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OPTIMAL PARTIAL REGULARITY OF MINIMIZERS OF QUASICONVEX VARIATIONAL INTEGRALS

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Abstract. We prove partial regularity with optimal Hölder exponent of vector-valued minimizers u of the quasiconvex variational integral $\int F(x,u,Du)\,\mathrm{d}x$ under polynomial growth. We employ the indirect method of the bilinear form.

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1. Introduction

We are interested in the regularity of the vector-valued minimizers $u \in W^{1,q}(\Omega, \mathbf{R}^N)$ of the variational integral

$$\mathcal{I}(u,\Omega) = \int_\Omega F(x,u(x),Du(x))\,\mathrm{d}x.$$

Here Ω is a bounded open subset of $\mathbf{R}^n$, $n \geq 2$, $N \geq 1$, $q \geq 2$, and $Du(x) \in \mathbf{R}^{N\times n}$ denotes the gradient of u at *a.e.* point $x \in \Omega$. The integral $\mathcal{I}(u,\Omega)$ is well-defined for $u \in W^{1,q}(\Omega,\mathbf{R}^N)$ if we admit as integrands Carathéodory functions $F(x,u,P) : \overline{\Omega} \times \mathbf{R}^N \times \mathbf{R}^{N\times n} \to \mathbf{R}$ of polynomial growth in P, *i.e.* functions which are measurable in x, continuous in (u,P) and which satisfy the growth condition

$$|F(x,u,P)| \leq c(1+|P|^q).$$

Definition 1. We say that $u \in W^{1,q}(\Omega,\mathbf{R}^N)$ is a *minimizer* of the variational integral $\mathcal{I}$ if

$$\mathcal{I}(u, \operatorname{supp}\varphi) \leq \mathcal{I}(u+\varphi, \operatorname{supp}\varphi)$$

for every $\varphi \in C_c^\infty(\Omega,\mathbf{R}^N)$.

The problem of regularity of a minimizer $u \in W^{1,q}(\Omega,\mathbf{R}^N)$ of the variational integral $\mathcal{I}$ has been intensively investigated over the last 23 years. As we know (see Exs. II.3.2 and II.3.4 of [6]), we can in general only expect *partial regularity* if $N > 1$, *i.e.* Hölder continuity of the gradient Du outside of a closed set of Lebesgue measure zero.

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The standard hypotheses are that $F(x,u,P)$ be uniformly strictly quasiconvex, Hölder continuous in (x,u) and of class C^2 in P, and to satisfy the coercivity and growth condition

$$\gamma|P|^q \leq F(x,u,P) \leq c(1+|P|^q).$$

Sometimes, a growth condition is also imposed on the second partial derivative $F_{PP}(x,u,P)$, *i.e.*

$$|F_{PP}(x,u,P)| \leq c(1+|P|^{q-2}). \tag{1}$$

We start with a short account of the development of the partial regularity theory for vector-valued minimizers of variational integrals (see also [7]). The general method of the proof is to compare the given minimizer u with a solution of a linear system with constant coefficients, for which standard elliptic estimates are available. For the direct approach, this comparison is carried out on an arbitrary ball either under a Dirichlet boundary condition, or with the so-called A-harmonic approximation (which is itself procured by a contradiction argument); for the indirect approach, it is shown that a sequence of blow-up functions $w_m \in W^{1,2}(B,\mathbf{R}^N)$, rescaled to the unit ball B, converges weakly to such a solution.

Following the direct approach, Giaquinta and Giusti [9, 10] showed partial regularity of the minimizers $u \in W^{1,2}(\Omega,\mathbf{R}^N)$ of the quadratic variational integral

$$\mathcal{Q}(u,\Omega) = \int_\Omega A(x,u)\cdot(Du,Du)\,\mathrm{d}x,$$

and of the variational integral $\mathcal{I}$ for $q=2$ under coercivity and strict convexity in P of the integrand $F(x,u,P)$.

The concept of quasiconvexity was introduced into this subject by Evans [3]. For quasiconvex integrands $F(P)$ depending solely on the variable P, he gave an indirect proof for partial regularity of minimizers $u \in W^{1,q}(\Omega,\mathbf{R}^N)$ (see also [4]). The same result holds true if the coercive and quasiconvex integrand $F(x,u,P)$ depends on all the variables. Various proofs were proposed, some direct [12, 13], some indirect [5], and some allowing for a generalized coercivity [20], or assuming no growth condition on F_{PP} [1, 8]. All proofs in the general situation are based on a reverse Hölder inequality with increasing supports for $Du - P_0$, for any constant $P_0 \in \mathbf{R}^{N\times n}$. This reverse Hölder inequality in turn is derived from Caccioppoli's second inequality by invoking the higher integrability theorem of Gehring, Giaquinta and Modica (see Th. 2).

As the higher integrability theorem is considered to be rather involved, it is desirable to find a simpler partial regularity proof which avoids the use of a reverse Hölder inequality. Such proofs are available in the following two special situations: (a) under a non-integrated version of convexity of the integrand $F(x,u,P)$ in the variable P, *i.e.* convexity or polyconvexity rather than quasiconvexity (see [15], Sect. 3, [17]); (b) for a variational integral $\int_\Omega F(x,Du)\,\mathrm{d}x$ whose integrand does not depend explicitly on the variable u (see [2–4]). Nevertheless, at the moment it seems that our main result, Theorem 1, is unattainable without resort to a reverse Hölder inequality.

Partial regularity with optimal Hölder exponent requires a more precise formulation of the continuity hypotheses on the integrand, namely that $u \mapsto F(x,u,P)$ and $(x,u) \mapsto F_P(x,u,P)$ be Hölder continuous with exponents δ and $\delta/(2-\delta)$ respectively. Thus we may admit discontinuities of $F(x,u,P)$ in x that do not propagate to $F_P(x,u,P)$ (*cf.* [15]). Indeed, such an integrand is of the unique form

$$F(x,u,P) = f(x,u,P) + g(x,u),$$

with $f(x,u,0)=0$, where $f(x,u,P)$ and $f_P(x,u,P)$ are of class $C^{0,\delta/(2-\delta)}$ in (x,u), and $f(x,u,P)$ and $g(x,u)$ are of class $C^{0,\delta}$ in u, but where $g(x,u)$ is only *measurable* in x. The optimal Hölder exponent of the gradient of the minimizer is then $\delta/(2-\delta)$. This was shown in the scalar case $N=1$ for $q=2$ by Giaquinta and Giusti [11] (see also [24] for a particular model). In the present paper we extend this result to the vectorial case $N \geq 1$ for $q \geq 2$. Moreover, as in [1, 8], we dispense with the growth condition on F_{PP}.

Our indirect proof of partial regularity employs the *method of the bilinear form*, which was introduced by Hamburger [15] in the context of minimizers of variational integrals in establishing convergence $w_m \to w$ in $W^{1,2}_{\mathrm{loc}}$

of a sequence of blow-up functions $w_m \in W^{1,2}(B, \mathbf{R}^N)$, which is known to converge only weakly. This technique has already been applied to solutions of nonlinear superelliptic and quasimonotone systems in [16, 18, 19], and to minimizers of convex, quasiconvex and polyconvex variational integrals in [15, 17]. Since we are not assuming any growth condition on F_{PP}, we need to define sets $E_{r,m} \subset B_r$, satisfying $\lim_{m\to\infty} |E_{r,m}| = 0$, where the functions w_m or Dw_m exceed a certain bound (*cf.* [8, 14]). A reverse Hölder inequality for $Du - P_0$, for any constant $P_0 \in \mathbf{R}^{N\times n}$, allows us to control the error integral of a rescaled power of $|Dw_m|$ over the set $E_{r,m}$. We show that the blow-up functions w_m are approximate minimizers of suitable rescaled variational integrals. This has two consequences. First, passing to the limit as $m \to \infty$ we infer that w solves a linear elliptic system with constant coefficients. Secondly, we derive the key estimate

$$\limsup_{m\to\infty} \int_{B_r} \eta^2 G(Y_0) \cdot (Dw_m, Dw_m)\,\mathrm{d}z \le \int_{B_r} \eta^2 G(Y_0) \cdot (Dw, Dw)\,\mathrm{d}z.$$

Here η is a cut-off function, the symmetric bilinear form $G(Y)$ depends continuously on Y, and the constant function Y_0 is the limit in L^2 of a suitable sequence of functions $\{Y_m\}$. We finally deduce from this estimate with the help of strict quasiconvexity that $w_m \to w$ in $W^{1,2}_{\mathrm{loc}}$. In this manner we achieve partial regularity with optimal Hölder exponent of minimizers of quasiconvex variational integrals.

