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UNIFORM ESTIMATES FOR THE PARABOLIC GINZBURG–LANDAU EQUATION

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Abstract. We consider complex-valued solutions u_ε of the Ginzburg–Landau equation on a smooth bounded simply connected domain Ω of $\mathbb{R}^N$, $N \geq 2$, where $\varepsilon > 0$ is a small parameter. We assume that the Ginzburg–Landau energy $E_\varepsilon(u_\varepsilon)$ verifies the bound (natural in the context) $E_\varepsilon(u_\varepsilon) \leq M_0|\log\varepsilon|$, where M_0 is some given constant. We also make several assumptions on the boundary data. An important step in the asymptotic analysis of u_ε, as $\varepsilon \to 0$, is to establish uniform L^p bounds for the gradient, for some $p > 1$. We review some recent techniques developed in the elliptic case in [7], discuss some variants, and extend the methods to the associated parabolic equation.

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1. Introduction

In many problems involving a small parameter ε (tending to zero), a crucial step in order to describe the asymptotic limit of the solutions is to establish uniform estimates, *i.e.* independent of ε. Of course, when the limit is singular these estimates may involve a function space which is larger than the energy space. A typical example is the Cahn–Hilliard (also called Modica–Mortola) functional, where the energy space is H^1, whereas the minimizers happen to be uniformly bounded in BV (see [29, 30]).

In this paper we will focus on uniform estimates for the complex-valued Ginzburg–Landau equation. Here again the energy space is H^1; however uniform estimates are established (in the elliptic case) in $W^{1,p}$ with $1 \leq p < \frac{N}{N-1} \leq 2$ (see [7, 9, 11, 26]). Our purpose is to review some new ideas introduced in [7] for the elliptic case, and then to extend these methods to the associated parabolic evolution problem.

More precisely, the following situation was analysed in [7]. Let N be an integer larger than two, and let Ω be a smooth bounded, simply connected domain in $\mathbb{R}^N$. For $0 < \varepsilon < 1$ a small parameter, consider solutions $u_\varepsilon : \Omega \to \mathbb{C}$ of the Ginzburg–Landau equation with Dirichlet data g_ε in $H^{1/2}(\partial\Omega; \mathbb{C})$:

$$(\mathrm{GL})_\varepsilon \qquad \begin{cases} -\Delta u_\varepsilon = \dfrac{1}{\varepsilon^2} u_\varepsilon (1 - |u_\varepsilon|^2) & \text{in } \Omega \\ u_\varepsilon = g_\varepsilon & \text{on } \partial\Omega. \end{cases}$$

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Assume moreover that there exist positive constants M_0 and M_1 such that

(H1) $$E_\varepsilon(u_\varepsilon) = \int_\Omega e_\varepsilon(u_\varepsilon) = \frac{1}{2}\int_\Omega |\nabla u_\varepsilon|^2 + \frac{1}{4\varepsilon^2}\int_\Omega (1-|u_\varepsilon|^2)^2 \le M_0|\log\varepsilon|,$$

(H2) $$||g_\varepsilon||^2_{H^{1/2}(\partial\Omega)} \le M_1,$$

(H3) $$|g_\varepsilon| = 1\,, \qquad \text{a.e. in } \partial\Omega.$$

The main result of [7] is the following:

Theorem 1 ([7], Th. 1)**.** *Let $1 \le p < \frac{N}{N-1}$. There exists a constant C_p depending only on M_0, M_1, Ω and p, but independent of ε, such that for any solution u_ε of $(GL)_\varepsilon$ verifying (H1), (H2) and (H3), we have*

$$\int_\Omega |\nabla u_\varepsilon|^p \le C_p. \tag{1}$$

The proof of Theorem 1 relies on a new result in [22] (see also [1]). Roughly speaking, this result shows that if, for $0 < \varepsilon < 1$, $v_\varepsilon : \Omega \to \mathbb{C}$ verifies the bound

$$E_\varepsilon(v_\varepsilon) \le M_0|\log\varepsilon|,$$

then the Jacobians $\{Jv_\varepsilon\}_{0<\varepsilon<1}$ are precompact in some weak norm (see Sect. 2 for a precise statement), whereas the family $\{v_\varepsilon\}_{0<\varepsilon<1}$ may not be compact in any reasonable norm. The Jacobian Jv (for a map $v : \Omega \to \mathbb{C}$) is defined as

$$Jv := \frac{1}{2}d(v\times dv) = \sum_{i<j}(v_{x_i}\times v_{x_j})dx_i \wedge dx_j$$

(here $a \times b := a_1b_2 - a_2b_1$ denotes the exterior product of two vectors $a, b \in \mathbb{R}^2 \simeq \mathbb{C}$). Moreover, if v_{ε_n} ($\varepsilon_n \to 0$) is a subsequence such that Jv_{ε_n} converges, then the limit J_* is a measure with the structure of an integer multiplicity rectifiable current of dimension $N-2$.

In many cases (and here specially in view of the parabolic equation considered later) it is natural to relax the condition $|g_\varepsilon| = 1$. Therefore, we will consider also the following variant of assumption (H3), namely we may assume instead that there exists a positive constant M_2 such that

(H3bis) $$\frac{1}{2}\int_{\partial\Omega} |\nabla g_\varepsilon|^2 + \frac{1}{4\varepsilon^2}\int_{\partial\Omega}(1-|g_\varepsilon|^2)^2 \le M_2|\log\varepsilon|.$$

We then have the following:

Theorem 1bis. *Let $1 \le p < \frac{N}{N-1}$. There exists a constant C_p depending only on M_0, M_1, M_2, Ω and p, but independent of ε, such that for any solution u_ε of $(GL)_\varepsilon$ verifying (H1), (H2), (H3bis) we have*

$$\int_\Omega |\nabla u_\varepsilon|^p \le C_p. \tag{2}$$

Estimate (1) was first considered in [9] for $N = 2$, where it was established in the case $g_\varepsilon = g$ is independent on ε and smooth (see also [6] for other references in case $N = 2$). It was then generalized under various restrictive assumptions on N, g_ε and u_ε (see [11, 26, 31] and [14]). Theorem 1 and Theorem 1bis cover all the above quoted results; however we expect that the same conclusion might be derived under milder assumptions on the boundary data g_ε.

As already mentioned, this kind of estimate is a crucial ingredient in the asymptotic analysis of solutions to equation $(\mathrm{GL})_\varepsilon$ as $\varepsilon \to 0$. The theory was developed during the last decade in [3,6,8–11,14,26,27,31,33,36]. In particular, the main result in [11] (Th. 1 there) can be derived under the assumptions considered here. More precisely, the following holds:

Theorem 2. *Let u_ε be a solution of (GL_ε) satisfying $(H1)$, $(H2)$, $(H3)$ or $(H3bis)$.*

Then, for a subsequence $\varepsilon_n \to 0$, there exist a map $u_* \in W^{1,p}(\Omega)$, $\forall\, 1 \le p < \frac{N}{N-1}$, and a map $g_* \in H^{1/2}(\partial\Omega)$ such that

i) $|u_*| = 1$ *on* Ω, $|g_*| = 1$, $u_* = g_*$ *on* $\partial\Omega$;

ii) $u_{\varepsilon_n} \to u_*$ *in* $W^{1,p}(\Omega)$, $g_{\varepsilon_n} \rightharpoonup g_*$ *in* $H^{1/2}(\partial\Omega)$;

iii) $\mathrm{div}(u_* \times \nabla u_*) = 0$ *in* Ω;

iv) $\dfrac{e_{\varepsilon_n}(u_{\varepsilon_n})}{|\log \varepsilon_n|} \rightharpoonup \mu_*$ *as measures, where* μ_* *is a bounded measure on* $\bar\Omega$.

Set $\mathcal{S} = \mathrm{supp}(\mu_*)$.

v) $\mathcal{S}$ *is a closed subset of* $\bar\Omega$ *with* $\mathcal{H}^{N-2}(\mathcal{S}) < +\infty$;

vi) $u_* \in C^\infty(\Omega\backslash\mathcal{S})$ *and for any ball* $B(x_0,r)$ *included in* $\Omega\backslash\mathcal{S}$ *there exists a function* $\varphi_* \in C^\infty(B(x_0,r))$, *such that* $\Delta\varphi_* = 0$, $u_* = \exp(i\varphi_*)$;

vii) $u_{\varepsilon_n} \to u_*$ *in* $C^k(K)$, *for any compact subset* K *of* $\Omega \setminus \mathcal{S}$;

viii) $\mathcal{S}$ *is* $\mathcal{H}^{N-2}$*-rectifiable;* μ_* *is a stationary varifold.*

The proof of Theorem 2 can be derived, following the same arguments as in Sections 4, 7, 8 and 9 of [11], from estimate (1) and the η-ellipticity property:

Theorem 3 ([11], Th. 2)**.** *Let u_ε be a solution of $(GL)_\varepsilon$ on the ball B_r. Then there exist constants $K > 0$, and $\alpha > 0$, depending only on N such that if*

$$E_\varepsilon(u_\varepsilon, B_r) \le \eta r^{N-2} \left|\log \frac{\varepsilon}{r}\right|, \tag{3}$$

with $\eta > 0$, then

$$|u_\varepsilon(0)| \ge \quad 1 - K\eta^\alpha. \tag{4}$$

Note that in [11], estimate (1) appeared as a consequence of Theorem 3 (together with covering arguments), whereas here the two properties happen to be completely independent results. In particular, for the proof of Theorem 2, this approach bypasses the (somewhat unpleasant) technicalities related to the analysis near the boundary.

