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AN APPROXIMATION THEOREM FOR SEQUENCES OF LINEAR STRAINS AND ITS APPLICATIONS

Kewei Zhang[1]

Abstract. We establish an approximation theorem for a sequence of linear elastic strains approaching a compact set in L^1 by the sequence of linear strains of mapping bounded in Sobolev space $W^{1,p}$. We apply this result to establish equalities for semiconvex envelopes for functions defined on linear strains *via* a construction of quasiconvex functions with linear growth.

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1. Introduction and main results

This paper establishes an approximation theorem for sequences of linear elastic strains approaching a compact set in the space of symmetric matrices. We apply the result to the study of equality of various semiconvex envelopes for functions defined on the space of linear strains. We show that under a simple coercivity condition, $Q_e(f) = C(f)$ if and only if $R_e(f) = C(f)$, where $Q_e(f)$ and $R_e(f)$ are the quasiconvex and rank-one convex envelopes of f on linear strains respectively. Before we state our main results, let us introduce some notation.

For $A \in M^{n\times n}$ – the space of $n \times n$ real matrices with the standard Euclidean inner product on $\mathbb{R}^{n^2}$, we let $e(A) = (A + A^T)/2$, where A^T is the transpose of A. If u is a smooth mapping from a domain $\Omega \subset \mathbb{R}^n$ to $\mathbb{R}^n$, we call $e(Du(x))$ the linear elastic strain of u where Du is the gradient of u. Let M^n_s and $(M^n_s)^\perp$ be the subspaces of symmetric and skew-symmetric matrices in $M^{n\times n}$ respectively, we see that $e(A) = P_{M^n_s}(A)$, where $P_{M^n_s}$ is the orthogonal projection from $M^{n\times n}$ onto M^n_s. Let $\mathrm{dist}(Y, K) = \inf_{X\in K} |Y - X|$ be the distance function from $Y \in M^n_s$ to a closed set $K \subset M^n_s$. The following is our approximation theorem.

Theroem 1. *Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain and $u_j \in C^\infty_0(\Omega, \mathbb{R}^n)$ such that*

$$\lim_{j\to\infty} \int_\Omega \mathrm{dist}(e(Du_j(x)), \bar{B}_R(0))\mathrm{d}x = 0, \tag{1.1}$$

where $\bar{B}_R(0)$ is the closed ball in M^n_s centered at 0 with radius R. Then there is a subsequence (u_{j_k}) of (u_j), and a sequence of Lipschitz mappings $v_k : \mathbb{R}^n \to \mathbb{R}^n$ such that for every $1 < p < \infty$,

$$\int_{\mathbb{R}^n} |e(Dv_k)|^p \mathrm{d}x \le C(n, p) < +\infty, \quad \text{and} \quad \lim_{k\to\infty} \int_\Omega |e(Du_{j_k}) - e(Dv_k)|\mathrm{d}x = 0. \tag{1.2}$$

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[1] School of Mathematical Sciences, University of Sussex Brighton, BN1 9QH, UK; e-mail: `k.zhang@sussex.ac.uk`

Note to Theorem 1. As kindly pointed to me by I. Fonseca and the referee, there is an alternative proof of Theorem 1 by using A-qausiconvexity. Further details can be found in the remark after the proof of Theorem 1.

Corollary 1. *Let (v_k) be the sequence of Lipschitz mappings given by Theorem 1. Then for each $1 < p < \infty$, there are a skew-symmetric matrix $A(p)_k \in (M^n_s)^\perp$ and some $x_k \in \mathbb{R}^n$ satisfying $\int_\Omega A(p)_k(x - x_k)\mathrm{d}x = 0$ such that the sequence $w_k : \Omega \to \mathbb{R}^n$ defined by*

$$w_k(x) = v_k - \frac{1}{\mathrm{meas}(\Omega)} \int_\Omega v_k(y)\mathrm{d}y - A_k(p)(x - x_k),\ x \in \Omega$$

is bounded in $W^{1,p}(\Omega, \mathbb{R}^n)$,

$$\int_\Omega |Dw_k|^p \mathrm{d}x \le C \int_\Omega |e(Dw_k)|^p \mathrm{d}x \le C_0(p) < +\infty, \qquad \int_\Omega w_k \mathrm{d}x = 0, \tag{1.3}$$

where C, $C_0(p)$ are positive constants independent of k and

$$\lim_{k\to\infty} \int_\Omega |e(Du_{j_k}) - e(Dw_k)|\mathrm{d}x = 0. \tag{1.4}$$

In Theorem 1, the ball $\bar{B}_R(0)$ can be replaced by any compact set $K \subset M^n_s$. Theorem 1 and Corollary 1 are motivated from an approximation result in [38] for $W^{1,1}_0$ approximating sequences of gradients approaching a compact set in $M^{N\times n}$ by a bounded $W^{1,\infty}$ sequence. The statements of Theorem 1 and Corollary 1 are almost optimal in the sense that they are false if $p = +\infty$ [24]. The main tool for establishing Theorem 1 is a generalized version of Liu's Luzin type theorem [25] to the space of bounded deformations $BD(\Omega)$ [12]. Combining the result in [12] with some classical estimates for standard singular integral operators [30] enables us to prove Theorem 1. Corollary 1 then follows from Poincaré's inequality and Korn's inequality [20].

The following is our main application of Theorem 1 and Corollary 1 to equalities of some semiconvex envelopes for functions defined on linear strains. The study of quasiconvex functions defined on linear elastic strains is closely related to the variational approach to material microstructure by using geometrically linear models [9,10,18]. Functions defined on M^n_s with linear growth are important in the theory of plasticity [3,21,35]. For a continuous function $f : M^n_s \to \mathbb{R}$ bounded below, let $Q_e(f)$ and $R_e(f)$ be the quasiconvex and rank-one convex envelopes of f respectively (see Sect. 2 for definitions). We denote by $C(f)$ the convex envelope of f.

Theroem 2. *Suppose $f : M^n_s \to \mathbb{R}$ is continuous and satisfies, for $A \in M^n_s$*

$$\lim_{|A|\to+\infty} \frac{f(A)}{|A|} = +\infty. \tag{1.5}$$

Then $Q_e(f) = C(f)$ if and only if $R_e(f) = C(f)$.

Theorem 2 was established for functions $f : M^{N\times n} \to \mathbb{R}$ [43] under the coercivity condition (1.5) for $A \in M^{N\times n}$. The difference between Theorem 2 and that in [43] is that in the present situation, the function $X \to f(e(X))$ is not coercive in the sense of (1.5) for $X \in M^{n\times n}$. A weaker version of Theorem 2 was proved in [44], *via* an elementary argument for functions defined on linear strains under the assumption that $f(A) \ge c(|A|^2 - 1)$.

The following result on the construction of quasiconvex functions on linear strains with linear growth will be used indirectly to establish Theorem 2 through some of its implications. The construction itself is of independent interest and its proof depends on Theorem 1 and Corollary 1.

Theroem 3. *Suppose $F : M^n_s \to \mathbb{R}$ is quasiconvex on linear strains and is bounded below. Assume that $0 \le F(Y) \le C_0(1 + |Y|^p)$ for $Y \in M^n_s$, and that for some real $\alpha \ge 0$, the sub-level set $K_\alpha := \{Y \in M^n_s : F(Y) \le \alpha\}$*

is compact. Then, for every $1 \le p < +\infty$, the quasiconvex function on linear strains $Q_e \operatorname{dist}^p(A, K_\alpha)$, $A \in M^n_s$ satisfies

$$C_0|Y|^p - C_1 \le Q_e \operatorname{dist}^p(Y, K_\alpha) \le C_2(1 + |Y|^p) \tag{1.6}$$

for $Y \in M^n_s$ and

$$K_\alpha = \{Y \in M^n_s : Q_e \operatorname{dist}^p(Y, K_\alpha) = 0\}, \tag{1.7}$$

where C_0, C_1, C_2 are positive constants and $Q_e \operatorname{dist}^p(Y, K_\alpha)$ is the quasiconvex envelope on linear strains of the p-distance function $\operatorname{dist}^p(Y, K_\alpha)$.

We need the following results on various semiconvex hulls of linear strains for compact sets in M^n_s and their properties to establish Theorem 2. For the $M^{N\times n}$ version of these results, see [33,37,39]. We define two types of quasiconvex hulls on linear strains for closed subsets of M^n_s.

Definition 1.1. Let $K \subset M^n_s$ be closed, for $1 \le p < \infty$, we defined the strong p-quasiconvex hull $\mathbb{Q}^e_p(K)$ and weak p-quasiconvex hull $Q^e_p(K)$ respectively as

$$\mathbb{Q}^e_p(K) = \left\{X \in M^n_s,\ f(A) \le \sup_{Y\in K} f(Y),\ f \text{ quasiconvex on linear strains },\ 0 \le f(A) \le C_f(1 + |A|^p)\right\}\cdot \tag{1.8}$$

$$Q^e_p(K) = \{A \in M^n_s,\ Q_e \operatorname{dist}^p(A,\ K) = 0\};$$

and the strong p-rank-one convex hull of linear strains as

$$\mathbb{R}^e_p(K) = \left\{X \in M^n_s,\ f(A) \le \sup_{Y\in K} f(Y),\ f \text{ rank-one convex on linear strains },\ 0 \le f(A) \le C_f(1 + |A|^p)\right\}\cdot \tag{1.9}$$

Clearly, one has $\mathbb{R}^e_p(K) \subset \mathbb{Q}^e_p(K) \subset Q^e_p(K)$, $\mathbb{R}^e_p(K) \subset \mathbb{R}^e_q(K)$ and $\mathbb{Q}^e_p(K) \subset \mathbb{Q}^e_q(K)$ for $1 \le q \le p < \infty$. We can also show that $Q^e_p(K) \subset Q^e_q(K)$ (see Proof of Th. 3 and (4.1)).

