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DEGENERATE EIKONAL EQUATIONS WITH DISCONTINUOUS REFRACTION INDEX *

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Abstract. We study the Dirichlet boundary value problem for eikonal type equations of ray light propagation in an inhomogeneous medium with discontinuous refraction index. We prove a comparison principle that allows us to obtain existence and uniqueness of a continuous viscosity solution when the Lie algebra generated by the coefficients satisfies a Hörmander type condition. We require the refraction index to be piecewise continuous across Lipschitz hypersurfaces. The results characterize the value function of the generalized minimum time problem with discontinuous running cost.

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Introduction

In this paper we study the following boundary value problem for the eikonal equation

$$\begin{cases} a_{ij}(x)U_{x_i}U_{x_j} = [f(x)]^2, & x \in \Omega; \\ u(x) = G(x), & x \in \partial\Omega, \end{cases} \tag{0.1}$$

where $\Omega \subset \mathbb{R}^N$ is open. Here and in the following we adopt the summation convention on the repeated indices. When the matrix $(a_{ij})_{i,j}$ is positive semidefinite, the equation describes the propagation of rays of light, see Courant and Hilbert [9]. In the classical case, the matrix (a_{ij}) is the identity matrix and f is a positive constant, so the partial differential equation in (0.1) reduces to

$$|DU(x)| = 1.$$

In this case light rays follow straight lines. In a inhomogeneous medium, either the refraction index $n(x) = f^2(x)$ or the matrix of the coefficients (a_{ij}) depend on the position. In the latter case, even when $n(\cdot)$ is constant the light follows the geodesics of the metric defined by the matrix (a_{ij}) which are no longer straight lines, in general. The case of discontinuous f produces the well known refraction phenomenon, described by Snell's law.

Keywords and phrases. Geometric optics, viscosity solutions, eikonal equation, minimum time problem, discontinuous coefficients.

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Geometric optics is not the only physical interpretation of problem (0.1), which instead appears often in models of mathematical physics describing for instance flame front propagation or the limiting behavior of singular perturbation problems. Also in optimal control theory, whose relation we make more explicit in the next section and will be used, problem (0.1) characterizes the value function of the generalized minimum time problem.

When f is continuous, problem (0.1) is well understood and rather complete existence and uniqueness results are known in the literature, in the framework of the theory of viscosity solutions, see *e.g.* Bardi [1], Bardi-Soravia [3], Soravia [21], or the books by Bardi and Capuzzo-Dolcetta [2], Barles [4] and the references therein. Here we will instead concentrate on the case of f discontinuous. While existence of a discontinuous, possibly extended real valued solution satisfying a weaker Dirichlet-type boundary condition is not a problem, being provided by the optimal control approach, problem (0.1) is not expected to have a unique solution in general, at least without a suitable definition of solution and appropriate conditions on f. In our previous paper [22], see also [11], we studied this problem and found explicit formulas for the minimal and maximal nonnegative viscosity solution, proving also a uniqueness result when $(a_{ij}) = I_{N\times N}$ and f is piecewise constant. The case of nondegenerate matrix (a_{ij}) was also studied, with different ideas, by Camilli-Siconolfi [7] who adopt a more stringent notion of solution and allow $f \in L^\infty$. Indeed they can characterize uniquely the solution in their sense which is the maximal among all viscosity solutions.

In this paper we allow the matrix (a_{ij}) to be degenerate and we stick to the standard definition of viscosity solution, which is easier to check and more stable to approximations. Therefore our previous results in [22] make it impossible to obtain uniqueness of solutions of (0.1) if the set of discontinuity points of f has nonempty interior. We will then restrict ourselves to the case when f is piecewise continuous across hypersurfaces. We obtain a comparison principle for the Dirichlet-type problem, a generalization of the classical Dirichlet problem (0.1), which is now classical in the theory of viscosity solutions. Our proof deals with the points of discontinuity of the coefficients in a similar way to the boundary points. The key technical ingredients are adopted from the classical paper on state constraints by Soner [19], as later developed to introduce the Dirichlet type problem by Barles-Perthame [5] and Ishii [13], see also [2, 4, 8, 10]. In particular the observation in [13] that only a nontangential continuity assumption at boundary points is required to one of the two functions to compare, later used in a crucial way also in Katsoulakis [14] for second order constrained problems, see also [4], is precisely what we need for using the Lie brackets. Among the main consequences of our version of the comparison principle, it allows us to obtain existence and uniqueness of a bounded from below and continuous viscosity solution of (0.1), taking up the Dirichlet boundary condition, when for instance the Lie algebra generated by the matrix (a_{ij}) satisfies the Hörmander condition (see Th. 3.4). We notice that such weak conditions for the comparison principle make the existence part rather straightforward while in the literature, even for a continuous f, this usually takes up some work. With more delicate arguments one can obtain the stronger Hölder continuity of the solution, see *e.g.* [20] and the references therein. However the literature in this direction is almost entirely devoted to the case $\Omega = \mathbb{R}^N \backslash \{x_o\}$.

Our results and methods extend in a straightforward way to more general Bellman equations of exit time control problems with discontinuous running cost such as those presented in [22]. In view of the physical interest of equation (0.1) and for the sake of clarity we will omit the more technical generality here. We refer however the reader to the other paper [23] where the uniqueness theory for continuous solutions is extended to a class of degenerate elliptic equations.

We want to mention that the study of discontinuous eikonal equations starts in the book by Lions [15], where it is shown the existence of a Lipschitz continuous almost everywhere solution to the problem with homogeneous boundary condition. Tourin [25] studied the case of the equation of shape-form-shading with homogeneous boundary condition and uniqueness of continuous viscosity solutions when f is piecewise Lipschitz continuous. Newcomb and Su [16] study the boundary value problem with f lower semicontinuous, using the stronger notion of Monge solution which is also explained in [22]. They prove a comparison principle and uniqueness in the class of continuous Monge solutions of the Dirichlet problem. Other results on the eikonal equation can be found in

Ostrov [17]. All of the previous results however exclude the degeneracy of the matrix of the coefficients (a_{ij}) which we take into account in this paper.

We finally want to recall that another chapter of the theory of viscosity solutions for equations with discontinuous coefficients concerns second order elliptic pdes. In the uniformly elliptic case, existence and uniqueness results are available in the literature. For this we just refer the reader to the early paper by Caffarelli, Crandall, Kocan, Swiech [6] where the theory is formulated, and to Swiech [24] and the references therein for further results.