For the integrand $F : \overline{\Omega} \times \mathbf{R}^N \times \mathbf{R}^{N\times n} \to \mathbf{R}$ we shall assume the following hypotheses, for an exponent $q \ge 2$.

Hypothesis 1. *We suppose that $F(x,u,P)$ is of class C^2 in P and of polynomial growth*

$$|F(x,u,P)| \le c(1 + |P|^q),$$

and we assume that F_{PP} is continuous.

Hypothesis 2. *We suppose that $u \mapsto (1+|P|^q)^{-1}F(x,u,P)$ and $(x,u) \mapsto (1+|P|^{q-1})^{-1}F_P(x,u,P)$ are Hölder continuous uniformly with respect to P, with exponents δ and $\delta/(2-\delta)$ respectively:*

$$\begin{aligned} |F(x,u,P) - F(x,v,P)| &\le (1 + |P|^q)\omega(|u|, |u-v|) \\ |F_P(x,u,P) - F_P(y,v,P)| &\le (1 + |P|^{q-1})\chi(|u|, |x-y| + |u-v|) \end{aligned}$$

for all $(x,u,P), (y,v,P) \in \overline{\Omega} \times \mathbf{R}^N \times \mathbf{R}^{N\times n}$. Here $\omega(s,t) = K(s)\min(t^\delta, 1)$ and $\chi(s,t) = K(s)\min(t^{\delta/(2-\delta)}, 1)$ for $0 < \delta < 1$ and for a nondecreasing function $K(s)$; we note that $\omega(s,t)$ and $\chi(s,t)$, for fixed s, are nondecreasing and bounded in t.

Hypothesis 3. *We suppose that F is uniformly strictly quasiconvex*

$$\int_{\mathbf{R}^n} (F(x_0,u_0,P_0 + D\varphi) - F(x_0,u_0,P_0))\,\mathrm{d}x \ge \gamma \int_{\mathbf{R}^n} (|D\varphi|^2 + |D\varphi|^q)\,\mathrm{d}x$$

for some $\gamma > 0$, and all $(x_0,u_0,P_0) \in \overline{\Omega} \times \mathbf{R}^N \times \mathbf{R}^{N\times n}$ and $\varphi \in C_c^\infty(\mathbf{R}^n, \mathbf{R}^N)$.

Hypothesis 4. *We suppose that*

$$F(x,u,P) \ge \tilde{F}(x,P)$$

for all $(x,u,P) \in \overline{\Omega} \times \mathbf{R}^N \times \mathbf{R}^{N\times n}$, and for some function $\tilde{F}(x,P)$, satisfying $|\tilde{F}(x,0)| \le c$, which is strictly quasiconvex at $P = 0$, and for which $(1 + |P|^q)^{-1}\tilde{F}(x,P)$ is continuous in x uniformly with respect to P:

$$\int_{\mathbf{R}^n} (\tilde{F}(x_0, D\varphi) - \tilde{F}(x_0, 0))\,\mathrm{d}x \ge \tilde{\gamma} \int_{\mathbf{R}^n} |D\varphi|^q\,\mathrm{d}x$$

for some $\tilde{\gamma} > 0$, and all $x_0 \in \overline{\Omega}$ and $\varphi \in C_c^\infty(\mathbf{R}^n, \mathbf{R}^N)$;

$$|\tilde{F}(x,P) - \tilde{F}(y,P)| \le (1 + |P|^q)\tilde{\omega}(|x-y|)$$

for all $x, y \in \overline{\Omega}$ *and* $P \in \mathbf{R}^{N\times n}$, *where* $\tilde{\omega}$ *is continuous and nondecreasing with* $\tilde{\omega}(0) = 0$.

Remark 1.

(a) Hypothesis 4 is fulfilled if $F(x,u,P)$ is coercive

$$F(x,u,P) \geq \gamma |P|^q.$$

(b) If $F(x,P)$ is independent of the variable u and continuous in x uniformly with respect to P then Hypothesis 4, with $\tilde{F}(x,P) = F(x,P)$, is already a consequence of Hypothesis 3.

(c) If we assume that $u \mapsto F(x,u,P)$ and $(x,u) \mapsto F_P(x,u,P)$ are Hölder continuous with exponents β and γ respectively then Hypothesis 2 holds with $\delta = \min(\beta, 2\gamma/(1+\gamma))$. In this case, Theorem 1 asserts that the gradient of the minimizer u is Hölder continuous in the regular set Ω_u with exponent $\delta/(2-\delta) = \min(\beta/(2-\beta), \gamma)$. Hölder continuity of $(x,u) \mapsto F(x,u,P)$ with exponent β together with growth condition (1) leads to the choice $\gamma = \beta/2$ and the conclusion that $u \in C^{1,\beta/2}(\Omega_u, \mathbf{R}^N)$. Indeed, for $q \geq 2$ the estimates

$$|F(x,u,P) - F(y,v,P)| \leq (1+|P|^q)\omega(|u|, |x-y| + |u-v|)$$

and (1) imply

$$|F_P(x,u,P) - F_P(y,v,P)| \leq c(1+|P|^{q-1})\omega^{1/2}(|u|, |x-y| + |u-v|).$$

This is shown for $q = 2$ on p. 247 of [11].

(d) Hypotheses 1 and 3 furnish the growth condition

$$|F_P(x,u,P)| \leq c(1+|P|^{q-1}). \tag{2}$$

To prove (2), we note that Hypothesis 3 implies rank-one convexity of F (see [14], Prop. 5.2). Therefore, by Hypothesis 1,

$$F_P(x,u,P)\cdot Q \leq F(x,u,P+Q) - F(x,u,P) \leq c(1+|P|^q + |Q|^q)$$

if rank $Q \leq 1$. Setting $Q = \pm(1+|P|)E$, where $E \in \mathbf{R}^{N\times n}$ is a unit matrix (*i.e.* one entry is 1, all others are 0) gives

$$|F_P(x,u,P)|(1+|P|) \leq c(1+|P|^q),$$

whence (2) follows.

Our main result is contained in the following

Theorem 1. *Let the integrand* F *satisfy Hypotheses* 1 *to* 4, *with exponent* $q \geq 2$, *and let* $u \in W^{1,q}(\Omega, \mathbf{R}^N)$ *be a minimizer of the variational integral* $\mathcal{I}$.

Then there exists an open set $\Omega_u \subset \Omega$, *whose complement has Lebesgue measure zero, such that the gradient* Du *is locally Hölder continuous in* Ω_u *with exponent* $\delta/(2-\delta)$, *for* $0 < \delta < 1$ *the exponent of Hypothesis* 2:

$$u \in C^{1,\delta/(2-\delta)}(\Omega_u, \mathbf{R}^N) \text{ and } \mathcal{L}^n(\Omega \setminus \Omega_u) = 0.$$

Moreover, the regular set is characterized by

$$\Omega_u = \left\{ x_0 \in \Omega : \sup_{r>0}(|u_{x_0,r}| + |Du_{x_0,r}|) < \infty \text{ and } \liminf_{r\searrow 0} ⨍_{B_r(x_0)} |Du - Du_{x_0,r}|^q \,\mathrm{d}x = 0 \right\}.$$

The example treated in [24] shows that the Hölder exponent $\delta/(2-\delta)$ is indeed optimal. For bounds on the Hausdorff dimension of the singular set $\Omega \setminus \Omega_u$ in certain cases, we refer to the interesting recent papers [21–23].

2. A decay estimate for the excess

In what follows, all constants c may depend on the data including the integrand F, on the number L from the proof of Proposition 1, and on the minimizer u itself. The Landau symbol $o(1)$ stands for any quantity for which $\lim_{m\to\infty} o(1) = 0$; this may in Section 4 also depend on the number $\beta > 0$ and the functions φ, w and ζ. We write $B_r(x_0) = \{x \in \mathbf{R}^n : |x - x_0| < r\}$, $B_r = B_r(0)$, and $B = B_1$ for the unit ball. We denote the mean of a function f on the ball $B_r(x_0)$ by

$$f_{x_0,r} = \fint_{B_r(x_0)} f \,\mathrm{d}x = \frac{1}{\mathcal{L}^n(B_r(x_0))} \int_{B_r(x_0)} f \,\mathrm{d}x.$$

In this section we assume Hypotheses 1 to 4 with $q \geq 2$, and we let $u \in W^{1,q}(\Omega, \mathbf{R}^N)$ be a minimizer of the variational integral $\mathcal{I}$. We define the excess of Du on the ball $B_r(x_0) \subset\subset \Omega$:

$$U(x_0, r) = \fint_{B_r(x_0)} (|Du - Du_{x_0,r}|^2 + |Du - Du_{x_0,r}|^q) \,\mathrm{d}x.$$

At first, we let $\alpha \leq \delta/(2-\delta)$ be the positive exponent appearing in Theorem 4. The conclusions of Theorem 1, as yet with exponent α instead of $\delta/(2-\delta)$, follow in a routine way from the next proposition (see [6], pp. 197–199; [8], Sect. 3; [14], pp. 349–352; [3], Sect. 7; [5], Sect. 6).