We recall also that statement viii) in Theorem 2 is a direct consequence of Theorem 3 and the analysis of [4] (rectifiability of $\mathcal{S}$ can be also deduced, as in [27], using the result of [25]).

In two dimensions, Theorem 3 originated simultaneously in [12, 36], and was used extensively for a large number of problems (see [6, 33]). In higher dimension, the first η-ellipticity result was given in [31] under the name η-compactness (for $N = 3$ and minimizing maps), then in [26] for minimizing maps in arbitrary dimension, in [27] for $N = 3$, u_ε not necessarily minimizing, and finally in [10, 11] in the general case.

Remark 1. From Theorem 2 we deduce directly that

$$Ju_{\varepsilon_n} \rightharpoonup J_* = Ju_* \qquad \text{in } \mathcal{D}'(\Omega).$$

It can be proved directly arguing as in Sections 5 and 6 of [11] (without the machinery of [1, 22]), that J_* is a bounded measure and that

$$\mathrm{supp}\,(J_*) \subset \mathcal{S} = \mathrm{supp}\,(\mu_*).$$

When g_ε varies with ε, then the two sets might be different. However, if $g_\varepsilon \equiv g$ is fixed (and $|g| = 1$), then it is not known if the two sets coincide; it is even not known if the rectifiable set supporting J_* is closed or not. Finally, we have (see Rem. 5.1)

$$u_* \in C^\infty(\Omega \setminus \mathrm{supp}\,(J_*))\,.$$

We turn next to the parabolic Ginzburg–Landau equation, which is the main focus of this paper:

$$(\mathrm{PGL})_\varepsilon \qquad \begin{cases} \dfrac{\partial u_\varepsilon}{\partial t} - \Delta u_\varepsilon = \dfrac{1}{\varepsilon^2} u_\varepsilon (1 - |u_\varepsilon|^2) & \text{in } \Omega \times (0, +\infty), \\ u_\varepsilon(x, 0) = u_\varepsilon^0(x) & \text{for a.e. } x \in \Omega\,, \\ u_\varepsilon(x,t) = g_\varepsilon(x) & \text{for a.e. } (x,t) \in \partial\Omega \times (0, +\infty). \end{cases}$$

Equations related to $(\mathrm{PGL})_\varepsilon$ appear in many applications, for instance as dynamical models in superconductivity. The equation $(\mathrm{PGL})_\varepsilon$ has been extensively studied in recent years. In particular the dynamics of vortices has been described in the two dimensional case, and some results have been obtained in higher dimensions (see *e.g.* [5, 20, 21, 23, 24]).

In this paper, we will make several assumptions on the initial data $u_\varepsilon^0 : \Omega \to \mathbb{C}$ and on the boundary condition $g_\varepsilon : \partial\Omega \to \mathbb{C}$, which is time independent. First, for g_ε, we assume (H2), (H3) or (H3bis) hold, as in the elliptic case. For u_ε^0 we assume that there exist positive constants M_0 and M_3 such that

$$\text{(H4)} \qquad E_\varepsilon(u_\varepsilon^0) \le M_0 |\log \varepsilon|\,, \qquad ||u_\varepsilon^0||^2_{H^{1/2}(\Omega)} \le M_3, \qquad |u_\varepsilon^0| \le 1.$$

Under these hypotheses, equation $(\mathrm{PGL})_\varepsilon$ admits, for fixed $\varepsilon > 0$, a unique solution $u_\varepsilon : \Omega \times [0, +\infty) \to \mathbb{C}$. Moreover, $u_\varepsilon \in C^\infty(\Omega \times (0,T))$ for each $T > 0$, and $|u_\varepsilon| \le 1$, by the maximum principle. For $t \ge 0$, let $u_\varepsilon^t : \Omega \to \mathbb{C}$ be the function defined by $u_\varepsilon^t(x) := u_\varepsilon(x,t)$, and recall the energy equality

$$E_\varepsilon(u_\varepsilon^T) + \int_0^T \int_\Omega \left| \frac{\partial u_\varepsilon}{\partial t} \right|^2 = E_\varepsilon(u_\varepsilon^0). \tag{5}$$

In particular we have the inequality

$$\int_0^T \int_\Omega \left| \frac{\partial u_\varepsilon}{\partial t} \right|^2 + |\nabla u_\varepsilon|^2 \le M_0 (T+1) |\log \varepsilon|, \tag{6}$$

and hence $u_\varepsilon \in H^1(\Omega \times [0,T])$, and the L^2 norm of its gradient (with respect to space-time variables) is bounded by a constant times $|\log \varepsilon|$, as in the elliptic case.

Our purpose is to extend some of the techniques developed in the elliptic framework: in particular, Hodge–de Rham decomposition, reduction to systems of linear equations, etc. One of the consequences of this analysis is the following estimate, which, to our knowledge, is new.

Theorem 4. *Let $1 \le p < \frac{N+1}{N}$, $T > 0$. There exists a constant C_p depending only on M_0, M_1, M_2, M_3, Ω, p and T, but independent of ε, such that for any solution u_ε of $(PGL)_\varepsilon$ with initial data u_ε^0 verifying (H4) and boundary data g_ε verifying (H2), and (H3) or (H3bis), we have*

$$\int_0^T \int_\Omega |\nabla u_\varepsilon|^p \le C_p. \tag{7}$$

A direct consequence of Theorem 4 and the analysis in Section 6 is the following:

Proposition 1. *Let $\{u_\varepsilon\}_{0<\varepsilon<1}$ be solutions of $(PGL)_\varepsilon$ satisfying (H4), (H2), and (H3) or (H3bis). Then, for a subsequence $\varepsilon_n \to 0$, there exists a map $u_* : \Omega \times [0, +\infty) \to \mathbb{C}$ such that, for every $1 \le p < \frac{N+1}{N}$ and every $T > 0$,*

i) $u_{\varepsilon_n} \rightharpoonup u_*$ in $L^p(\Omega \times [0,T])$, $\nabla u_{\varepsilon_n} \rightharpoonup \nabla u_*$ in $L^p(\Omega \times [0,T])$;

ii) $u_* \in C^\infty(\Omega \times [0,+\infty) \setminus \text{supp}\,(\tilde{J}_*); S^1)$.

Here $\tilde{J}_*$ is the weak limit of $\tilde{J}u_{\varepsilon_n}$ in the sense of [1, 22] (see Sect. 2).

As one easily sees, Proposition 1 gives only partial results concerning the convergence of u_ε as ε goes to zero. Note in particular, that at this stage we cannot even exclude the fact that the support of $\tilde{J}_*$ has positive measure. A further step for the asymptotic analysis of u_ε would be to derive the analogous of Theorem 3 for the parabolic case. In this context, a result in case $\Omega = \mathbb{R}^3$ is provided in [28].

The relation of the (possible) asymptotic behavior of solutions to $(\text{PGL})_\varepsilon$ with motion by mean curvature, in Brakke's (weak) formulation, has been shown in [4], under some additional assumption on the solutions, which is conjectured there. This assumption can be proved in the elliptic case, yielding, as already mentioned, statement viii) of Theorem 2.

Finally, in the case the initial condition u_ε^0 has some special properties (in particular, the concentration set of u_ε^0 is a smooth $(N-2)$-dimensional manifold), then convergence to motion by mean curvature (for the concentration set), up to appearance of singularities, is established in [21] (see also [24]). The techniques there, rely on a careful analysis on the concentration of the energy density $e_\varepsilon(u_\varepsilon)$.

The outline of the paper is as follows.

The next section is devoted to an important estimate first derived by Jerrard and Soner [22] for Jacobians of maps $v_\varepsilon : \Omega \to \mathbb{C}$ verifying the bound $E_\varepsilon(v_\varepsilon) \leq M_0|\log\varepsilon|$. We present several variants of this estimate (most of the ideas are from [7]), which take into account the boundary data. These estimates are the main ingredient in the proofs of Theorem 1, Theorem 1bis and Theorem 4. Section 3 is concerned with the Hodge–de Rham decomposition of $u \times \nabla u$, and its interplay with the results of Section 2. Section 4 deals with a similar issue, for a situation specially adapted for the parabolic problem. In Section 5 we give the proofs of Theorem 1 and Theorem 1bis. Finally, in Section 6 we prove Theorem 4.

2. Uniform bounds for Jacobians

In this section, we will describe and extend a new estimate for Jacobians of maps in H^1 with some control on the Ginzburg–Landau energy. This estimate is actually in the same spirit as, in the scalar case (*i.e.* for real-valued maps v), the famous estimate (see [29, 30])

$$\int_\Omega |\nabla\Phi(v)| \leq \sqrt{2}\varepsilon E_\varepsilon(v), \tag{2.1}$$

where $\Phi(v) = \frac{v^2}{2} - \frac{v^3}{3}$ is a primitive of $(1-v^2)$. Estimate (2.1) is the starting point of the strong L^1 compactness of sequences of functions v_ε satisfying $\varepsilon E_\varepsilon(v_\varepsilon) \leq C$ (and not only of solutions of $(GL)_\varepsilon$!).