In order to state Theorem 5, we need to define the so-called closed lamination convex hull $L^e_c(K)$ on linear strains for a compact set $K \subset M^n_s$ as follows [44].

Notice that the subspace $(M^n_s)^\perp$ of skew-symmetric matrices does not have rank-one matrices. We say that $A \in M^n_s$ is a compatible linear strain if $\operatorname{span}[A] \oplus (M^n_s)^\perp$ has rank-one matrices. For example, the identity matrix $I \in M^n_s$ is incompatible. Two linear strains A, $B \in M^n_s$ are called compatible if $A - B$ is a compatible linear strain and we call $\{A,\ B\}$ a compatible pair. Clearly, $A \in M^n_s$ is compatible if and only if $\{0,\ A\}$ is a compatible pair. This definition of compatibility is equivalent to that in [22], that is, A, $B \in M^n_s$ are compatible if either $A - B$ is a rank-one matrix or $A - B$ is of rank two and the two non-zero eigenvalues have opposite signs.

A set $K \subset M^n_s$ is called lamination convex on linear strains if for every compatible pair $\{A,\ B\} \subset K$, one has $\{tA + (1-t)B,\ 0 \le t \le 1\} \subset K$. For a compact set $K \subset M^n_s$, the closed lamination convex hull on linear strains $L^e_c(K)$ is the smallest closed lamination convex set on linear strains that contains K.

The closed laminated convex hull for a compact set $K \subset M^{N\times n}$ was defined in [40] motivated from [28]. It is easy to see that $K \subset L^e_c(K) \subset \mathbb{R}^e_p(K) \subset \mathbb{Q}^e_p(K) \subset C(K)$.

Theroem 4. *Let $K \subset M^n_s$ be non-empty and compact. Then $1 \le p < \infty$,*

$$\mathbb{Q}^e_p(K) = Q^e_p(K) = Q^e_1(K). \tag{1.10}$$

The following result is an implication of the results established in [44]. For the convenience of the reader, we give a proof in the Appendix.

Theroem 5. *Let $K \subset M^n_s$ be compact. Then $L^e_c(K) = C(K)$ if and only if $\mathbb{Q}^e_2(K) = C(K)$.*

From Theorem 4 we see that we may replace $\mathbb{Q}^e_2$ by any Q^e_p for $1 \le p < \infty$. Also $\mathbb{R}^e_2(K) = C(K)$ if and only if $\mathbb{Q}^e_2(K) = C(K)$.

In Section 2, we give notation and preliminaries which are needed for establishing our main results. We prove Theorem 1 and Corollary 1 in Section 3. In Section 4 we establish Theorems 2–4 by applying Theorem 1 directly or indirectly followed by some examples. In the Appendix we give a proof of Theorem 5 and establish an elementary result Lemma 4.1 which is needed in the proof of Theorem 3.

2. Notation and preliminaries

Throughout the rest of this paper Ω denotes a bounded open subset of $\mathbb{R}^n$. We denote by $M^{N\times n}$ the space of real $N \times n$ matrices, with norm $|P| = (\operatorname{tr} P^T P)^{1/2}$. In this paper we are mainly interested in the case when $N = n \ge 2$. We let $C^\infty_0(\Omega, \mathbb{R}^n)$ be the space of smooth functions $\phi : \Omega \to \mathbb{R}^n$ having compact support in Ω. We denote the Lebesgue spaces $L^p(\Omega, \mathbb{R}^n)$ and Sobolev spaces $W^{1,p}(\Omega, \mathbb{R}^n)$ and $W^{1,p}_0(\Omega, \mathbb{R}^n)$ for vector-valued functions $u : \Omega \to \mathbb{R}^n$ as usual [2]. As in Section 1, we let M^n_s and $(M^n_s)^\perp$ be the subspaces of symmetric and skew-symmetric matrices respectively. We see that these two subspaces are orthogonal to each other. For $X \in M^{n\times n}$, we let $e(X) = (X + X^T)/2 = P_{M^n_s}(X)$ where $P_{M^n_s}$ is the orthogonal projection from $M^{n\times n}$ to M^n_s. We denote weak convergence of sequences by $\rightharpoonup$. The Lebesgue measure of a measurable set S in $\mathbb{R}^n$ is $\operatorname{meas}(S)$ while the complement of a set $S \subset \mathbb{R}^n$ is S^c. We use various C's to denote positive constants such as $C(n, p)$. In later sections, two $C(n, p)$'s in the same line may not be the same. They just mean positive constants depending only on n and p.

We define the p-distance function from $Y \in M^n_s$ to a set $K \subset M^n_s$ by $\operatorname{dist}^p(Y, K) := \inf_{A\in K} |Y - A|^p$. The following are some results we need later.

Definition 2.1. (See [4, 26].) A continuous function $f : M^{N\times n} \to \mathbb{R}$ is quasiconvex if

$$\int_U f(P + D\phi(x))\,\mathrm{d}x \ge f(P)\operatorname{meas}(U)$$

for every $P \in M^{N\times n}$, $\phi \in C^\infty_0(U; \mathbb{R}^N)$, and every bounded open subset $U \subset \mathbb{R}^n$. A function $f : M^{N\times n} \to \mathbb{R}$ is rank-one convex if for any $A, B \in M^{N\times n}$ with B a rank-one matrix, the function $t \to f(A + tB)$ is convex.

It is well-known now that quasiconvexity implies rank-one convexity [4, 11, 26] while the converse is not true [32]. To construct quasiconvex functions, we need the following

Definition 2.2. (See [11].) Suppose $f : M^{N\times n} \to \mathbb{R}$ is a continuous function. The quasiconvex envelope (rank-one convex envelope, respectively) $Q(f)$ ($R(f)$ respectively) of f is defined by

$$Q(f) = \sup\{g \le f;\ g \text{ quasiconvex}\}, \qquad R(f) = \sup\{g \le f;\ g \text{ rank-one convex}\}\cdot$$

It is well-known [11] that $C(f) \le Q(f) \le R(f) \le f$ and the quasiconvex envelope Qf can be calculated by

$$Qf(P) = \inf_{\phi\in C^\infty_0(\Omega;R^N)} \frac{1}{\operatorname{meas}(\Omega)} \int_\Omega f(P + D\phi(x))\,\mathrm{d}x, \tag{2.1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain. In particular the infimum in (2.1) is independent of the choice of Ω.

For a continuous function $f : M^n_s \to \mathbb{R}$, we say that f is quasiconvex (rank-one convex respectively) on linear strains, if the function $F : M^{n\times n} \to \mathbb{R}$ defined by $F(X) = f(e(X))$ is a quasiconvex (rank-one convex respectively) function.

We define the quasiconvex envelope $Q_e(f)$ and rank-one convex envelope $R_e(f)$ on linear strains for a continuous function $f : M_s^n \to \mathbb{R}$ by

$$Q_e(f) = \sup\{g \le f;\ g \text{ quasiconvex on linear strains}\}$$
$$R_e(f) = \sup\{g \le, f;\ g \text{ rank-one convex on linear strains}\}\cdot$$

The following simple statement is easy to prove.

Proposition 2.1. *For a continuous function $f : M_s^n \to \mathbb{R}$, let $F(X) = f(e(X))$ for $X \in M^{n\times n}$. Then $Q(F(X)) = Q_e(f(e(X)))$.*

Proof. Clearly $Q_e(f(e(X))) \le Q(F(X)) = \sup\{g \le f;\ g \text{ quasiconvex }\}$. However, if we let $D \subset \mathbb{R}^n$ be the unit cube, then

$$Q(F(X)) = \inf_{\phi\in C_0^\infty(D;\mathbb{R}^n)} \int_D F(X + D\phi)\mathrm{d}x = \inf_{\phi\in C_0^\infty(D;\mathbb{R}^n)} \int_D f(e(X + D\phi))\mathrm{d}x := h(e(X)),$$

depending only on $e(X)$. Note that $X \to h(e(X)) = Q(F(X))$ is quasiconvex, hence h is quasiconvex on linear strains and $h \le f$. By definition, $h \le Q_e(f)$. The proof is finished. □

We will use the following theorem concerning the existence and properties of Young measures [5, 19, 34].

Proposition 2.2. *Let (z_j) be a bounded sequence in $L^1(\Omega;\mathbb{R}^s)$. Then there exist a subsequence (z_{j_k}) of (z_j) and a family $(\nu_x)_{x\in\Omega}$ of probability measures on $\mathbb{R}^s$, depending measurably on $x \in \Omega$, such that*

$$f(z_{j_k}) \rightharpoonup \int_{\mathbb{R}^s} f(\lambda)\mathrm{d}\nu_x(\lambda), \qquad \text{in } L^1(\Omega)$$

for every continuous function $f : \mathbb{R}^s \to \mathbb{R}$ such that $(f(z_{j_k}))$ is sequentially weakly relatively compact in $L^1(\Omega)$.

If the sequence z_j is in the form $z_j = Du_j$, where $\Omega \subset \mathbb{R}^n$ is open and bounded, and (u_j) is a bounded sequence in $W^{1,p}(\Omega, \mathbb{R}^N)$ for some $1 < p \le \infty$, then the corresponding Young measure ν_x is called p-gradient Young measures (see [10, 19, 23]). The Young measure is *trivial* if ν_x is a Dirac measure for *a.e.* x. In this case there exists a function u such that ν_x is the Dirac measure at $Du(x)$, and up to a subsequence, $Du_k \to Du$ almost everywhere.

One of the restrictions of p-gradient Young measures [19] is that for every quasiconvex function $f : M^{N\times n} \to \mathbb{R}$, satisfying $|f(X)| \le C(1 + |X|^p)$ when $1 < p < \infty$,

$$\int_{M^{N\times n}} f(\lambda)\mathrm{d}\nu_x \ge f\left(\int_{M^{N\times n}} \lambda\mathrm{d}\nu_x\right) \tag{2.2}$$

for almost every $x \in \Omega$, (see for example, [8, 10, 19]).