Showing that $u \in C^{1,\delta/(2-\delta)}(\Omega_u, \mathbf{R}^N)$ with the optimal exponent $\delta/(2-\delta)$ requires a second step. As soon as we know that $u \in C^{0,1}(\Omega_u, \mathbf{R}^N)$, we consider the restriction $u \in C^{0,1}(\overline{\Sigma}, \mathbf{R}^N)$ to any open set $\Sigma \subset\subset \Omega_u$, for which the next proposition is now valid with exponent $\alpha = \delta/(2-\delta)$ (see Rem. 2). We thus conclude that $u \in C^{1,\delta/(2-\delta)}(\Sigma, \mathbf{R}^N)$.

Proposition 1. *Let $L > 0$ and $\tau \in \,]0,1[$ be given. Then there exist positive constants $c_1(L)$, $H(L,\tau)$ and $\epsilon(L,\tau)$ such that if*

$$B_r(x_0) \subset\subset \Omega,\ |u_{x_0,r}| \leq L,\ |Du_{x_0,r}| \leq L \text{ and } U(x_0,r) \leq \epsilon$$

then

$$U(x_0, \tau r) \leq c_1 \tau^2 U(x_0, r) + H r^{2\alpha}.$$

Proof. We will determine the constant c_1 later on. If the proposition were not true then there would exist a sequence of balls $B_{r_m}(x_m) \subset\subset \Omega$ such that, setting

$$u_m = u_{x_m,r_m},\ P_m = Du_{x_m,r_m},\ \lambda_m^2 = U(x_m, r_m), \tag{3}$$

we have

$$|u_m| \leq L,\ |P_m| \leq L,\ \lambda_m \searrow 0, \tag{4}$$

but

$$U(x_m, \tau r_m) > c_1 \tau^2 \lambda_m^2 + m r_m^{2\alpha}. \tag{5}$$

Since (5) implies $\lambda_m > 0$, we can define the rescaled functions

$$w_m(z) = \frac{u(x_m + r_m z) - u_m - r_m P_m \cdot z}{r_m \lambda_m}$$

for $z \in B$. We notice that

$$Dw_m(z) = \frac{Du(x_m + r_m z) - P_m}{\lambda_m}, \tag{6}$$

$$(w_m)_{0,1} = 0,\ (Dw_m)_{0,1} = 0. \tag{7}$$

Then (3) and (5) become

$$\fint_B |Dw_m|^2 \,\mathrm{d}z + \lambda_m^{q-2} \fint_B |Dw_m|^q \,\mathrm{d}z = 1, \tag{8}$$

$$c_1\tau^2 + m\lambda_m^{-2} r_m^{2\alpha} < \fint_{B_\tau} |Dw_m - (Dw_m)_{0,\tau}|^2 \,\mathrm{d}z + \lambda_m^{q-2} \fint_{B_\tau} |Dw_m - (Dw_m)_{0,\tau}|^q \,\mathrm{d}z. \tag{9}$$

From (8), (7) and the Poincaré inequality we immediately have

$$\|w_m\|_{W^{1,2}(B)} \le c,\ \lambda_m^{(q-2)/q} \|w_m\|_{W^{1,q}(B)} \le c. \tag{10}$$

We infer from (9), (8) and (4) that

$$\lambda_m^{-1} r_m^\alpha \searrow 0 \text{ and } r_m \searrow 0. \tag{11}$$

It follows from (10) and (4) that, on passing to a subsequence and relabelling, we have

$$\begin{array}{ll} Dw_m \rightharpoonup Dw & \text{weakly in } L^2(B, \mathbf{R}^{N\times n}),\\ w_m \to w & \text{in } L^2(B, \mathbf{R}^N),\\ \lambda_m Dw_m \to 0 & \text{in } L^2(B, \mathbf{R}^{N\times n});\\[1ex] \lambda_m^{(q-2)/q} Dw_m \rightharpoonup 0 & \text{weakly in } L^q(B, \mathbf{R}^{N\times n}),\\ \lambda_m^{(q-2)/q} w_m \to 0 & \text{in } L^q(B, \mathbf{R}^N) \text{ (for } q>2);\\[1ex] (x_m, u_m, P_m) \to (x_0, u_0, P_0) & \text{in } \overline{\Omega} \times \mathbf{R}^N \times \mathbf{R}^{N\times n}. \end{array} \tag{12}$$

Now suppose that we can show that $w \in W^{1,2}(B, \mathbf{R}^N)$ is a weak solution of the following linear system with constant coefficients:

$$\operatorname{div}(F_{PP}(x_0, u_0, P_0) \cdot Dw) = 0. \tag{13}$$

We infer from Hypothesis 1 and (4) that

$$|F_{PP}(x_0, u_0, P_0)| \le c,$$

and from Hypothesis 3 that (13) is uniformly elliptic (see [14], Prop. 5.2):

$$F_{PP}(x_0, u_0, P_0) \cdot (\eta \otimes \xi, \eta \otimes \xi) \ge \gamma |\eta \otimes \xi|^2 \text{ for all } \eta \in \mathbf{R}^N,\ \xi \in \mathbf{R}^n.$$

Hence, from the relevant regularity theory (see [6], Th. III.2.1, Rems. III.2.2, III.2.3) we conclude that w is smooth and

$$\fint_{B_\tau} |Dw - Dw_{0,\tau}|^2 \,\mathrm{d}z \le c_2 \tau^2 \fint_B |Dw - Dw_{0,1}|^2 \,\mathrm{d}z, \tag{14}$$

where by (12), (7) and (8)

$$Dw_{0,1} = 0 \text{ and } \fint_B |Dw|^2 \,\mathrm{d}z \le \liminf_{m\to\infty} \fint_B |Dw_m|^2 \,\mathrm{d}z \le 1. \tag{15}$$

On the other hand, if we also know that

$$Dw_m \to Dw \quad \text{in } L^2_{\mathrm{loc}}(B, \mathbf{R}^{N\times n}), \tag{16}$$

$$\lambda_m^{(q-2)/q} Dw_m \to 0 \quad \text{in } L^q_{\mathrm{loc}}(B, \mathbf{R}^{N\times n}) \text{ (for } q>2) \tag{17}$$

then it would follow from (9) and (10) that

$$c_1\tau^2 \leq \fint_{B_\tau} |Dw - Dw_{0,\tau}|^2 \,\mathrm{d}z.$$

If we now choose $c_1 = 2c_2$, we obtain a contradiction to (14) and (15). This proves the proposition. □

The remainder of this work is devoted to showing (13), and (16), (17), which are the assertions of Lemmas 4 and 5 respectively.

We introduce some further notation. We set

$$\begin{aligned} F_m(z,w,R) = \lambda_m^{-2}(&F(x_m + r_m z,\ u_m + r_m P_m \cdot z + r_m\lambda_m w,\ P_m + \lambda_m R) \\ &-F(x_m + r_m z,\ u_m + r_m P_m \cdot z + r_m\lambda_m w,\ P_m) \\ &-F_P(x_m + r_m z,\ u_m + r_m P_m \cdot z + r_m\lambda_m w,\ P_m)\cdot \lambda_m R) \end{aligned}$$

for $(z,w,R) \in \overline{B} \times \mathbf{R}^N \times \mathbf{R}^{N\times n}$. By Hypothesis 1, Remark 1(d) and (4), we note the estimate

$$|F_m(z,w,R)| \leq c(\lambda_m^{-2} + \lambda_m^{q-2}|R|^q), \tag{18}$$

and we also define the corresponding variational integrals

$$\mathcal{I}_m(w,U) = \int_U F_m(z,w,Dw)\,\mathrm{d}z$$

for $w \in W^{1,q}(B,\mathbf{R}^N)$ and measurable subsets $U \subset B$. We define the set

$$\mathfrak{Y} = \overline{\Omega} \times \mathbf{R}^N \times \mathbf{R}^{N\times n} \times \mathbf{R}^{N\times n}.$$

Next, we define a symmetric bilinear form $G(Y)$ on $\mathbf{R}^{N\times n}$, for $Y = (x,u,P,Q) \in \mathfrak{Y}$, by

$$G(Y) = \int_0^1 (1-s)F_{PP}(x,u,P+sQ)\,\mathrm{d}s.$$

By Hypothesis 1, the bilinear form $G(Y)$ depends continuously on $Y \in \mathfrak{Y}$. We observe that

$$F_m(z,w,R) = G(x_m + r_m z,\ u_m + r_m P_m\cdot z + r_m\lambda_m w,\ P_m,\ \lambda_m R)\cdot(R,R). \tag{19}$$

We end this section by showing that w_m is an approximate minimizer of the rescaled variational integral $\mathcal{I}_m$.