For complex-valued maps $v \in H^1(\Omega; \mathbb{C})$ satisfying the even stronger bound (but natural in this context) $E_\varepsilon(v) \leq C|\log\varepsilon|$, we can not expect similar compactness properties. A simple example is provided by maps presenting wild oscillations in the phase, for instance take $v_\varepsilon = \exp(i|\log\varepsilon|^{1/2}\phi)$, where $\phi : \Omega \to \mathbb{R}$ is an arbitrary smooth function. Note that compactness cannot be expected even for solutions of $(\text{GL})_\varepsilon$ (see [15]).

However, oscillations in the phase of v are not seen by the Jacobian of v, which, we recall, is defined as the two-form

$$Jv := \frac{1}{2}d(v \times dv) = \sum_{i<j}(v_{x_i} \times v_{x_j})dx_i \wedge dx_j. \tag{2.2}$$

In particular, $v_{x_i} \times v_{x_j} = 0$ whenever v_{x_i} and v_{x_j} are colinear. Hence, when $|v| = 1$ as in the example above, we have $Jv \equiv 0$.

It turns out that Jacobians possess compactness properties in some weak norm, as was first shown by Jerrard and Soner in [22]. More precisely, from the computations in [22] we deduce immediately:

Theorem 2.1. *Let $\alpha > 0$, $M > 0$, and U be a smooth bounded domain in $\mathbb{R}^N$. There exists a constant $K_0 > 0$, depending only on U, M and α, but independent of ε, such that for every map $v_\varepsilon \in H^1(U,\mathbb{C})$ verifying the bound*

$$E_\varepsilon(v_\varepsilon, U) = \frac{1}{2}\int_U |\nabla v_\varepsilon|^2 + \frac{1}{4\varepsilon^2}\int_U (1-|v_\varepsilon|^2)^2 \leq M|\log\varepsilon|, \tag{2.3}$$

we have

$$||Jv_\varepsilon||_{[C_c^{0,\alpha}(U)]^*} \leq K_0. \tag{2.4}$$

Remark 2.1. Theorem 2.1 extends immediately to the case U is a compact manifold. Note that in case U is a compact manifold without boundary, then $C_c^{0,\alpha}(U) = C^{0,\alpha}(U)$.

The proof of Theorem 2.1 relies on reduction to the two dimensional case by slicing arguments, and then on lower bounds established in [19] (see also [34]) which generalize earlier results in [9, 17].

A different proof, which works also for $\mathbb{R}^k$-valued maps, has been derived independently in [1] using Geometric Measure Theory: the strategy is to approximate the Jacobian of v_ε by an integer multiplicity polyhedral current with uniformly bounded mass, and then apply Federer–Fleming compactness theorem (see [16, 35]).

Finally, a totally different approach is provided in [2]: it relies on the parabolic regularization of v_ε, as in [3], and on the regularity theory of [11].

In order to prove Theorem 1 and Theorem 1bis, we will make use of the following variant of Theorem 2.1.

Proposition 2.1. *Let $\alpha > 0$ and let Ω be a smooth domain. Let $g_\varepsilon : \partial\Omega \to \mathbb{C}$ satisfying either*
i) $(H2)$ *and* $(H3)$
or
ii) $(H3bis)$.
Then there exists a constant $K_1 > 0$ depending on α, either M_1 or M_2, Ω, but independent of ε, such that for every map $v_\varepsilon \in H^1(\Omega,\mathbb{C})$ verifying (2.3) and $v_\varepsilon = g_\varepsilon$ on $\partial\Omega$ we have

$$||Jv_\varepsilon||_{[C^{0,\alpha}(\bar\Omega)]^*} \leq K_1. \tag{2.5}$$

Remark that condition i) yields some compactness on g_ε, whereas this is not the case for condition ii). In case assumption i) holds, the proof of Proposition 2.1 was given in [7]. We will give here the proof under the assumption ii) (it is actually even simpler than under hypothesis i)), and then recall some elements of the proof in case i).

The idea, in both cases, is to extend the map v_ε to some larger domain G containing Ω, in such a way that the Ginzburg–Landau energy as well as the Jacobian of the extension remain controlled. Then we apply Theorem 2.1 to this particular extension of v_ε on the larger domain G.

Proof of Proposition 2.1 assuming i). For $x \in \mathbb{R}^N$ set $d(x) := \operatorname{dist}(x,\Omega)$. Consider, for $\delta > 0$ the set $W_\delta = \{x \in \mathbb{R}^N \setminus \bar\Omega,\ d(x) < \delta\}$. For $\delta_0 > 0$ sufficiently small, on $W_0 \equiv W_{\delta_0}$ the nearest-point projection $\pi : W_0 \to \partial\Omega$ is well-defined and smooth, and its restriction to each level set $d^{-1}(t)$ in W_0 gives rise to diffeomorphisms $\pi_t : d^{-1}(t) \to \partial\Omega$.

Remark 2.2. Recall that for each $x \in W_0$, d is differentiable at x and $|\nabla d(x)| = 1$. Moreover, $\pi(x) = x - d(x)\nabla d(x)$, hence for each level t, $||\nabla\pi_t - I||_{L^\infty(W_0)} \leq Ct$, where the constant C depends only on the curvature of $\partial\Omega$.

Set $G = W_0 \cup \bar\Omega$. We extend v_ε to a map $\tilde v_\varepsilon$ defined on G by setting

$$\begin{cases} \tilde v_\varepsilon(x) = v_\varepsilon(x) & \text{for } x \in \bar\Omega\,, \\ \tilde v_\varepsilon(x) = g_\varepsilon(\pi(x)) & \text{for } x \in W_0\,. \end{cases}$$

For the Ginzburg–Landau energy $E_\varepsilon(\tilde v_\varepsilon, G)$ of $\tilde v_\varepsilon$ on G we have the straightforward estimate (by (H3bis) and Rem. 2.2)

$$E_\varepsilon(\tilde v_\varepsilon, G) = E_\varepsilon(v_\varepsilon, \Omega) + E_\varepsilon(\tilde v_\varepsilon, W_0) \le (M_0 + C\delta_0 M_2)|\log\varepsilon|, \tag{2.6}$$

with C depending on the curvature of $\partial\Omega$.

We turn now to the Jacobian $J\tilde v_\varepsilon$. Observe first that $J\tilde v_\varepsilon = \pi_t^* Jg_\varepsilon$ on each level set $d^{-1}(t) \subset W_0$, while obviously $J\tilde v_\varepsilon = Jv_\varepsilon$ in Ω.

Let $f : [0, \delta_0] \to [0, 1]$ a smooth cut-off function such that $f(t) = 1$ for $t < \delta_0/3$, $f(t) = 0$ for $t > 2\delta_0/3$. We will extend any smooth 2-form $\zeta \in C^\infty(\bar\Omega; \Lambda^2\mathbb{R}^N)$ to a (Lipschitz continuous) 2-form $\tilde\zeta$ compactly supported in G as follows: on $\partial\Omega$ decompose $\zeta = \zeta_\top + \zeta_N$ where $\zeta_\top$ and ζ_N denote respectively the tangential and the normal part of ζ with respect to $\partial\Omega$ (see for instance the Appendix of [11] for notations). Then, for a fixed $0 < t < \delta_0$, set

$$\tilde\zeta(x) = f(t)\left[(\pi_t^{-1})^*\zeta_\top(x) + \zeta_N(\pi(x))\right], \qquad \text{for each } x \in d^{-1}(t). \tag{2.7}$$

A simple calculation yields, for each $\alpha > 0$, $||\tilde\zeta||_{C^{0,\alpha}(G)} \le C||\zeta||_{C^{0,\alpha}(\bar\Omega)}$, where C depends only on $||f||_{C^{0,\alpha}}$ and the curvature of $\partial\Omega$.

In view of (2.6) and Theorem 2.1 we have

$$\left|\int_G J\tilde v_\varepsilon \cdot \tilde\zeta\right| \le ||J\tilde v_\varepsilon||_{[C_c^{0,\alpha}(G)]^*}||\tilde\zeta||_{C^{0,\alpha}(G)} \le C_1||\zeta||_{C^{0,\alpha}(\bar\Omega)}. \tag{2.8}$$

We compute, using the coarea formula and Remark 2.2,

$$\begin{aligned}\left|\int_{W_0} J\tilde v_\varepsilon \cdot \tilde\zeta\right| &= \left|\int_{W_0} |\nabla d|\pi^* Jg_\varepsilon \cdot \tilde\zeta\right| = \left|\int_0^{\delta_0} \mathrm{d}t \int_{d^{-1}(t)} f(t)\pi_t^* Jg_\varepsilon \cdot (\pi_t^{-1})^*\zeta_\top\right| \\ &\le C\delta_0\left|\int_{\partial\Omega} Jg_\varepsilon \cdot \zeta_\top\right| \le C\delta_0||Jg_\varepsilon||_{[C^{0,\alpha}(\partial\Omega)]^*}||\zeta||_{C^{0,\alpha}(\partial\Omega)} \\ &\le C_2||\zeta||_{C^{0,\alpha}(\bar\Omega)},\end{aligned} \tag{2.9}$$

where the last inequality follows from assumption (H3bis) and Remark 2.1 in the case $U = \partial\Omega$. Combining (2.8) and (2.9) we finally deduce

$$\left|\int_\Omega Jv_\varepsilon \cdot \zeta\right| \le \left|\int_G J\tilde v_\varepsilon \cdot \tilde\zeta\right| + \left|\int_{W_0} J\tilde v_\varepsilon \cdot \tilde\zeta\right| \le (C_1 + C_2)||\zeta||_{C^{0,\alpha}(\bar\Omega)}, \tag{2.10}$$

and the conclusion follows.

