 + \left|\partial_x^4 h\right|_2^2 + \left|\partial_x^3 h\right|_6^2\right).$$

We conclude as for I_7.

It remains I_9 which is less or equal than $|h|_{24}^2 \left|\partial_x h\right|_6 \left|\partial_x^3 h\right|_4 \left|\partial_x^2 h\right|_4^2$. By using Proposition 4.9 and the bound of $\left|\partial_x^3 h\right|_4$, we get

$$I_9 \leqslant M |h|_{24}^2 \left|\partial_x h\right|_6 \left|\partial_x h\right|_4 \left|\partial_x^2 h\right|_2^{\frac{1}{4}} \left|\partial_x^4 h\right|_2^{\frac{1}{4}} \left|\partial_x^3 h\right|_6^{\frac{3}{2}}.$$

As usual, $|h|_{24}^2 \left|\partial_x h\right|_6 \left|\partial_x h\right|_4$ is in a bounded set of $L^\infty(0,T)$ and $I_9 \leqslant M \left(\left|\partial_x^2 h\right|_2^2 + \left|\partial_x^4 h\right|_2^2 + \left|\partial_x^3 h\right|_6^2\right)$. We conclude as for I_7 and the proof is finished.

$\square$

Proof of Proposition 5.4. From Lemma 5.5 (with $p_1 = p_2 = 2$, $H_1 = H(\Omega)$, $H_2 = H(\Omega)'$, $H = L^2(\Omega)$), it's easy to see that we only need that

- For all $T > 0$, $\partial_t v_n$ and $\partial_t h_n$ are bounded in $L^2\left(0,T;H(\Omega)'\right)$:

  Let us fix $T > 0$. (37) implies that v_n, $\partial_x^2 v_n$ and $\partial_z^2 v_n$ are in a bounded set of $L^2\left((0,T),H(\Omega)'\right)$.

  As z lies in $(0,1)$, the same conclusion remains for $\frac{z}{\lambda}\partial_x v_n$.

  Thanks to (38) and (40), $\frac{z}{\lambda}\partial_x h_n$, $\partial_x^2 h_n$, $\partial_z^2 h_n$ and $\partial_x^4 h_n$ remain in a bounded set of $L^2\left((0,T)\times\Omega\right)$ which is continuously embedded in $L^2\left((0,T);H(\Omega)'\right)$.

  It only remains the nonlinear terms. Lemma 5.6 shows that they are in a bounded set of $L^2((0,T)\times\Omega)$, therefore in a bounded set of $L^2\left((0,T);H(\Omega)'\right)$.

  Then, up to an extraction of a subsequence, for all $T > 0$, $v_n \to v$ strong in $L^2\left((0,T)\times\Omega\right)$ and a.e. Obviously, the same result holds on h. So (45) and (46) are proved.
- For all T positive real, $\partial_t \partial_x h_n$ and $\partial_t \partial_z h_n$ are in a bounded set of $L^2\left(0,T;H(\Omega)'\right)$:

  Lemma 5.6 shows that they are in a bounded set of $L^2((0,T)\times\Omega)$, therefore their derivatives with respect to x and z are in a bounded set of $L^2\left((0,T);H(\Omega)'\right)$.

  So, as for v_n, we conclude that, for all $T > 0$, $\partial_x h_n \to \partial_x h$ and $\partial_z h_n \to \partial_z h$ strong in $L^2\left((0,T)\times\Omega\right)$ and a.e., and (47) and (48) are proved.

$\square$

Remark 5.7. v and h depend *a priori* on T. With a diagonal extraction, we eliminate this fact (up to an extraction of a subsequence...).