Lemma 1. *Suppose that F satisfies Hypotheses 1 to 4. For $\varphi \in W_0^{1,q}(B,\mathbf{R}^N)$ and $E \subset B$, we then have*

$$\begin{aligned} \mathcal{I}_m(w_m, B\setminus E) \quad &\leq \quad \mathcal{I}_m(w_m+\varphi, B\setminus E) \\ &\quad +c\int_E (\lambda_m^{-2} + \lambda_m^{q-2}|Dw_m|^q + \lambda_m^{q-2}|D\varphi|^q)\,\mathrm{d}z + o(1)(1+\|\varphi\|^2_{W^{1,2}}). \end{aligned} \tag{20}$$

Proof. On rescaling, we find from the minimality of u that

$$\begin{aligned} &\int_B F(x_m + r_m z,\ u_m + r_m P_m\cdot z + r_m\lambda_m w_m,\ P_m + \lambda_m Dw_m)\,\mathrm{d}z \\ &\qquad \leq \int_B F(x_m + r_m z,\ u_m + r_m P_m\cdot z + r_m\lambda_m (w_m+\varphi),\ P_m + \lambda_m (Dw_m + D\varphi))\,\mathrm{d}z \end{aligned}$$

for every $\varphi \in C_c^\infty(B, \mathbf{R}^N)$. So it follows from (19) that

$$\begin{aligned}
\mathcal{I}_m(w_m, B\setminus E) \le \mathcal{I}_m(w_m+\varphi, B\setminus E)&\\
&+\lambda_m^{-2}\int_B (F(x_m+r_m z,\ u_m+r_mP_m\cdot z+r_m\lambda_m(w_m+\varphi),\ P_m)\\
&\qquad -F(x_m+r_m z,\ u_m+r_mP_m\cdot z,\ P_m))\,\mathrm{d}z\\
&-\lambda_m^{-2}\int_B (F(x_m+r_m z,\ u_m+r_mP_m\cdot z+r_m\lambda_m w_m,\ P_m)\\
&\qquad -F(x_m+r_m z,\ u_m+r_mP_m\cdot z,\ P_m))\,\mathrm{d}z\\
&+\lambda_m^{-1}\int_B (F_P(x_m+r_m z,\ u_m+r_mP_m\cdot z+r_m\lambda_m(w_m+\varphi),\ P_m)\\
&\qquad -F_P(x_m,u_m,P_m))\cdot Dw_m\,\mathrm{d}z\\
&-\lambda_m^{-1}\int_B (F_P(x_m+r_m z,\ u_m+r_mP_m\cdot z+r_m\lambda_m w_m,\ P_m)\\
&\qquad -F_P(x_m,u_m,P_m))\cdot Dw_m\,\mathrm{d}z\\
&+\lambda_m^{-1}\int_B (F_P(x_m+r_m z,\ u_m+r_mP_m\cdot z+r_m\lambda_m(w_m+\varphi),\ P_m)\\
&\qquad -F_P(x_m,u_m,P_m))\cdot D\varphi\,\mathrm{d}z\\
&-\mathcal{I}_m(w_m,E)+\mathcal{I}_m(w_m+\varphi,E)\\
&=\mathcal{I}_m(w_m+\varphi, B\setminus E)+(\mathrm{I})+(\mathrm{II})+(\mathrm{III})+(\mathrm{IV})+(\mathrm{V})+(\mathrm{VI}).
\end{aligned}$$

By virtue of Hypothesis 2, we estimate term (I) as follows using (4), (10), (11), the Hölder inequality and the fact that $\alpha \le \delta/(2-\delta)$

$$\begin{aligned}
(\mathrm{I}) &\le \lambda_m^{-2}(1+|P_m|^q)K(|u_m|+r_m|P_m|)\int_B (r_m\lambda_m|w_m+\varphi|)^\delta\,\mathrm{d}z\\
&\le c(\lambda_m^{-1}r_m^{\delta/(2-\delta)})^{2-\delta}(1+\|w_m+\varphi\|_{L^2}^\delta)\le o(1)(1+\|\varphi\|_{L^2}^\delta).
\end{aligned}$$

For term (V), we have

$$\begin{aligned}
(\mathrm{V}) &\le \lambda_m^{-1}(1+|P_m|^{q-1})K(|u_m|)\int_B (r_m+r_m|P_m|+r_m\lambda_m|w_m+\varphi|)^{\delta/(2-\delta)}|D\varphi|\,\mathrm{d}z\\
&\le c\lambda_m^{-1}r_m^{\delta/(2-\delta)}(1+\|w_m+\varphi\|_{L^2}^{\delta/(2-\delta)})\,\|\varphi\|_{W^{1,2}}\le o(1)(1+\|\varphi\|_{W^{1,2}}^2).
\end{aligned}$$

We estimate (II), (III) and (IV) in a similar manner. Finally, by Hypothesis 1 and Young's inequality, we infer that

$$(\mathrm{VI}) \le c\int_E (\lambda_m^{-2}+\lambda_m^{q-2}|Dw_m|^q+\lambda_m^{q-2}|D\varphi|^q)\,\mathrm{d}z. \qquad \square$$

3. Caccioppoli and reverse Hölder inequalities

We recall a simple algebraic lemma (see [6], Lem. V.3.1; [14], Lem. 6.1) and the higher integrability theorem of Gehring, Giaquinta and Modica (see [6], Prop. V.1.1; [14], Th. 6.6).

Lemma 2. *Let $f(t)$ be a bounded nonnegative function defined for $R/2 \le t \le R$. Suppose that*

$$f(t) \le \theta f(s) + A(s-t)^{-2} + B(s-t)^{-q} + C$$

for $R/2 \le t < s \le R$, where θ, A, B, C are nonnegative constants with $\theta < 1$. Then

$$f\left(\frac{R}{2}\right) \le c(\theta, q)(AR^{-2} + BR^{-q} + C).$$

Theorem 2. *Let Ω be a bounded open subset of $\mathbf{R}^n$, and let $f \in L^1_{\rm loc}(\Omega)$ and $g \in L^t_{\rm loc}(\Omega)$ be nonnegative functions with $0 < s < 1 < t < \infty$. Suppose that*

$$⨍_{B_{R/2}(x_0)} f \,\mathrm{d}x \le b \left\{ ⨍_{B_R(x_0)} f^s \,\mathrm{d}x \right\}^{1/s} + ⨍_{B_R(x_0)} g \,\mathrm{d}x \tag{21}$$

for every ball $B_R(x_0) \subset\subset \Omega$ with $R \le R_0$. Then $f \in L^{1+\epsilon}_{\rm loc}(\Omega)$ for any $0 \le \epsilon < \epsilon_0$, and

$$\left\{ ⨍_{B_{\mu R}(x_0)} f^{1+\epsilon} \,\mathrm{d}x \right\}^{1/(1+\epsilon)} \le c ⨍_{B_R(x_0)} f \,\mathrm{d}x + c \left\{ ⨍_{B_R(x_0)} g^{1+\epsilon} \,\mathrm{d}x \right\}^{1/(1+\epsilon)}$$

for every ball $B_R(x_0) \subset\subset \Omega$ with $R \le R_0$, and $0 < \mu < 1$, where $\epsilon_0 = \epsilon_0(n, s, t, b)$ and $c = c(n, s, t, b, \mu, \epsilon)$ are positive constants.

For $P_0 \in \mathbf{R}^{N\times n}$ with $|P_0| \le L$, and $(x, u, P) \in \overline{\Omega} \times \mathbf{R}^N \times \mathbf{R}^{N\times n}$, we set

$$\bar{F}(x, u, P) = F(x, u, P_0 + P) - F(x, u, P_0) - F_P(x, u, P_0) \cdot P.$$

Clearly, $\bar{F}$ is strictly quasiconvex, and $\bar{F}(x, u, 0) = 0$.

The first part of the next lemma is contained in [8], Lemma 2.1 and [14], Lemma 9.1.

Lemma 3. *There exists a constant c such that the estimates*

$$|\bar{F}(x, u, P)| \le c(|P|^2 + |P|^q), \tag{22}$$

$$|\bar{F}_P(x, u, P)| \le c(|P| + |P|^{q-1}), \tag{23}$$

$$|\bar{F}(x, u, P + Q) - \bar{F}(x, u, P)| \le c(|P| + |P|^{q-1} + |Q| + |Q|^{q-1})|Q| \tag{24}$$

hold for all $(x, u, P) \in \overline{\Omega} \times \mathbf{R}^N \times \mathbf{R}^{N\times n}$ with $|u| \le L$, and $Q \in \mathbf{R}^{N\times n}$.