Proof of Proposition 2.1 assuming ii). The argument is based also on an extension of v_ε to a larger domain G. However, the construction is different and is more involved. It yields a control of the Jacobian of the extension in the (stronger) L^1 norm. It is based on the following lemma, proved in [7].

Lemma 2.1 ([7], Prop. 4)**.** *Let U be a smooth bounded domain of $\mathbb{R}^N$. There exists a constant $K_2 > 0$ depending only on U such that for every $\gamma \in H^{1/2}(\partial U, S^1)$ there exists $w_\varepsilon \in H^1(U, \mathbb{R}^2)$ verifying*

$$w_\varepsilon = \gamma \qquad \text{on } \partial U \tag{2.11}$$

$$E_\varepsilon(w_\varepsilon, U) \le K_2||\gamma||^2_{H^{1/2}(\partial U)}|\log\varepsilon| \tag{2.12}$$

$$||Jw_\varepsilon||_{L^1(U)} \le K_2||\gamma||^2_{H^{1/2}(\partial U)}. \tag{2.13}$$

We recall that a map w_ε verifying (2.11, 2.12) was already constructed in [13] (Th. 5). Using a projection argument of [18], a different construction was provided in [32]. The construction in [7] follows the ideas of [18,32].

With Lemma 2.1 at our disposal, the proof is completed as follows: define $G = W_0 \cup \bar{\Omega}$ as in case i), and set $U = W_0$, so that $\partial U = \partial\Omega \cup \partial G$. Let γ be the function defined on ∂U by $\gamma \equiv g$ on $\partial\Omega$ and $\gamma \equiv 1$ on ∂G, so that $||\gamma||_{H^{1/2}(\partial U)} \leq C||g||_{H^{1/2}(\partial\Omega)}$. Let w_ε be the function defined on U as in Lemma 2.1, and set

$$\begin{cases} \tilde{v}_\varepsilon(x) = v_\varepsilon(x) & \text{for } x \in \Omega, \\ \tilde{v}_\varepsilon(x) = w_\varepsilon(x) & \text{for } x \in U. \end{cases}$$

In particular $\tilde{v}_\varepsilon \in H^1(G, \mathbb{C})$, and clearly

$$E_\varepsilon(\tilde{v}_\varepsilon, G) \leq E_\varepsilon(v_\varepsilon, \Omega) + E_\varepsilon(w_\varepsilon, U) \leq C(M_0 + M_1)|\log\varepsilon|. \tag{2.14}$$

Let ζ and $\tilde{\zeta}$ be as in case i). In view of (2.14) and Theorem 2.1 we deduce

$$\left|\int_G J\tilde{v}_\varepsilon \cdot \tilde{\zeta}\right| \leq K||\tilde{\zeta}||_{C_c^{0,\alpha}(G)} \leq CK||\zeta||_{C^{0,\alpha}(\bar{\Omega})}, \tag{2.15}$$

where K depends on $\delta_0, M_0, M_1, \alpha$. On the other hand $||Jw_\varepsilon||_{L^1(U)} \leq C||g||^2_{H^{1/2}} \leq CM_1$ by (2.13), so that

$$\left|\int_U Jw_\varepsilon \cdot \tilde{\zeta}\right| \leq CM_1||\tilde{\zeta}||_{L^\infty(U)} \leq CM_1||\tilde{\zeta}||_{C_c^{0,\alpha}(G)} \leq CK||\zeta||_{C^{0,\alpha}(\bar{\Omega})}, \tag{2.16}$$

and the conclusion follows arguing as for (2.10) using (2.15) and (2.16).

Next, we adapt the previous discussion to a situation which we will encounter in the parabolic case. For that purpose, let $T > 0$ and set

$$\Lambda_T = \Omega \times [0, T] \subset \mathbb{R}^{N+1}.$$

For $0 \leq t \leq T$, we consider the slices $\Omega_t = \Omega \times \{t\}$, so that

$$\partial\Lambda_T = \partial\Omega \times [0, T] \cup \Omega_0 \cup \Omega_T.$$

In what follows, ∇v_ε denotes the gradient of v_ε with respect to spatial variables, whereas $\tilde{\nabla} v_\varepsilon$ represents the gradient with respect to all space-time variables $(x^1, \ldots, x^N, t)$. Similarly, Jv_ε (resp. $\tilde{J}v_\varepsilon$) will denote the spatial component of the Jacobian (resp. the Jacobian with respect to all variables $x^1, \ldots, x^N, t$).

Let $g_\varepsilon : \partial\Omega \to \mathbb{C}$ be given. We consider functions $v_\varepsilon : \Lambda_T \to \mathbb{C}$ verifying

$$v_\varepsilon(x, t) = g_\varepsilon(x) \qquad \text{in } \partial\Omega \times [0, T]. \tag{2.17}$$

We will assume that for some positive constant M_0

$$\int_{\Omega_t} |\nabla v_\varepsilon|^2 + \frac{1}{2\varepsilon}(1 - |v_\varepsilon|^2)^2 \leq M_0|\log\varepsilon| \qquad \text{for } t = 0, T, \tag{2.18}$$

$$\int_{\Lambda_T} |\tilde{\nabla} v_\varepsilon|^2 + \frac{1}{2\varepsilon}(1 - |v_\varepsilon|^2)^2 \leq M_0(T + 1)|\log\varepsilon|. \tag{2.19}$$

Proposition 2.2. *Let $\alpha > 0$, $T > 0$, and $v_\varepsilon : \Lambda_T \to \mathbb{C}$. Assume that v_ε verifies (2.17, 2.18) and (2.19). Assume moreover that g_ε verifies either (H2) and (H3), or (H3bis). Then there exists a constant $K_2 > 0$ depending on M_0, either M_1 or M_2, α, Ω and T but independent on ε, such that*

$$||\tilde{J}v_\varepsilon||_{[C^{0,\alpha}(\bar{\Lambda}_T)]^*} \leq K_2. \tag{2.20}$$

Proof. Let W_0 and G be as in the proof of Proposition 2.1 and consider the domain $Q_T = G \times (-1, T+1)$, so that $\Lambda_T \subset\subset Q_T$. We construct an extension $\check{v}_\varepsilon$ of v_ε to Q_T setting,

$$\begin{cases} \check{v}_\varepsilon(x,t) = \tilde{v}_\varepsilon^t(x)\,, & \forall\, t \in [0,T],\ \forall\, x \in W_0 \\ \check{v}_\varepsilon(x,t) = \tilde{v}_\varepsilon^0(x), & \forall\, t \in (-1,0),\ \forall x \in W_0 \\ \check{v}_\varepsilon(x,t) = \tilde{v}_\varepsilon^T(x)\,, & \forall\, t \in (T,T+1)\ \ \forall x \in W_0, \end{cases}$$

where, for a map $w : \Omega \to \mathbb{C}$, $\tilde{w}$ defines its extension to the domain G as in the proof of Proposition 2.1 (the definition is different in case (H3) and in case (H3bis)). Similarly, for a test function $\zeta \in C^{0,\alpha}(\bar{\Lambda}_T, \Lambda^2\mathbb{R}^{N+1})$, we define its extension $\check{\zeta}$ to the larger domain Q_T, by

$$\begin{cases} \check{\zeta}(x,t) = \tilde{\zeta}^t(x) & \forall\, t \in [0,T]\,,\ \forall x \in W_0 \\ \check{\zeta}(x,t) = \chi(t)\tilde{\zeta}^0(x) & \forall\, t \in (-1,0)\,,\ \forall x \in W_0 \\ \check{\zeta}(x,t) = \chi(t)\tilde{\zeta}^T(x) & \forall\, t \in (T,T+1)\,,\ \forall x \in W_0. \end{cases}$$

Here, for a test function $\psi \in C^{0,\alpha}(\bar{\Omega}; \Lambda^2\mathbb{R}^{N+1})$, $\tilde{\psi}$ denotes its extension to the domain G as in the proof of Proposition 2.1 (again, the definition is different in case (H3) and in case (H3bis)). The function $\chi : \mathbb{R} \to \mathbb{R}^+$ denotes a cut-off function such that $\chi(t) = 0$ if $t \le -1$ or $t \ge T+1$.

We then have

$$E_\varepsilon(\check{v}_\varepsilon, Q_T) \le C E_\varepsilon(v_\varepsilon, \Lambda_T) + E_\varepsilon(v_\varepsilon^0, \Omega) + E_\varepsilon(v_\varepsilon^T, \Omega) \le C M_0 (T+1) |\log \varepsilon|.$$

Therefore, we may apply Theorem 2.1 to $\check{v}_\varepsilon$ on Q_T, to assert that

$$\left| \int_{Q_T} J\check{v}_\varepsilon \cdot \check{\zeta} \right| = \left| \int_{\Lambda_T} J v_\varepsilon \cdot \zeta + \int_{Q_T \setminus \Lambda_T} J\check{v}_\varepsilon \cdot \check{\zeta} \right| \le C ||\check{\zeta}||_{C^{0,\alpha}(Q_T)} \le C ||\zeta||_{C^{0,\alpha}(\bar{\Lambda}_T)}.$$

Arguing as in the proof of Proposition 2.1, we estimate the integral of $J\check{v}_\varepsilon \cdot \check{\zeta}$ on the three components of $Q_T \setminus \Lambda_T = W_0 \times [0,T] \cup G \times (-1,0) \cup G \times (T, T+1)$, and complete the proof as above.