5.4. **Conclusion of the proof of Theorem 2.2**

We first show that $\partial_t v_n$ tends to $\partial_t v$ so that equation (3) holds:

Proposition 5.8. *Let $T > 0$. Then, $\partial_t v_n$ tends towards $\partial_t v$ in $\mathrm{L}^2\left(0, T; \mathrm{H}(\Omega)'\right)$ and v and h satisfy the equation (3). In addition, $v \in \mathrm{C}^0\left([0, T]; \mathrm{L}^2(\Omega)\right)$.*

Proof. Let $N \in \mathbb{N}$, $\Phi : [0, T] \times \Omega \to \mathbb{R}$, C^∞ with respect to time, so that $\Phi(T) = \Phi(0) = 0$ and, for all $t \in [0, T]$, $\Phi(t, .) \in E_N$. Here, $(.,.)$ will denote the inner product on $\mathrm{L}^2(\Omega)$ and n will be an integer greater than N. The linear terms of $\int_0^T (\partial_t v_n, \Phi)$ do not hold any problem (we use (41–44, 45) and Lebesgue's theorem). Let us focus on the nonlinear terms of $\int_0^T (\partial_t v_n, \Phi)$, which are $\int_0^T \int_\Omega \partial_x (\partial_x h_n \partial_z h_n) \Phi \, \mathrm{d}t$, $\int_0^T \int_\Omega \partial_z \left((\partial_x h_n)^2\right) \Phi \, \mathrm{d}t$ and $\int_0^T \int_\Omega \partial_x \left((\partial_x h_n)^3\right) \Phi \, \mathrm{d}t$. Thanks to (47, 48) and Lebesgue's theorem, we have:

$$\int_0^T \int_\Omega \partial_x h_n \partial_z h_n \partial_x \Phi \, \mathrm{d}t \xrightarrow[n \to \infty]{} \int_0^T \int_\Omega \partial_x h \partial_z h \partial_x \Phi \, \mathrm{d}t,$$

$$\int_0^T \int_\Omega (\partial_x h_n)^2 \partial_z \Phi \, \mathrm{d}t \xrightarrow[n \to \infty]{} \int_0^T \int_\Omega (\partial_x h)^2 \partial_z \Phi \, \mathrm{d}t$$

and

$$\int_0^T \int_\Omega (\partial_x h_n)^3 \partial_x \Phi \, \mathrm{d}t \xrightarrow[n \to \infty]{} \int_0^T \int_\Omega (\partial_x h)^3 \partial_x \Phi \, \mathrm{d}t.$$

As, $(\partial_t v_n, \Phi) = <\partial_t v_n, \Phi>_{\mathrm{H}(\Omega)', \mathrm{H}(\Omega)}$ and

$$\int_0^T (\partial_t v_n, \Phi) \, \mathrm{d}t = -\int_0^T (v_n, \partial_t \Phi) \, \mathrm{d}t \xrightarrow[n \to \infty]{} -\int_0^T (v, \partial_t \Phi) \, \mathrm{d}t.$$

We have perform the limit for each Φ in E_N and for all N, this means that v and h verify the equation (3) in a weak sense, in $\mathrm{L}^2\left(0, T; \mathrm{H}(\Omega)'\right)$.

Finally, Lemma 3.10 involves $v \in \mathrm{C}^0\left([0, T]; \mathrm{L}^2(\Omega)\right)$, which concludes the proof of the proposition. $\square$

The initial conditions are treated in the usual way and we have $v(0) = v_0$.

The same result holds for equation (4):

Proposition 5.9. *Let $T > 0$. Then, $\partial_t h_n$ tends to $\partial_t h$ in $\mathrm{L}^2\left(0, T; \mathrm{H}(\Omega)'\right)$ and v and h verify the equation (4). In addition, $h \in \mathrm{C}^0\left([0, T]; \mathrm{L}^2(\Omega)\right)$.*

Proof. We choose N and Φ as in Proposition 5.8. The linear terms do not hold any problem: we use (45) and Lebesgue's theorem. Then, v and h verify the equation (4) in a weak sense. We apply Lemma 3.10 to conclude that $h \in \mathrm{C}^0\left([0, T]; \mathrm{L}^2(\Omega)\right)$, hence the proof is finished. $\square$

Finally, the initial condition of h is treated as for v and we have $h(0) = h_0$. This concludes the proof of Theorem 2.2.