Moreover, the estimate

$$|\bar{F}(x, u, P) - \bar{F}(y, v, P)| \le c\chi(|u|, |x - y| + |u - v|)(1 + |P|^{q-1})|P| \tag{25}$$

holds for all $(x, u, P), (y, v, P) \in \overline{\Omega} \times \mathbf{R}^N \times \mathbf{R}^{N\times n}$.

Proof. We let $K = \sup_{\mathfrak{Z}_L} |F_{PP}|$ for the compact set

$$\mathfrak{Z}_L = \{(x, u, P) \in \overline{\Omega} \times \mathbf{R}^N \times \mathbf{R}^{N\times n} : |u|, |P| \le L + 1\}.$$

For $|P| \le 1$, we then have

$$|\bar{F}(x, u, P)| = \left| \int_0^1 (1 - s) F_{PP}(x, u, P_0 + sP) \cdot (P, P) \,\mathrm{d}s \right| \le K|P|^2,$$

while for $|P| \ge 1$,

$$|\bar{F}(x, u, P)| \le c(1 + |P|^q) \le c|P|^q.$$

These two estimates furnish (22). By quasiconvexity of $\bar{F}$, (22) implies (23) (*cf.* Rem. 1(d)). Further, (24) is an immediate consequence of (23). To see (25), we write

$$\bar{F}(x,u,P) = \int_0^1 (F_P(x,u,P_0+sP) - F_P(x,u,P_0)) \cdot P \, \mathrm{d}s,$$

and similarly for $\bar{F}(y,v,P)$, and we apply Hypothesis 2 to their difference. □

We first prove a reverse Hölder inequality for Du (*cf.* [8], Prop. 2.2 and (2.18); [14], Th. 6.7 and Prop. 9.1; [9], Th. 4.1).

Theorem 3. *Let $u \in W^{1,q}(\Omega, \mathbf{R}^N)$ be a minimizer of the variational integral $\mathcal{I}$, whose integrand F satisfies Hypotheses 1 and 4.*

Then $Du \in L^{q(1+\epsilon')}_{loc}(\Omega, \mathbf{R}^{N\times n})$ for any $0 \le \epsilon' < \epsilon_1$, and

$$\left\{ \fint_{B_{\mu R}(x_0)} (1+|Du|^q)^{1+\epsilon'} \, \mathrm{d}x \right\}^{1/(1+\epsilon')} \le c \fint_{B_R(x_0)} (1+|Du|^q) \, \mathrm{d}x$$

for every ball $B_R(x_0) \subset\subset \Omega$ with $R \le R_0$, and $0<\mu<1$, where ϵ_1, R_0 and $c(\mu,\epsilon')$ are positive constants.

Proof. We fix some ball $B_R(x_0) \subset\subset \Omega$ with $R \le R_0$, and we set $u_0 = u_{x_0,R}$. For $R/2 \le t < s \le R$, we let $\zeta \in C_c^\infty(B_s(x_0))$ be a cut-off function with $0 \le \zeta \le 1$, $\zeta = 1$ on $B_t(x_0)$ and $|D\zeta| \le c(s-t)^{-1}$. We set $\varphi = \zeta(u-u_0)$. By Hypothesis 4 and the minimality of u, we then have

$$\begin{aligned}
\tilde{\gamma} \int_{B_s} |D\varphi|^q \, \mathrm{d}x &\le \int_{B_s} (\tilde{F}(x_0, D\varphi) - \tilde{F}(x_0,0)) \, \mathrm{d}x \\
&\le \int_{B_s} \tilde{F}(x, D\varphi) \, \mathrm{d}x + \tilde{\omega}(R_0) \int_{B_s} |D\varphi|^q \, \mathrm{d}x + cR^n \\
&\le \int_{B_s} F(x,u,D\varphi) \, \mathrm{d}x + \tilde{\omega}(R_0) \int_{B_s} |D\varphi|^q \, \mathrm{d}x + cR^n \\
&\le \int_{B_s} F(x, u-\varphi, Du - D\varphi) \, \mathrm{d}x + \int_{B_s} (F(x,u,D\varphi) - F(x,u,Du)) \, \mathrm{d}x \\
&\quad + \tilde{\omega}(R_0) \int_{B_s} |D\varphi|^q \, \mathrm{d}x + cR^n.
\end{aligned}$$

Thus, by Hypothesis 1 and the fact that $D\varphi = Du$ on $B_t(x_0)$, we obtain for sufficiently small R_0 that

$$\int_{B_t} |Du|^q \, \mathrm{d}x \le c_1 \int_{B_s \setminus B_t} |Du|^q \, \mathrm{d}x + c(s-t)^{-q} \int_{B_R} |u-u_0|^q \, \mathrm{d}x + cR^n.$$

We now "fill the hole", that is, we add c_1 times the left-hand side to both sides and we divide the resulting inequality by $1+c_1$. This yields

$$\int_{B_t} |Du|^q \, \mathrm{d}x \le \theta \int_{B_s} |Du|^q \, \mathrm{d}x + c(s-t)^{-q} \int_{B_R} |u-u_0|^q \, \mathrm{d}x + cR^n,$$

where $\theta = c_1/(1+c_1) < 1$. By an application of Lemma 2, we arrive at *Caccioppoli's first inequality*

$$\int_{B_{R/2}} |Du|^q \, \mathrm{d}x \le cR^{-q} \int_{B_R} |u-u_0|^q \, \mathrm{d}x + cR^n.$$

By the Poincaré-Sobolev inequality, we deduce estimate (21) with $f = 1 + |Du|^q$, $g = 0$ and $s = q_*/q = n/(n+q) < 1$. The result now follows by Theorem 2. □

We are ready for a reverse Hölder inequality for $Du - P_0$ plus an error term (*cf.* [8], Ths. 2.2 and 2.5). It provides a uniform bound in $L^{2(1+\epsilon)}_{\text{loc}}$ for the gradients of the blow-up functions w_m (see Cor. 1).

Theorem 4. *Let $u \in W^{1,q}(\Omega, \mathbf{R}^N)$ be a minimizer of the variational integral $\mathcal{I}$, whose integrand F satisfies Hypotheses* 1 *to* 4.

Then there exist positive constants ϵ, $\alpha \le \delta/(2-\delta)$, R_0 and $c(\mu)$ such that

$$\left\{ ⨍_{B_{\mu R}(x_0)} (|Du - P_0|^2 + |Du - P_0|^q)^{1+\epsilon}\,\mathrm{d}x \right\}^{1/(1+\epsilon)} \le c ⨍_{B_R(x_0)} (|Du - P_0|^2 + |Du - P_0|^q)\,\mathrm{d}x + cR^{2\alpha} \left\{ ⨍_{B_R(x_0)} (1 + |Du|^q)\,\mathrm{d}x \right\}^{1+2\alpha/q}$$

holds for every ball $B_R(x_0) \subset\subset \Omega$, $0 < \mu < 1$ and $P_0 \in \mathbf{R}^{N\times n}$, with $R \le R_0$, $|u_{x_0,R}| \le L$ and $|P_0| \le L$.

Proof. We fix $B_R(x_0) \subset\subset \Omega$ and $P_0 \in \mathbf{R}^{N\times n}$, subject to the conditions $R \le R_0$, $|u_0| \le L$ and $|P_0| \le L$, where $u_0 = u_{x_0,R}$.