3. Hodge–de Rham decomposition

3.1. Splitting of the energy

Let $x_0 \in \Omega$, and assume that $v : \Omega \to \mathbb{C}$ is smooth, and $v(x_0) \ne 0$. Then $v \ne 0$ in some open neighborhood U of x_0, so that we may write

$$v(x) = \rho(x) \exp(i\phi(x))\,, \qquad \text{for } x \in U, \tag{3.1}$$

where $\rho = |v|$ and ϕ is a real-valued function on U, defined up to an integer multiple of 2π. Moreover,

$$\nabla v = \exp(i\phi)\nabla\rho + i\rho\exp(i\phi)\nabla\phi \tag{3.2}$$

and $|\nabla v|^2$ splits as

$$|\nabla v|^2 = |\nabla\rho|^2 + \rho^2 |\nabla\phi|^2.$$

From (3.2) we also notice that

$$v \times \nabla v = \rho^2 \nabla\phi.$$

When v vanishes somewhere in Ω, we are not able to define the function ϕ to the whole of Ω. However, $v \times \nabla v$ is globally well-defined on Ω, and we have, as above, the identity

$$|v|^2|\nabla v|^2 = |v|^2|\nabla|v||^2 + |v \times \nabla v|^2,$$

and hence

$$|\nabla v|^2 = |\nabla|v||^2 + |v \times \nabla v|^2 + (1-|v|^2)(|\nabla v|^2 - |\nabla|v||^2).$$

Since $|\nabla v|^2 \geq |\nabla|v||^2$, this yields

$$|\nabla v|^2 \leq |\nabla|v||^2 + |v \times \nabla v|^2 + |1-|v|^2||\nabla v|^2,$$

so that

$$|\nabla v|^2 \leq |\nabla|v||^2 + |v \times \nabla v|^2 + \sqrt{2}\varepsilon \left[\frac{(1-|v|^2)^2}{4\varepsilon^2} + \frac{|\nabla v|^2}{2}\right],$$

i.e.

$$|\nabla v|^2 \leq |\nabla|v||^2 + |v \times \nabla v|^2 + \sqrt{2}\varepsilon\, e_\varepsilon(v). \tag{3.3}$$

In view of (3.3), for every $1 \leq p \leq 2$ there exists some constant C_p depending on p and Ω such that

$$\int_\Omega |\nabla v|^p \leq C_p \left[\int_\Omega |\nabla|v||^p + \int_\Omega |v \times \nabla v|^p + (\varepsilon E_\varepsilon(v))^{p/2}\right]. \tag{3.4}$$

Remark 3.1. Assume v verifies $E_\varepsilon(v) \leq M_0|\log\varepsilon|$. Then, for $0 < \varepsilon < 1$ we have $\varepsilon|\log\varepsilon| \leq 1$, and consequently

$$\varepsilon E_\varepsilon(v) \leq M_0.$$

3.2. Hodge–de Rham decomposition for $v \times \nabla v$

The Hodge–de Rham decomposition asserts that every l-form μ on a simply connected domain Ω can be decomposed as

$$\mu = dH + d^*\Phi,$$

where H is a $(l-1)$-form on Ω , Φ represents a $(l+1)$-form, d represents the exterior derivative, and $d^* = \pm \star d \star$ (here $\star$ denotes the Hodge operator). In general there is no uniqueness of such a decomposition. We may therefore impose auxiliary conditions, in particular on the boundary. A common choice of auxiliary conditions is

$$\begin{cases} d^*H &= 0\,, \qquad d\Phi = 0 \quad \text{in } \Omega, \\ H_\top &= 0\,, \qquad \Phi_\top = 0 \quad \text{on } \partial\Omega. \end{cases}$$

These conditions ensure uniqueness of the decomposition. Moreover, for any $1 < p < +\infty$ there exists a constant C_p depending on p and Ω, such that

$$||H||_{W^{1,p}} + ||\Phi||_{W^{1,p}} \leq C_p||\mu||_{L^p}. \tag{3.5}$$

Next let $v : \Omega \to \mathbb{C}$ be a function in H^1. We apply the previous decomposition to the 1-form $v \times dv$, where $dv = \sum_{i=1}^N \partial_i v\, dx^i$. Therefore,

$$v \times dv = d\varphi \; + d^*\psi, \tag{3.6}$$

where φ is a real-valued function in Ω (and hence $d^*\varphi = 0$) and ψ is a 2-form in Ω, such that

$$\begin{cases} d\psi = 0 & \text{in } \Omega, \\ \psi_\top = 0 \quad \varphi = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.7}$$

Applying the d operator to (3.6), we obtain

$$2Jv = d(v \times dv) = dd^*\psi = dd^*\psi + d^*d\psi = -\Delta\psi \qquad \text{in } \Omega.$$

Hence ψ is solution to the boundary problem

$$\begin{cases} -\Delta\psi = 2Jv & \text{in } \Omega \\ \psi_\top = 0\,, \quad (d^*\psi)_\top = (v \times dv)_\top & \text{on } \partial\Omega. \end{cases} \tag{3.8}$$

This elliptic problem determines the 2-form ψ uniquely as a function of Jv and the boundary value of v on $\partial\Omega$.

3.3. $W^{1,p}$ estimates for ψ

In this section we prove:

Proposition 3.1. *Let $1 \le p < \frac{N}{N-1}$, and assume $v \equiv v_\varepsilon : \Omega \to \mathbb{C}$ verifies (H1), (H2), and (H3) or (H3bis), and let $\psi_\varepsilon \equiv \psi$ be given by (3.6, 3.7). Then we have*

$$\int_\Omega |\nabla\psi_\varepsilon|^p \le K_p$$

where K_p is a constant depending only on p, M_0, M_1 and M_2 in case (H3bis).

Remark 3.2. Here we **do not** assume that v_ε is a solution of $(\mathrm{GL})_\varepsilon$. Proposition 3.1 ensures compactness of the "$d^*\psi$" component of $v \times dv$. This part accounts in particular for topological obstructions to the lifting property (3.1).

In order to prove Proposition 3.1, we need the following linear estimate related to (3.8) (this estimate is standard for functions).

Lemma 3.1. *Let $1 < p < +\infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Let $l \in \mathbb{N}$, $1 \le l \le N$. Let φ and ω be l-forms on Ω, and A be an $(l-1)$-form on $\partial\Omega$. Assume that*

$$\begin{cases} -\Delta\varphi = \omega & \textit{in } \Omega \\ \varphi_\top = 0, \qquad (d^*\varphi)_\top = A & \textit{on } \partial\Omega. \end{cases}$$

There exists some constant C depending only on Ω and p such that

$$||\varphi||_{W^{1,p}(\Omega)} \le C\left(||\omega||_{[W^{1,q}(\Omega)]^*} + ||A||_{[W^{1-\frac{1}{q},q}(\partial\Omega)]^*}\right).$$

We apply Lemma 3.1 to $\omega = 2Jv$ and $A = (v \times dv)_\top = g_\varepsilon \times dg_\varepsilon$. Since $1 \le p < \frac{N}{N-1}$, we have $q > N$, and, for $\alpha = 1 - \frac{N}{p}$, we recall the embedding $W^{1,q}(\Omega) \hookrightarrow C^{0,\alpha}(\bar{\Omega})$. By duality we therefore have the embedding

$$[C^{0,\alpha}(\bar{\Omega})]^* \hookrightarrow [W^{1,q}(\Omega)]^*. \tag{3.9}$$

If v_ε verifies (H1), (H2), (H3) or (H3bis), by Proposition 2.1 we have $||Jv_\varepsilon||_{[C^{0,\alpha}(\bar\Omega)]^*} \le C$, hence by (3.9) we have

$$||Jv_\varepsilon||_{[W^{1,q}(\Omega)]^*} \le C_p. \tag{3.10}$$

In order to apply Lemma 3.1 it remains to estimate $||g_\varepsilon \times dg_\varepsilon||_{[W^{1-\frac{1}{q},q}(\partial\Omega)]^*}$.