1. Preliminaries and relationships with optimal control

We start this section presenting the main general assumptions that we adopt below. Other key assumptions are discussed later. The boundary data

$$G : \partial\Omega \to [0, +\infty[\text{ is continuous.} \tag{1.1}$$

The matrix of the coefficients satisfies

$$(a_{ij}) = (\sigma_{ik}) \cdot (\sigma^t_{kj}), \tag{1.2}$$

thus it is symmetric, positive semidefinite but possibly degenerate, where $(M \leq N)$

$$\sigma(\cdot) \equiv (\sigma_{ik})_{i=1,\ldots N;k=1,\ldots M} : \overline{\Omega} \to \mathbb{R}^{NM} \text{ is L-Lipschitz continuous.} \tag{1.3}$$

Moreover the function $f : \mathbb{R}^N \to [\rho, +\infty[$, $\rho > 0$ is Borel measurable and possibly discontinuous.

It is sometimes convenient for calculations to rewrite the differential operator in equation (1.1) by exploiting a square, namely by observing that $(p \in \mathbb{R}^N)$

$$a_{ij}(x)p_ip_j = \sum_{k=1}^{M} (p \cdot \sigma_k(x))^2 = |p \cdot \sigma(x)|^2 ,$$

where we indicated as vector fields $\sigma_k : \overline{\Omega} \to \mathbb{R}^N$, $k = 1, \ldots, M$ the columns of the matrix $(\sigma_{ik})_{i,k}$. In this way the eikonal equation (0.1) turns out to be equivalent to the Bellman equation (for $b = (b^1, \ldots, b^M) \in \mathbb{R}^M$)

$$\max_{|b|\leq 1} \left\{ -DU(x) \cdot \sum_{k=1}^{M} b^k \sigma_k(x) - f(x) \right\} = 0, \tag{1.4}$$

associated to a symmetric optimal control system

$$\dot{y} = \sum_{k=1}^{M} b^k \sigma_k(y), \quad y(0) = x \tag{1.5}$$

where controls are measurable functions $b : [0, +\infty[\to \{b \in \mathbb{R}^M : |b| \leq 1\}$ and whose solution we indicate as $y(\cdot) \equiv y_x(\cdot, b)$. Time optimal trajectories of system (1.5) are the geodesics corresponding to the metric defined by the matrix (a_{ij}), and they are straight lines when this matrix is the identity. Solutions of equation (1.4) and problem (0.1) are instead related to the optimal control problem

$$J(x, b(\cdot)) = \int_0^{\tau_x} f(y(t))\, \mathrm{d}t + G(y(\tau_x)) \to \min, \tag{1.6}$$

where $\tau_x(b(\cdot)) = \inf\{t : y_x(t, b) \notin \Omega\}$. We showed in [22] that (0.1) with $f \in L^\infty$ may have multiple viscosity solutions, which may be discontinuous, extended real-valued and possibly satisfy the boundary condition in a weaker Dirichlet-type sense (or in the sense of lower semicontinuous solutions, see Sect. 4). Most notably, we found explicit representation formulas for the minimal and maximal nonnegative solution under appropriate assumptions. They are, respectively

$$V_m(x) = \min_{b(\cdot)} \int_0^{\tau_x} f_*(y(t))\,\mathrm{d}t + G(y(\tau_x)),$$
$$V_M(x) = \inf_{b(\cdot)} \int_0^{\tau_x} f^*(y(t))\,\mathrm{d}t + G(y(\tau_x)),$$

where

$$f_*(x) = \lim_{r\to 0^+} \inf\{f(y) : |y-x| \le r\}, \quad f^*(x) = \lim_{r\to 0^+} \sup\{f(y) : |y-x| \le r\}, \tag{1.7}$$

see [22] for details. The question that remains to be solved is when $V_m \equiv V_M$, which is therefore equivalent to uniqueness. It follows that with the notion of viscosity solution uniqueness is impossible if the set

$$\Gamma = \{x \in \overline{\Omega} : f \text{ is discontinuous at } x\} \tag{1.8}$$

has nonempty interior. For instance, in [7] the notion of solution has to be suitably strengthened to produce uniqueness results for $f \in L^\infty$.

In [22] we proved a necessary and sufficient condition for $V_m \equiv V_M$ which reads as a geometric property of optimal trajectories. To better understand the sense of the results of this paper, we mention here a slightly more stringent sufficient condition. The precise results can be found below in Sections 3 and 4 but for a complete discussion we refer the reader to [22]. Suppose that at $x \in \Omega$ we can find an optimal control $\hat{b}$ for the value $V_m(x)$, whose corresponding trajectory solution of (1.5) is *transversal* to the set Γ in (1.8), namely such that

$$|\{t : y(t) \in \Gamma\}| = 0.$$

Then $V_m(x) = V_M(x)$. If this happens for all $x \in \Omega$, then $V_m \equiv V_M$. The difficulty is that proving the above transversality condition directly (or the necessary and sufficient condition) is not a trivial task. In this paper we will proceed differently by proving a comparison theorem and uniqueness results for problem (0.1).

We now introduce precisely the notion of viscosity solution for (0.1). The role of the stars as super or subscripts in the definition is as in (1.7). The definition follows Ishii [13].

Definition 1.1. A lower (resp. upper) semicontinuous function $U : \Omega \to \mathbb{R} \cup \{+\infty\}$ (resp. $U : \Omega \to \mathbb{R}$) is a viscosity super- (resp. sub-) solution of the equation in (0.1) if for all $\varphi \in C^1(\Omega)$, $U(x) < +\infty$, and $x \in \operatorname{argmin}_{x\in\Omega}(U-\varphi)$, (resp. $x \in \operatorname{argmax}_{x\in\Omega}(U-\varphi)$), we have

$$a_{ij}(x)\varphi_{x_i}(x)\varphi_{x_j}(x) \ge [f_*(x)]^2, \quad (\text{resp. } a_{ij}(x)\varphi_{x_i}(x)\varphi_{x_j}(x) \le [f^*(x)]^2).$$

We also put $D\varphi(x) \in D^-U(x)$ ($D\varphi(x) \in D^+U(x)$, resp.).

A function U is a discontinuous viscosity solution of the equation in (0.1) if U^* is a subsolution and U_* is a supersolution. A viscosity solution of problem (0.1) is a (discontinuous) viscosity solution of the equation, continuous at the points of $\partial\Omega$ that attains the boundary condition.

Although the equation has discontinuous coefficients, solutions may be Lipschitz continuous. For instance the problem

$$\begin{cases} |U'(x)| = 1 + \chi_{\mathbb{R}_+}, & \mathbb{R}\backslash\{-1\}, \\ U(-1) = 0, \end{cases}$$

where $\chi_{\mathbb{R}_+}$ indicates the characteristic function of positive reals, is solved uniquely by the function

$$\begin{cases} |x+1|, & x \le 0, \\ 2x+1, & x \ge 0. \end{cases}$$

However solutions may be infinitely many, in general. For instance the problem

$$\begin{cases} |U'(x)| = 1 + \chi_{\mathbb{Q}}, & \mathbb{R}\backslash\{0\}, \\ U(0) = 0, \end{cases}$$

is solved by $U_1(x) = |x|$ and by $U_2(x) = 2|x|$ and these are the extremal solutions. The reader can find all of the other solutions.