For $r > 0$ and $x \in \mathbb{R}^n$, let $B_r(x) = \{y \in \mathbb{R}^n : |y - x| < r\}$ and $\mathrm{meas}(B_r(x)) = \omega_n r^n$, we have ([30]),

Definition 2.3. (The Maximal Function) Let $f \in L^1_{loc}(\mathbb{R}^n)$, we define

$$(Mf)(x) = \sup_{r>0} \frac{1}{\omega_n r^n} \int_{B_r(x)} |f(y)|\,\mathrm{d}y,$$

where ω_n is the volume of the n dimensional unit ball.

We have the following weak-$(1,1)$ and strong-(p,p) estimates [30][p. 5, Th. 1(b)]:

Proposition 2.3. *If $f \in L^1(\mathbb{R}^n)$, then for every $\lambda > 0$,*

$$\mathrm{meas}(\{x \in \mathbb{R}^n : (Mf)(x) > \lambda\}) \leq \frac{C(n)}{\lambda} \int_{\mathbb{R}^n} |f| \, \mathrm{d}x.$$

If $f \in L^p(\mathbb{R}^n)$ for $1 < p < \infty$, there is a constant $C(n,p) > 0$ such that

$$\|(Mf)\|_{L^p(\mathbb{R}^n)} \leq C(n)\|f\|_{L^p(\mathbb{R}^n)},$$

which also implies the weak-(p,p) estimate

$$\mathrm{meas}(\{x \in \mathbb{R}^n : (Mf)(x) > \lambda\}) \leq \frac{C(n,p)}{\lambda^p} \int_{\mathbb{R}^n} |f|^p \, \mathrm{d}x.$$

The following results on convolution operators can be found in ([30], Ch. II., Ths. 3 and 4)

Proposition 2.4. *Let $K : \mathbb{R}^n \to \mathbb{R}$ be a 0-homogeneous function, smooth with mean value zero on the unit sphere S^{n-1}. Then for $f \in L^p(\mathbb{R}^n)$, $1 \leq p < \infty$, define*

$$T_\epsilon(f)(x) = \int_{|y-x|\geq\epsilon} \frac{K(y-x)}{|y-x|^n} f(y)\mathrm{d}y, \qquad \epsilon > 0. \qquad \textit{Then}$$

(a) *there exists a constant $A_p > 0$ independent of $\epsilon > 0$ so that*

$$\|T_\epsilon(f)\|_{L^p(\mathbb{R}^n)} \leq A_p\|f\|_{L^p(\mathbb{R}^n)}, \qquad 1 < p < \infty;$$

(b) $\lim_{\epsilon\to 0} T_\epsilon(f) = T(f)$ *exists in $L^p(\mathbb{R}^n)$ norm $(1 < p < \infty)$ while* $\lim_{\epsilon\to 0} T_\epsilon(f)(x)$ *exists for almost every $x \in \mathbb{R}^n$ when $1 \leq p < \infty$ and*

$$\|T(f)\|_{L^p(\mathbb{R}^n)} \leq A_p\|f\|_{L^p(\mathbb{R}^n)}, \quad 1 < p < \infty;$$

(c) *let $T^*(f)(x) = \sup_{\epsilon>0} |T_\epsilon(f)(x)|$. If $f \in L^1(\mathbb{R}^n)$, then the mapping $f \to T^*(f)$ is of weak type-$(1,1)$, that is,*

$$\mathrm{meas}(\{x \in \mathbb{R}^n : (T^*f)(x) > \lambda\}) \leq \frac{C(n)}{\lambda} \int_{\mathbb{R}^n} |f| \, \mathrm{d}x, \qquad \textit{for all } \lambda > 0; \tag{2.3}$$

(d) *if $1 < p < \infty$, then $\|T^*(f)\|_{L^p(\mathbb{R}^n)} \leq A_p\|f\|_{L^p(\mathbb{R}^n)}$, which implies the weak-(p,p) estimate*

$$\mathrm{meas}(\{x \in \mathbb{R}^n : (T^*f)(x) > \lambda\}) \leq \frac{C(n,p)}{\lambda^p} \int_{\mathbb{R}^n} |f|^p \, \mathrm{d}x. \tag{2.4}$$

The following are some useful estimates for functions in the space of bounded deformations $BD(\Omega)$ and $BD(\mathbb{R}^n)$ [3,12,21]. To simplify the statements, we only state the results for functions in $W^{1,1}$ which is contained in BD.

Let $\mathcal{R}$ be the class of rigid motions in $\mathbb{R}^n$, that is, affine functions of the form $Ax + d$ with A skew-symmetric $n \times n$ matrix and $d \in \mathbb{R}^n$. The following Poincaré type inequality is in [21] for functions in $BD(\Omega)$, however, we only consider functions in $W^{1,1}(\Omega, \mathbb{R}^n) \subset BD(\Omega)$.

Proposition 2.5. *Let $\Omega \subset \mathbb{R}^n$ be a bounded, connected open set with Lipschitz boundary and let $R : BD(\Omega) \to \mathcal{R}$ be a continuous linear mapping which leaves the elements of $\mathcal{R}$ fixed. Then there exists a constant $C(\Omega, R) > 0$ such that*

$$\int_\Omega |u - R(u)|\mathrm{d}x \leq C(\Omega, R) \int_\Omega |e(Du)|\mathrm{d}x, \quad \textit{for all } u \in W^{1,1}(\Omega, \mathbb{R}^n).$$

When $\Omega \subset \mathbb{R}^n$ is an open ball, there is a precise representation of the rigid motion $R(u)$ given by the following result [3] (we still state it for $W^{1,1}$ functions).

Proposition 2.6. *Let $u \in W^{1,1}(\mathbb{R}^n, \mathbb{R}^n)$, $x \in \mathbb{R}^n$ and $\epsilon > 0$. Then there exist a vector $d_\epsilon(e(Du))(x)$ and a skew-symmetric matrix $A_\epsilon(e(Du))(x)$, such that*

$$\int_{B_\epsilon(x)} |u(y) - A_\epsilon(e(Du))(x)(y-x) - d_\epsilon(e(Du))(x)|\mathrm{d}y \leq C(n)\epsilon \int_{B_\epsilon(x)} |e(Du(y))|\mathrm{d}y,$$

where $C(n) > 0$ is a constant. Furthermore $d_\epsilon(\cdot)$ and $A_\epsilon(\cdot)$ can be represented as singular integrals

$$d^i_\epsilon(F)(x) = \sum_{l,m=1}^{n} \int_{|y-x|\geq\epsilon} \frac{\Lambda^i_{lm}(y-x)}{nw_n|y-x|^n} F_{lm}(y)\mathrm{d}y,$$
$$A^{ij}_\epsilon(F)(x) = \sum_{l,m=1}^{n} \int_{|y-x|\geq\epsilon} \frac{\Gamma^{ij}_{lm}(y-x)}{2w_n|y-x|^{n+2}} F_{lm}(y)\mathrm{d}y,$$

where $F \in L^p(\mathbb{R}^n, M^n_s)$, with $1 \leq p < \infty$, Λ and Γ are third and forth order smooth tensors with zero average on S^{n-1}, and Λ is 0-homogeneous and Γ 2-homogeneous respectively.

If we define $A^*(F)(x) = \sup_{\epsilon>0} |A_\epsilon(F)(x)|$, we see from Proposition 2.5 that A^* have the weak-$(1,1)$, strong-(p,p) and weak-(p,p) estimates for $F \in L^p(\mathbb{R}^n, M^n_s)$, $1 \leq p < \infty$.

The following is a simple variation of ([12], Th. 3.1) – the Luzin type theorem for BD functions. In the original statement in [12], it was stated for the case $\lambda = \tau$ with our notation. However, by examining the proof, it is easy to see that the following can be deduced by using the original proof. We also notice that $S_u = \emptyset$ in our setting, where S_u is the singular part of u (see [3]).

Proposition 2.7. *Let $u \in C^\infty_0(\mathbb{R}^n, \mathbb{R}^n)$. We define for any $\lambda > 0$ and $\tau > 0$*

$$E^\lambda = \{x \in \mathbb{R}^n,\ M(e(Du))(x) < \lambda\}, \quad H^\tau = \{x \in \mathbb{R}^n,\ A^*(e(Du))(x) < \tau\},$$

where A^* is defined as above. Then there is a Lipschitz mapping $v_{\lambda,\tau} : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$|v_{\lambda,\tau}(x) - v_{\lambda,\tau}(y)| \leq C(n)(\lambda+\tau)|x-y|, \quad x,\ y \in \mathbb{R}^n$$

such that $u(x) = v_{\lambda,\tau}(x)$ on $W_{\lambda,\tau} = E^\lambda \cap H^\tau$.

We conclude this section by stating a special form of Korn's inequality ([20], Th. 8):

Proposition 2.8. *Let $\Omega \subset \mathbb{R}^n$ be as in Proposition 2.5 and let $u \in W^{1,p}(\Omega, \mathbb{R}^n)$, $1 < p < \infty$. Then there is a skew-symmetric matrix A_u such that*

$$\int_\Omega |Du(x) - A_u|^p \mathrm{d}x \leq C \int_\Omega |e(Du(x))|^p \mathrm{d}x,$$

where $C > 0$ is a constant independent of u and A_u.

3. Proof of Theorem 1 and Corollary 1

We prove Theorem 1 through Lemma 3.1 to Lemma 3.5.