Proposition 3.2. *Let $q > N$ and $g \in H^{1/2}(\partial\Omega;\mathbb{C})$ such that $||g||_{L^\infty(\partial\Omega)} \le 1$. Then there is a constant $C_q > 0$ depending only on q and Ω such that*

$$||g \times dg||_{[W^{1-\frac{1}{q},q}(\partial\Omega)]^*} \le C_q \left(||g||^2_{H^{1/2}(\partial\Omega)} + ||g||_{H^{1/2}(\partial\Omega ega)}\right).$$

Proof. Let $\xi \in W^{1-\frac{1}{q},q}(\partial\Omega)$. Note that we have the embedding $W^{1-\frac{1}{q},q}(\partial\Omega) \hookrightarrow C^0 \cap H^{1/2}(\partial\Omega)$. On the other hand, we recall that $H^{1/2} \cap L^\infty(\partial\Omega)$ is an algebra, therefore since $|g| \le 1$, ξg belongs to $H^{1/2} \cap L^\infty(\partial\Omega)$ and

$$\begin{aligned}||\xi g||_{H^{1/2}(\partial\Omega)} &\le C\left(||\xi||_{L^\infty}||g||_{H^{1/2}} + ||g||_{L^\infty}||\xi||_{H^{1/2}}\right)\\ &\le C\left(||\xi||_{W^{1-\frac{1}{q},q}(\partial\Omega)}||g||_{H^{1/2}(\partial\Omega)} + ||\xi||_{W^{1-\frac{1}{q},q}(\partial\Omega)}\right)\\ &\le ||\xi||_{W^{1-\frac{1}{q},q}(\partial\Omega)}\left(1 + ||g||_{H^{1/2}(\partial\Omega)}\right).\end{aligned}$$

Finally, we have

$$\begin{aligned}\left|\int_{\partial\Omega} \xi(g \times dg)\right| &\le ||dg||_{H^{-1/2}(\partial\Omega)}||\xi g||_{H^{1/2}(\partial\Omega)}\\ &\le C||g||_{H^{1/2}(\partial\Omega)}||\xi||_{W^{1-\frac{1}{q},q}(\partial\Omega)}\left(1 + ||g||_{H^{1/2}(\partial\Omega)}\right),\end{aligned} \tag{3.11}$$

and the conclusion follows:

Proposition 3.2bis. *Let $q > N$ and $g \in H^{1/2}(\partial\Omega;\mathbb{C})$ such that*

$$\frac{1}{2}\int_{\partial\Omega} |\nabla g_\varepsilon|^2 + \frac{1}{4\varepsilon^2}\int_{\partial\Omega}(1 - |g_\varepsilon|^2)^2 \le M_2|\log\varepsilon|.$$

Then for some constant $K_q > 0$ depending only on q, Ω, M_2 and $||g||_{H^{1/2}(\partial\Omega)}$, we have

$$||g \times dg||_{[W^{1-\frac{1}{q},q}(\partial\Omega)]^*} \le K_q.$$

Proof. Consider the function $\tilde g$ defined on $\partial\Omega$ by $\tilde g = g$ if $|g| \le 1$, $\tilde g = g/|g|$ otherwise. We have $\tilde g \times d\tilde g = g \times dg$ if $|g| \le 1$, and $\tilde g \times d\tilde g = \frac{1}{|g|^2} g \times dg$, if $|g| \ge 1$. Therefore

$$|g \times dg - \tilde g \times d\tilde g| \le ||g|^2 - 1||dg| \le \sqrt{2}\varepsilon\left(\frac{(1 - |g_\varepsilon|^2)^2}{4\varepsilon^2} + \frac{|\nabla g|^2}{2}\right). \tag{3.12}$$

Let ξ be as in Proposition 3.2. We have, as in (3.11),

$$\left|\int_{\partial\Omega} \xi(\tilde g \times d\tilde g)\right| \le C||\xi||_{W^{1-\frac{1}{q},q}(\partial\Omega)}\left(||g||^2_{H^{1/2}(\partial\Omega)} + ||g||_{H^{1/2}(\partial\Omega)}\right). \tag{3.13}$$

On the other hand, in view of (3.12), we deduce

$$\left|\int_{\partial\Omega} \xi(\tilde g \times d\tilde g - g \times dg)\right| \leq C||\xi||_{W^{1-\frac{1}{q},q}(\partial\Omega)} M_2\varepsilon|\log\varepsilon|. \tag{3.14}$$

Combining (3.13) with (3.14) we are led to

$$\left|\int_{\partial\Omega} \xi(g \times dg)\right| \leq C||\xi||_{W^{1-\frac{1}{q},q}(\partial\Omega)} \left(||g||^2_{H^{1/2}(\partial\Omega)} + ||g||_{H^{1/2}(\partial\Omega)} + M_2\varepsilon|\log\varepsilon|\right),$$

and the conclusion follows.

Proof of Proposition 3.1. In view of Lemma 3.1 and (3.8), we have

$$||\psi_\varepsilon||_{W^{1,p}(\Omega)} \leq C\left(||Jv_\varepsilon||_{[W^{1,q}(\Omega)]^*} + ||g_\varepsilon \times dg_\varepsilon||_{W^{1-\frac{1}{q},q}(\partial\Omega)}\right),$$

where q is such that $\frac{1}{p} + \frac{1}{q} = 1$. The conclusion follows from (3.10), and Proposition 3.2 (in case (H3bis), from Prop. 3.2bis).

4. Hodge–de Rham decomposition on Λ_T

We will consider here a situation specially tailored for the parabolic case.

As in Section 2 consider, for $T > 0$, the cylinder $\Lambda_T = \Omega \times [0,T] \subset \mathbb{R}^{N+1}$. Let g_ε satisfy (H2), and (H3) or (H3bis). We will consider maps $v_\varepsilon : \Lambda_T \to \mathbb{C}$; recall that for $0 \leq t \leq T$, we have defined

$$v_\varepsilon^t : \Omega \to \mathbb{C}, \qquad v_\varepsilon^t(x) = v_\varepsilon(x,t).$$

We will assume throughout this section that v_ε verifies (2.17–2.19), and

$$|v_\varepsilon| \leq 1, \tag{4.1}$$

$$||v_\varepsilon^0||_{H^{1/2}(\Omega)} \leq M_3. \tag{4.2}$$

We also recall that $\tilde\nabla$ represents the gradient in $\mathbb{R}^{N+1}$, and we denote by δ the exterior derivative in $\mathbb{R}^{N+1}$, and $\delta^* = \pm \star \delta \star$, where $\star$ is the Hodge operator on $\mathbb{R}^{N+1}$.

Proposition 4.1. *Let v_ε be as above. Then there exist a function Φ, a 1-form χ and a 2-form Ψ on Λ_T, such that*

$$\begin{cases} v_\varepsilon \times \delta v_\varepsilon = \delta\Phi + \delta^*\Psi + \chi, \\ \delta\Psi = 0 \quad \text{in } \Lambda_T, \\ \Phi = 0 \quad \text{on } \Omega_0 \cup \partial\Omega \times [0,T], \qquad \Psi_\top = 0 \quad \text{on } \partial\Lambda_T. \end{cases} \tag{4.3}$$

Moreover, for $1 \leq p < \frac{N+1}{N}$, there exist constants C_p and $0 < \alpha < 1$, depending on p, T, Ω, such that

$$||\tilde\nabla\Psi||_{L^p(\Lambda_T)} \leq C_p\,, \quad ||\chi||_{L^p(\Lambda_T)} \leq C_p\varepsilon^\alpha. \tag{4.4}$$

Comment. Although the statement of Proposition 4.1 looks very similar to that of Proposition 3.1, we have to point out a major difference: on $\Omega_T \subset \partial\Lambda_T$ no uniform bound (and hence no compactness) is assumed for v_ε. In particular, this is the reason why we **do not** impose $\Phi = 0$ on Ω_T.

Proof of Proposition 4.1. We decompose the proof into two steps. The first step corresponds to a Hodge–de Rham decomposition of v_ε on $\Omega_T \subset \partial\Lambda_T$: this allows to analyse the possible lack of compactness on this portion of $\partial\Lambda_T$. In the second step, we use a gauge transformation to remove this possible lack of compactness.

Step 1: HdR decomposition on Ω_T. In view of the results of Section 3, we write

$$v_\varepsilon^T \times dv_\varepsilon^T = d\varphi^T + d^*\psi^T \qquad \text{in } \Omega, \tag{4.5}$$

where φ^T (resp. ψ^T) is a function (resp. a 2-form) defined on Ω, satisfying

$$d\psi^T = 0 \quad \text{in } \Omega\,, \qquad \varphi^T = 0 \text{ and } \psi^T_\top = 0 \qquad \text{on } \partial\Omega.$$

By Proposition 3.1 and assumption (2.19), we have, for $1 \le p < \frac{N}{N-1}$,

$$||\nabla\psi^T||^p_{L^p(\Omega)} \le C_p\,, \qquad ||\nabla\psi^T||^2_{L^2(\Omega)} + ||\nabla\varphi^T||^2_{L^2(\Omega)} \le CM_0|\log\varepsilon|. \tag{4.6}$$

We consider the harmonic extension Φ_0 of φ^T to Λ_T defined by

$$\begin{cases} \tilde\Delta\Phi_0 = 0 & \text{in } \Lambda_T, \\ \Phi_0 = 0 & \text{on } \partial\Lambda_T \setminus \Omega_T, \\ \Phi_0(x,T) = \varphi^T(x) & \text{on } \Omega_T. \end{cases}$$

By (4.6) and standard estimates, we have

$$||\tilde\nabla\Phi_0||^2_{L^2(\Lambda_T)} \le CM_0|\log\varepsilon|. \tag{4.7}$$

Step 2: "Gauge transformation" of v_ε. On Λ_T we consider the map $w_\varepsilon : \Lambda_T \to \mathbb{C}$ defined by

$$w_\varepsilon = v_\varepsilon \exp(-i\Phi_0) \qquad \text{in } \Lambda_T.$$

Note that $|w_\varepsilon| = |v_\varepsilon|$. The computations in Section 3 yield

$$w_\varepsilon \times \delta w_\varepsilon = v_\varepsilon \times \delta v_\varepsilon - |v_\varepsilon|^2\delta\Phi_0 = v_\varepsilon \times \delta v_\varepsilon - \delta\Phi_0 + (1-|v_\varepsilon|^2)\delta\Phi_0 \qquad \text{in } \Lambda_T. \tag{4.8}$$