We just remark that in general Lipschitz continuous almost everywhere solutions would not provide a good notion of weak solution for our problem. Indeed existence may fail in general if f is not lower semicontinuous, and in fact Lipschitz continuous viscosity solutions do not necessarily satisfy the equation almost everywhere. This problem happens when f has a set of discontinuities with positive measure. On the other hand, functions V_m, V_M when finite always solve problem (0.1), although without appropriate assumptions the boundary condition might be attained in the weaker Dirichlet-type sense, that we introduce in the next section. As a matter of fact, no (finite) solution will exist if $V_m(x) = +\infty$ for some $x \in \Omega$. For more information about the previous comments, we refer the reader to [7, 15, 22].

2. A comparison principle and uniqueness of continuous solutions

Before introducing the main result of this section, we need a further preliminary discussion. In order to obtain the uniqueness results, we will assume that the data f be piecewise continuous in the following sense. We need the notion of Lipschitz hypersurface.

Definition 2.1. The set $\Gamma \subset \mathbb{R}^N$ is said to be a Lipschitz hypersurface if for all $\hat{x} \in \Gamma$ one of its neighborhoods is partitioned by Γ into two connected open sets Ω^+, Ω^- and Γ itself, and we can find a *transversal* unit vector $\eta^+ \in \mathbb{R}^N$, $|\eta^+| = 1$, with the following property: there are $c, r > 0$ such that and if $x \in B(\hat{x}, r) \cap \overline{\Omega^\pm}$ then $B(x \pm t\eta^+, ct) \subset \Omega^\pm$ for all $0 < t \leq c$, respectively. We will say that an open set Ω is a Lipschitz domain if $\partial\Omega$ is a Lipschitz hypersurface. In this case if for $\hat{x} \in \partial\Omega$ and transversal unit vector η we have $\Omega^+ \subset \Omega$, then we call $\eta = \eta_\Omega$ an inward unit vector.

Now we present the key assumption on the coefficient f.

Assumption (DC). The set

$$\Gamma = \{x \in \mathbb{R}^N : f \text{ is discontinuous at } x\}$$

is the disjoint union of a finite family of connected Lipschitz hypersurfaces and f is piecewise continuous across Γ. In particular, for $\hat{x} \in \Gamma$ we can find $c, r > 0$, open connected sets Ω^+, Ω^- and inward unit vectors $\eta^+, \eta^- = -\eta^+$ to $\Omega^\pm$ respectively as in Definition 2.1. We may also suppose that $\Omega^\pm \subset \Omega$ if $\hat{x} \in \Omega$ and we assume that the discontinuous coefficient f is continuous in each component $\Omega^\pm$ with a continuous extension in $\overline{\Omega^\pm}$, and that if $x \in \Gamma$

$$f(x) \in \left[\lim_{\Omega^- \ni y \to x} f(y), \lim_{\Omega^+ \ni y \to x} f(y)\right],$$

where it is assumed that the above limits exist and the notation $\pm$ is introduced in such a way that the interval is well defined.

If $\hat{x} \in \Gamma \cap \partial\Omega$ we assume in addition that in the above we can choose c, r, η^+, η^- in such a way that η^+, η^- are also both inward for Ω *i.e.* for instance

$$B(x + t\eta^+, ct) \subset \Omega \cap \Omega^+, \quad B(x + t\eta^-, ct) \subset \Omega \cap \Omega^-$$

for all $x \in B(\hat{x}, R) \cap \overline{\Omega \cap \Omega^\pm}$ and $0 < t \leq c$, respectively. In this case however we have to allow $\eta^+ \neq -\eta^-$, in general.

To proceed, we need to allow that the boundary condition in (0.1) be satisfied in a weaker sense. We introduce the following definition.

Definition 2.2. We say that an upper semicontinuous function $U : \overline{\Omega} \to \mathbb{R}$, subsolution of the equation in (0.1), satisfies the Dirichlet type boundary condition in the viscosity sense

$$U \le G \text{ or } a_{ij}(x)U_{x_i}U_{x_j} \le [f^*(x)]^2, \quad \text{on } \partial\Omega$$

if for all $\varphi \in C^1(\mathbb{R}^N)$ and $x \in \partial\Omega$, $x \in \operatorname{argmax}_{x\in\overline{\Omega}}(U - \varphi)$ such that $U(x) > G(x)$, then we have

$$a_{ij}(x)\varphi_{x_i}(x)\varphi_{x_j}(x) \le [f^*(x)]^2.$$

Lower semicontinuous functions that satisfy a Dirichlet type boundary condition of the form

$$u \ge G \text{ or } a_{ij}(x)U_{x_i}U_{x_j} \ge [f_*(x)]^2, \quad \text{on } \partial\Omega$$

are defined accordingly.

Related to Lipschitz domains and surfaces, we introduce the following regularity property of functions.

Definition 2.3. Given a Lipschitz surface $\Gamma \subset \mathbb{R}^N$ with transversal unit vector η, we say that a function $u : \Omega \to \mathbb{R}$, is nontangentially continuous at $\hat{x} \in \Gamma$ in the direction of η if there are sequences $t_n \to 0^+$, and $p_n \to 0$, $p_n \in \mathbb{R}^N$, such that

$$\lim_{n\to+\infty} u(\hat{x} + t_n\eta + t_np_n) = u(\hat{x}).$$

We are now ready to state the comparison principle for (0.1).

Theorem 2.4. *Let Ω be an open domain with Lipschitz boundary. On the data assume (1.1), (1.2) and (1.3). Let us suppose that the assumption (DC) is satisfied. Let $U : \overline{\Omega} \to \mathbb{R}$, $V : \overline{\Omega} \to \mathbb{R} \cup \{+\infty\}$ be bounded from below and respectively an upper semicontinuous subsolution or a lower semicontinuous supersolution of*

$$a_{ij}(x)U_{x_i}U_{x_j} = [f(x)]^2, \quad x \in \Omega.$$

Let us assume that u, v satisfy the Dirichlet type boundary conditions

$$U \le G \text{ or } a_{ij}(x)U_{x_i}U_{x_j} \le [f^*(x)]^2, \quad \text{on } \partial\Omega,$$
$$V \ge G \text{ or } a_{ij}(x)U_{x_i}U_{x_j} \ge [f_*(x)]^2, \quad \text{on } \partial\Omega.$$

Suppose that U, V are nontangentially continuous on $\partial\Omega\backslash\Gamma$ in the inward direction η_Ω, and on $\Gamma \cap \partial\Omega$ in the directions η^-, η^+ respectively, with the same sequence t_n. Assume moreover that at each point of $\Gamma\cap\Omega$, either U is nontangentially continuous in the direction of η^- or V is nontangentially continuous in the direction of η^+. Then $U \le V$ in $\overline{\Omega}$.