Proof of Theorem 1. The idea of the proof is the following. We use Lemma 3.1 and Lemma 3.2 to show that $e(Du_j)$ is "essentially" bounded in L^∞. Then for a carefully chosen subsequence u_{j_k} we use the Luzin type theorem for BD functions to obtain a sequence (v_k) of Lipschitz mappings such that $v_k(x) = u_{j_k}(x)$

on a large subset W_{j_k} of Ω while the Lipschitz constants of (v_k) are unbounded (Lem. 3.3). Then we show (Lem. 3.4) that the linear strains $e(Dv_k)$ of these Lipschitz functions are in fact, bounded in L^p by using the estimates on A^* defined in Proposition 2.6 and the maximal function of $e(Du_j)$. Finally we prove that $\lim_{k\to\infty}\int_\Omega |e(Du_{j_k} - Dv_k)| = 0$ (Lem. 3.5).

Lemma 3.1. *Under the assumptions of Theorem 1 we have*

$$\lim_{j\to\infty}\int_{\{x\in\Omega,\,|e(Du_j(x))|\geq 2R\}} |e(Du_j(x))|\mathrm{d}x = 0, \tag{3.1}$$

and $|e(Du_j(x))|$ *is equi-integrable on* Ω*.*

Proof. Since $\lim_{j\to\infty}\int_\Omega \mathrm{dist}(e(Du_j(x)), \bar{B}_R(0))\mathrm{d}x = 0$, we see that

$$\begin{aligned}
0 &= \lim_{j\to\infty}\int_\Omega \mathrm{dist}(e(Du_j(x)), \bar{B}_R(0))\mathrm{d}x\\
&\geq \limsup_{j\to\infty}\int_{\{x\in\Omega,\,|e(Du_j(x))|\geq 2R\}} \mathrm{dist}(e(Du_j(x)), \bar{B}_R(0))\mathrm{d}x\\
&\geq \limsup_{j\to\infty} R\,\mathrm{meas}\left(\{x\in\Omega,\ |e(Du_j(x))|\geq 2R\}\right),
\end{aligned}$$

where we have used the fact that when $|e(Du_j(x))|\geq 2R$, $\mathrm{dist}(e(Du_j(x)), \bar{B}_R(0))\geq R$. Next since we have

$$\mathrm{dist}(e(Du_j(x)), \bar{B}_R(0)) \geq |e(Du_j(x))| - R, \tag{3.2}$$

whenever $|e(Du_j(x))| > R$, we see that

$$\begin{aligned}
0 &= \lim_{j\to\infty}\int_\Omega \mathrm{dist}(e(Du_j(x)), \bar{B}_R(0))\mathrm{d}x\\
&\geq \limsup_{j\to\infty}\int_{\{x\in\Omega,\,|e(Du_j(x))|\geq 2R\}} \mathrm{dist}(e(Du_j(x)), \bar{B}_R(0))\mathrm{d}x\\
&\geq \limsup_{j\to\infty}\int_{\{x\in\Omega,\,|e(Du_j(x))|\geq 2R\}} \left(|e(Du_j(x))| - R\right)\mathrm{d}x,
\end{aligned}$$

hence the first conclusion follows. The second claim is easy to prove because $\bar{B}_R(0)$ is compact. In fact, if we let $b_j = \int_\Omega \mathrm{dist}(e(Du_j(x)), \bar{B}_R(0))\mathrm{d}x$, then $b_j \to 0$ as $j\to\infty$. For any measurable subset G of Ω, we have, from (3.2) that

$$b_j \geq \int_G \mathrm{dist}(e(Du_j(x)), \bar{B}_R(0))\mathrm{d}x \geq \int_G |e(Du_j(x))|\mathrm{d}x - R\,\mathrm{meas}(G)$$

so that $\int_G |e(Du_j(x))|\mathrm{d}x \leq b_j + R\,\mathrm{meas}(G)$. The equi-integrability of $|e(Du_j(x))|$ on Ω then follows easily from this inequality. In fact, for any $\epsilon > 0$, if we first choose $\delta_1 = \epsilon/(2R)$, then there is some $N > 0$, such that $b_j < \epsilon/2$ and $\int_G |e(Du_j(x))|\mathrm{d}x \leq \epsilon$ whenever $j > N$ and $\mathrm{meas}(G) < \delta_1$. Let $\delta_2 > 0$ be such that $\int_G |e(Du_j(x))|\mathrm{d}x \leq \epsilon$ when $\mathrm{meas}(G) < \delta_2$ and $1\leq j\leq N$. Now if we take $\delta = \min\{\delta_1, \delta_2\}$, the second conclusion then follows. □

Now, if

$$a_j = \int_{\{x\in\Omega,\,|e(Du_j(x))|\geq 2R\}} |e(Du_j(x))|\mathrm{d}x, \tag{3.3}$$

we see that $a_j \to 0$ as $j\to\infty$ which follows from the proof of Lemma 3.1.

Now we extend u_j to be defined on $\mathbb{R}^n$ by zero, then we may consider the maximal function $M\left(|e(Du_j)|\right)(x)$ of $|e(Du_j(x))|$. We have

Lemma 3.2. *For $\lambda > 2R$, let*

$$E_j^\lambda = \{x \in \mathbb{R}^n,\ M\left(|e(Du_j)|\right)(x) < \lambda\},\ \textit{hence}\ \left(E_j^\lambda\right)^c = \{x \in \mathbb{R}^n,\ M\left(|e(Du_j)|\right)(x) \geq \lambda\}\cdot$$

Let a_j be as defined in (3.3), then

$$\operatorname{meas}\left(\left(E_j^\lambda\right)^c\right) \leq \frac{C(n)a_j}{\lambda - 2R} \to 0 \qquad \textit{as} \qquad j \to \infty. \tag{3.4}$$

Proof. The proof of Lemma 3.2 is very similar to that of ([38], Lem. 3.1). We define

$$h(t) = \begin{cases} 0, & \text{if , } |t| \leq 2R, \\ |t| - 2R, & \text{if , } |t| \geq 2R. \end{cases}$$

Then $h : \mathbb{R} \to \mathbb{R}_+$ is a continuous function. We claim that

$$\{x \in \mathbb{R}^n,\ M\left(|e(Du_j)|\right)(x) \geq \lambda\} \subset \{x \in \mathbb{R}^n,\ M\left(h(|e(Du_j)|)\right)(x) \geq \lambda - 2R\}\cdot \tag{3.5}$$

We prove (3.5) as follows. If $M\left(|e(Du_j)|\right)(x) \geq \lambda$, by definition, there is a sequence of positive numbers $\epsilon_k > 0$ and $r_k > 0$ with $\epsilon_k \to 0$ as $k \to \infty$ such that

$$\frac{1}{\operatorname{meas}(B_{r_k}(x))} \int_{B_{r_k}(x)} |e(Du_j(y))| \mathrm{d}y \geq \lambda - \epsilon_k.$$

Since

$$\begin{aligned} M\left(h(|e(Du_j)|)\right)(x) &\geq \frac{1}{\operatorname{meas}(B_{r_k}(x))} \int_{B_{r_k}(x)} h(|e(Du_j(y))|) \mathrm{d}y \\ &= \frac{1}{\operatorname{meas}(B_{r_k}(x))} \int_{\{y \in B_{r_k}(x),\ |e(Du_j(y))| \geq 2R\}} \left(|e(Du_j(y))| - 2R\right) \mathrm{d}y \\ &= \frac{1}{\operatorname{meas}(B_{r_k}(x))} \int_{B_{r_k}(x)} |e(Du_j(y))| \mathrm{d}y - \frac{1}{\operatorname{meas}(B_{r_k}(x))} \int_{\{y \in B_{r_k}(x),\ |e(Du_j(y))| \geq 2R\}} 2R \mathrm{d}y \\ &\quad - \frac{1}{\operatorname{meas}(B_{r_k}(x))} \int_{\{y \in B_{r_k}(x),\ |e(Du_j(y))| < 2R\}} |e(Du_j(y))| \mathrm{d}y \geq \lambda - \epsilon_k - 2R. \end{aligned}$$

Passing to the limit $k \to \infty$ we obtain $M\left(h(|e(Du_j)|)\right)(x) \geq \lambda - 2R$. Thus (3.5) is proved.

Now from the weak-$(1,1)$ estimate of the maximal function (Prop. 2.4), we have

$$\begin{aligned} \operatorname{meas}\left(\{x \in \mathbb{R}^n,\ M\left(h(|e(Du_j)|)\right)(x) \geq \lambda - 2R\}\right) &\leq \frac{C(n)}{\lambda - 2R} \int_{\mathbb{R}^n} h(|e(Du_j(y))|) \mathrm{d}y \\ &\leq \frac{C(n)}{\lambda - 2R} \int_{\{y \in \Omega,\ |e(Du_j(y))| \geq 2R\}} |e(Du_j(y))| \mathrm{d}y = \frac{C(n)}{\lambda - 2R} a_j \to 0, \end{aligned}$$

as $j \to \infty$, where a_j is defined by (3.3). The proof is finished. □

Since the sequence (a_j) defined by (3.3) converges to zero as $j \to \infty$, we may find a subsequence (a_{j_k}) such that

$$a_{j_k} \leq \mathrm{e}^{-(k+1)}. \tag{3.6}$$

Recall the operator $A^*(e(Du))$ defined following Proposition 2.6. Now we apply Proposition 2.7 to our sequence u_{j_k}.