6. Proof of the local existence

We will use exactly the same arguments as for the global existence.

6.1. **Approximation of the solution**

As in Section 5.1, we find v_n and h_n in $\mathrm{C}^\infty\left([0, t_n); E_n\right)$, unique maximal solution of (29) and (30), verifying $v_n(0) = \Pi_n(v_0)$ and $h_n(0) = \Pi_n(h_0)$. Of course, v_n and h_n are 2-periodic in x and vanish at $z = 0, 1$.

6.2. *A priori* **estimates**

We denote $N_n(t)$ the quantity $|v_n|_2^2 + \epsilon_1 k^4 |h_n|_2^2 + |h_n|_2^2$. As for global existence, this section shows that N_n is uniformly bounded in n:

Proposition 6.1. *Let v_0 in $L(\Omega)$, h_0, in $K(\Omega)$. There exists $T_e > 0$ such that, for all T in $[0, T_e)$, there exists $M > 0$ such that, for all n in $\mathbb{N}$, for all t in $[0, T]$, $N_n(t) \leqslant N_o M$.*

Proof. We make the same computations as in Theorem 4.2 and this gives Q (1-valuated polynomial with positive coefficients) such that $N_n'(t) \leqslant Q(N_n(t))$, whose time existence is t_n. As for global existence, let us consider the maximal solution z of the differential equation

$$\begin{cases} z'(t) = Q(z(t)) \\ z(0) = |v_0|_2\, 2 + \epsilon_1 \left|\partial_x^2 h_0\right|_2^2 + |\partial_z h_0|_2^2, \end{cases}$$

whose time definition is denoted T_e. By classical results on differential equations, $t_n \geqslant T_e$ and $N_n(t) \leqslant z(t)$, for all t in $[0, T_e[$. Let $T \in]0, T_e[$, $\tilde{Q}$ such that $Q(X) = X\tilde{Q}(X)$ (possible since Q is a 1-valuated polynomial), $M_1(T) = \sup_{[0,T]} z(t)$, $M(T) = \tilde{Q}(M_1(T))$ and $M = \exp(TM(T))$. Since $N_n(t) \leqslant z(t)$, we have

$$N_n'(t) \leqslant N_n \tilde{Q}(N_n) \leqslant N_n \tilde{Q}(z) \leqslant y = N_n \tilde{Q}(M_1(T)) \leqslant N_n M(T).$$

By Gronwall's lemma, for all t in $[0, T]$,

$$N_n(t) \leqslant N_n(0) \exp(TM(T)) \leqslant z(0) \exp(TM(T)).$$

The same arguments as in Theorem 5.3 give $N_n(0) \leqslant N_0 = z(0)$. So

$$\sup_{t \in [0,T]} (N_n(t)) \leqslant N_0 M.$$

$\square$

This result and the viscous terms given by the Theorem 4.2 imply that, for all $T < T_e$,

$$v_n \text{ is in a bounded set of } L^\infty(0,T; L(\Omega)) \cap L^2(0,T; H(\Omega)), \tag{52}$$

$$h_n \text{ is in a bounded set of } L^\infty(0,T; K(\Omega)) \cap L^2\left(0,T; H^2(\Omega) \cap H(\Omega)\right), \tag{53}$$

$$\partial_x^2 \partial_z h_n \text{ is in a bounded set of } L^2((0,T) \times \Omega), \tag{54}$$

$$\partial_x^4 h_n \text{ is in a bounded set of } L^2((0,T) \times \Omega). \tag{55}$$