Proof. To proceed with the proof, we first modify the equation. We introduce the increasing change of variables $u = 1-\mathrm{e}^{-r} = \psi(r)$ and recall that by standard calculations if W is a viscosity super/sub solution of the equation in (0.1) then $w = \psi(W)(\le 1)$ is a super/sub solution of

$$|Dw(x) \cdot \sigma(x)| = f(x)(1 - w(x)), \tag{2.1}$$

in the following sense: if $\varphi \in C^1$ and $x \in \operatorname{argmax}(w - \varphi)$ (resp. $x \in \operatorname{argmin}(w - \varphi)$) then

$$|D\varphi(x) \cdot \sigma(x)| \le f^*(x)(1 - w(x)), \quad (\text{resp. } |D\varphi(x) \cdot \sigma(x)| \ge f_*(x)(1 - w(x)).$$

The only slightly new point is the case when W is a supersolution and $W(\hat{x}) = +\infty$. In that case, if we set $w = \psi(W)$ and $w - \varphi$ attains a local minimum point at $\hat{x}$ with $w(\hat{x}) = 1$, then

$$|D\varphi(\hat{x}) \cdot \sigma(\hat{x})| = \sqrt{a_{ij}(\hat{x})\varphi_{x_i}(\hat{x})\varphi_{x_j}(\hat{x})} \ge f_*(\hat{x})(1 - w(\hat{x})) = 0.$$

With the notation of our statement, we define the bounded functions

$$u(x) = \psi(U(x)), \quad v(x) = \psi(V(x)).$$

In view of the change of variables, functions u, v satisfy corresponding assumptions of the statement for the equation (2.1) with boundary data $g = \psi(G)$.

We now fix $r, \beta > 0$, $0 < k < 1$ and assume by contradiction that there is $x_o \in \overline{\Omega}$ such that $u(x_o) - v(x_o) > 0$. For convenience of notation, we will set $x_o = 0$. Then we can find $\hat{x} \in \overline{\Omega}$ such that

$$u(\hat{x}) - v(\hat{x}) \geq u(\hat{x}) - v(\hat{x}) - \beta\langle \hat{x}\rangle^k = \max_{x\in\overline{\Omega}} u(x) - v(x) - \beta\langle x\rangle^k = 2\gamma > 0, \tag{2.2}$$

where we denoted $\langle x\rangle = (1 + |x|^2)^{1/2}$ and fix β sufficiently small. Note that $\langle x\rangle^k$ is a k-Lipschitz continuous function.

We remark that the proof below will be local in a neighborhood of the point $\hat{x}$. If $\hat{x} \in \partial\Omega$, we start by checking the boundary condition and observe that it holds

$$\text{either} \quad u(\hat{x}) > g(\hat{x}) \quad \text{or} \quad v(\hat{x}) < g(\hat{x}). \tag{2.3}$$

We will suppose below, just to fix the ideas, that the former is attained. For the same reason, we also suppose that if $\hat{x} \in \Gamma \cap \Omega$ then v is nontangentially continuous in the direction η^+. For $\varepsilon > 0$ we now introduce the function

$$w^\varepsilon(x, y) = u(x) - v(y) - \frac{\gamma}{2}\left|\frac{x-y}{\varepsilon} + \eta\right|^2 - \frac{r}{2}|x - \hat{x}|^2 - \beta\langle x\rangle^k,$$

where we choose $\eta = 0$ if $\hat{x} \in \Omega\backslash\Gamma$, $\eta = \eta_\Omega$ if $\hat{x} \in \partial\Omega\backslash\Gamma$ and $\eta = \eta^+$ as in (DC) if $\hat{x} \in \Gamma$.

On $\partial\Omega\cup\Gamma$, we will use the nontangential continuity of v in the direction of η, and at least along a subsequence $\varepsilon_n \to 0^+$ that we think as fixed from now on, although we drop the subscript in the notation, we find that

$$\lim_{\varepsilon\to 0^+} v(\hat{x} + \varepsilon(\eta + p_\varepsilon)) = v(\hat{x}), \tag{2.4}$$

for some $p_\varepsilon \to 0$. Notice that formally we may choose $p_\varepsilon \equiv 0$ if $\eta = 0$ for $\hat{x} \in \Omega\backslash\Gamma$, thus (2.4) always holds.

Let us pick up $(x_\varepsilon, y_\varepsilon) \in \overline{\Omega} \times \overline{\Omega}$ such that

$$w^\varepsilon(x_\varepsilon, y_\varepsilon) = \max_{\overline{\Omega}\times\overline{\Omega}} w^\varepsilon.$$

We may also suppose that, at least along a subsequence (which we drop in the notation),

$$\lim_{\varepsilon\to 0^+} (x_\varepsilon, y_\varepsilon) = (\overline{x}, \overline{y}). \tag{2.5}$$

In particular from $w^\varepsilon(x_\varepsilon, y_\varepsilon) \geq w^\varepsilon(\hat{x}, \hat{x})$ we find that

$$w^\varepsilon(x_\varepsilon, y_\varepsilon) \geq u(\hat{x}) - v(\hat{x}) - \frac{\gamma}{2} - \beta\langle \hat{x}\rangle^k \geq \gamma \geq 0, \tag{2.6}$$

then we have $\langle x_\varepsilon\rangle^k \le \frac{C}{\beta}$, $|x_\varepsilon - y_\varepsilon| \le C\varepsilon$ for some constant $C > 0$. By taking $\varepsilon \to 0^+$, u, v bounded imply that $\overline{x} = \overline{y}$. We now compute, also by (2.4),

$$\begin{aligned} u(\overline{x}) - v(\overline{x}) - \beta\langle\overline{x}\rangle^k &\ge \limsup_{\varepsilon\to 0^+} u(x_\varepsilon) - v(y_\varepsilon) - \beta\langle x_\varepsilon\rangle^k \\ &\ge \liminf_{\varepsilon\to 0^+} u(x_\varepsilon) - v(y_\varepsilon) - \beta\langle x_\varepsilon\rangle^k \ge \liminf_{\varepsilon\to 0^+} w^\varepsilon(x_\varepsilon, y_\varepsilon) + \frac{r}{2}|x_\varepsilon - \hat{x}|^2 \\ &\ge \lim_{\varepsilon\to 0^+} \left[u(\hat{x}) - v(\hat{x} + \varepsilon(\eta + p_\varepsilon)) - \beta\langle\hat{x}\rangle^k - \frac{\gamma}{2}|p_\varepsilon|^2 + \frac{r}{2}|x_\varepsilon - \hat{x}|^2\right] \\ &= u(\hat{x}) - v(\hat{x}) - \beta\langle\hat{x}\rangle^k + \frac{r}{2}|\overline{x} - \hat{x}|^2. \end{aligned} \tag{2.7}$$