Lemma 3.3. *Let $\lambda = 4R$ in Lemma 3.2 and let $\tau_k = 4kR$, for $k = 1, 2, \ldots$ Let*

$$H_{j_k} = \{x \in \mathbb{R}^n,\ A^*(e(Du_{j_k}))(x) < \tau_k\}\cdot$$

Then u_{j_k} is a Lipschitz mapping on the set

$$W_{j_k} = E^{\lambda}_{j_k} \cap H_{j_k} = \{x \in \mathbb{R}^n\ M\left(|e(Du_j)|\right)(x) < \lambda, \quad A^*(e(Du_{j_k}))(x) < \tau_k\}, \tag{3.7}$$

satisfying

$$|u_{j_k}(x) - u_{j_k}(y)| \le C(n)(\lambda + \tau_k)|x - y| = C(n)4(1 + k)R|x - y|. \tag{3.8}$$

From Lemma 3.3 and Kirszbraun's theorem [36], there is a Lipschitz extension v_k of u_{j_k} to $\mathbb{R}^n$ such that

$$|v_k(x) - v_k(y)| \le C(n)4(1 + k)R|x - y|, \tag{3.9}$$

for all x, $y \in \mathbb{R}^n$ and

$$|Dv_k(x)| \le C(n)4(1 + k)R, \tag{3.10}$$

for almost every $x \in \mathbb{R}^n$ and $Dv_k(x) = Du_{j_k}(x)$ almost everywhere on W_{j_k} ([16], Lem. 7.7).

Lemma 3.4. *There is a constant $C_0 > 0$ independent of v_k such that*

$$\int_{\mathbb{R}^n} |e(Dv_k(x))|^p \mathrm{d}x \le C_0.$$

Proof. For a measurable set $S \subset \mathbb{R}^n$, we let

$$J_p(w, S) = \int_S |e(Dw(x))|^p \mathrm{d}x, \qquad 1 \le p < \infty \tag{3.11}$$

as long as the right hand side of (3.11) is finite. We then have

$$\int_{\mathbb{R}^n} |e(Dv_k(x))|^p \mathrm{d}x = J_p(v_k, \mathbb{R}^n) = J_p(v_k, W_{j_k}) + J_p(v_k, (W_{j_k})^c).$$

We see that

$$J_p(v_k, W_{j_k}) = J_p(u_{j_k}, W_{j_k}) \le \lambda^p \operatorname{meas}(\Omega) = (4R)^p \operatorname{meas}(\Omega).$$

This follows from the fact that $Dv_k = Du_{j_k}$ almost everywhere on W_{j_k}, u_{j_k} is supported in Ω, $W_{j_k} \subset E^{\lambda}_{j_k}$ and $|e(Du_{j_k}(x))| \le M(|e(Du_{j_k})|)(x) \le \lambda = 4R$ on $E^{\lambda}_{j_k}$.

We also have

$$J_p(v_k, (W_{j_k})^c) = J_p(v_k, (H_{j_k} \cap E^{\lambda}_{j_k})^c) \le J_p(v_k, (H_{j_k})^c \cup (E^{\lambda}_{j_k})^c) \le J_p(v_k, (H_{j_k})^c) + J_p(v_k, (E^{\lambda}_{j_k})^c).$$

Notice that $|Dv_k(x)| \le C(n)4R(k + 1)$ and $\lambda = 4R$, we have

$$\begin{aligned} J_p(v_k, (E^{\lambda}_{j_k})^c) &\le [C(n)4R(k + 1)]^p \operatorname{meas}((E^{\lambda}_{j_k})^c) \le \frac{C(n, p)R^p(k + 1)^p}{(2R)^p} a_{j_k} \\ &\le \frac{C(n, p)R^p(k + 1)^p}{(2R)^p} \mathrm{e}^{-(k+1)} \le C(n, p)(k + 1)^p \mathrm{e}^{-(k+1)} \le C(n, p). \end{aligned} \tag{3.12}$$

To estimate $J_p(v_k,(H_{j_k})^c)$, we write $e(Du_{j_k}(x)) = e_1(Du_{j_k}(x)) + e_2(Du_{j_k}(x))$, where

$$e_1(Du_{j_k}(x)) = \begin{cases} e(Du_{j_k}(x)), & \text{if } |e(Du_{j_k}(x))| < 4R, \\ 0, & \text{if } |e(Du_{j_k}(x))| \geq 4R; \end{cases}$$
$$e_2(Du_{j_k}(x)) = \begin{cases} e(Du_{j_k}(x)), & \text{if } |e(Du_{j_k}(x))| \geq 4R, \\ 0, & \text{if } |e(Du_{j_k}(x))| < 4R. \end{cases}$$

Then from

$$A^*(e(Du_{j_k}))(x) = A^*(e_1(Du_{j_k}) + e_2(Du_{j_k}))(x) \leq A^*(e_1(Du_{j_k}))(x) + A^*(e_2(Du_{j_k}))(x),$$

we have

$$(H_{j_k})^c \subset \{x \in \mathbb{R}^n,\ A^*(e_1(Du_{j_k}))(x) \geq \tau_k/2\} \cup \{x \in \mathbb{R}^n,\ A^*(e_2(Du_{j_k}))(x) \geq \tau_k/2\}\cdot$$

Since Du_{j_k} is supported in Ω and $e_1(Du_{j_k})(x)$ is a bounded sequence in $L^\infty(\mathbb{R}^n)$, we have from the weak-(p,p) estimate of operator A^* (Props. 2.4 and 2.6) that

$$\begin{aligned} \operatorname{meas}\left(\{x \in \mathbb{R}^n,\ A^*(e_1(Du_{j_k}))(x) \geq \tau_k/2\}\right) &\leq \frac{C(n,p)}{(\tau_k)^p} \int_{\mathbb{R}^n} |e_1(Du_{j_k})(y)|^p \mathrm{d}y \\ &= \frac{C(n,p)}{(4Rk)^p} \int_\Omega (4R)^p \mathrm{d}y \leq \frac{C(n,p)}{k^p} \operatorname{meas}(\Omega). \end{aligned}$$

Since $e_2(Du_{j_k})$ is bounded in $L^1(\mathbb{R}^n)$ and is also supported in Ω, we apply the weak-$(1,1)$ estimate to operator A^*:

$$\begin{aligned} \operatorname{meas}\left(\{x \in \mathbb{R}^n,\ A^*(e_2(Du_{j_k}))(x) \geq \tau_k/2\}\right) &\leq \frac{C(n)}{\tau_k} \int_{\mathbb{R}^n} |e_2(Du_{j_k}(y))| \mathrm{d}y \\ &= \frac{C(n)}{4Rk} \int_{\{x\in\Omega,\ |e(Du_{j_k}(y))| \geq 4R\}} |e(Du_{j_k}(y))| \mathrm{d}y \leq \frac{C(n)}{4Rk} a_{j_k} \leq \frac{C(n)}{4Rk\mathrm{e}^{k+1}}\cdot \end{aligned}$$

Thus

$$\operatorname{meas}((H_{j_k})^c) \leq \frac{C(n,p)}{k^p} + \frac{C(n)}{Rke^{(k+1)}} \to 0. \tag{3.13}$$

Consequently,

$$J_p(v_k,(H_{j_k})^c) \leq C(n,p) R^p (1+k)^p \operatorname{meas}((H_{j_k})^c) \leq C(n,p) \left[R^p \left(\frac{1+k}{k}\right)^p + \frac{k^{p-1}}{\mathrm{e}^{k+1}} \right] \leq C(n,p,R),$$

which follows from the simple facts that $(k+1)/k \leq 2$ for $k = 1,2,\ldots$ and $k^{p-1}/\mathrm{e}^{k+1} \to 0$ as $k \to \infty$. Finally, we sum up these inequalities to obtain

$$J_p(v_k,\mathbb{R}^n) = J_p(v_k,\Omega \cap W_{j_k}) + J_p(v_k,(W_{j_k})^c) \leq C(n,p,R) := C_0.$$

The proof is then finished. □

Note from (3.4) and (3.13), we have $\operatorname{meas}(W^c_{j_k}) \to 0$ as $k \to \infty$.

Lemma 3.5. *Let v_k be defined as above, then*

$$\lim_{k\to\infty} \int_\Omega |e(Dv_k(x)) - e(Du_{j_k}(x))| \mathrm{d}x = 0.$$

Proof of Lemma 3.5. Since $e(Dv_k(x)) = e(Du_{j_k}(x))$ almost everywhere on W_{j_k} defined by (3.8), we have

$$\int_\Omega |e(Dv_k(x)) - e(Du_{j_k}(x))|\mathrm{d}x \le \int_{(W_{j_k})^c} |e(Dv_k(x)) - e(Du_{j_k}(x))|\mathrm{d}x$$
$$\le J_1(v_k, (W_{j_k})^c) + J_1(u_{j_k}, \Omega \setminus W_{j_k}).$$

Since $\mathrm{meas}((W_{j_k})^c) \to 0$ as $j \to \infty$, and $J_p(v_k, (W_{j_k})^c)$ is bounded, we have, from Hölder's inequality that

$$0 \le J_1(v_k, (W_{j_k})^c) \le (J_p(v_k, (W_{j_k})^c)^{1/p} \left(\mathrm{meas}\,((W_{j_k})^c)\right)^{\frac{p-1}{p}} \to 0,$$

as $k \to \infty$.

By Lemma 3.1, $|e(Du_{j_k}(x))|$ is equi-integrable on Ω. We see that $J_1(u_{j_k}, \Omega \setminus W_{j_k}) \to 0$ as $k \to \infty$. The conclusion follows. □

Remark. Recently, I. Fonseca and the referee both explained to me that Theorem 1 may also be proved by using the general theory of $\mathcal{A}$-quasiconvex functions, in particular, the work by Fonseca and Müller ([15], Cor. 2.18), where a second order operator $\mathcal{A}$ was proposed to treat the linear elastic strain ([15], Ex. 3.10(e)). In fact, for the orthogonal complement $E^\perp$ of a general subspace $E \subset M^{N\times n}$ without rank-one matrices, if $K \subset E^\perp$ is compact and $\mathrm{dist}(P_{E^\perp}(Du_j), K) \to 0$ in $L^1(\Omega)$, is seems a more promising approach by using the A-quasiconvexity method than the one used here. The only problem is to work out algebraically the operator A (see [15] for details).