We next fix some ball $B_r(y_0) \subset\subset B_R(x_0)$. For $r/2 \le t < s \le r$, we let $\zeta \in C_c^\infty(B_s(y_0))$ be a cut-off function with $0 \le \zeta \le 1$, $\zeta = 1$ on $B_t(y_0)$ and $|D\zeta| \le c(s-t)^{-1}$. We set

$$P(x) = u_{y_0,r} + P_0 \cdot (x - y_0),\ \varphi = \zeta(u - P),\ \psi = (1-\zeta)(u - P),$$

for which

$$\varphi + \psi = u - P,\ D\varphi + D\psi = Du - P_0.$$

The strict quasiconvexity of $\bar{F}$ at $P = 0$, (24), (25) and Young's inequality assert that

$$\begin{aligned}\gamma \int_{B_s} (|D\varphi|^2 + |D\varphi|^q)\,\mathrm{d}x &\le \int_{B_s} \bar{F}(x_0, u_0, D\varphi)\,\mathrm{d}x \le \int_{B_s} \bar{F}(x_0, u_0, Du - P_0)\,\mathrm{d}x \\ &+ c \int_{B_s \setminus B_t} (|D\varphi| + |D\varphi|^{q-1} + |D\psi| + |D\psi|^{q-1})|D\psi|\,\mathrm{d}x \le \int_{B_s} \bar{F}(x, u, Du - P_0)\,\mathrm{d}x \\ + c \int_{B_s} \chi(|u_0|, |x - x_0| + |u - u_0|)&(1 + |Du|^{q-1})|D\varphi + D\psi|\,\mathrm{d}x + c \int_{B_s \setminus B_t} (|D\varphi|^2 + |D\varphi|^q + |D\psi|^2 + |D\psi|^q)\,\mathrm{d}x.\end{aligned} \tag{26}$$

On the other hand, by the minimality of u

$$\int_{B_s} F(x, u, Du)\,\mathrm{d}x \le \int_{B_s} F(x, u - \varphi, Du - D\varphi)\,\mathrm{d}x,$$

and thus

$$\begin{aligned}\int_{B_s}\bar F(x,u,Du-P_0)\,\mathrm{d}x \quad\le\quad & \int_{B_s}\bar F(x,u-\varphi,D\psi)\,\mathrm{d}x\\ &+\int_{B_s}(F(x,u-\varphi,P_0)-F(x,u,P_0))\,\mathrm{d}x\\ &+\int_{B_s}(F_P(x,u-\varphi,P_0)-F_P(x,u,P_0))\cdot D\psi\,\mathrm{d}x\\ &+\int_{B_s}(F_P(x_0,u_0,P_0)-F_P(x,u,P_0))\cdot D\varphi\,\mathrm{d}x.\end{aligned}$$

By (25), (22), Hypothesis 2, the Cauchy inequality and the estimate $\chi^2\le c\omega$, we deduce

$$\begin{aligned}\int_{B_s}\bar F(x,u,Du-P_0)\,\mathrm{d}x &\le \int_{B_s}\bar F(x,u_0,D\psi)\,\mathrm{d}x + c\int_{B_s}\chi(|u_0|,|u-u_0|+|\varphi|)(1+|D\psi|^{q-1})|D\psi|\,\mathrm{d}x\\ &\quad + c\int_{B_s}\omega(|u_0|,|u-u_0|+|\varphi|)\,\mathrm{d}x + c\int_{B_s}\chi(|u_0|,|x-x_0|+|u-u_0|)|D\varphi|\,\mathrm{d}x\\ &\le c\int_{B_s}(|u-u_0|^\delta+|\varphi|^\delta+R^{2\delta/(2-\delta)})\,\mathrm{d}x + c\int_{B_s\setminus B_t}(|D\psi|^2+|D\psi|^q)\,\mathrm{d}x+\frac{\gamma}{4}\int_{B_s}|D\varphi|^2\,\mathrm{d}x. \qquad (27)\end{aligned}$$

Using the Young and the Poincaré inequality for φ on $B_s(y_0)$ we estimate the term

$$c\int_{B_s}|\varphi|^\delta\,\mathrm{d}x \le \int_{B_s}(\epsilon s^{-2}|\varphi|^2 + c(\epsilon)s^{2\delta/(2-\delta)})\,\mathrm{d}x \le \frac{\gamma}{4}\int_{B_s}|D\varphi|^2\,\mathrm{d}x + cs^{n+2\delta/(2-\delta)}. \qquad (28)$$

By combining (26), (27) and (28), we conclude that

$$\int_{B_t} f\,\mathrm{d}x \le c_1\int_{B_s\setminus B_t} f\,\mathrm{d}x + c(s-t)^{-2}\int_{B_r}|u-P|^2\,\mathrm{d}x + c(s-t)^{-q}\int_{B_r}|u-P|^q\,\mathrm{d}x + c\int_{B_r} g\,\mathrm{d}x$$

for the functions

$$\begin{aligned} f &= |Du-P_0|^2+|Du-P_0|^q,\\ g &= \chi^{q/(q-1)}(|u_0|,|x-x_0|+|u-u_0|)(1+|Du|^q)+|u-u_0|^\delta+R^{2\delta/(2-\delta)}.\end{aligned}$$

We note that the definitions of f and g do not involve y_0 or r. “Filling the hole” and applying Lemma 2, with $\theta=c_1/(1+c_1)<1$, results in *Caccioppoli's second inequality*

$$\int_{B_{r/2}} f\,\mathrm{d}x \le cr^{-2}\int_{B_r}|u-P|^2\,\mathrm{d}x + cr^{-q}\int_{B_r}|u-P|^q\,\mathrm{d}x + c\int_{B_r} g\,\mathrm{d}x.$$

By means of the Poincaré-Sobolev and Hölder inequalities, we deduce, for $s=2_*/2=n/(n+2)<1$, that

$$⨍_{B_{r/2}(y_0)} f\,\mathrm{d}x \le c\left\{⨍_{B_r(y_0)} f^s\,\mathrm{d}x\right\}^{1/s} + c⨍_{B_r(y_0)} g\,\mathrm{d}x$$

for all $B_r(y_0)\subset\subset B_R(x_0)$. Invoking Theorem 2 we finally arrive at

$$\left\{⨍_{B_{\mu R}(x_0)} f^{1+\epsilon}\,\mathrm{d}x\right\}^{1/(1+\epsilon)} \le c⨍_{B_{\nu R}(x_0)} f\,\mathrm{d}x + c\left\{⨍_{B_{\nu R}(x_0)} g^{1+\epsilon}\,\mathrm{d}x\right\}^{1/(1+\epsilon)} \qquad (29)$$

for some exponent $0 < \epsilon < \epsilon'$ and $0 < \mu < \nu < 1$. Here the constant c also depends on μ/ν and ϵ, and ϵ' is the exponent from Theorem 3.

We next set

$$2\alpha = \min\left(\frac{q\delta}{(q-1)(2-\delta)}, \delta, \frac{q^*(\epsilon'-\epsilon)}{(1+\epsilon')(1+\epsilon)}\right)$$

for the Sobolev exponent $q^* = nq/(n-q)$ if $q < n$, and $q^* \in]0, \infty[$ if $q \geq n$. By the estimate $\chi^{q/(q-1)}(L,t) \leq ct^{2\alpha}$, we have

$$g \leq c(R + |u-u_0|)^{2\alpha}(1+|Du|^q) + |u-u_0|^\delta.$$

Then, using the Hölder and Poincaré-Sobolev inequalities and Theorem 3 we control the last term of (29) by

$$\begin{aligned}\left\{\fint_{B_{\nu R}} g^{1+\epsilon}\,\mathrm{d}x\right\}^{1/(1+\epsilon)} &\leq c\left\{\fint_{B_{\nu R}} (R+|u-u_0|)^{q^*}\,\mathrm{d}x\right\}^{2\alpha/q^*}\left\{\fint_{B_{\nu R}} (1+|Du|^q)^{1+\epsilon'}\,\mathrm{d}x\right\}^{1/(1+\epsilon')} \\ &\quad + c\left\{\fint_{B_R} |u-u_0|^{\delta(1+\epsilon')}\,\mathrm{d}x\right\}^{1/(1+\epsilon')} \leq c\left(R^{2\alpha} + \left\{\fint_{B_R} |u-u_0|^{q^*}\,\mathrm{d}x\right\}^{2\alpha/q^*}\right)\fint_{B_R}(1+|Du|^q)\,\mathrm{d}x \\ &\quad + cR^\delta\left\{\fint_{B_R}|Du|^q\,\mathrm{d}x\right\}^{\delta/q} \leq cR^{2\alpha}\left\{\fint_{B_R}(1+|Du|^q)\,\mathrm{d}x\right\}^{1+2\alpha/q}. \quad \square\end{aligned}$$

Remark 2. In the special case that $u \in C^{0,1}(\overline{\Omega}, \mathbf{R}^N)$, Theorem 4 is valid with exponent $\alpha = \delta/(2-\delta)$ and a constant c also depending on $\|u\|_{C^{0,1}(\overline{\Omega},\mathbf{R}^N)}$. In order to see this, we estimate the second last term of (26) by

$$\begin{aligned}&c\int_{B_s} \chi(|u_0|, |x-x_0| + |u-u_0|)(1+|Du|^{q-1})|D\varphi + D\psi|\,\mathrm{d}x \\ &\qquad \leq c\int_{B_s} R^{\delta/(2-\delta)}|D\varphi + D\psi|\,\mathrm{d}x \leq c\int_{B_s}(R^{2\delta/(2-\delta)} + |D\psi|^2)\,\mathrm{d}x + \frac{\gamma}{4}\int_{B_s}|D\varphi|^2\,\mathrm{d}x.\end{aligned}$$

Moreover, for estimate (27) we substitute

$$\begin{aligned}\int_{B_s}\bar{F}(x,u,Du-P_0)\,\mathrm{d}x &\leq \int_{B_s}\bar{F}(x,u,D\psi)\,\mathrm{d}x + c\int_{B_s}\chi(|u|,|\varphi|)(1+|D\psi|^{q-1})|D\psi|\,\mathrm{d}x \\ &\quad + c\int_{B_s}\omega(|u|,|\varphi|)\,\mathrm{d}x + c\int_{B_s}\chi(|u_0|,|x-x_0|+|u-u_0|)|D\varphi|\,\mathrm{d}x \\ &\leq c\int_{B_s}(|\varphi|^\delta + R^{2\delta/(2-\delta)})\,\mathrm{d}x \\ &\quad + c\int_{B_s\setminus B_t}(|D\psi|^2 + |D\psi|^q)\,\mathrm{d}x + \frac{\gamma}{4}\int_{B_s}|D\varphi|^2\,\mathrm{d}x.\end{aligned}$$

Setting $g = R^{2\delta/(2-\delta)}$ the proof is then already complete with (29).