From (2.2) and (2.7) we deduce that $\overline{x} = \hat{x}$ and then also

$$\lim_{\varepsilon\to 0^+} u(x_\varepsilon) - v(y_\varepsilon) = u(\hat{x}) - v(\hat{x}). \tag{2.8}$$

From (2.8) and the semicontinuity of u, v notice that

$$\begin{aligned} u(\hat{x}) \ge \limsup_{\varepsilon\to 0^+} u(x_\varepsilon) \ge \liminf_{\varepsilon\to 0^+} u(x_\varepsilon) &= \liminf_{\varepsilon\to 0^+} (u(x_\varepsilon) - v(y_\varepsilon)) + v(y_\varepsilon) \\ &\ge (u(\hat{x}) - v(\hat{x})) + v(\hat{x}) = u(\hat{x}), \end{aligned}$$

and then

$$\lim_{\varepsilon\to 0^+} u(x_\varepsilon) = u(\hat{x}), \qquad \lim_{\varepsilon\to 0^+} v(x_\varepsilon) = v(\hat{x}). \tag{2.9}$$

From $w^\varepsilon(x_\varepsilon, y_\varepsilon) \ge w^\varepsilon(\hat{x}, \hat{x} + \varepsilon(\eta + p_\varepsilon))$ we also get that

$$\frac{\gamma}{2}\left|\frac{x_\varepsilon - y_\varepsilon}{\varepsilon} + \eta\right|^2 \le u(x_\varepsilon) - v(y_\varepsilon) - (u(\hat{x}) - v(\hat{x} + \varepsilon(\eta + p_\varepsilon))) + \frac{\gamma}{2}|p_\varepsilon|^2 + \beta k|x_\varepsilon - \hat{x}|,$$

and then by (2.4) and (2.8) we obtain

$$\lim_{\varepsilon\to 0^+} \left|\frac{x_\varepsilon - y_\varepsilon}{\varepsilon} + \eta\right| = 0. \tag{2.10}$$

From (2.10) it also follows that, when ε is sufficiently small,

$$|x_\varepsilon - y_\varepsilon + \varepsilon\eta| < c\varepsilon, \tag{2.11}$$

where $c > 0$ is that of assumptions (DC).

We now want to apply the definition of viscosity sub and supersolution at the points $x_\varepsilon, y_\varepsilon$. Notice for instance that by definition of $(x_\varepsilon, y_\varepsilon)$, (if $(x_\varepsilon, y_\varepsilon) \in \Omega\times\Omega$)

$$\begin{aligned} \gamma\left(\frac{x_\varepsilon - y_\varepsilon + \varepsilon\eta}{\varepsilon^2}\right) + r(x_\varepsilon - \hat{x}) + \beta D\langle\cdot\rangle^k(x_\varepsilon) &\in D^+u(x_\varepsilon), \\ \gamma\left(\frac{x_\varepsilon - y_\varepsilon + \varepsilon\eta}{\varepsilon^2}\right) &\in D^-v(y_\varepsilon). \end{aligned} \tag{2.12}$$

We have to consider a few separate cases. If $\hat{x} \in \Omega$, by (2.5) we may suppose that, for ε sufficiently small $x_\varepsilon, y_\varepsilon \in \Omega$. If instead $\hat{x} \in \partial\Omega$, then by our choice of $\eta = \eta_\Omega$ or $\eta = \eta^+$, and (2.11) we have that at least $y_\varepsilon \in \Omega$, while $x_\varepsilon \in \overline{\Omega}$. Thus we might need to use the boundary condition for u at x_ε. Observe that by (2.3), (2.9) and continuity of g, we may suppose that, for ε sufficiently small, if $x_\varepsilon \in \partial\Omega$ then we have

$$u(x_\varepsilon) > g(x_\varepsilon).$$

We are then always allowed to use the definition of viscosity solution in equation (2.1) at the points $x_\varepsilon, y_\varepsilon$ for u and v respectively, and calculate

$$\begin{aligned} f^*(x_\varepsilon)(1-u(x_\varepsilon)) &\geq \left|\left(\gamma\frac{x_\varepsilon - y_\varepsilon + \varepsilon\eta}{\varepsilon^2} + r(x_\varepsilon - \hat{x}) + \beta k\frac{x_\varepsilon}{(1+|x_\varepsilon|^2)^{1-k/2}}\right)\cdot\sigma(x_\varepsilon)\right| \\ &\geq \left|\gamma\frac{x_\varepsilon - y_\varepsilon + \varepsilon\eta}{\varepsilon^2}\cdot\sigma(y_\varepsilon)\right| - L\gamma\frac{|x_\varepsilon - y_\varepsilon|}{\varepsilon}\left|\frac{x_\varepsilon - y_\varepsilon + \varepsilon\eta}{\varepsilon}\right| - Mr|x_\varepsilon - \hat{x}| - M\beta k, \end{aligned} \tag{2.13}$$

where M is a local bound for σ. Similarly we obtain

$$f_*(y_\varepsilon)(1-v(y_\varepsilon)) \leq \left|\gamma\frac{x_\varepsilon - y_\varepsilon + \varepsilon\eta}{\varepsilon^2}\cdot\sigma(y_\varepsilon)\right|. \tag{2.14}$$

Subtracting (2.14) from (2.13) we get

$$f^*(x_\varepsilon)(1-u(x_\varepsilon)) - f_*(y_\varepsilon)(1-v(y_\varepsilon)) \geq -L\gamma\frac{|x_\varepsilon - y_\varepsilon|}{\varepsilon}\left|\frac{x_\varepsilon - y_\varepsilon + \varepsilon\eta}{\varepsilon}\right| - Mr|x_\varepsilon - \hat{x}| - M\beta k. \tag{2.15}$$

By taking the lim sup as $\varepsilon \to 0$ we find out that the right hand side of (2.15) goes to $-M\beta k$ by (2.10).