6.3. **Existence of the limit**

Up to an extraction of a subsequence, equations (52–55) imply that there exists

$$v \text{ in } L^\infty(0,T; L(\Omega)) \cap L^2(0,T; H(\Omega))$$

and

$$h \text{ in } L^\infty(0,T; K(\Omega)) \cap L^2\left(0,T; H^2(\Omega) \cap H(\Omega)\right)$$

such that $\partial_x^2 \partial_z h \in L^2 \left((0,T) \times \Omega\right)$ and $\partial_x^4 h \in L^2 \left((0,T) \times \Omega\right)$ and

$$v_n \rightharpoonup v \text{ weak * in } L^\infty \left(0, T; L(\Omega)\right) \text{ and weakly in } L^2 \left(0, T; H(\Omega)\right), \tag{56}$$

$$h_n \rightharpoonup h \text{ weak * in } L^\infty \left(0, T; K(\Omega)\right) \text{ and}$$
$$\text{weakly in } L^2 \left(0, T; H^2 \left(\Omega\right) \cap H(\Omega)\right), \tag{57}$$

$$\partial_x^2 \partial_z h_n \rightharpoonup \partial_x^2 \partial_z h \text{ weakly in } L^2 \left((0,T) \times \Omega\right), \tag{58}$$

$$\partial_x^4 h_n \rightharpoonup \partial_x^4 h \text{ weakly in } L^2 \left((0,T) \times \Omega\right). \tag{59}$$

As for the global existence, v and h will verify the equations (3) and (4) by using compactness in the time spaces (Arguments of Prop. 5.4 remain). Hence we conclude that (45–47) and (48) hold.

Remark 6.2. Of course it is possible to find v and h independant of T by using a diagonal extraction.

6.4. Conclusion

The arguments given in the proof of the global existence are valid but instead of using (41–44), we use (56–59). The case of the initial condition ($t = 0$) is proved in the same way. So the proof of Theorem 2.4 is completed.

7. Conclusion

The mathematical results exposed in this paper explain the stabilization of lamellar phases observed in the experiments (Th. 2.2): instabilities does not develop. More over, the linear system is controlled by a nonconstant parameter and is stable at high shear, *i.e.* when this parameter is high (Th. 2.1). It shows that small scales are always stable and that the linear system tends to its rest state.

References

[1] O.V. Besov, V.P. Il'in and S.M. Nikol'skii, *Integral representations of functions and embeddings theorems*, Vol. 1. V.H. Winston and Sons.

[2] A. Babin, B. Nicolaenko and A. Mahalov, Regularity and integrability of $3D$ Euler and Navier–Stokes equations for rotating fluids. *Asymptot. Anal.* **15** (1997) 103-150.

[3] A. Colin, T. Colin, D. Roux and A.S. Wunenburger, Undulation instability under shear: A model to explain the different orientation of a lamellar phase under shear. *European J. Soft Condensed Matter* (to appear).

[4] A. de Bouard and J.C. Saut, Solitary waves of generalized Kadomtsev-Petviashvili equations. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **14** (1997) 211-236.

[5] O. Diat, D. Roux and F. Nallet. *J. Phys. II France* **3** (1993) 1427.

[6] O. Diat and D. Roux. *J. Phys. II France* **3** (1993) 9.

[7] I. Gallagher, Asymptotic of solutions of hyperbolic equations with a skew-symmetric perturbation. *J. Differential Equations* **150** (1998) 363-384.

[8] E. Grenier, Oscillatory perturbations of the Navier–Stokes equations. *J. Math. Pures Appl.* **76** (1997) 477-498.

[9] O.A. Ladyzhenskaya, *The Mathematical Theory of Viscous Incompressible Flow*. Gordon and Breach (1963).

[10] J.L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaires*. Dunod (1969).

[11] J.L. Lions and E. Magenes, *Problèmes aux limites non homogènes et applications*. Dunod (1968).

[12] J. Simon, Compact sets in the Spaces $L^p \left(0, T; B\right)$. *Ann. Mat. Pura Appl.* **146** (1987) 65-96.

[13] R. Temam, Infinite-dimensional dynamical systems in mechanics and physics. Springer-Verlag, *Appl. Math. Sci.* **68** (1997).