Corollary 1. *In terms of $w_m \in W^{1,q}(B, \mathbf{R}^N)$, we have, for $0 < r < 1$,*

$$\left\{\fint_{B_r}(|Dw_m|^2 + \lambda_m^{q-2}|Dw_m|^q)^{1+\epsilon}\,\mathrm{d}z\right\}^{1/(1+\epsilon)} \leq c(r).$$

Proof. Substituting (6), $x_0 = x_m$, $R = r_m$, $\mu = r$ and $P_0 = P_m$ in Theorem 4, and using (4) and (10) yields

$$\left\{ \fint_{B_r} (|Dw_m|^2 + \lambda_m^{q-2}|Dw_m|^q)^{1+\epsilon}\,\mathrm{d}z \right\}^{1/(1+\epsilon)}$$
$$\le c(r) \fint_B (|Dw_m|^2 + \lambda_m^{q-2}|Dw_m|^q)\,\mathrm{d}z$$
$$+ c(r)\lambda_m^{-2} r_m^{2\alpha} \left\{ \fint_B (1 + |P_m + \lambda_m Dw_m|^q)\,\mathrm{d}z \right\}^{1+2\alpha/q} \le c(r). \quad \square$$

4. Convergence of the blow-up functions

For $0 < r < 1$ and $\beta > 0$, we define the sets

$$E_{r,m} = \{z \in B_r : \lambda_m(|w_m| + |Dw_m|) \ge \beta\}.$$

By (10) and (4), we infer that

$$|E_{r,m}| \le \beta^{-2}\lambda_m^2 \int_{E_{r,m}} (|w_m| + |Dw_m|)^2\,\mathrm{d}z \le c\beta^{-2}\lambda_m^2 = o(1). \tag{30}$$

Moreover, by the Hölder inequality, Corollary 1 and (30), we deduce

$$\int_{E_{r,m}} \lambda_m^{q-2}|Dw_m|^q\,\mathrm{d}z \le \left\{ \int_{B_r} (\lambda_m^{q-2}|Dw_m|^q)^{1+\epsilon}\,\mathrm{d}z \right\}^{1/(1+\epsilon)} |E_{r,m}|^{\epsilon/(1+\epsilon)} = o(1).$$

In summary,

$$\int_{E_{r,m}} (\lambda_m^{-2} + \lambda_m^{q-2}|Dw_m|^q)\,\mathrm{d}z \le c\beta^{-2} + o(1). \tag{31}$$

By choosing for φ a test function in Lemma 1 and sending $m \to \infty$, we derive

Lemma 4. *The function $w \in W^{1,2}(B, \mathbf{R}^N)$ is a weak solution of the linear elliptic system with constant coefficients*

$$\operatorname{div}(F_{PP}(x_0, u_0, P_0) \cdot Dw) = 0.$$

In particular, we conclude that w is smooth.

Proof. We fix $\varphi \in C_c^\infty(B_r, \mathbf{R}^N)$ with $0 < r < 1$, and $\beta > 0$, and we define the functions

$$\begin{aligned} Y_m &= (x_m + r_m z,\ u_m + r_m P_m \cdot z + r_m\lambda_m w_m,\ P_m,\ \lambda_m Dw_m), \\ \bar{Y}_m &= Y_m + (0,\ r_m\lambda_m\varphi,\ 0,\ \lambda_m D\varphi). \end{aligned}$$

By virtue of (4), (11) and (12), we notice the limits

$$Y_m, \bar{Y}_m \to Y_0 = (x_0, u_0, P_0, 0) \text{ in } L^2(B, \mathfrak{Y}). \tag{32}$$

We define the compact set

$$\mathfrak{Y}_\beta = \{(x, u, P, Q) \in \mathfrak{Y} : |u|, |P|, |Q| \le 2L + \beta + \|\varphi\|_{W^{1,\infty}(B_r)}\}$$

and note that $Y_m(z), \bar{Y}_m(z) \in \mathfrak{Y}_\beta$ for *a.e.* $z \in B_r \setminus E_{r,m}$. Therefore

$$\sup_{B_r \setminus E_{r,m}} \{|G(Y_m)|, |G(\bar{Y}_m)|\} \le \sup_{\mathfrak{Y}_\beta} |G| = c(\beta). \tag{33}$$

We claim by Lebesgue's dominated convergence that

$$(1 - \chi_{E_{r,m}}) G(Y_m) \;\to\; G(Y_0) \text{ in } L^p(B_r), \tag{34}$$

$$(1 - \chi_{E_{r,m}}) G(\bar{Y}_m) \;\to\; G(Y_0) \text{ in } L^p(B_r), \tag{35}$$

for $1 \le p < \infty$, where $\chi_{E_{r,m}}$ is the characteristic function of the set $E_{r,m}$. Indeed, the left-hand sides of (34) and (35) are bounded and converge pointwise *a.e.* on B_r, by (30), which asserts that $\chi_{E_{r,m}} \to 0$ in $L^1(B_r)$, and by (32), (33) and the continuity of G.

We then infer by Hölder's inequality and Corollary 1 that

$$\begin{aligned} &\left| \int_{B_r \setminus E_{r,m}} G(Y_m) \cdot (Dw_m, Dw_m)\, \mathrm{d}z - \int_{B_r} G(Y_0) \cdot (Dw_m, Dw_m)\, \mathrm{d}z \right| \\ &\qquad \le \left\{ \int_{B_r} |(1 - \chi_{E_{r,m}}) G(Y_m) - G(Y_0)|^{(1+\epsilon)/\epsilon}\, \mathrm{d}z \right\}^{\epsilon/(1+\epsilon)} \left\{ \int_{B_r} |Dw_m|^{2(1+\epsilon)}\, \mathrm{d}z \right\}^{1/(1+\epsilon)} \\ &\qquad = o(1). \end{aligned} \tag{36}$$

Similarly, we obtain

$$\int_{B_r \setminus E_{r,m}} G(\bar{Y}_m) \cdot (Dw_m + D\varphi, Dw_m + D\varphi)\, \mathrm{d}z = \int_{B_r} G(Y_0) \cdot (Dw_m + D\varphi, Dw_m + D\varphi)\, \mathrm{d}z + o(1). \tag{37}$$

For $\varphi \in C_c^\infty(B_r, \mathbf{R}^N)$ and $E = E_{r,m}$, estimate (20) in combination with (4), (31), (19) and the definition of Y_m and $\bar{Y}_m$ is easily seen to give

$$\int_{B_r \setminus E_{r,m}} G(Y_m) \cdot (Dw_m, Dw_m)\, \mathrm{d}z \le \int_{B_r \setminus E_{r,m}} G(\bar{Y}_m) \cdot (Dw_m + D\varphi, Dw_m + D\varphi)\, \mathrm{d}z + c\beta^{-2} + o(1).$$

As a consequence of (36) and (37), we deduce

$$\int_{B_r} G(Y_0) \cdot (Dw_m, Dw_m)\, \mathrm{d}z \le \int_{B_r} G(Y_0) \cdot (Dw_m + D\varphi, Dw_m + D\varphi)\, \mathrm{d}z + c\beta^{-2} + o(1),$$

which we write as

$$0 \le 2 \int_{B_r} G(Y_0) \cdot (Dw_m, D\varphi)\, \mathrm{d}z + \int_{B_r} G(Y_0) \cdot (D\varphi, D\varphi)\, \mathrm{d}z + c\beta^{-2} + o(1).$$

We conclude by (12), scaling of φ and since $\beta > 0$ was arbitrary that

$$0 \le 2 \int_{B_r} G(Y_0) \cdot (Dw, D\varphi)\, \mathrm{d}z,$$

and the result follows by replacing φ by $-\varphi$, and noting that $F_{PP}(x_0, u_0, P_0) = 2G(Y_0)$. □

Lemma 5. *We have the limits*

$$Dw_m \to Dw \quad \text{in } L^2_{\text{loc}}(B, \mathbf{R}^{N\times n}), \tag{38}$$

$$\lambda_m^{(q-2)/q} Dw_m \to 0 \quad \text{in } L^q_{\text{loc}}(B, \mathbf{R}^{N\times n}) \text{ (for } q>2\text{)}. \tag{39}$$

In the proof we shall make use of the fact that, by Lemma 4, the function w and its gradient Dw are locally bounded on B.