In order to handle the discontinuous term, notice that if $\hat{x} \in \overline{\Omega}\backslash\Gamma$, by continuity of f and (2.8), (2.9) and (2.2), the left hand side of (2.15) tends to $-(u(\hat{x}) - v(\hat{x}))f(\hat{x}) \leq -2\gamma\rho$ providing a contradiction for k small enough. Thus we are left with considering the possibility that $\hat{x} \in \Gamma$. Notice that we may restrict ourselves to subsequences $\varepsilon_n \to 0$ such that it always happens either $x_\varepsilon \in \Omega^-$ or $x_\varepsilon \in \Omega^+ \cup \Gamma$ for n sufficiently large. These two cases are eventually dealt with similarly. In the latter however we use our choice of the test function in order to make sure that then $y_\varepsilon \in \Omega^+$ for n large enough, thus $x_\varepsilon, y_\varepsilon$ stay on the same side. This is obtained by combining (2.11) and (DC).

In order to fix the ideas, let us suppose now that $x_\varepsilon \in \Omega^-$ applies to our case. To simplify notations below, the choice of the subsequences will not appear explicitly. By assumption (DC), we can then estimate the limit of the left hand side of (2.15) as follows

$$\begin{aligned} \limsup_{\varepsilon\to 0^+} f^*(x_\varepsilon)(1-u(x_\varepsilon)) - f_*(y_\varepsilon)(1-v(y_\varepsilon)) &= \limsup_{\varepsilon\to 0^+}(f(x_\varepsilon) - f_*(y_\varepsilon))(1-u(x_\varepsilon)) - f_*(y_\varepsilon)(u(x_\varepsilon) - v(y_\varepsilon)) \\ &\leq (f_*(\hat{x}) - f_*(\hat{x}))(1-u(\hat{x})) - (u(\hat{x}) - v(\hat{x}))f_*(\hat{x}) \leq -2\gamma\rho, \end{aligned} \tag{2.16}$$

where the last inequality uses (2.2), (2.8) and (2.9). Again (2.16), and (2.15) provide a contradiction when k is chosen sufficiently small. □

In view of Theorem 2.4, we can improve regularity of solutions of the Dirichlet problem. The following Corollary gives us a rather general set of sufficient conditions for the existence of a continuous viscosity solution of a boundary value problem and we will use it in the existence result in the next section.

Corollary 2.5. *Assume (1.1), (1.2) and (1.3). Let Ω be open and $U : \overline{\Omega} \to \mathbb{R}$ be a bounded from below viscosity solution of*

$$a_{ij}(x)U_{x_i}U_{x_j} = [f(x)]^2, \quad x \in \Omega.$$

Let us suppose that the assumption (DC) is satisfied. If U is continuous on $\partial\Omega$ and either U^* or U_* is nontangentially continuous in the direction of η^+, η^-, respectively, at each point of $\Gamma \cap \Omega$, then $U \in \mathcal{C}(\overline{\Omega})$.

Proof. Apply Theorem 2.4 to the supersolution U_* and the subsolution U^* and $G = U$ on $\partial\Omega$. Then $U^* \le U_*$ implies $U^* \equiv U_* \equiv U$. □

The following variant of the comparison theorem also holds. Notice that only the nontangential continuity of one of the two functions that we want to compare is required at the boundary.

Theorem 2.6. *Let Ω be an open domain with Lipschitz boundary. Assume (1.1), (1.2) and (1.3) and let us suppose that the assumption (DC) is satisfied. Let $U, V : \overline{\Omega} \to \mathbb{R}$ be respectively an upper and a lower-semicontinuous function, bounded from below, respectively a subsolution and a supersolution of*

$$a_{ij}(x)U_{x_i}U_{x_j} = [f(x)]^2, \quad x \in \Omega.$$

Let us assume that V restricted on $\partial\Omega$ is continuous and that U satisfies the Dirichlet type boundary condition

$$U \le V \text{ or } a_{ij}(x)U_{x_i}U_{x_j} \le [f^*(x)]^2, \qquad \text{on } \partial\Omega.$$

Suppose moreover that V is nontangentially continuous on $\partial\Omega\backslash\Gamma$ in the inward direction η_Ω and on $\Gamma \cap \overline{\Omega}$ in the direction of η^+. Then $U \le V$ in $\overline{\Omega}$.

Proof. The same proof as that of Theorem 2.4 applies also to this case, with the choice $G = V$ on $\partial\Omega$. □

The following is an obvious consequence of the previous statement.

Corollary 2.7. *Assume (1.1), (1.2) and (1.3) and let us suppose that the assumption (DC) is satisfied. Let $U : \overline{\Omega} \to \mathbb{R}$ be a continuous, bounded from below viscosity solution of the Dirichlet boundary value problem (0.1). Then U is unique in the class of discontinuous solutions of the corresponding Dirichlet type problem.*

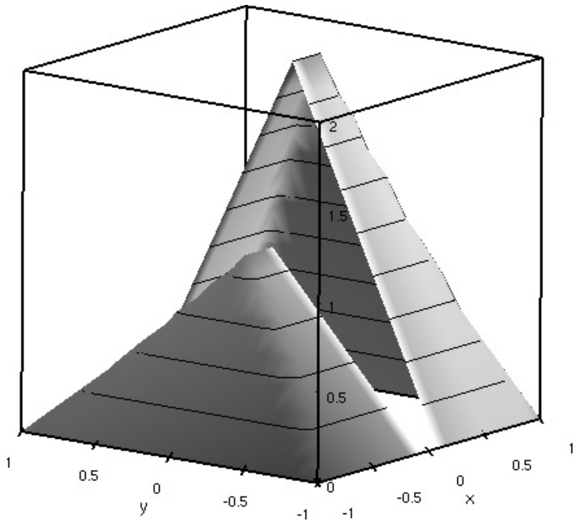
Example 2.8. The following example shows that discontinuous solutions may exist, without contradicting the uniqueness result of Corollary 2.7 which therefore does not cover the whole scope of possible problems. Let us consider the boundary value problem

$$\begin{cases} x^2(U_x)^2 + (U_y)^2 = f(x,y), &]-1,1[\times]-1,1[, \\ U(\pm 1, y) = U(x, \pm 1) = 0, \quad x, y \in [-1,1], & \end{cases}$$

where $f(x,y) = 2$, for $x > 0$, and $f(x,y) = 1$ for $x \le 0$. It is easy to verify that the piecewise continuous function,

$$U(x,y) = \begin{cases} 2(1-|y|), & x > 0,\ |y| \ge 1 + \ln x, \\ -2\ln x, & x > 0,\ |y| \le 1 + \ln x, \\ U(-x,y)/2, & x \le 0, \end{cases}$$

is a viscosity solution of the problem. Corollary 2.7 then implies that there is no continuous solution. However by the results in [22] it is easy to show that all discontinuous solutions have U as lower semicontinuous envelope.