Since $|v_\varepsilon| \le 1$ by assumption (4.1), we have

$$|\nabla w_\varepsilon| \le |\nabla v_\varepsilon| + |\nabla\Phi_0|,$$

hence it follows, from (4.7) and (2.19),

$$||\nabla w_\varepsilon||^2_{L^2(\Lambda_T)} + \varepsilon^{-2}||(1-|w_\varepsilon|^2)||^2_{L^2(\Lambda_T)} \le CM_0|\log\varepsilon|. \tag{4.9}$$

Note also that, since $|w_\varepsilon| \le 1$ by (4.1) and $||(1-|v_\varepsilon|^2)||^2_{L^2(\Lambda_T)} \le M_0(T+1)\varepsilon^2|\log\varepsilon|$, it follows by Hölder's inequality and (4.7) that, for $1 \le p < 2$,

$$||(1-|v_\varepsilon|^2)\delta\Phi_0||^p_{L^p(\Lambda_T)} \le CM_0(T+1)\varepsilon^{2-p}|\log\varepsilon|, \tag{4.10}$$

and similarly

$$||(1-|v_\varepsilon|^2)d\varphi^T||^p_{L^p(\Omega)} \le CM_0\varepsilon^{2-p}|\log\varepsilon|. \tag{4.11}$$

Next, we apply the HdR decomposition to w_ε on Λ_T, so that

$$w_\varepsilon \times dw_\varepsilon = \delta\Phi_1 + \delta^*\Psi \,,$$

where Φ_1 (resp. Ψ) is a function (resp. a 2-form) defined on Λ_T such that

$$\delta\Psi = 0 \quad \text{in } \Lambda_T \,, \qquad \Phi_1 = 0 \text{ and } \Psi_\top = 0 \quad \text{on } \partial\Lambda_T. \tag{4.12}$$

The equation for Ψ is then $-\tilde{\Delta}\Psi = 2\tilde{J}w_\varepsilon$ in Λ_T, with boundary condition $\Psi_\top = 0$ on $\partial\Lambda_T$, and

$$(\delta^*\Psi)_\top = w_\varepsilon \times dw_\varepsilon = v_\varepsilon \times dv_\varepsilon - |v_\varepsilon|^2 d\varphi^T = d^*\Psi^T + (1-|v_\varepsilon|^2)d\varphi^T \text{ on } \Omega_T.$$

In view of (4.6) and (4.11), we have for $1 \le p < \frac{N+1}{N} < \frac{N}{N-1}$,

$$||(\delta^*\Psi)_\top||_{L^p(\Omega_T)} \le C_p \,,$$

and hence, since $L^p(\Omega_T) = [L^q(\Omega_T)]^* \subset [W^{1-1/q,q}(\Omega_T)]^*$, for $\frac{1}{p}+\frac{1}{q}=1$, it follows using the same arguments as in the proof of Proposition 3.1,

$$||\tilde{\nabla}\Psi||_{L^p(\Lambda_T)} \le C_p \,, \qquad \forall\, 1 \le p < \frac{N+1}{N} \,. \tag{4.13}$$

Finally, going back to (4.8), we have

$$v_\varepsilon \times \delta v_\varepsilon = w_\varepsilon \times dw_\varepsilon - \delta\Phi_0 + (1-|v_\varepsilon|^2)\delta\Phi_0 = \delta(\Phi_0+\Phi_1) + \delta^*\Psi + (1-|v_\varepsilon|^2)\delta\Phi_0.$$

We set

$$\Phi = \Phi_0 + \Phi_1 \,, \qquad \chi = (1-|v_\varepsilon|^2)\delta\Phi_0 \,,$$

so that $v_\varepsilon \times \delta v_\varepsilon = \delta\Phi + \delta^*\Psi + \chi$. This completes the proof, in view of (4.10, 4.12) and (4.13).

5. The elliptic equation

In this section, we turn to solutions $u \equiv u_\varepsilon$ of $(\mathrm{GL})_\varepsilon$ verifying (H1), (H2), (H3) or (H3bis). It follows from the analysis in Section 3.1, equations (3.4) and (3.6), that

$$\int_\Omega |\nabla u|^p \le C_p \left[\int_\Omega |\nabla\rho|^p + \int_\Omega |\nabla\varphi|^p + \int_\Omega |\nabla\psi|^p + (\varepsilon E_\varepsilon(u))^{p/2} \right], \tag{5.1}$$

where $\rho = |u|$, and $\varphi \equiv \varphi_\varepsilon$ and $\psi \equiv \psi_\varepsilon$ are such that

$$\begin{cases} u \times du = d\varphi + d^*\psi & \text{in } \Omega, \\ d\psi = 0 & \text{in } \Omega, \\ \varphi = 0 & \psi_\top = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{5.2}$$

In order to prove Theorem 1 and Theorem 1bis it suffices, in view of (5.1), Remark 3.1 and Proposition 3.1, to bound φ and ρ.

5.1. φ vanishes!

As a consequence of $(\mathrm{GL})_\varepsilon$, it turns out that $\varphi = 0$. Indeed, taking the exterior product of $(\mathrm{GL})_\varepsilon$ by the solution u, we derive

$$\operatorname{div}(u \times \nabla u) = u \times \Delta u = 0.$$

In the formalism of differential forms, we may rewrite previous identity as $d^*(u \times du) = 0$. Since $u \times du = d\varphi + d^*\psi$, it follows that φ verifies

$$\begin{cases} -\Delta\varphi = d^*d\varphi = 0 & \text{in } \Omega \\ \varphi = 0 & \text{on } \partial\Omega\,, \end{cases}$$

so that $\varphi = 0$ in Ω.

5.2. $W^{1,p}$ estimates for ρ

The equation for $\rho^2 = |u|^2$ is

$$-\Delta\rho^2 + 2|\nabla u|^2 = \frac{2}{\varepsilon^2}\rho^2(1-\rho^2). \tag{5.3}$$

Proposition 5.1. *Let u be a solution of $(GL)_\varepsilon$ verifying (H1). Let $1 \leq p < 2$. There exists some constant K_p and $0 < \alpha < 1$ depending only on p, M_0 and Ω, such that, for $0 < \varepsilon < 1$,*

$$\int_\Omega |\nabla\rho|^p \leq K_p\varepsilon^\alpha\,.$$

Proof. It is similar to the proof of ([11], *Erratum*). We introduce the set

$$A = \{x \in \Omega,\ \rho(x) > 1 - \varepsilon^{1/2}\}$$

and the function

$$\bar\rho = \max\{\rho, 1 - \varepsilon^{1/2}\},$$

so that $\bar\rho = \rho$ on A and $0 \leq 1 - \bar\rho \leq \varepsilon^{1/2}$ in Ω.

Next let ζ_ε be a function in $C_c^\infty(\Omega)$ such that $0 \leq \zeta_\varepsilon \leq 1$ on Ω, $\zeta_\varepsilon \equiv 1$ on $\Omega_\varepsilon \equiv \{x \in \Omega,\ \operatorname{dist}(x, \partial\Omega) \geq \varepsilon^{1/2}\}$, and $|\nabla\zeta_\varepsilon| \leq C\varepsilon^{-1/2}$, where the constant C depends only on Ω.

Finally, we multiply equation (5.1) by $\zeta_\varepsilon(\bar\rho^2 - 1)$ (which is compactly supported in Ω), and integrate over Ω. We obtain

$$\int_\Omega \nabla\rho^2\nabla\bar\rho^2\zeta_\varepsilon + \int_\Omega \frac{2\rho(1-\rho^2)(1-\bar\rho^2)}{\varepsilon^2}\zeta_\varepsilon = \int_\Omega (1-\bar\rho^2)|\nabla u|^2 + \int_\Omega \nabla\rho^2\nabla\zeta_\varepsilon(1-\bar\rho^2).$$

It follows that on the set $A_\varepsilon = \Omega_\varepsilon \cap A$ we have

$$\begin{aligned} \int_{A_\varepsilon} |\nabla\rho^2|^2 = \int_{A_\varepsilon} \nabla\rho^2\nabla\bar\rho^2 &\leq \varepsilon^{1/2}\int_\Omega |\nabla u|^2 + \frac{2C}{\varepsilon^{1/2}}\int_\Omega |\nabla\rho||1-\rho^2| \\ &\leq \varepsilon^{1/2}\int_\Omega |\nabla u|^2 + C(2\varepsilon)^{1/2}\left[\int_\Omega |\nabla\rho|^2 + \int_\Omega \frac{(1-\rho^2)^2}{2\varepsilon^2}\right], \end{aligned}$$

hence, since $\rho \geq 1 - \varepsilon^{1/2}$ on A_ε, we have, for $\varepsilon \leq 1/4$,

$$\int_{A_\varepsilon} |\nabla\rho|^2 \leq 4\int_{A_\varepsilon} |\nabla\rho^2|^2 \leq 16C\varepsilon^{1/2}E_\varepsilon(u) \leq 16CM_0\varepsilon^{1/2}|\log\varepsilon|. \tag{5.4}$$

Set $W_\varepsilon = \Omega \setminus \Omega_\varepsilon$, $B = \Omega \setminus A$, so that

$$\Omega = B \cup A_\varepsilon \cup W_\varepsilon. \tag{5.5}$$

From (H1) we deduce $\int_B (1-\rho^2)^2 \le 4M_0\varepsilon^2|\log\varepsilon|$ and hence, since $(1-\rho) \ge \varepsilon^{1/2}$ on B, it follows $|B| \le 4M_0\varepsilon|\log\varepsilon|$. Thus

$$\int_B |\nabla\rho|^p \le \left(\int_\Omega |\nabla\rho|^2\right)^{p/2} |B|^{1-p/2} \le C|\log\varepsilon|^{p/2}(\varepsilon|\log\varepsilon|)^{1-p/2},$$

i.e.

$$\int_B |\nabla\rho|^p \le C\varepsilon^{1-p/2}|\log\varepsilon|. \tag{5.6}$$

Finally, we turn to W_ε. Clearly, by construction $|W_\varepsilon| \le C\varepsilon^{1/2}$. Hence

$$\int_{W_\varepsilon} |\nabla\rho|^p \le \left(\int_\Omega |\nabla\rho|^2\right)^{p/2} |W|^{1-p/2} \le C\varepsilon^{1/2-p/4}|\log\varepsilon|^{p/2}. \tag{5.7}$$

Combining (5.5) with (5.4, 5.6) and (5.7) we derive the desired conclusion.