We fix $0<s<r<1$ and $\beta>0$, we let $\zeta \in C_c^\infty(B_r)$ be a cut-off function with $0 \le \zeta \le 1$ and $\zeta = 1$ on B_s, and we also define the function $\eta = \zeta(2-\zeta^2)^{1/2}$. We note, since $2-\zeta^2 \ge 1$, that η has the same properties as ζ, *i.e.* it is a cut-off function $\eta \in C_c^\infty(B_r)$ with $0 \le \eta \le 1$ and $\eta = 1$ on B_s. Moreover, $|D\eta| \le 2^{1/2}|D\zeta|$. Next, we set $\varphi = \zeta^2(w-w_m)$ and we define the functions

$$\begin{aligned} Y_m &= (x_m + r_m z,\ u_m + r_m P_m \cdot z + r_m \lambda_m w_m,\ P_m,\ \lambda_m Dw_m), \\ \bar{Y}_m &= Y_m + (0,\ r_m\lambda_m \varphi,\ 0,\ \lambda_m D\varphi), \\ \tilde{Y}_m &= (x_m,\ u_m,\ P_m,\ \lambda_m \eta(Dw_m - Dw) + \lambda_m (w_m - w)\otimes D\eta). \end{aligned}$$

By virtue of (4), (11) and (12), we notice the limits

$$Y_m, \bar{Y}_m, \tilde{Y}_m \to Y_0 = (x_0, u_0, P_0, 0) \text{ in } L^2(B, \mathfrak{Y}).$$

We define the compact set

$$\mathfrak{Y}_\beta = \{(x,u,P,Q) \in \mathfrak{Y} : |u|, |P|, |Q| \le 2L + 2(1+\beta+\|w,\zeta\|_{W^{1,\infty}(B_r)})^2\}$$

and note that $Y_m(z), \bar{Y}_m(z), \tilde{Y}_m(z) \in \mathfrak{Y}_\beta$ for *a.e.* $z \in B_r \setminus E_{r,m}$. Therefore

$$\sup_{B_r \setminus E_{r,m}} \{|G(Y_m)|, |G(\bar{Y}_m)|, |G(\tilde{Y}_m)|\} \le \sup_{\mathfrak{Y}_\beta} |G| = c(\beta). \tag{40}$$

By the same argument as that for (34) and (35), we show that

$$\begin{aligned} (1-\chi_{E_{r,m}})G(Y_m) &\to G(Y_0) \text{ in } L^p(B_r), \\ (1-\chi_{E_{r,m}})G(\bar{Y}_m) &\to G(Y_0) \text{ in } L^p(B_r), \\ (1-\chi_{E_{r,m}})G(\tilde{Y}_m) &\to G(Y_0) \text{ in } L^p(B_r), \end{aligned}$$

for $1 \le p < \infty$. Similarly to (36) and (37), we then obtain

$$\int_{B_r \setminus E_{r,m}} G(Y_m)\cdot(Dw_m, Dw_m)\,\mathrm{d}z = \int_{B_r} G(Y_0)\cdot(Dw_m, Dw_m)\,\mathrm{d}z + o(1), \tag{41}$$

$$\int_{B_r \setminus E_{r,m}} G(\bar{Y}_m)\cdot(Dw_m + D\varphi, Dw_m + D\varphi)\,\mathrm{d}z = \int_{B_r} G(Y_0)\cdot(Dw_m + D\varphi, Dw_m + D\varphi)\,\mathrm{d}z + o(1), \tag{42}$$

$$\int_{B_r \setminus E_{r,m}} \eta^2 G(\tilde{Y}_m)\cdot(Dw_m - Dw, Dw_m - Dw)\,\mathrm{d}z = \int_{B_r} \eta^2 G(Y_0)\cdot(Dw_m - Dw, Dw_m - Dw)\,\mathrm{d}z + o(1). \tag{43}$$

We now insert $\varphi = \zeta^2(w-w_m) \in W_0^{1,q}(B_r, \mathbf{R}^N)$ and $E = E_{r,m}$ in (20), for which

$$Dw_m + D\varphi = \zeta^2 Dw + (1-\zeta^2)Dw_m + 2\zeta(w-w_m)\otimes D\zeta.$$

By (4), (10), (12), (31), (40), (19) and the definition of Y_m and $\bar{Y}_m$, this easily yields

$$\int_{B_r\setminus E_{r,m}} G(Y_m)\cdot(Dw_m, Dw_m)\,\mathrm{d}z \le \int_{B_r\setminus E_{r,m}} G(\bar{Y}_m)\cdot(Dw_m + D\varphi, Dw_m + D\varphi)\,\mathrm{d}z + c\beta^{-2} + o(1).$$

As a consequence of (41), (42), (40) and (12), we deduce

$$\begin{aligned}\int_{B_r} G(Y_0)\cdot(Dw_m, Dw_m)\,\mathrm{d}z &\le \int_{B_r} G(Y_0)\cdot(Dw_m + D\varphi, Dw_m + D\varphi)\,\mathrm{d}z + c\beta^{-2} + o(1)\\ &= \int_{B_r} G(Y_0)\cdot(\zeta^2 Dw + (1-\zeta^2)Dw_m, \zeta^2 Dw + (1-\zeta^2)Dw_m)\,\mathrm{d}z + c\beta^{-2} + o(1),\end{aligned}$$

which by (12) immediately implies the key estimate

$$\limsup_{m\to\infty}\int_{B_r}\eta^2 G(Y_0)\cdot(Dw_m, Dw_m)\,\mathrm{d}z \le \int_{B_r}\eta^2 G(Y_0)\cdot(Dw, Dw)\,\mathrm{d}z + c\beta^{-2} \tag{44}$$

with the function $\eta = \zeta(2-\zeta^2)^{1/2}$.

According to Hypotheses 1 and 3, Young's inequality, (18), (19) and the definition of $\tilde{Y}_m$, we have

$$\begin{aligned}\gamma\int_{B_r}(|D\varphi|^2 + \lambda_m^{q-2}|D\varphi|^q)\,\mathrm{d}z &\le \int_{B_r} F_m(0,0,D\varphi)\,\mathrm{d}z\\ &\le \int_{B_r\setminus E_{r,m}} G(\tilde{Y}_m)\cdot(D\varphi, D\varphi)\,\mathrm{d}z + c\int_{E_{r,m}}(\lambda_m^{-2} + \lambda_m^{q-2}|D\varphi|^q)\,\mathrm{d}z,\end{aligned}$$

where now $\varphi = \eta(w_m - w)$. By (31), (4), (12), (43) and (40), this gives

$$\gamma\int_{B_s}(|Dw_m - Dw|^2 + \lambda_m^{q-2}|Dw_m - Dw|^q)\,\mathrm{d}z \le \int_{B_r}\eta^2 G(Y_0)\cdot(Dw_m - Dw, Dw_m - Dw)\,\mathrm{d}z + c\beta^{-2} + o(1).$$

Thus we infer, using (44) and (12), that

$$\gamma\limsup_{m\to\infty}\int_{B_s}(|Dw_m - Dw|^2 + \lambda_m^{q-2}|Dw_m - Dw|^q)\,\mathrm{d}z \le (1-2+1)\int_{B_r}\eta^2 G(Y_0)\cdot(Dw, Dw)\,\mathrm{d}z + c\beta^{-2}.$$

Bearing in mind that $\beta > 0$ was arbitrary we conclude that

$$\lim_{m\to\infty}\int_{B_s}|Dw_m - Dw|^2\,\mathrm{d}z = 0,\ \lim_{m\to\infty}\lambda_m^{q-2}\int_{B_s}|Dw_m - Dw|^q\,\mathrm{d}z = 0.$$

The last equation implies

$$\lim_{m\to\infty}\lambda_m^{q-2}\int_{B_s}|Dw_m|^q\,\mathrm{d}z = 0,$$

and we have shown that (38) and (39) hold. □

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