The punch line of this section is that we may use Corollary 2.5 to simplify the construction of a continuous solution of (0.1) and then this turns out to be unique in the class of discontinuous solutions by Corollary 2.7. This is the path we follow in the next section.

3. Existence of continuous solutions

We proved in Corollary 2.7 that bounded from below and continuous viscosity solutions of the Dirichlet problem (0.1) are unique (in a wider class). However Example 2.8 showed that existence of continuous solutions is not always ensured by the problem. In this section we give sufficient conditions to prove that problem (0.1) has indeed a solution $U \in C(\overline{\Omega})$ bounded from below. The main point is that comparison principle will again be helpful in view of Corollary 2.5 to reduce our task. As we mentioned in Section 1, standard theory provides us with candidate solutions for (0.1). These are the value functions V_m, V_M earlier introduced. As shown in [22], in general, these functions are lower and upper semicontinuous, respectively and when finite, they are solutions of the Dirichlet type problem corresponding to (0.1). In order to get their continuity, we then only need to show the appropriate nontangential continuity properties on $\Gamma \cup \partial\Omega$, in order to apply the comparison principle.

The first result concerns the value function V_M.

Proposition 3.1. *Let Ω be an open domain and assume (1.1), (1.2) and (1.3). Let us suppose that the assumption (DC) holds. At $\hat{x} \in \Gamma \cap \Omega$ let us suppose that:*

$$\begin{aligned}&\text{there is a sequence of control functions } b_n(\cdot), \text{ positive numbers } t_n \to 0^+,\\ &k > 0 \text{ and vector } v \in \mathbb{R}^N \text{ inward } \Omega^- \text{ such that}\\ &x_n = y_{\hat{x}}(t_n, b_n) = \hat{x} + (t_n)^k v + o((t_n)^k), \quad \text{as } n \to +\infty.\end{aligned} \tag{3.1}$$

Then the value function V_M is nontangentially continuous at $\hat{x}$ in the direction of v.

Proof. By the Dynamic Programming Principle in optimal control, *e.g.* see [2], for n sufficiently large, we may write

$$V_M(\hat{x}) \le \int_0^{t_n} f^*(y(t, b_n))\,\mathrm{d}t + V_M(y(t_n, b_n)) \le M t_n + V_M(x_n),$$

where M is a local bound for f. As $n \to +\infty$ we obtain

$$V_M(\hat{x}) \leq \liminf_{n\to+\infty} V_M(x_n) \leq \limsup_{n\to+\infty} V_M(x_n) \leq V_M(\hat{x}),$$

since V_M is upper semicontinuous in Ω as shown in [22].

The conclusion comes by construction of the sequence x_n. □

The second statement is similar but concerns instead the boundary condition.

Proposition 3.2. *Let Ω be an open domain with Lipschitz boundary and assume (1.1), (1.2) and (1.3). Let us suppose that the assumption (DC) holds. At $\hat{x} \in \partial\Omega$ let v be a unit inward vector to Ω (in the sense of Def. 2.1). Suppose that:*

$$\begin{aligned} &\text{there is a sequence of control functions } b_n(\cdot), \text{ positive numbers } t_n \to 0^+,\\ &k > 0 \text{ such that}\\ &x_n = y_{\hat{x}}(-t_n, b_n) = \hat{x} + (t_n)^k v + o((t_n)^k), \quad \text{as } n \to +\infty. \end{aligned} \tag{3.2}$$

Then both value functions V_m, V_M satisfy

$$\limsup_{n\to+\infty} V(x_n) \leq G(\hat{x}).$$

Proof. The two proofs are identical. Consider the trajectory $y_{x_n}(\cdot, \hat{b}_n)$ solution of (1.5) with $x = x_n$ and $b \equiv \hat{b}_n$ where $\hat{b}_n(t) = b_n(t_n - t)$ for $t \in [0, t_n]$. We take for instance V_M and by the Dynamic Programming Principle we may write

$$V_M(x_n) \leq \int_0^{t_n\wedge\tau_{x_n}} f^*(y_{x_n}(t, \hat{b}_n))\, \mathrm{d}t + G(y_{x_n}(t_n \wedge \tau_{x_n}, \hat{b}_n)) \leq Mt_n + G(y_{x_n}(t_n \wedge \tau_{x_n}, \hat{b}_n)),$$

where M is a local bound for f. From here the conclusion as before. □

We want to discuss conditions (3.1), (3.2). We will suppose below that the vector fields σ_m, defined as the columns of the matrix σ in (1.2), have all the regularity that we need. Suppose for instance that at $\hat{x} \in \Gamma \cap \Omega$ we can find $m \in \{1, \ldots, M\}$ such that $\sigma_m(\hat{x})$ is inward Ω^-. Then by choosing $v = \sigma_m(\hat{x})$, $b_n(t) \equiv e_m$, $t_n = \frac{1}{n}$ we obtain

$$x_n = \hat{x} + \frac{1}{n}v + o\left(\frac{1}{n}\right)\cdot$$

Here and below, vectors e_m are the standard basis vectors of $\mathbb{R}^M$. Thus (3.1) is satisfied. Similarly for (3.2) if $v = \sigma_m(\hat{x})$ is inward Ω and we assume appropriate conditions on $\sigma_m(\hat{x})$.

To obtain versions of (3.1), (3.2) with larger exponent k, one can use the properties of the Lie algebra generated by the vector fields σ_m. In fact those formulas are precisely the expansions of the trajectories as stated for instance in Haynes-Hermes [12]. For a more up to date discussion and refinements see also the work by Rampazzo-Sussmann [18]. To describe (3.1), let us suppose for instance that the Lie bracket $[\sigma_m(\cdot), \sigma_r(\cdot)](\hat{x})$ is inward Ω^- at $\hat{x} \in \Gamma \cap \Omega$. Then by setting $v = [\sigma_m(\cdot), \sigma_r(\cdot)](\hat{x})$, $k = 2$, $t_n = 4/n$, we can find a sequence of controls $b_n(\cdot)$ such that (3.1) is satisfied. The sequence can be explicitly described by setting $b_n(t) = e_m$ for $t \in [0, \frac{1}{n}[$, $b_n(t) = e_r$ for $t \in [\frac{1}{n}, \frac{2}{n}[$, $b_n(t) = -e_m$ for $t \in [\frac{2}{n}, \frac{3}{n}[$, $b_n(t) = -e_r$ for $t \in [\frac{3}{n}, \frac{4}{n}]$. The same condition on Lie brackets of higher order just require to increase in (3.1) the exponent k to the order of the bracket.

The next statement concerns value function V_m.