Proof of Corollary 1. By Korn's inequality (Prop. 2.8), we have

$$\int_\Omega |Dv_k(x) - A_{v_k}|^p\mathrm{d}x \le C(n,p)\int_\Omega |e(Dv_k(x))|^p\mathrm{d}x. \tag{3.14}$$

Since

$$w_k(x) = v_k(x) - \frac{1}{\mathrm{meas}(\Omega)}\int_\Omega v_k(y)\mathrm{d}y - A_{v_k}(x - x_k),$$

where $x_k \in \mathbb{R}^n$ is such that $\int_\Omega A_{v_k}(x - x_k)\mathrm{d}x = 0$, we see from Poincaré's inequality and (3.14), we see that (w_k) is bounded in $W^{1,p}(\Omega,\mathbb{R}^n)$. Also because $e(Dw_k(x)) = e(Dv_k(x))$ in Ω, the conclusion follows from Lemma 3.5. □

4. Proof of Theorems 2–4 and examples

Proof of Theorem 3. Let us consider the quasiconvex function on linear strains $Q_e\,\mathrm{dist}(A, K_\alpha)$, $A \in M^n_s$. We first prove that $K := \{A \in M^n_s,\ Q_e\,\mathrm{dist}(A, K_\alpha) = 0\} = K_\alpha$. Obviously we have $K_\alpha \subset K$. We only need to show that $K \subset K_\alpha$. Let $A_0 \in K$. We have from Proposition 2.1 for quasiconvex envelope on linear strains that there is a sequence $(\phi_j) \subset C^\infty_0(D,\mathbb{R}^n)$ such that

$$\lim_{j\to\infty}\int_D \mathrm{dist}(A_0 + e(D\phi_j(x)),\ K_\alpha)\mathrm{d}x = 0,$$

where $D \subset \mathbb{R}^n$ is the unit cube. Let $K_{\alpha,A_0} = \{Y - A_0,\ Y \in K_\alpha\}$, then K_{α,A_0} is still compact and $\lim_{j\to\infty}\int_D \mathrm{dist}(e(D\phi_j(x)),\ K_{\alpha,A_0})\mathrm{d}x = 0$. We then have, from Lemma 3.1 that $|e(D\phi_j)|$ is equi-integrable on D, hence up to a subsequence (still denoted by the same subscripts) $e(D\phi_j)$ converges weakly in $L^1(D)$. For each fixed j, we extend ϕ_j to be defined in $\mathbb{R}^n$ as a periodic function and then let

$$u_j(x) = \frac{1}{j}\phi_j(jx)$$

for $j = 1, 2, \dots$ and for $x \in D$. We see that $u_j \in C_0^\infty(D, \mathbb{R}^n)$. Furthermore, $|e(Du_j)|$ is equi-integrable in Ω, up to a subsequence $e(Du_j) \rightharpoonup 0$ in $L^1(D, M_s^n)$ as $j \to 0$ and

$$\lim_{j\to\infty} \int_D \operatorname{dist}(e(Du_j(x)),\ K_{\alpha,A_0})\mathrm{d}x = 0.$$

The properties of u_j mentioned above should be well-known and can be verified easily. However, for the convenience of the reader, we state them in the following slightly general way (Lem. 4.1 below) and give a proof in the Appendix. Note that we also have $\int_D e(D\phi_j)\mathrm{d}x = 0$, hence we only need to prove the claims for each component of $e(D\phi_j)$.

Lemma 4.1. *Suppose $f_j \rightharpoonup f$ in $L^1(D)$ as $j \to \infty$ and $\int_D f_j \mathrm{d}x = 0$, where $D \subset \mathbb{R}^n$ is the (closed) unit cube. Then if we extend f_j to $\mathbb{R}^n$ periodically and let $u_j(x) = f_j(jx)$, $x \in D$, then (u_j) is equi-integrable on D, and up to a subsequence $u_j \rightharpoonup 0$ in $L^1(D)$ as $j \to 0$. Furthermore, for any continuous function $W : \mathbb{R} \to \mathbb{R}$ with linear growth, that is, $|W(t)| \le C(|t| + 1)$, we have $\int_D W(f_j)\mathrm{d}x = \int_D W(u_j)\mathrm{d}x$.*

Proof of Theorem 3 (continued). Now, since $\lim_{j\to\infty} \int_D \operatorname{dist}(e(Du_j(x)),\ K_{\alpha,A_0})\mathrm{d}x = 0$, we see, from Theorem 1 and Corollary 1 that there is a subsequence (u_{j_k}) of (u_j) and a bounded sequence (w_k) in $W^{1,2p}(D, \mathbb{R}^n)$ (hence $|Dw_k|^p$ is equi-integrable on Ω) such that $\|e(Du_{j_k}) - e(Dw_k)\|_{L^1(D)} \to 0$ as $k \to \infty$. We assert that $e(Dw_k) \rightharpoonup 0$ in L^p as $k \to \infty$. If we let ν_x be the family of Young measures corresponding to Dw_k, we have

$$0 = \int_{M^{n\times n}} e(\tau)\mathrm{d}\nu_x(\tau) = e\left(\int_{M^{n\times n}} \tau \mathrm{d}\nu_x(\tau)\right),$$

which follows from the Young measure representation of the weak limit and the fact that $e(\cdot)$ is a linear mapping on $M^{n\times n}$. We also have $\int_{M^{n\times n}} \operatorname{dist}(e(\tau),\ K_{\alpha,A_0})\mathrm{d}\nu_x(\tau) = 0$ for almost every $x \in \Omega$. Hence on the support of ν_x, $A_0 + e(\tau) \in K_\alpha$, ν_x – almost everywhere. Let $F^{(\alpha)}(Y) = \max\{F(Y) - \alpha,\ 0\}$, then $F^{(\alpha)}(\cdot)$ is still a quasiconvex function on linear strains with p-th growth at infinity, while $|e(Dw_k)|^p$ is equi-integrable on D, therefore by the lower semicontinuity theorem [1] and the Young measure representation, we have, up to a subsequence,

$$\begin{aligned} 0 = \int_D \int_{M^{n\times n}} F^\alpha(A_0 + e(\tau))\mathrm{d}\nu_x(\tau)\mathrm{d}x &= \lim_{k\to\infty} \int_D F^\alpha(A_0 + e(w_k(x)))\mathrm{d}x \\ \ge \int_D F^\alpha\left(A_0 + e\left(\int_{M^{n\times n}} \tau \mathrm{d}\nu_x\right)\right)\mathrm{d}x &= \int_D F^\alpha(A_0)\mathrm{d}x \ge 0. \end{aligned}$$

Hence $F^\alpha(A_0) = 0$ and $A_0 \in K_\alpha$.

Finally we show that the quasiconvex function on linear strains $Q_e \operatorname{dist}^p(\cdot,\ K_\alpha)$ satisfies the requirements (1.5) and (1.6). We observe that for $Y \in M_s^n$,

$$(Q_e \operatorname{dist}(Y,\ K_\alpha))^p \le Q_e \operatorname{dist}^p(Y,\ K_\alpha). \tag{4.1}$$

In fact, let $\phi_j \in C_0^\infty(D, \mathbb{R}^n)$ be a minimizing sequence such that

$$Q_e \operatorname{dist}^p(e(Y),\ K_\alpha) = \lim_{j\to\infty} \int_D \operatorname{dist}^p(Y + e(D\phi_j),\ K_\alpha)\mathrm{d}x,$$

where D is the unit cube in $\mathbb{R}^n$. For each $j > 0$, we apply Hölder's inequality to obtain

$$\begin{aligned} Q_e \operatorname{dist}(Y,\ K_\alpha) &\le \int_D Q_e \operatorname{dist}(Y + e(D\phi_j),\ K_\alpha)\mathrm{d}x \\ &\le \int_D \operatorname{dist}(Y + e(D\phi_j),\ K_\alpha)\mathrm{d}x \le \left(\int_D \operatorname{dist}^p(Y + e(D\phi_j),\ K_\alpha)\mathrm{d}x\right)^{1/p}. \end{aligned}$$

Hence (4.1) follows by letting $j \to \infty$. We see that the zero set of $Q_e \operatorname{dist}^p(Y, K_\alpha)$ is contained in that of $Q_e \operatorname{dist}(Y, K_\alpha)$ which is K_α. On the other hand, it is obvious that the zero set of $Q \operatorname{dist}^p(Y, K_\alpha)$ contains K_α. The conclusion then follows. □

Proof of Theorem 4. Since by definition, we have $\mathbb{Q}_p^e(K) \subset Q_p^e(K) \subset Q_1^e(K)$, we only need to prove that $Q_1^e(K) \subset \mathbb{Q}_p^e(K)$ for every $1 \le p < \infty$.

Let $f : M_s^n \to \mathbb{R}$ be quasiconvex on linear strains and $0 \le f(Y) \le C_f(1 + |Y|^p)$ for $Y \in M_s^n$. We let $\alpha = \max_{Y \in K} f(Y)$ and define $K_\alpha = \{Y \in M_s^n,\ f(Y) \le \alpha\}$, then we see that $\mathbb{Q}_p^e(K)$ is the intersection of all such K_α. From Theorem 2, we see that

$$K_\alpha = \{Y \in M_s^n,\ Q_e \operatorname{dist}(Y, K_\alpha) = 0\}\cdot$$

On the other hand, since it is obviously true that $K \subset K_\alpha$, we have $\operatorname{dist}(Y, K_\alpha) \le \operatorname{dist}(Y, K)$ for all $Y \in M_s^n$ so that $0 \le Q_e \operatorname{dist}(Y, K_\alpha) \le Q_e \operatorname{dist}(Y, K)$ hence $Q_1^e(K) \subset K_\alpha$ and $Q_1^e(K) \subset \mathbb{Q}_p^e(K)$. The proof is finished. □

The following lemma will be used in the proof of Theorem 2.