5.3. **Proof of Theorem 1 and Theorem 1bis completed**

Combining the results of Proposition 5.1, Proposition 3.1, Remark 3.1, and the fact that $\varphi = 0$, we deduce from (5.1) the conclusions of Theorem 1 and Theorem 1bis.

Remark 5.1. Let $\varepsilon_n \to 0$ be a subsequence such that $u_{\varepsilon_n} \rightharpoonup u_*$ in $W^{1,p}(\Omega)$ for every $1 \le p < \frac{N}{N-1}$, $g_{\varepsilon_n} \rightharpoonup g_*$ in $H^{1/2}(\partial\Omega)$, $\psi_{\varepsilon_n} \rightharpoonup \psi_*$ in $W^{1,p}(\Omega; \Lambda^2\mathbb{R}^N)$. Passing to the limit in (5.2), we have

$$u_* \times du_* = d^*\psi_*.$$

We may also pass to the limit in the equation for ψ_ε, so that we are led to

$$\begin{cases} -\Delta\psi_* = 2Ju_* & \text{in } \Omega, \\ \psi_\top = g_* \times dg_* & \text{on } \partial\Omega. \end{cases}$$

In particular, ψ_*, and hence u_*, belong to $C^\infty(\Omega \setminus \operatorname{supp}(Ju_*))$.

6. The parabolic equation

In this section, we turn to solutions $u \equiv u_\varepsilon$ of $(\mathrm{PGL})_\varepsilon$ verifying (H4), (H2), and (H3) or (H3bis). Applying Proposition 4.1 to u, we write

$$u \times \delta u = \delta\Phi + \delta^*\Psi + \chi \qquad \text{in } \Lambda_T, \tag{6.1}$$

where Φ is a function, χ a 1-form, Ψ a 2-form defined on Λ_T, such that $\delta\Psi = 0$ in Λ_T, $\Psi_\top = 0$ on $\partial\Lambda_T$,

$$\Phi = 0 \qquad\qquad \text{on } \Omega_0 \cup \partial\Omega \times [0,T], \tag{6.2}$$

and for $1 \leq p < \frac{N+1}{N}$, there are constants $C_p > 0$, $0 < \alpha_p < 1$, such that

$$||\tilde{\nabla}\Psi||_{L^p(\Lambda_T)} \leq C_p\,, \qquad ||\chi||_{L^p(\Lambda_T)} \leq C_p \varepsilon^{\alpha_p}. \tag{6.3}$$

Moreover, we have, for $1 \leq p < 2$,

$$\int_{\Lambda_T} |\nabla u|^p \leq C_p \left(M_0 (T+1)^{p/2} + \int_{\Lambda_T} |\nabla \rho|^p + |\nabla \Phi|^p + |\tilde{\nabla}\Psi|^p + |\chi|^p \right), \tag{6.4}$$

so that in order to prove Theorem 4 it suffices to bound ρ and Φ.

6.1. L^p estimates for $\nabla \rho$

The equation for ρ^2 is

$$\frac{\partial \rho^2}{\partial t} - \Delta \rho^2 + 2|\nabla u|^2 = \frac{2}{\varepsilon^2}\rho^2(1-\rho^2) \quad \text{in } \Lambda_T. \tag{6.5}$$

We then have:

Proposition 6.1. *Let $u = u_\varepsilon$ be a solution of $(GL)_\varepsilon$ verifying (H4), (H2), and (H3) or (H3bis). Let $\rho = |u|$, and $1 \leq p < 2$. There exists some constant $K_p > 0$ and $0 < \alpha < 1$ such that, for every $0 < \varepsilon < 1$,*

$$\int_{\Lambda_T} |\nabla \rho|^p \leq K_p \varepsilon^\alpha.$$

Proof. It is similar to the proof of Proposition 5.1. One introduces $\bar{\rho} = \max\{\rho, 1-\varepsilon^{1/2}\}$ on Λ_T, and $\zeta_\varepsilon : \Omega \to \mathbb{R}^+$, as there. Multiplying (6.5) by $\bar{\rho}\zeta_\varepsilon$ on Λ_T one obtains, computing as for (5.4),

$$\int_{A_\varepsilon} |\nabla \rho^2|^2 \leq \varepsilon^{1/2} \int_{\Lambda_T} |\nabla u|^2 + C\varepsilon^{1/2} \int_{\Lambda_T} |\nabla \rho|^2 + \frac{(1-\rho^2)^2}{2\varepsilon^2} + \int_{\Lambda_T} \left| \frac{\partial \rho}{\partial t}(\bar{\rho} - 1) \right|, \tag{6.6}$$

where $A_\varepsilon = \{y \in \Lambda_T\,,\ \rho(y) \geq 1 - \varepsilon^{1/2}\}$. The last term in inequality (6.6) can be bounded by $C\varepsilon \int_{\Lambda_T} |\tilde{\nabla}\rho|^2 + \frac{(1-\rho^2)^2}{4\varepsilon^2}$. Hence

$$\int_{A_\varepsilon} |\nabla \rho|^2 \leq C M_0 (T+1) \varepsilon^{1/2} |\log \varepsilon|. \tag{6.7}$$

The proof is then completed as in the proof of Proposition 5.1.

6.2. L^p estimates for $\nabla \Phi$

Taking the exterior product of $(\mathrm{PGL})_\varepsilon$ with u, we are led to

$$u \times \frac{\partial u}{\partial t} - \operatorname{div}(u \times \nabla u) = 0 \qquad \text{in } \Lambda_T. \tag{6.8}$$

We have, in view of (6.1),

$$\begin{cases} u \times du &= d\Phi + (\delta^*\Psi + \chi) - P_t(\delta^*\Psi + \chi)dt\,, \\ u \times \dfrac{\partial u}{\partial t} &= \dfrac{\partial \Phi}{\partial t} + P_t(\delta^*\Psi + \chi). \end{cases} \tag{6.9}$$

Here, for a 1-form ω on Λ_T, we denote $P_t(\omega)$ its dt component, *i.e.* if $\omega = \omega_1 dx^1 + \cdots + \omega_N dx^N + \omega_t dt$, then $P_t(\omega) = \omega_t$.

Combining (6.8) and (6.9) we are led to the equation for Φ:

$$\frac{\partial \Phi}{\partial t} - \Delta \Phi = d^*(\delta^* \Psi + \chi - P_t(\delta^* \Psi + \chi) dt) - P_t(\delta^* \Psi + \chi) \qquad \text{in } \Lambda_T. \tag{6.10}$$

Set $f = (f_1, \dots, f_N)$, where

$$f_1 dx^1 + \cdots + f_N dx^N = \delta^* \Psi + \chi - P_t(\delta^* \Psi + \chi) dt \,,$$

and let $g = -P_t(\delta^* \Psi + \chi)$. In view of (6.2) and (6.10) we obtain the following initial and boundary value parabolic problem for Φ:

$$\begin{cases} \dfrac{\partial \Phi}{\partial t} - \Delta \Phi & = \operatorname{div} f + g \qquad \text{in } \Lambda_T \,, \\ \Phi(x,0) & = 0 \qquad\qquad\qquad \forall\, x \in \Omega \,, \\ \Phi(x,t) & = 0 \qquad\qquad\qquad \forall\, x \in \partial\Omega \,,\ \forall\, t \in [0,T], \end{cases} \tag{6.11}$$

where, for every $1 \le p < \frac{N+1}{N}$,

$$||f||_{L^p(\Lambda_T;\mathbb{R}^N)} + ||g||_{L^p(\Lambda_T)} \le C_p \,,$$

for some constant C_p independent of ε.

Note that (6.11) is a well posed parabolic problem for Φ. In view of standard parabolic estimates, we deduce

$$\int_{\Lambda_T} |\nabla \Phi|^p \le C_p(T). \tag{6.12}$$

6.3. **Proof of Theorem 4 completed**

Combining (6.3) with (6.12) and the result of Proposition 6.1, we deduce from (6.4) the conclusion of Theorem 4.

6.4. **Proof of Proposition 1**

Statement i) is straightforward. For statement ii) we argue as in Remark 5.1.

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