Proposition 3.3. *Let us suppose that the assumption (DC) holds. At $\hat{x} \in \Gamma \cap \Omega$ let us suppose that:*

$$\begin{aligned}
&\text{we can find an optimal control } \hat{b}(\cdot), \text{ for } V_m(\hat{x}), \text{positive numbers } t_n \to 0^+, \\
&\text{and an inward unit vector } v \text{ to } \Omega^+ \text{ such that} \\
&x_n = y_{\hat{x}}(t_n, \hat{b}) = \hat{x} + t_n v + o(t_n), \quad \text{as } n \to +\infty.
\end{aligned} \tag{3.3}$$

Then the value function V_m is nontangentially continuous at $\hat{x}$ in the direction of v.

Proof. We use again the Dynamic Programming Principle for V_m and deduce that, for n sufficiently large

$$V_m(\hat{x}) = \int_0^{t_n} f_*(y(t, \hat{b}))\ \mathrm{d}t + V_m(x_n) \geq V_m(x_n).$$

As $n \to +\infty$ we obtain

$$V_m(\hat{x}) \geq \limsup_{n \to +\infty} V_m(x_n) \geq \liminf_{n \to +\infty} V_m(x_n) \geq V_m(\hat{x}),$$

since V_m is lower semicontinuous as shown in [22]. □

To comment on Proposition 3.3, observe that

$$\frac{x_n - \hat{x}}{t_n} = \frac{1}{t_n} \int_0^{t_n} \sum_{k=1}^{M} \hat{b^k}(t) \sigma_k(y(t))\ \mathrm{d}t \leq \frac{1}{t_n} \int_0^{t_n} \sum_{k=1}^{M} \hat{b^k}(t) \sigma_k(\hat{x})\ \mathrm{d}t + LMt_n = v_n + o(1),$$

as $n \to +\infty$, where M is a local bound for f and $v_n \in co\ \{\sigma_k(\hat{x}) : k = 1, \ldots, M\}$. This implies that at least along a subsequence, $v_n \to v \in \overline{co}\ \{\sigma_k(\hat{x}) : k = 1, \ldots, M\}$. If Γ is smooth, a necessary condition for (3.3) is then that there exists $k \in \{1, \ldots, M\}$ such that $\sigma_k(\hat{x})$ is inward Ω^+. A clearly sufficient condition is instead that all the vectors in $\{\sigma_k(\hat{x}) : k = 1, \ldots, M\}$ are inward Ω^+ at $\hat{x}$.

We summarize all of the previous propositions in the following existence result.

Theorem 3.4. *Let Ω be an open domain with Lipschitz boundary. On the data assume (1.1), (1.2) and (1.3) and that the vector fields σ_k have the regularity needed in the next assumptions. Let us suppose that the assumption (DC) is satisfied. Suppose that the value function V_M is finite in $\overline{\Omega}$ and assume that the Dirichlet boundary value problem (0.1) admits a continuous viscosity subsolution $G : \overline{\Omega} \to \mathbb{R}$. Assume that at each $\hat{x} \in (\Gamma \cap \Omega)$ either there is a Lie bracket generated by the family of vector fields $\{\sigma_k\}$ which is inward Ω^- or that all of the vector fields $\sigma_k(\hat{x})$ are inward Ω^+. Assume moreover that at each $\hat{x} \in \partial\Omega$ there is a Lie bracket which is inward Ω. If $\hat{x} \in \Gamma$ we need to find Lie brackets which are inward also to $\Omega \cap \Omega^-$ and to $\Omega \cap \Omega^+$. Then (0.1) has a unique continuous viscosity solution (in the class of discontinuous solutions of the Dirichlet-type problem) and such solution is either V_M or V_m which thus coincide.*

Proof. We first use the comparison principle Theorem 2.6 to compare the continuous subsolution G and the two value functions V_m, V_M. This gives $G \leq V_m \leq V_M$ in $\overline{\Omega}$. From the assumption and Proposition 3.2, we then obtain that value function V_M is nontangentially continuous on $\partial\Omega$ in the direction of an appropriate Lie bracket. At this point the assumption on $\Gamma \cap \Omega$ allows us to use either Proposition 3.1 or Proposition 3.3 to achieve the correct nontangential continuity that allows us to compare directly the subsolution V_M and the supersolution V_m by applying Theorem 2.4. Thus $V_M \leq V_m$, they are therefore equal, and thus continuous solutions of (0.1). Finally Corollary 2.7 extends uniqueness to the class of discontinuous solutions of the Dirichlet type problem. □

Notice how by using Corollary 2.5 the existence of a continuous solution becomes almost straightforward, but this is the power of the comparison theorem, in particular the fact that nontangential continuity is precisely compatible with the expansion formulas of the trajectories of control problems and the properties of the Lie algebra. In the statement of Theorem 3.4 there are two assumptions that we did not comment yet. The first is that V_M is finite in $\overline{\Omega}$. This means that we can find a trajectory of the control system (1.5) such that $\tau_x < +\infty$, *i.e.* the boundary $\partial\Omega$ can be reached in finite time. A sufficient condition explicitly given on the vector fields σ_k is provided by Chow's Theorem, requiring that the Lie algebra generated by the vector fields spans $\mathbb{R}^N$ at every point in $\overline{\Omega}$ (Hörmander's condition). This condition is however way too strong for our needs.

The other assumption is that the boundary condition in (0.1) is the restriction of a continuous subsolution of the problem and it is clearly a necessary condition for its solvability. It is well known that in general the Dirichlet boundary value problem for a degenerate equation cannot be solved. This assumption is standard and called compatibility of the boundary condition, see also the way the problem for discontinuous solutions is defined in the next section. The role of the compatibility condition is discussed in detail in the book by Lions [15]. We just notice that for instance the problem with homogeneous boundary condition admits $G \equiv 0$ as subsolution.

As we mentioned in Section 1, a byproduct of the uniqueness result Theorem 3.4 is the following geometric property on the control problem (1.5), (1.6), which is equivalent to uniqueness as shown in [22].

Corollary 3.5. *Under the assumptions of Theorem 3.4, then at each $x \in \Omega$ we can find a minimizing sequence of controls for $V_m(x)$, i.e. $b_n : [0, +\infty[\to B_1(0) \subset \mathbb{R}^N$ measurable and satisfying (here $y_n(\cdot) = y_x(\cdot, b_n)$)*

$$V_m(x) = \lim_{n\to+\infty} \int_0^{\tau_x} f_*(y_n(t))\, \mathrm{d}t + G(y_n(\tau_x)),$$

such that

$$\lim_{n\to+\infty} \int_0^{\tau_x} (f^*(y_n(t)) - f_*(y_n(t)))\, \mathrm{d}t = 0.$$

In particular, if $V_M(x)$ admits an optimal trajectory, this must be transversal to Γ.

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