Lemma 4.2. *Suppose $F : M_s^n \to \mathbb{R}$ is continuous and non-negative with $F^{-1}(0) = K$ a non-empty and compact set. Furthermore, assume that F satisfies the coercivity condition $\lim_{|A| \to \infty} F(A)/|A| = +\infty$. Let $(Q_e(F))^{-1}(0) = \{A \in M_s^n,\ Q_e(f(A)) = 0\}$. Then $(Q_e(F))^{-1}(0) \subset Q_1^e(K)$.*

If we take Theorem 4 into account, clearly, Lemma 4.1 also implies $(Q_e(F))^{-1}(0) \subset \mathbb{Q}_p^e(K)$ for all $1 \le p < \infty$.

Proof of Lemma 4.2. Let $A \in (Q_e(F))^{-1}(0)$, then there is a sequence $\phi_j \in C_0^\infty(D, \mathbb{R}^n)$ such that $\lim_{j \to \infty} \int_D F(A + e(D\phi_j))\mathrm{d}x = 0$. It is then easy to see, from the coercivity condition that up to a subsequence, $\operatorname{dist}(A + e(D\phi_j), K) \to 0$ almost everywhere and $e(D\phi_j)$ is equi-integrable. Hence $\int_D \operatorname{dist}(A + e(D\phi_j), K)\mathrm{d}x \to 0$ as $j \to \infty$. By definition, $A \in Q_1^e(K)$, the proof is finished. □

With the help of Theorems 3–5, we now prove Theorem 2.

Proof of Theorem 2. Since $C(f) \le Q_e(f) \le R_e(f)$, we only need to show that if $R_e(f) \ne Cf$ then $Q_e(f) \ne Cf$.

For $f : M_s^n \to \mathbb{R}$, let $\operatorname{epi}(f) = \{(A, t) \in M_s^n \times \mathbb{R},\ f(A) \ge t\}$ be the epi-graph of f. First we claim that there is a supporting plane E (see [29]) of $\operatorname{epi}(Cf)$ in M_s^n such that $K_0 = \operatorname{epi}(R_e f) \cap E$ is not convex while $CK_0 = \operatorname{epi}(C(f)) \cap E$. If this is not true, we can easily see that $R_e(f) = C(f)$ on $K = \mathcal{P}(\mathcal{K}_\prime) = \mathcal{P}(\mathcal{C}(\mathcal{K}_\prime))$, where $K = \mathcal{P}(\mathcal{K}_\prime)$ is the orthogonal projection of $K_0 \subset M_s^n \times \mathbb{R}$ to M_s^n, so that $R_e(f) \equiv C(f)$ (see [29]), and we reach a contradiction.

Now, we use the supporting plane E to construct a non-negative rank-one convex function F on linear strains, vanishing exactly on K with superlinear growth (1.5) so that K is also compact.

Since the plane E is the graph of a real-valued affine function $L(\cdot)$ defined on M_s^n, we see that $R_e(f(\cdot)) - L(\cdot) \ge 0$ and $R_e(f(A)) - L(A) = 0$ if and only if $A \in K$. We also see that K is compact because $R_e(f) \ge Cf$ hence $R_e(f)$ satisfies

$$\lim_{|A| \to \infty} \frac{R_e(f(A))}{|A|} = +\infty. \tag{4.2}$$

Let us consider $F(A) = R_e(f(A)) - L(A)$, for $A \in M_s^n$. Then $X \to F(e(X))$ is rank-one convex, $F \ge 0$, and $F(A) = 0$ if and only if $A \in K$, and

$$\lim_{|A| \to \infty} \frac{F(A)}{|A|} = +\infty. \tag{4.3}$$

Next we show that $Q_e(f(P)) > Cf(P)$ for a certain matrix $P \in M_s^n$, if $Q_e(F(P)) > 0$ holds.

From Proposition 2.1 and the fact that L is affine, we see that

$$Q_e(F(\cdot)) = Q_e[R_e(f(\cdot)) - L(\cdot)] = Q_e(f(\cdot)) - L(\cdot).$$

Therefore, we only need to prove that $Q_e(F)$ is not convex. Since $F \geq 0$, it suffices to show that the zero set of $Q_e(F)$

$$Q_eF^{-1}(0) = \{A \in M^n_s,\, Q_e(F(A)) = 0\}$$

is not convex.

From Theorem 5, we see that $\mathbb{Q}^e_2(K)$ is not convex because the non-convex set K is the zero set of the non-negative rank-one convex function F on linear strains, so it is a closed lamination convex set. We also notice that $C(f(\cdot)) - L(\cdot) \geq 0$ and $C(f(A)) - L(A) = 0$ if and only if $A \in C(K)$.

Let $P \in C(K) \setminus \mathbb{Q}^e_2(K)$. From Theorem 4, we see that $\mathbb{Q}^e_2(K) = \mathbb{Q}^e_1(K) = Q^e_1(K)$. From Lemma 4.2 we see further that $(Q_e(F))^{-1}(0) \subset Q^e_1(K)$. Thus $Q_e(F(P)) > 0$ which implies

$$Q_e(F(P)) = Q_e(f(P)) - L(P) > 0 = Cf(P) - L(P)$$

and $Q_e(f(P)) > Cf(P)$. The proof is finished. □

Now let us examine some examples of quasiconvex functions defined on linear strains with linear growth.

Example 4.1. Let $I \in M^{n\times n}$ be the unit matrix. It was proved in [13] following an argument in [31], that for

$$K_0 = \{-I,\, I\} \subset M^{n\times n},$$

$Q\,\mathrm{dist}(e(X),\, K_0)$ does not vanish at $X = 0$, hence $Q\,\mathrm{dist}(e(X),\, K_0)$ is not convex. In fact, it was established earlier in [41] that for any closed set $K \subset E_\partial$ – the subspace of conformal matrices (or $K \subset E_{\bar\partial}$ – the subspace of anti-conformal matrices) of $M^{2\times 2}$, there is a constant $c > 0$ independent of K such that

$$Q\,\mathrm{dist}(X,\, K) \geq c\,\mathrm{dist}(X,\, K) \tag{4.4}$$

for all $X \in M^{2\times 2}$. This result was generalized in [17] to the case when $K \subset E$ where $E \subset M^{N\times n}$ is a subspace without rank-one matrices, that is

$$Q\,\mathrm{dist}(X,\, K) \geq c(E)\,\mathrm{dist}(X,\, K) \tag{4.5}$$

where $c(E) > 0$ depends only on E. Therefore the following much improved estimate of the above result in [13] for K_0 can be deduced from these earlier results:

$$Q_e\,\mathrm{dist}(Y,\, K_0) \geq c(n)\,\mathrm{dist}(Y,\, K_0). \tag{4.6}$$

In the 2×2 case we observe that $K_0 \times (M^2_s)^\perp \subset E_\partial$, thus from (4.4), we have

$$Q_e\,\mathrm{dist}(e(X),\, K_0) = Q\,\mathrm{dist}(X,\, K_0 \times (M^2_s)^\perp) \geq c\,\mathrm{dist}(X,\, K_0 \times (M^2_s)^\perp) = c\,\mathrm{dist}(e(X),\, K_0). \tag{4.7}$$

For the $n \times n$ case, since it is known [10, 22] that the subspace $E = \mathrm{span}\{I\} \oplus (M^n_s)^\perp$ does not have rank-one matrices, so, we see from (4.4) that (4.5) holds if we replace E_∂ by E.

Example 4.2. From a special case of the explicit calculation of the quasiconvex relaxation for the two linear strain energy [22], we see that the quasiconvex function on linear strains $Q_e\,\mathrm{dist}^2(Y,\, K_0)$ satisfies that

$$Q_e\,\mathrm{dist}^2(Y,\, K_0) = \mathrm{dist}^2(Y,\, K_0)$$

when $\mathrm{dist}^2(Y,\, K_0)$ is small, where $K_0 = \{-I,\, I\} \subset M^n_s$ is as defined in Example 4.1. We see that the sub-level set

$$K_{\alpha^2} = \left\{Y \in M^n_s,\, Q_e\,\mathrm{dist}^2(e(Y),\, K_0) \leq \alpha^2\right\} = \bar{B}_\alpha(I) \cup \bar{B}_\alpha(-I)$$

when $\alpha > 0$ is small, where $\bar{B}_\alpha(I)$ and $\bar{B}_\alpha(-I)$ are closed balls in M^n_s centered at I and $-I$ respectively with radius α. We may also make the two closed balls disjoint. We see that $K_{\alpha^2} = Q^e_1(K_{\alpha^2})$ hence the zero set of the following quasiconvex function on linear strains with linear growth $Q_e\,\mathrm{dist}(Y,\, K_{\alpha^2})$ is K_{α^2} itself.

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Appendix

Proof of Lemma 4.1. Since $f_j \rightharpoonup f$ in $L^1(D)$, we have, from Dunford–Pettis Theorem [14], $|f_j|$ is equi-integrable on D. Thus for every $\epsilon > 0$, there is some $\delta > 0$, such that $\int_E |f_j| \mathrm{d}x \le \epsilon$ for all $j > 0$ whenever $E \subset D$ is measurable and $\mathrm{meas}(E) \le \delta$. We let $jE - m = \{jx - m,\ x \in E\}$ for $m \in \mathbb{R}^n$. Then we can decompose jD into j^n unit cubes (D_s) whose vertex that is closest to the origin m_s, – a vector with integer components between 0 and $j^n - 1$. Also the intersection between different D_s's is only on the boundary. We have $jD = \cup_{s=1}^{j^n} D_s$. Thus, since f_j is now periodic with period D,

$$\begin{aligned}\int_E |u_j(x)| \mathrm{d}x &= \int_E |f_j(jx)| \mathrm{d}x = \frac{1}{j^n} \int_{jE} |f_j(y)| \mathrm{d}y \\ &= \frac{1}{j^n} \sum_{s=1}^{j^n} \int_{jE \cap D_s} |f_j(y)| \mathrm{d}y = \frac{1}{j^n} \sum_{s=1}^{j^n} \int_{(jE \cap D_s) - m_s} |f_j(y)| \mathrm{d}y.\end{aligned}$$

Since $\mathrm{meas}(E) \le \delta$, $\mathrm{meas}(jE) \le j^n \delta$, $\sum_{s=1}^{j^n} \mathrm{meas}[(jE \cap D_s) - m_s] = \mathrm{meas}(jE) \le j^n \delta$ and $(jE \cap D_s) - m_s \subset D$. We may write, for each s, $\mathrm{meas}[(jE \cap D_s) - m_s] = j_s \delta + r_s \delta$ where $j_s \ge 0$ is an integer and $0 \le r_s < 1$. Thus

$$\sum_{s=1}^{j^n} j_s \delta \le \sum_{s=1}^{j^n} (j_s \delta + r_s \delta) \le j^n \delta$$

so that $\sum_{s=1}^{j^n} j_s \le j^n$. We also have $\mathrm{meas}[(jE \cap D_s) - m_s] \le (j_s + 1)\delta$, hence it is easy to prove that $\int_{(jE \cap D_s) - m_s} |f_j(y)| \mathrm{d}y \le (j_s + 1)\epsilon$, which implies

$$\frac{1}{j^n} \sum_{s=1}^{j^n} \int_{(jE \cap D_s) - m_s} |f_j(y)| \mathrm{d}y \le 2\epsilon.$$

Therefore $|u_j|$ is equi-integrable on D.

Again from Dunford–Pettis Theorem, up to a subsequence $u_j \rightharpoonup u$ in $L^1(D)$ for some $u \in L^1(D)$. We only need to show that $u = 0$ almost everywhere. Since we have assumed that D is closed, for each $\phi \in C_0(D)$, we consider

$$\int_D u_j \phi \mathrm{d}x = \int_D f_j(jx)\phi(x) \mathrm{d}x = \frac{1}{j^n} \int_{jD} f_j(y) \phi\left(\frac{y}{j}\right) \mathrm{d}y = \int_D f_j(y) \left(\sum_{s=1}^{j^n} \phi\left(\frac{y}{j} - m_s\right) \frac{1}{j^n}\right) \mathrm{d}y.$$

Then, from the boundedness of f_j in $L^1(D)$, the assumption $\int_D f_j \mathrm{d}x = 0$ and the definition of Riemann integral, we see that $\int_D u_j \phi \mathrm{d}x \to 0$ as $j \to \infty$ hence $u = 0$.

The last claim that $\int_D W(u_j(x)) \mathrm{d}x = \int_D W(f_j(x)) \mathrm{d}x$ can be easily checked by changing the variable $jx = y$ and the periodicity assumption on f_j. □

The following is the proof of Theorem 5 extracted from [44]. We need some preparations.

A quadratic function $q : M_s^n \to \mathbb{R}$ is called a rank-one convex quadratic function on linear strains if $X \to q(e(X)$ is a rank-one convex quadratic function defined on $M^{n \times n}$. We denote by RC_e the set of all rank-one convex quadratic functions on linear strains.

Definition A.1. The quadratic rank-one convex hull $qr^e(K)$ of a compact set $K \subset M^n_s$ is defined by

$$qr^e(K) = \left\{ X \in M^n_s,\ q(X) \leq \sup_{Y \in K} q(Y),\ q \in RC_e \right\}.$$

From the definition of $qr^e(K)$, one can easily see that

$$K \subset L^e_c(K) \subset \mathbb{R}^e_2(K) \leq \mathbb{Q}^e_2(K) \subset qr^e(K) \subset C(K).$$

Therefore if we can prove that $qr^e(K) = C(K)$ if and only if $L^e_c(K) = C(K)$, Theorem 5 will then follows.

Let $E \subset M^n_s$ be a linear subspace without compatible matrices, and $E^\perp$ being its orthogonal complement of E in M^n_s. Let

$$q(A) = |P_{E^\perp}(A)|^2 - \lambda_E |P_E(A)|^2, \tag{A.1}$$

where $P_{E^\perp}$ and P_E are orthogonal projections to $E^\perp$ and E respectively, and $\lambda_E > 0$ is the largest positive number such that the quadratic form q is a rank-one convex (so is quasiconvex) on linear strains. The constant λ_E can be defined as follows. Since E does not have compatible strains, $E \oplus (M^n_s)^\perp$ does not have rank-one matrices. Note that $P_{E^\perp}(a \otimes b) = P_{E^\perp}(e(a \otimes b))$ and $|P_{E^\perp}(a \otimes b)|^2 > 0$ for any nonzero rank-one matrix $a \otimes b$. Let $a,\ b \in \mathbb{R}^n$, we then define

$$\frac{1}{\lambda_E} = \sup_{|a|=|b|=1} \frac{|P_E(e(a \otimes b))|^2}{|P_{E^\perp}(e(a \otimes b))|^2} < +\infty \tag{A.2}$$

and $\lambda_E > 0$ satisfies the requirement.

If E_1 is a plane in M^n_s parallel to E and $X \in E_1$, then

$$q_X(A) = |P_{E^\perp}(A)|^2 - \lambda_E |P_E(A - X)|^2 \tag{A.3}$$

is a quadratic rank-one convex function reaching its strict maximum at X in E_1 with $q_X(X) = 0$ and $q_X(A) < 0$ for $A \in E_1 \setminus \{X\}$. We have

Lemma A.1. *Suppose $E \subset M^n_s$ is a linear subspace without compatible matrices and E_1 is a plane parallel to E. Then any closed subset $K \subset E_1$ is a quadratic rank-one convex set of linear strains, that is, $qr^e(K) = K$.*

Proof. If $K \neq E_1$, then for any $X \in E_1 \setminus K$, we consider q_X defined by (A.3), then $q_X \in RC_e$ and $q_X(X) = 0 > \sup_{A \in K} q_X(A)$. Therefore $X \notin qr^e(K)$. The proof is then finished. □

Proof of Theorem 5. We first show that if E_1 is a supporting plane (see [29]) of $C(K)$ then

$$qr^e(K) \cap E_1 = qr^e(K \cap E_1). \tag{A.4}$$

Let E be the plane in M^n_s containing $C(K)$ with the same dimension as $C(K)$ (see [29]). Obviously, $qr^e(K \cap E_1) \subset qr^e(K) \cap E_1$. Let $X \in qr^e(K) \cap E_1$. There is an affine function l defined on M^n_s such that $l < 0$ on the open half space in E containing $C(K) \setminus E_1$, $l = 0$ on E_1 and $l > 0$ on the opposite half space to $C(K)$ in E. We also define $E_1(\epsilon) = \{A \in E,\ \mathrm{dist}(A, E_1) \leq \epsilon,\ l(A) \leq 0\}$ which is a set on the same side as $C(K)$ in E, where $\mathrm{dist}(A, E_1)$ is the euclidean distance from A to E_1. For any fixed $q \in RC_e$ we consider, for every integer $n > 0$ the quadratic function $q(\cdot) + nl(\cdot) \in RC_e$. Since for any $A \in K \cap E_1$, $l(A) = 0$, we have, for every fixed point $X \in qr^e(K) \cap E_1$,

$$q(X) = q(X) + nl(X) \leq \sup_{A \in K} [q(A) + nl(A)].$$

Since $q + nl$ is continuous and K compact, the maximum is attained at some $A_n \in K$, that is, $\sup_{A \in K}[q(A) + nl(A)] = q(A_n) + nl(A_n)$, so that $q(X) \leq q(A_n) + nl(A_n)$. Since K is compact there is a subsequence $A_{n_k} \to$

$A_0 \in K$ as $k \to 0$. Notice that $l(A_n) \leq 0$ for all n. If we let $k \to \infty$ we see that $\delta_k := \mathrm{dist}(A_{n_k}, E_1) \to 0$. Otherwise $q(X)$ cannot be finite. Now we have

$$q(X) \leq q(A_{n_k}) + n_k l(A_{n_k}) \leq \sup\{q(A),\, A \in K \cap E_1(\delta_k)\}\cdot \tag{A.5}$$

Again the "sup" in (A.5) can be reached by, say $B_k \in K \cap E_1(\delta_l)$, and up to a subsequence $B_k \to B_0 \in K \cap E_1$ as $k \to \infty$.

Passing to the limit $k \to 0$ on both side of the inequality $q(X) \leq q(B_k)$ and noticing that $B_0 \in K \cap E_1$, we have $q(X) \leq q(B_0) \leq \sup_{A \in K \cap E_1} q(A)$, hence $X \in qr^e(K \cap E_1)$, (A.4) is proved. Notice also that $C(K) \cap E_1 = C(K \cap E_1)$.

Now suppose $K \subset M_s^n$ is compact while $L_c^e(K) \neq C(K)$, but $qr^e(K) = C(K)$. We may assume that K is a closed laminated convex set. Then among all these K's there is one for which the affine dimension $\dim C(K) \geq 1$ of $C(K)$ is the smallest. For such K we claim that the plane E in M_s^n spanned by $C(K)$ does not have compatible pairs. Otherwise it is easy to see that there is a supporting plane E_1 of $C(K)$ such that $E_1 \cap K$ is still a closed laminated convex set on linear strains while $qr^e(K \cap E_1) = qr^e(K) \cap E_1 = C(K) \cap E_1$ is convex. This contradicts to the fact that the dimension $\dim C(K)$ is the smallest. Now since $C(K) \subset E$ and E does not have compatible pairs, there is some $X \in C(K) \neq K$. If we define q_X as in Lemma A.1, then there is $\delta > 0$, such that $q_X(X) = 0 > -\delta = \sup_{A \in K \subset E} q(A)$. Hence $X \notin qr^e(K)$ and $qr^e(K) \neq C(K)$, a contradiction. $\square$

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